An Introduction to 3D Stochastic Fluid Dynamics

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1 Introduction

We consider a viscous, constant density, Newtonian fluid described by the stochastic Navier–Stokes equations on the torus $\mathcal{T} = [0, L]^3$, $L > 0$,

$$\frac{\partial u}{\partial t} + (u \cdot \nabla) u + \nabla p = \nu \triangle u + \sum_{i=1}^{\infty} \sigma_i h_i (x) \dot{\beta}_i (t) \tag{1}$$

with $\mathsf{div} u = 0$ and periodic boundary conditions, with suitable fields $h_i (x)$ and independent Brownian motions $\beta_i (t)$. The notation $(u \cdot \nabla) u$ stands for the vector field with components $[(u \cdot \nabla) u]_j = \sum_{k=1}^{d} u_k \partial_k u_j$; the operation $\triangle u$ has to be understood componentwise. The fluid is described by the velocity field $u = u (t, x)$ (a random vector field) and the pressure field $p = p (t, x)$ (a random scalar field). The fluid in a torus is an artificial model, but the topics we are going to investigate are so poorly understood that it is meaningful to idealize the mathematics as much as possible, preserving only those aspects that we believe to be essential. The random white noise force $\sum_{i=1}^{\infty} \sigma_i h_i (x) \dot{\beta}_i (t)$ is also part of the idealization, not only for its specific form but more basically because body forces are usually either absent or gradient-like (as the gravitational force field) and not so roughly varying. The complex phenomena (related to turbulence) we want to investigate are usually caused by complicate boundary effects (think to the fluid below a grid), too difficult to be dealt with at present. Thus we hope that a white noise force may both simplify the investigation and produce some phenomena similar to those of more realistic fluid systems.

The parameter $\nu > 0$ is called the kinematic viscosity. We shall investigate sometimes the limit as $\nu \to 0$, without any rescaling with ν of the random force: this is a singular limit problem, hopefully similar to the more realistic boundary layer ones. The limit as $\nu \to 0$ with constant-amplitude force essentially corresponds to the limit of infinite Reynolds number.

The white noise force is assumed, for sake of simplicity, to be the superposition of independent perturbations of various modes: we shall assume that the $h_i(x)$'s are eigenfunctions of the Stokes operator and the $\beta_i(t)$'s are independent scalar Brownian motions. The most interesting physical situation is the case when only a few large modes are activated, those with smaller wave length. In such a case L has the meaning of length scale of the action of the external force. A force with a typical length scale is a model for a fluid which interacts with a boundary or an object. However, in some cases one is able to deal only with the case of several or infinitely many modes, for mathematical reasons.

Section 3 is devoted to a finite dimensional model that captures several features of equations (1). It covers the Galerkin approximations of (1), so its analysis represents a main step in view of the infinite dimensional system; even the results of Section 5 are only based on the finite dimensional facts of Section 3.

In Section 4 we take the limit as the dimension goes to infinity and treat the 3D stochastic Navier–Stokes system. We define solutions to the martingale problem, prove their existence, and (partially) describe how to extract Markov selections. Finally, as a short introduction to the theory of Da Prato and Debussche [23], we prove that every Markov process composed of martingale solutions has a Strong Feller like property, under the assumptions on the noise imposed in [23]. This is a property of continuous dependence on initial condition that represents a striking step forward with respect to the deterministic theory. Relevant references on the topics of this section are, among others, [8], [12], [15], [16], [18], [19], [24], [25], [26], [33], [35], [36], [40], [41], [42], [45], [57], [58], [59], [61], [63], [64], [66], [67].

Section 5 deals with turbulence, restricting the attention to the so called K41 theory. A definition of K41 scaling law is given and investigated, disproved in 2D, shown to be equivalent to hopefully more manageable properties in 3D, that could be better analyzed in the future to understand whether they are true or, more likely, how they should be modified. These notes are restricted to the 3D case, where we aim to describe a few first steps in the direction of relevant open problems. The theory in the 2D case is richer of well posedness results, even for cylindrical noise, ergodicity, control, non-viscous case and limit, see among others references [2], [3], [4], [6], [8], [9], [11], [13], [14], [17], [21], [22], [23], [28], [30], [31], [32], [47], [49], [50], [51], [55], [56], [62].

2 Abstract Framework and General Preliminaries

We describe here a minimal amount of preliminaries on spaces and operators appearing in fluid dynamics; see for instance [53] and [65] for extensive discussions. Let $\mathbb{L}^2(\mathcal{T})$ be the space of vector fields $u : \mathcal{T} \to \mathbb{R}^3$ with $L^2(\mathcal{T})$-components. For every $\alpha > 0$, let $\mathbb{H}^\alpha(\mathcal{T})$ be the space of fields $u \in \mathbb{L}^2(\mathcal{T})$ with components in the Sobolev space $H^\alpha(\mathcal{T}) = W^{\alpha,2}(\mathcal{T})$.

Let $\mathcal{D}^\infty$ be the space of infinitely differentiable divergence free periodic fields u on $\mathcal{T}$, with zero mean:

$$\int_{\mathcal{T}} u(x)\, dx = 0.$$

This zero mean condition plays somewhat the role of a boundary condition. Let H be the closure of $\mathcal{D}^\infty$ in the $\mathbb{L}^2(\mathcal{T})$-topology; it is the space of all fields $u \in \mathbb{L}^2(\mathcal{T})$ such that $\mathsf{div} u = 0$, $u \cdot n$ on the boundary is periodic (one can show that for divergence free fields the trace $u \cdot n$ on the boundary is a well defined $H^{-1/2}$-distribution), $\int_{\mathcal{T}} u(x)\, dx = 0$. We endow H with the inner product

$$\langle u, v\rangle_H = \frac{1}{L^3}\int_{\mathcal{T}} u(x) \cdot v(x)\, dx$$

and the associated norm $|.|_H$.

Let V (resp. $D(A)$) be the closure of $\mathcal{D}^\infty$ in the $\mathbb{H}^1(\mathcal{T})$-topology (resp. $\mathbb{H}^2(\mathcal{T})$-topology); it is the space of divergence free, zero mean, periodic elements of $\mathbb{H}^1(\mathcal{T})$ (resp. of $\mathbb{H}^2(\mathcal{T})$). The spaces V and $D(A)$ are dense and compactly embedded in H (Rellich theorem). Due to the zero mean condition we also have

$$\int_{\mathcal{T}} |Du(x)|^2\, dx \geq \lambda \int_{\mathcal{T}} |u(x)|^2\, dx$$

for every $u \in V$, for some positive constant λ (Poincarè inequality). So we may endow V with the norm

$$|u|_V^2 := \int_{\mathcal{T}} |Du(x)|^2\, dx$$

where $|Du(x)|^2 = \sum_{i,j=1}^{3}\left(\frac{\partial u_i(x)}{\partial x_j}\right)^2$.

Let $A : D(A) \subset H \to H$ be the operator $Au = -\triangle u$ (componentwise). Notice that $\triangle u \in H$ because we are in the periodic case (otherwise we would need a projection on divergence free fields). Since A is a selfadjoint positive (unbounded) operator in H, there is a complete orthonormal system $\{h_i\}_{i\in\mathbb{N}} \subset H$ made of eigenfunctions of A, with eigenvalues $0 < \lambda_1 \leq \lambda_2 \leq \ldots$ ($Ah_i = \lambda_i h_i$). Notice that the positivity is due to the zero mean condition. We may take the Poincaré constant λ above equal to λ_1. Notice that we have

$$\langle Au, u\rangle_H = |u|_V^2$$

for every $u \in D(A)$, so in particular

$$\langle Au, u\rangle_H \geq \lambda |u|_H^2 .$$

On the torus we know explicitly eigenfunctions and eigenvalues (see example 3.1 below), and often it is useful to parametrize them by vector wave numbers instead of the index $i \in \mathbb{N}$.

Let V' be the dual of V; with proper identifications we have $V \subset H \subset V'$ with continuous dense injections, and the scalar product $\langle\cdot,\cdot\rangle_H$ extends to the dual pairing $\langle\cdot,\cdot\rangle_{V,V'}$ between V and V'. Denote the norms in H and V by $|\cdot|_H$ and $\|\cdot\|_V$ respectively.

Let $B(\cdot,\cdot) : V \times V \to V'$ be the bilinear operator defined as

$$\langle w, B(u,v)\rangle_{V,V'} = \sum_{i,j=1}^{3} \int_{\mathcal{T}} u_i \frac{\partial v_j}{\partial x_i} w_j dx$$

for every $u, v, w \in V$. To clarify the definition, first take $u, v, w \in \mathcal{D}^\infty$ and notice that by Hölder inequality

$$\left| \sum_{i,j=1}^{3} \int_{\mathcal{T}} u_i \frac{\partial v_j}{\partial x_i} w_j dx \right| \leq \sum_{i,j=1}^{3} |u_i|_{L^{\alpha_1}(\mathcal{T})} |v_j|_{W^{1,\alpha_2}(\mathcal{T})} |w_j|_{L^{\alpha_3}(\mathcal{T})}$$

where $\alpha_i > 1$, $\sum_{i=1}^{3} \frac{1}{\alpha_i} = 1$. Sobolev embedding theorem says

$$|u|_{L^q(\mathcal{T})} \leq C(s,p,\mathcal{T}) |u|_{W^{s,p}(\mathcal{T})}, \quad \frac{1}{q} = \frac{1}{p} - \frac{s}{d}$$

Hence, for $p = 2$,

$$\left| \sum_{i,j=1}^{3} \int_{\mathcal{T}} u_i \frac{\partial v_j}{\partial x_i} w_j dx \right| \leq C \sum_{i,j=1}^{3} |u_i|_{W^{s_1,2}(\mathcal{T})} |v_j|_{W^{1+s_2,2}(\mathcal{T})} |w_j|_{W^{s_3,2}(\mathcal{T})}$$

($C = C(s_1, s_2, s_3, \mathcal{T})$) for $s_i \geq 0$ such that $\frac{3}{2} - \frac{1}{d} \sum_{i=1}^{3} s_i = 1$, namely $\sum_{i=1}^{3} s_i = \frac{d}{2}$. In particular

$$\left| \sum_{i,j=1}^{3} \int_{\mathcal{T}} u_i \frac{\partial v_j}{\partial x_i} w_j dx \right| \leq C \sum_{i,j=1}^{3} |u_i|_{W^{s_1,2}(\mathcal{T})} |v_j|_{W^{1,2}(\mathcal{T})} |w_j|_{W^{s_3,2}(\mathcal{T})}$$

with $s_1 + s_3 = \frac{d}{2}$. For $d = 3$ (but also $d = 2$ and 4) we may take $s_1 = s_3 = \frac{d}{4}$ and confirm that $B(\cdot,\cdot)$ can be extended to a bilinear mapping $B(\cdot,\cdot) : V \times V \to V'$.

We every $u, v \in \mathcal{D}^\infty$ we have

$$\langle B(u,v), v\rangle_H = \frac{1}{2} L^{-3} \int_{\mathcal{T}} (u(x) \cdot \nabla) |v(x)|^2 dx = 0$$

since $\mathsf{div} u = 0$, and this property extends from $\mathcal{D}^\infty$ to the various spaces of vector fields used in the sequel.

To get further estimates, it is useful to recall that the function $\alpha \mapsto \log |u|_{W^{\alpha,2}(\mathcal{T})}$ is convex:

$$|u|_{W^{\alpha s_1 + (1-\alpha) s_2, 2}(\mathcal{T})} \leq |u|^{\alpha}_{W^{s_1,2}(\mathcal{T})} |u|^{(1-\alpha)}_{W^{s_2,2}(\mathcal{T})}$$

for $\alpha \in [0,1]$ (easy by Fourier analysis).

Among the infinitely many consequences of the previous computations we have

$$\left|\langle B(u,v),w\rangle\right| \leq C(\mathcal{T}) \sum_{i,j=1}^{d} |u_i|_{W^{\frac{d}{4},2}(\mathcal{T})} |v_j|_{W^{1,2}(\mathcal{T})} |w_j|_{W^{\frac{d}{4},2}(\mathcal{T})}$$

and

$$|u|_{W^{\frac{d}{4},2}(\mathcal{T})} \leq |u|_{L^2(\mathcal{T})}^{1-\frac{d}{4}} |u|_{W^{1,2}(\mathcal{T})}^{\frac{d}{4}}$$

thus

$$\left|\langle B(u,v),w\rangle\right| \leq C |u|_H^{1-\frac{d}{4}} \|u\|_V^{\frac{d}{4}} \|v\|_V |w|_H^{1-\frac{d}{4}} \|w\|_V^{\frac{d}{4}}.$$

We have proved:

Lemma 2.1. *In $d=3$,*

$$\left|\langle B(u,v),w\rangle\right| \leq C |u|_H^{1/4} \|u\|_V^{3/4} \|v\|_V |w|_H^{1/4} \|w\|_V^{3/4}.$$

We also need the following inequalities.

Lemma 2.2. *In $d=3$,*

$$\left|\langle Ax, B(x,x)\rangle_H\right| \leq C \|x\|_V^{3/2} |Ax|_H^{3/2}$$

for every $x \in D(A)$.

Proof. Due to the periodicity of vector fields that allows us to drop the boundary terms in the integrations by parts, for every $u, v \in \mathcal{D}^\infty$ we have

$$\begin{aligned}
\langle Au, B(v,u)\rangle_H &= \sum_{i,j=1}^{3} \int_{\mathcal{T}} v_i \partial_i u_j \triangle u_j = -\sum_{i,j,k=1}^{3} \int_{\mathcal{T}} \partial_k (v_i \partial_i u_j) \partial_k u_j \\
&= -\sum_{i,j,k=1}^{3} \int_{\mathcal{T}} \partial_k v_i \partial_i u_j \partial_k u_j - \frac{1}{2} \sum_{i,j,k=1}^{3} \int_{\mathcal{T}} v_i \partial_i (\partial_k u_j)^2 \\
&= -\sum_{i,j,k=1}^{3} \int_{\mathcal{T}} \partial_k v_i \partial_i u_j \partial_k u_j
\end{aligned}$$

since $\mathsf{div} v = 0$ and thus

$$\begin{aligned}
\langle Au, B(u,u)\rangle_{H_L} &\leq C \sum_{i,k=1}^{3} \left(\int_{\mathcal{T}} |\partial_k u_i|^3 \right) \leq C |Du|_{L^3(\mathcal{T})}^3 \\
&\leq C |Du|_{W^{1/2,2}(\mathcal{T})}^3 \leq C |Du|_{L^2(\mathcal{T})}^{3/2} |Du|_{W^{1,2}(\mathcal{T})}^{3/2}.
\end{aligned}$$

Lemma 2.3.

$$|AB(u,v)|_H \leq C\left(|Au| \|Av\|_V + |Av| \|Au\|_V\right)$$

$$\left|A^{1/2} B(u,v)\right|_H \leq C |Au| |Av|$$

$$|B(u,v)|_H \leq C \left(|Au| \left|A^{1/2} v\right| \wedge \left|A^{1/2} u\right| |Av|\right)$$

and for every $\gamma \in (0, 1/2)$

$$\left| A^{\gamma} B(u,v) \right|_H \leq C \left(|Au|^2 \left| A^{\gamma+\frac{1}{2}} v \right|^2 \wedge |Av|^2 \left| A^{\gamma+\frac{1}{2}} u \right|^2 \right).$$

Proof. Up to multiplicative constants that we omit,

$$\begin{aligned} |AB(u,v)|_H &\leq \sqrt{\sum_{i,j=1}^{3} \int_{\mathcal{T}} \left[\triangle (u_i \partial_i v_j) \right]^2 dx} \\ &\leq |Dv|_{\infty} |Au| + |Du|_{\infty} |Av| + |u|_{\infty} \|Av\|_V \\ &\leq |Au| \|Av\|_V + |Av| \|Au\|_V \end{aligned}$$

$$\begin{aligned} \left| A^{1/2} B(u,v) \right|_H &\leq \sqrt{\sum_{i,j=1}^{3} \int_{\mathcal{T}} \left[D (u_i \partial_i v_j) \right]^2 dx} \\ &\leq |Du|_4 |Dv|_4 + |u|_{\infty} |Av| \end{aligned}$$

$$|B(u,v)|_H^2 \leq \sum_{i,j=1}^{3} \int_{\mathcal{T}} \left[u_i \partial_i v_j \right]^2 dx \leq |u|_4^2 |Dv|_4^2 \leq \left| A^{1/2} u \right| |Av|$$

$$|B(u,v)|_H^2 \leq |u|_{\infty}^2 |Dv|_2^2 \leq |Au| \left| A^{1/2} v \right|$$

and the last one follows by interpolation.

With the previous notations in mind, we (formally) rewrite equations (1) of Section 1 as an abstract stochastic evolution equation in H

$$du(t) + \left[\nu A u(t) + B(u(t), u(t)) \right] dt = \sum_{i=1}^{\infty} \sigma_i h_i d\beta_i(t). \tag{1}$$

The rigorous definition of solution will be given in Section 4 and is not entirely trivial; here we only anticipate that, as in the deterministic case, we have to interpret expressions in integral and weak form over test functions

$$\begin{aligned} &\langle u(t), \varphi \rangle_H + \int_0^t \nu \langle u(s), A\varphi \rangle_H ds - \int_0^t \langle B(u(s), \varphi), u(s) \rangle_H ds \\ &= \langle u_0, \varphi \rangle_H + \sum_{i=1}^{\infty} \sigma_i \langle h_i, \varphi \rangle_H \beta_i(t) \end{aligned} \tag{2}$$

with $\varphi \in \mathcal{D}^{\infty}$. As a last general remark, we shall always assume at least

$$\sum_{i=1}^{\infty} \sigma_i^2 < \infty$$

(H-valued Brownian motion), but some of the most interesting results will require

$$\sum_{i=1}^{\infty} \lambda_i \sigma_i^2 < \infty$$

to have certain regularities in $D(A)$.

3 Finite Dimensional Models

The reason for this Section is twofold: first we may illustrate a number of basic facts and open problems in a simple setting where the rigor is easy to control; second, most of these results are a preliminary technical step for the analysis of Sections 4 and 5.

3.1 Introduction and Examples

Consider a real *finite* dimensional Hilbert space H,

$$\dim H < \infty$$

endowed with norm $|.|_H$ and inner product $\langle ., . \rangle_H$. We stress that H is *not* the space introduced above but it is finite dimensional; we should write H_n to avoid misunderstandings, and similarly we should write A_n, B_n etc., but in the whole Section we never take the limit as $n \to \infty$ (except in very few well advertized places) so we drop the subscript n for sake of simplicity.

About the various constants involved in the following estimates, we say that a constant is *not* universal if it depends on $\dim H$, ν, the norm in H of A, or constants related to the continuity properties of B. When a constant is independent of these quantities, we call it *universal* and denote it generically by $C > 0$. The non-universal constants are not stable in the limit of the stochastic Navier–Stokes equations (Section 4) or in the limit as $\nu \to 0$ (Section 5).

Let A be a positive definite symmetric linear mapping in H,

$$\langle Ax, x \rangle_H \geq \lambda |x|_H^2$$

for every $x \in H$, where $\lambda > 0$ is a universal constant (Poincaré constant in our applications); $B(.,.) : H \times H \to H$ a bilinear mapping such that

$$\langle B(x,x), x \rangle_H = 0 \tag{1}$$

for every $x \in H$, Q a semi-definite symmetric matrix in H, $(W_t)_{t \geq 0}$ a Brownian motion in H defined on a filtered probability space $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P\right)$.

Remark 3.1. Sometimes we need a stronger version of (1):

$$\langle B(y,x), x\rangle_H = 0 \tag{2}$$

for every $x, y \in H$. This is equivalent to

$$\langle B(y,x), z\rangle_H = -\langle B(y,z), x\rangle_H \tag{3}$$

for every $x, y, z \in H$: if (2) holds, then

$$\begin{aligned} 0 &= \langle B(y, x+z), x+z\rangle_H \\ &= \langle B(y,z), x\rangle_H + \langle B(y,x), z\rangle_H \end{aligned}$$

so (3) is true. If (3) holds then, taking $z = x$, we get (2).

Since A is positive definite, $(u,v) \mapsto \langle Au, v\rangle_H$ is another inner product in H and

$$\|u\|_V := \sqrt{\langle Au, u\rangle_H}$$

is a norm in H and we have

$$\lambda |x|_H^2 \le \|u\|_V^2 \le C_A |x|_H^2$$

for a non universal constant C_A (the norm of A in H); the lower bound is universal.

Consider the stochastic differential equation (SDE) in H, with $\nu > 0$,

$$dX_t = [-\nu A X_t - B(X_t, X_t)]\, dt + \sqrt{Q} dW_t, \quad t \ge 0 \tag{4}$$

with initial condition given by an $\mathcal{F}_0$-measurable random variable $X_0 : \Omega \to H$. As usual, we interpret the equation in the integral sense

$$X_t = X_0 + \int_0^t [-\nu A X_s - B(X_s, X_s)]\, ds + \sqrt{Q} W_t. \tag{5}$$

Example 3.1. Our main example is the Galerkin approximation of equation (1) of Section 1. Given $L > 0$, on the torus $\mathcal{T} = [0, L]^3$, consider the complexification of the infinite dimensional spaces and operators introduced in the previous section, that we denote just in this example by $\mathcal{H}$, $\mathcal{V}$, $\mathcal{D}(\mathcal{A})$, $\mathcal{A}$, $\mathcal{B}$ to avoid superposition with the notations of the present chapter. Define the index sets

$$\Lambda^{(n)} = \left\{ (k, \alpha) \in \left(\frac{2\pi}{L}\mathbb{Z}^3\right) \times \{1,2\} : 0 < |k|^2 \le \left(\frac{2\pi}{L} n\right)^2 \right\}$$

and $\Lambda^{(\infty)} = \cup_n \Lambda^{(n)}$. The eigenvectors of $\mathcal{A}$ are given by

$$h_{k,\alpha}(x) := a_{k,\alpha} e^{ik\cdot x}, \quad x \in \mathcal{T}, \quad (k, \alpha) \in \Lambda^{(\infty)}$$

(with eigenvalues $\lambda_{k,\alpha} = |k|^2$), where, for every $k \in \mathbb{R}^3 \diagdown 0$, we have to choose an orthonormal basis $a_{k,\alpha}$, $\alpha = 1, 2$, of the orthogonal space to k (the space generated by the vectors $\left(e_\alpha - \frac{k_\alpha k}{|k|^2}\right)$, $\alpha = 1, 2, 3$). Let

$$\mathcal{H}^{(n)} = span\left\{h_{k,\alpha}; (k,\alpha) \in \Lambda^{(n)}\right\}$$

and let $\pi^{(n)}$ be the orthogonal projection of $\mathcal{H}$ on $\mathcal{H}^{(n)}$, which commutes with $\mathcal{A}$. With these notations, our main example of finite dimensional system is defined by the objects

$$\mathcal{H}^{(n)}, \mathcal{A}|_{\mathcal{H}^{(n)}}, \pi^{(n)}\mathcal{B}(.,.)|_{\mathcal{H}^{(n)}\times\mathcal{H}^{(n)}}$$

that we simply denote by H, A, $B(.,.)$.

Example 3.2. The famous Lorenz system in $\mathbb{R}^3$, with parameters $a, b, c > 0$,

$$\begin{cases} dx + (ax - ay)\,dt = \sigma_1 d\beta_1 \\ dy + (-bx + y + xz)\,dt = \sigma_2 d\beta_2 \\ dz + (cz - xy)\,dt = \sigma_3 d\beta_3 \end{cases}$$

fits into the framework of this section if $b = a \in (0,1]$. The same is true for the Minea system:

$$\begin{cases} dx + \left(x + \delta\left(y^2 + z^2\right)\right)dt = \sigma_1 d\beta_1 \\ dy + (y - \delta xy)\,dt = \sigma_2 d\beta_2 \\ dz + (z - xz)\,dt = \sigma_3 d\beta_3. \end{cases}$$

Example 3.3. Another interesting example is the GOY model (from Gledzer, Ohkitani, Yamada), a particular case of the so called "shell model". See [44] for an introduction and references. It is a simplified Fourier system where the interaction between different modes is preserved only between neighbor modes, and the complex valued variables $u_n(t) = u_{n,1}(t) + iu_{n,2}(t)$ are summaries of the Fourier coefficients. The finite dimensional model is defined, for $n = -1, 0, 1..., N, N+1, N+2$, by the constraints

$$u_{-1}(t) = u_0(t) = u_{N+1}(t) = u_{N+2}(t) = 0$$

and the equations for $n = 1, ..., N$

$$\begin{aligned} &du_n + \nu k_n^2 u_n dt + ik_n\left(\frac{1}{4}\overline{u}_{n-1}\overline{u}_{n+1} - \overline{u}_{n+1}\overline{u}_{n+2} + \frac{1}{8}\overline{u}_{n-1}\overline{u}_{n-2}\right) \\ &= \sigma_n d\beta_n \end{aligned}$$

where $k_n = 2^n k_0$, $k_0 > 0$ given. Some foundational results on the related infinite dimensional system can be found in [4].

3.2 A Priori Bounds

Throughout this section we assume that $(X_t)_{t\geq 0}$ is a continuous adapted solution of equation (4). In this section it is sufficient to work under condition (1) on B.

Lemma 3.1. *[L^2 bounds and energy equality]Assume $E\left|X_0\right|_H^2 < \infty$. Then, for every $T > 0$, we have*

$$E\left(\sup_{t\in[0,T]} |X_t|_H^2 + \nu \int_0^T \|X_s\|_V^2 \, ds\right) \leq C_1\left(E\left|X_0\right|_H^2, TrQ, T\right) \tag{6}$$

where $C_1\left(E\left|X_0\right|_H^2, TrQ, T\right)$ is given by (12),

$$|X_t|_H^2 + 2\nu \int_0^t \|X_s\|_V^2 \, ds = |X_0|_H^2 + TrQ\, t + M_t \tag{7}$$

where M_t is a square integrable martingale, and

$$\frac{1}{2}E\left|X_T\right|_H^2 + \nu E \int_0^T \|X_s\|_V^2 \, ds = \frac{1}{2}E\left|X_0\right|_H^2 + \frac{1}{2}TrQ\, T. \tag{8}$$

Proof. **Step 1**. The function $f(x) = |x|_H^2$ has derivatives

$$Df(x) = 2x, \quad D^2f(x) = 2 \cdot \mathrm{Id}$$

hence Itô formula and property (1) give us

$$|X_t|_H^2 = |X_0|_H^2 - \int_0^t 2\nu \left\langle AX_s, X_s\right\rangle_H ds + M_t + TrQ\, t \tag{9}$$

where

$$M_t = 2\int_0^t \left\langle X_s, \sqrt{Q}dW_s\right\rangle_H$$

is a local martingale.

Step 2. Let us prove that

$$E\int_0^T |X_t|_H^2 \, dt < \infty \tag{10}$$

for every $T > 0$. The reader not interested in details but only in the main concepts may drop this technical point and go to step 3. To simplify, one may think that (10) has been imposed as an additional assumption in the lemma. However, it is conceptually interesting to notice that the assumptions of the lemma do not include any quantitative bound on the solution, but only on the data (the integrability of the initial condition and the gaussianity of the forcing term), and it is the equation itself, with its particular algebraic structure, that produce bounds on the solution.

We may localize M_t by an increasing sequence of stopping times (τ_n): $\tau_n \to \infty$ a.s. and $t \mapsto M_{t\wedge\tau_n}$ is a square integrable martingale for every n. To be more specific, we may take

$$\tau_n = \inf\left\{t \geq 0 : |X_t|_H^2 = n\right\}.$$

From (9) and the positivity of A we have

$$|X_{t\wedge\tau_n}|_H^2 \leq |X_0|_H^2 + M_{t\wedge\tau_n} + TrQ\ (t\wedge\tau_n) \tag{11}$$

and thus

$$E\left[|X_{t\wedge\tau_n}|_H^2\right] \leq E\,|X_0|_H^2 + TrQ\ t.$$

We also have

$$\int_0^{T\wedge\tau_n} |X_t|_H^2\,dt = \int_0^{T\wedge\tau_n} |X_{t\wedge\tau_n}|_H^2\,dt \leq \int_0^T |X_{t\wedge\tau_n}|_H^2\,dt$$

hence

$$E\int_0^{T\wedge\tau_n} |X_t|_H^2\,dt \leq \int_0^T E\,|X_{t\wedge\tau_n}|_H^2\,dt \leq T\left(E\,|X_0|_H^2 + TrQ\ T\right).$$

By the monotone convergence theorem we get (10).

Step 3. Having proved (10), M_t is now a square integrable martingale. Then we have (7) and then (8), which also implies the second part of the bound (6). To prove the first part of (6) we use the bound

$$|X_t|_H^2 \leq |X_0|_H^2 + |M_t| + TrQ\ t$$

coming from (7) due to the positivity of A. We have

$$\sup_{t\in[0,T]} |X_t|_H^2 \leq |X_0|_H^2 + 1 + \sup_{t\in[0,T]} |M_t|^2 + TrQ\ T$$

hence, by Doob's inequality $E\sup_{t\in[0,T]} |M_t|^2 \leq 4E\,|M_T|^2$, we have

$$\begin{aligned} &E\sup_{t\in[0,T]} |X_t|_H^2 \\ &\leq E\,|X_0|_H^2 + 1 + 16E\int_0^T \langle QX_s, X_s\rangle\,ds + TrQ\ T. \end{aligned}$$

From (8) (with t in place of T) and the positivity of A we already know that

$$\sup_{t\in[0,T]} E\left[|X_t|_H^2\right] \leq E\,|X_0|_H^2 + TrQ\ T.$$

Hence we have (6) with

$$C_1\left(E\left|X_0\right|_H^2, TrQ, T\right) := E\left|X_0\right|_H^2 + 1 + 16TrQ\left(E\left|X_0\right|_H^2 + TrQ\,T\right) + TrQ\,T. \tag{12}$$

The proof is complete.

Remark 3.2. The bound (6) tells us the basic topologies where we have to look for solutions. With others below, it will give us the bounds, on the Galerkin approximations of the stochastic Navier–Stokes equations, needed to extract subsequences that converge in a proper way to pass to the limit in the equations.

Remark 3.3. The two identities (7) and (8) express energy balance laws. Let us comment the second one: we may think that the term $\frac{1}{2}TrQ$ is the mean rate of energy (mean energy per unit of time) injected into the system, $\nu E\int_0^T \|X_s\|_V^2\,ds$ is the mean energy dissipated on $[0,T]$, $\frac{1}{2}E\left|X_T\right|_H^2$ is the mean (kinetic) energy of the system.

We say that a stochastic process $(X_t)_{t\geq 0}$ is *stationary* if given any $0 \leq t_1 < ... < t_n$ and $s \geq 0$, the law of the r.v. $(X_{t_1+s}, ..., X_{t_n+s})$ (r.v. in H^n) is independent of s. In the following corollary in fact we just need that the covariance of X_t is independent of t.

Notice that in principle we should assume a quantitative bound like $E\left|X_0\right|_H^2 < \infty$ to start with, like in Lemma 3.1; but stationarity provides itself a mechanism for proper estimates (the property of stationarity is like an a priori bound itself). Somewhat related results can be found in [10] by a different approach.

A technical remark: in the proof of the following corollary there is a step where we use the fact that equation (4) has a unique solution for every square integrable $\mathcal{F}_0$-measurable initial condition X_0. This fact will be proved in a subsequent section. So, from a logical viewpoint, we should state this corollary only later on. We anticipate here to avoid repetitions.

Corollary 3.1. *If $(X_t)_{t\geq 0}$ is stationary then*

$$E\left\|X_t\right\|_V^2 = \frac{TrQ}{2\nu} \tag{13}$$

for every $t \geq 0$ (in particular $E\left\|X_t\right\|_V^2 < \infty$ for every stationary solution).

Proof. If $E\left|X_0\right|_H^2 < \infty$, from (8) and the stationarity we first have

$$2\nu\int_0^T E\left\|X_s\right\|_V^2\,ds = TrQ\,T$$

but also $E\left\|X_s\right\|_V^2$ is independent of t, whence the result. Therefore, in order to complete the proof, we have only to show that $E\left|X_0\right|_H^2 < \infty$ is true for every stationary solution.

Given $\varepsilon > 0$, let $R_\varepsilon > 0$ be such that $P\left(|X_0|_H^2 > R_\varepsilon\right) < \varepsilon$. Let $\Omega_\varepsilon \in \mathcal{F}$ be defined as $\Omega_\varepsilon = \left\{|X_0|_H^2 \leq R_\varepsilon\right\}$; we have $P(\Omega_\varepsilon) \geq 1 - \varepsilon$. Define $X_0^{(\varepsilon)}$ as X_0 on Ω_ε, 0 otherwise. Let $\left(X_t^{(\varepsilon)}\right)_{t\geq 0}$ be the unique solution of equation (4) with initial condition $X_0^{(\varepsilon)}$ (theorem 3.1 below). Just looking at the integral form of (4) (which has an elementary pathwise meaning) it is easy to realize that $X_\cdot^{(\varepsilon)}(\omega) = X_\cdot(\omega)$ for P-a.e. $\omega \in \Omega_\varepsilon$. For $\left(X_t^{(\varepsilon)}\right)_{t\geq 0}$ we have (8), hence

$$\frac{1}{T}\int_0^T E\left\|X_s^{(\varepsilon)}\right\|_V^2 ds \leq \frac{R_\varepsilon}{2\nu T} + \frac{TrQ}{2\nu}$$

(in a sense, we use here an idea of Chow and Hasminski [20]). Then, given $N > 0$,

$$\begin{aligned}
&E\left(\|X_0\|_V^2 \wedge N\right) \\
&= \frac{1}{T}\int_0^T E\left[\|X_s\|_V^2 \wedge N\right] ds \\
&= \frac{1}{T}\int_0^T E\left[1_{\Omega_\varepsilon}\left(\|X_s\|_V^2 \wedge N\right)\right] ds + \frac{1}{T}\int_0^T E\left[1_{\Omega_\varepsilon^c}\left(\|X_s\|_V^2 \wedge N\right)\right] ds \\
&\leq \frac{1}{T}\int_0^T E\left[1_{\Omega_\varepsilon}\left(\left\|X_s^{(\varepsilon)}\right\|_V^2 \wedge N\right)\right] ds + N\varepsilon \\
&\leq \frac{1}{T}\int_0^T E\left[\left\|X_s^{(\varepsilon)}\right\|_V^2\right] ds + N\varepsilon \\
&\leq \frac{R_\varepsilon}{2\nu T} + \frac{TrQ}{2\nu} + N\varepsilon.
\end{aligned}$$

It is now sufficient to take first the limit as $T \to \infty$, then as $\varepsilon \to 0$, finally as $N \to \infty$. The proof is complete.

Remark 3.4. Quantitative knowledge of the statistics of the stationary regime of a turbulent fluid is one of the most important and open problems of fluid dynamics (due in great part to the fact that a turbulent fluid is a non-equilibrium, although possibly stationary, system, so there are no general paradigm as the Gibbs one to describe its stationary regime; in mathematical terms, equation (4) is not gradient like, hence we do not know the density of its invariant measure explicitly in a simple Gibbs form). The identity (13) is a positive example in this direction. It has a very interesting interpretation in connection with the experimental fact that the rate of energy dissipation of a turbulent fluid has a finite limit when the viscosity goes to zero.

In the proof of existence of solutions we need a stopped version of part of the previous result. This is the only point in this section where we do not assume that X is a solution over the whole half-line $[0, \infty)$, but only on a random time interval.

Lemma 3.2. *Let $\tau \geq 0$ be a stopping time and $(X_t)_{t\geq 0}$ a continuous adapted process that P-a.s. satisfies (5) for $t \in [0, \tau(\omega)]$. Assume $E|X_0|_H^2 < \infty$. Then, for every $T > 0$, we have*

$$E\left(\sup_{t\in[0,T]} |X_{t\wedge\tau}|_H^2\right) \leq C_1\left(E|X_0|_H^2, TrQ, T\right).$$

Proof. We may repeat step by step the previous proof, substituting everywhere the process $X_{t\wedge\tau}$ to X_t and stating every identity or inequality for $t \in [0, \tau(\omega)]$ only. But we profit of this repetition to give an alternative proof. Let

$$\tau_R' = \inf\left\{t \geq 0 : |X_t|_H^2 = R\right\}, \quad \tau_R = \tau_R' \wedge \tau$$

and notice that $\tau_R' \uparrow \tau$ as $R \to \infty$. We have (also for τ in place of τ_R)

$$\begin{aligned}
X_{t\wedge\tau_R} &= X_0 + \int_0^{t\wedge\tau_R} \left[-\nu AX_s - B(X_s, X_s)\right] ds + \sqrt{Q} W_{t\wedge\tau_R} \\
&= X_0 + \int_0^t \left[-\nu AX_{s\wedge\tau_R} - B(X_{s\wedge\tau_R}, X_{s\wedge\tau_R})\right] 1_{s\leq\tau_R} ds \\
&+ \int_0^t 1_{s\leq\tau_R}\sqrt{Q} dW_s.
\end{aligned}$$

Apply Itô formula to $|X_{t\wedge\tau_R}|_H^2$ and get

$$\begin{aligned}
|X_{t\wedge\tau_R}|_H^2 = |X_0|_H^2 - 2\int_0^t \left\langle \nu AX_{s\wedge\tau_R} - B(X_{s\wedge\tau_R}, X_{s\wedge\tau_R}), X_{s\wedge\tau_R}\right\rangle_H 1_{s\leq\tau_R} ds \\
+ \widetilde{M}_t + TrQ\ (t\wedge\tau_R)
\end{aligned}$$

and thus

$$|X_{t\wedge\tau_R}|_H^2 + 2\nu\int_0^t \|X_{s\wedge\tau_R}\|_V^2 1_{s\leq\tau_R} ds = |X_0|_H^2 + \widetilde{M}_t + TrQ\ (t\wedge\tau_R)$$

where

$$\widetilde{M}_t = 2\int_0^t \left\langle X_{s\wedge\tau_R}, 1_{s\leq\tau_R}\sqrt{Q} dB_s\right\rangle_H.$$

Then, by Doob's inequality along with the isometry formula of Itô integrals

$$\begin{aligned}
E\sup_{t\in[0,u]} |X_{t\wedge\tau_R}|_H^2 &\leq E|X_0|_H^2 + 1 + E\sup_{t\in[0,u]}\left|\widetilde{M}_t\right|_H^2 + TrQ\ u \\
&\leq E|X_0|_H^2 + 1 + + TrQ\ u \\
&+ C\cdot TrQ\cdot E\int_0^u 1_{s\leq\tau_R} |X_{s\wedge\tau_R}|_H^2 ds
\end{aligned}$$

Therefore

$$E \sup_{t\in[0,u]} |X_{t\wedge\tau_R}|_H^2 \le C + C\int_0^u E \sup_{t\in[0,s]} |X_{t\wedge\tau_R}|_H^2\, ds$$

which implies the result by Gronwall lemma applied to the function $f(t) = E\sup_{t\in[0,u]} |X_{t\wedge\tau_R}|_H^2$ and the independence of the constants of R. The proof is complete.

The statement of the following lemma is not the strongest possible one (because we claim (15) only for $p^* < p$), see the next remark. However we restrict ourselves to this result since its proof is now elementary, being very similar to the one just given above. Moreover, it will be sufficient for our purposes.

Lemma 3.3 (L^p estimates). *Assume $E\,|X_0|_H^p < \infty$ for some $p > 2$. Then, for every $T > 0$,*

$$E\int_0^T |X_s|_H^p\, ds < C_2\left(p, E\,|X_0|_H^p, TrQ, T\right) \tag{14}$$

and

$$E \sup_{t\in[0,T]} |X_t|_H^{p^*} < C_3\left(p, E\,|X_0|_H^p, TrQ, T\right) \tag{15}$$

where $p^* = p/2 + 1$ (which is also greater than 2) and the constants C_2 and C_3 are given in the proof.

Proof. **Step 1**. We consider now the function

$$f(x) = |x|_H^p = \left(|x|_H^2\right)^{p/2}$$

which has derivatives

$$\begin{aligned} Df(x) &= p\,|x|_H^{p-2}\, x, \\ D^2 f(x) &= p\,(p-2)\,|x|_H^{p-4}\, x\otimes x + p\,|x|_H^{p-2}\cdot \mathrm{Id} \end{aligned}$$

with the property

$$Tr\left[QD^2 f(x)\right] = p\,(p-1)\,|x|_H^{p-2}\, TrQ.$$

Itô formula and property (1) give us now

$$\begin{aligned} &|X_t|_H^p + p\nu\int_0^t |X_s|_H^{p-2}\langle AX_s, X_s\rangle_H\, ds \\ &= |X_0|_H^p + M_t^{(p)} + \frac{p\,(p-1)\,TrQ}{2}\int_0^t |X_s|_H^{p-2}\, ds \end{aligned} \tag{16}$$

where

$$M_t^{(p)} = p \int_0^t |X_s|_H^{p-2} \left\langle X_s, \sqrt{Q} dW_s \right\rangle_H$$

is a local martingale.

Step 2. The proof of (14) is now the same as the proof of (10), plus a simple iterative argument. Let p' be any number between 2 and the value of p declared in the assumptions of the lemma. By the same localization argument used in the previous proof, we get

$$E\left[|X_{t\wedge\tau_n}|_H^{p'}\right] \leq E\left[|X_0|_H^{p'}\right] + \frac{p'(p'-1)\,TrQ}{2} E\int_0^T |X_s|_H^{p'-2}\, ds$$

and thus

$$E\int_0^{T\wedge\tau_n} |X_t|_H^{p'}\, dt \leq E\int_0^T |X_{t\wedge\tau_n}|_H^{p'}\, dt \leq TE\left[|X_0|_H^{p'}\right] + T\frac{p'(p'-1)\,TrQ}{2} E\int_0^T |X_s|_H^{p'-2}\, ds$$

which implies

$$E\int_0^T |X_t|_H^{p'}\, dt \leq TE\left[|X_0|_H^{p'}\right] + T\frac{p'(p'-1)\,TrQ}{2} E\int_0^T |X_s|_H^{p'-2}\, ds \tag{17}$$

by the monotone convergence theorem. This inequality allows us to iterate a bound of the form (14) from a smaller value of p to a larger one, starting from $p = 2$ given in the previous lemma. More formally, let Π be the set of $p' \in [2, p]$ (p given in the claim of the lemma), such that, for some constant $C\left(p, E|X_0|_H^p, TrQ, T\right)$, we have

$$E\int_0^T |X_t|_H^{p'}\, dt \leq C\left(p, E|X_0|_H^p, TrQ, T\right)$$

for all $T \geq 0$. The set Π is non empty ($2 \in \Pi$ by the previous lemma) and has the property that $a \in \Pi$, $a < p$ implies that $b = (a+2) \wedge p \in \Pi$ (from (17)). Then $p \in \Pi$, so (14) is proved.

Step 3. Unfortunately (14) does not imply that $M_t^{(p)}$ is a square integrable martingale, but this is true for $M_t^{(p^*)}$, since

$$(p^*)^2 \int_0^t |X_s|_H^{2(p^*-2)} \langle QX_s, X_s\rangle_H\, ds$$
$$\leq (p^*)^2 |Q| \int_0^t |X_s|_H^{2(p^*-1)}\, ds = (p^*)^2 |Q| \int_0^t |X_s|_H^p\, ds.$$

We can now repeat the proof of step 3 of the previous lemma:

$$\sup_{t\in[0,T]} |X_t|_H^{p^*} \leq |X_0|_H^{p^*} + 1 + \sup_{t\in[0,T]} \left| M^{(p^*)} \right|^2 + \frac{p^*(p^*-1)\,TrQ}{2} \int_0^T |X_s|_H^{p^*-2}\, ds$$

hence, by Doob's inequality,

$$E \sup_{t\in[0,T]} |X_t|_H^{p^*} \leq E\,|X_0|_H^{p^*} + 1 + 4\,(p^*)^2 \int_0^T |X_s|_H^{2(p^*-2)} \langle QX_s, X_s\rangle_H\, ds + \frac{p^*(p^*-1)\,TrQ}{2} \int_0^T |X_s|_H^{p^*-2}\, ds$$

which is easily bounded by a constant $C_3\left(p, E\,|X_0|_H^p, TrQ, T\right)$ due to (14). The proof is complete.

Corollary 3.2. *For a stationary solutions we have* $E\left[|X_t|_H^{p-2}\,\|X_t\|_V^2\right] < \infty$ *(hence in particular* $E\,|X_t|_H^p < \infty$*) for every* $p \geq 2$ *and*

$$E\left[|X_t|_H^{p-2}\,\|X_t\|_V^2\right] = \frac{(p-1)\,TrQ}{2\nu} E\left[|X_t|_H^{p-2}\right].$$

Proof. The proof is the same given above for $p = 2$ and it is based on identity (16), that one has to iterated in p.

Remark 3.5. Under the same assumptions we have the expected estimate

$$E \sup_{t\in[0,T]} |X_t|_H^p \leq C\left(p, E\,|X_0|_H^p, TrQ, T\right)$$

for every $T > 0$. Its proof follows the lines of the proof of lemma 3.2 and makes use of Burkholder-Davis-Gundy inequality. See Flandoli and Gaterek [34], Appendix 1.

We have seen that the regularity of solutions can be improved in the direction of p-integrability on Ω. Less easy is to improve it in the direction of stronger topologies (in fact in finite dimensions they are all equivalent, but the equivalence is not stable in the passage to the limit). A natural question would be whether we have an estimate of the form

$$E\left(\sup_{t\in[0,T]} \|X_t\|_V^2 + \nu \int_0^T |AX_s|_H^2\, ds\right) \leq C_1\left(E\,\|X_0\|_V^2, TrQ, T\right). \tag{18}$$

This is an open problem (the answer is positive in the 2D case). Nevertheless, under assumptions inspired to the 3D case we can state at least one result on the time integrability of $|AX_s|_H^2$, proved by [41] in the deterministic case (see a related result with a different proof in [52], Thm 3.6) and extended to the stochastic case by Da Prato and Debussche [23]. To avoid unnecessary complications due to the generality, let us assume that A and Q commute, so there exists a common orthonormal system $\{e_i\}$ of eigenvectors, with $Ae_i = \lambda_i e_i$, $Qe_i = \sigma_i^2 e_i$.

To understand assumption (19) below, notice that in a finite dimensional space H we always have $\langle Ax, B(x,x)\rangle_H \leq C_{A,B} |x|_H^2$ for a suitable nonuniversal constant $C_{A,B}$. On the contrary, the constant C in (19) is universal, see lemma 2.2. In 2D (always with periodic boundary conditions) the situation is entirely different since we have $\langle Ax, B(x,x)\rangle_H = 0$ (vorticity conservation for $\nu = 0$).

Lemma 3.4 (bounds on $|AX_t|_H$). *Assume that*

$$\langle Ax, B(x,x)\rangle_H \leq C \|x\|_V^{3/2} |Ax|_H^{3/2}, \quad x \in H \tag{19}$$

and

$$\sum_i \sigma_i^2 \lambda_i < \infty.$$

Then

$$E\int_0^T \frac{|AX_s|_H^2}{\left(1+\|X_s\|_V^2\right)^2} ds \leq C\frac{1}{\nu^4} E\int_0^T \|X_s\|_V^2 + \frac{1}{\nu}\left(1 + C\sum_i \sigma_i^2\lambda_i\right)$$

$$\sqrt[3]{\nu} E\int_0^T |AX_s|_H^{2/3} ds \leq C\left(T, \sum_i \sigma_i^2\lambda_i\right)\left(1 + \frac{1}{\nu}E\int_0^T \|X_s\|_V^2\right)$$

where, concerning the term $E\int_0^T \|X_s\|_V^2 ds$, we recall the bound (6).

Proof. Introduce the function $f : H \to \mathcal{R}$ defined as

$$f(x) = \frac{1}{1+\|x\|_V^2} = (1 + \langle Ax, x\rangle_H)^{-1}.$$

We have

$$Df(x) = -\left(1+\|x\|_V^2\right)^{-2} D\langle Ax, x\rangle_H = -2\frac{Ax}{\left(1+\|x\|_V^2\right)^2}$$

$$D^2 f(x) = 8\frac{Ax\otimes Ax}{\left(1+\|x\|_V^2\right)^3} - 2\frac{A}{\left(1+\|x\|_V^2\right)^2}.$$

Hence

$$\frac{1}{1+\|X_t\|_V^2} = \frac{1}{1+\|X_0\|_V^2} + 2\int_0^t \frac{\langle AX_s, \nu AX_s + B(X_s,X_s)\rangle_H}{\left(1+\|X_s\|_V^2\right)^2} ds$$
$$+ \widetilde{M}_t + \frac{1}{2}\int_0^t g(X_s)\, ds$$

where

$$\widetilde{M}_t = -2 \int_0^t \frac{\langle AX_s, \sqrt{Q} dW_s \rangle_H}{\left(1 + \|X_s\|_V^2\right)^2}$$

is a local martingale and

$$g(x) = \sum_i \sigma_i^2 \left[8 \frac{\langle Ax, e_i \rangle_H^2}{\left(1 + \|x\|_V^2\right)^3} - 2 \frac{\langle Ae_i, e_i \rangle_H}{\left(1 + \|x\|_V^2\right)^2} \right].$$

In fact $\widetilde{M}_t$ is a square integrable martingale, because

$$\int_0^t \frac{\langle QAX_s, AX_s \rangle_H}{\left(1 + \|X_s\|_V^2\right)^4} ds \leq \int_0^t \frac{\left|A^{1/2} Q A^{1/2}\right| \left|A^{1/2} X_s\right|_H^2}{\left(1 + \|X_s\|_V^2\right)^4} ds$$

$$\leq \sum_i \sigma_i^2 \lambda_i \int_0^t \frac{\|X_s\|_V^2}{\left(1 + \|X_s\|_V^2\right)^4} ds \leq \sum_i \sigma_i^2 \lambda_i < \infty.$$

We also have $|g(x)| \leq C \sum_i \sigma_i^2 \lambda_i$, so, from

$$2\nu \int_0^t \frac{|AX_s|_H^2}{\left(1 + \|X_s\|_V^2\right)^2} ds \leq \frac{1}{1 + \|X_t\|_V^2} - 2 \int_0^t \frac{\langle AX_s, B(X_s, X_s) \rangle_H}{\left(1 + \|X_s\|_V^2\right)^2} ds$$
$$- \widetilde{M}_t - \frac{1}{2} \int_0^t g(X_s) \, ds$$

the assumption on B and the martingale property of $\widetilde{M}_t$ we have

$$2\nu E \int_0^t \frac{|AX_s|_H^2}{\left(1 + \|X_s\|_V^2\right)^2} ds$$
$$\leq 1 + 2CE \int_0^t \frac{\|X_s\|_V^{3/2} |AX_s|_H^{3/2}}{\left(1 + \|X_s\|_V^2\right)^2} ds + \frac{C}{2} \sum_i \sigma_i^2 \lambda_i.$$

Moreover,

$$E \int_0^T \frac{\|X_s\|_V^{3/2} |AX_s|^{3/2}}{\left(1 + \|X_s\|_V^2\right)^2} dt \leq \varepsilon \nu E \int_0^T \frac{|AX_s|^2}{\left(1 + \|X_s\|_V^2\right)^2} + C_\varepsilon \frac{1}{\nu^3} E \int_0^T \|X_s\|_V^2$$

for every $\varepsilon > 0$ and for a suitable constant C_ε, due to the following Young inequality $(f, g \geq 0)$

$$fg \leq \varepsilon f^p + \frac{C}{\varepsilon^{1/(p-1)}} g^{p'}, p' = \frac{p}{p-1}.$$

With a universal choice of $\varepsilon > 0$ we have

$$\nu E \int_0^t \frac{|AX_s|_H^2}{\left(1 + \|X_s\|_V^2\right)^2} ds \leq 1 + C \frac{1}{\nu^3} E \int_0^t \|X_s\|_V^2 + C \sum_i \sigma_i^2 \lambda_i.$$

This implies the first inequality of the lemma. The second one simply follows from the following inequalities:

$$\sqrt[3]{\nu} E \int_0^T |AX_s|_H^{2/3} ds \leq \left(E \int_0^T \left(\nu \frac{|AX_s|_H^2}{\left(1 + \|X_s\|_V^2\right)^2} \right) dt \right)^{1/3} \cdot \left(E \int_0^T \left(1 + \|X_s\|_V^2\right) dt \right)^{2/3}$$

$$\leq \left(1 + C \frac{1}{\nu^3} E \int_0^t \|X_s\|_V^2 + C \sum_i \sigma_i^2 \lambda_i \right)^{1/3} \left(T + E \int_0^T \|X_s\|_V^2 dt \right)^{2/3}.$$

The proof is complete.

Remark 3.6. Under the assumption $E|X_0|_H^2 < \infty$ we know that $E \int_0^t \|X_s\|_V^2 ds$ is bounded by a universal constant, so the same is true for $E \int_0^T |AX_s|_H^{2/3} ds$. This implies $E\left[|AX_t|_H^{2/3}\right] < \infty$ for almost every t. If in addition the process is stationary, we have $E\left[|AX_t|_H^{2/3}\right] < \infty$ for every t. In terms of invariant measures μ of the limit infinite dimensional problem this will imply that $\mu(D(A)) = 1$.

Remark 3.7. For stationary X_s,

$$2E \frac{\langle AX_s, \nu AX_s + B(X_s, X_s) \rangle_H}{\left(1 + \|X_s\|_V^2\right)^2} + \frac{1}{2} Eg(X_s) = 0.$$

Notice that $|g(x)| \leq C \sum_i \sigma_i^2 \lambda_i$, so under the assumption that this quantity is finite and given, we may heuristically think that $Eg(X_s)$ converges to a nonzero value g_0 as $\nu \to 0$. Then we have

$$E \frac{|AX_s|_H^2}{\left(1 + \|X_s\|_V^2\right)^2} \sim \frac{1}{\nu} \left(\frac{g_0}{4} + E \frac{\langle AX_s, B(X_s, X_s) \rangle_H}{\left(1 + \|X_s\|_V^2\right)^2} \right).$$

Remark 3.8. Let us briefly understand that under assumption (19) it is not possible to obtain a bound of the form (18). Without all the details, from Itô formula for $d\|X_t\|_V^2$, we have

$$d\|X_t\|_V^2 + 2\nu |AX_t|_H^2 \le |\langle AX_t, B(X_t, X_t)\rangle_H| \text{ plus other terms}$$

and

$$\begin{aligned} |\langle AX_t, B(X_t, X_t)\rangle_H| &\le C\|X_t\|_V^{3/2} |AX_t|_H^{3/2} \\ &\le \nu |AX_t|_H^2 + C\|X_t\|_V^6 \end{aligned}$$

so we meet the differential inequality

$$d\|X_t\|_V^2 \le C\|X_t\|_V^6 \text{ plus other terms}$$

that cannot be closed on a global time interval.

3.3 Comparison of Two Solutions and Pathwise Estimates

Having assumed an additive noise, it disappears when we write the equation for the difference of two solutions; this has some advantages. We can reach similar advantages for a single solution with the following trick: we consider the difference between the solution and an auxiliary process, usually the solution of the associated linear equation. Let us perform some of these computations in this section.

However, here we assume the stronger algebraic condition (2) on B.

Lemma 3.5. *Let* $\left(X_t^{(1)}\right)$ *and* $\left(X_t^{(2)}\right)$ *be two solutions on some interval* $[0, T]$ *and let us set* $V_t = X_t^{(1)} - X_t^{(2)}$. *Let* C_B *be a constant such that*

$$\langle B(x, y), x\rangle_H \le C_B |x|_H^2 |y|_H$$

for every $x, y \in H$. *Then*

$$|V_t|_H \le |V_0|_H e^{C_B \int_0^t \left|X_s^{(2)}\right|_H ds}.$$

Proof. We have

$$\frac{dV_t}{dt} + \nu AV_t + B\left(X_t^{(1)}, V_t\right) + B\left(V_t, X_t^{(2)}\right) = 0$$

whence

$$\frac{1}{2}\frac{d|V_t|_H^2}{dt} + \nu \langle AV_t, V_t\rangle_H = -\left\langle B\left(V_t, X_t^{(2)}\right), V_t\right\rangle_H \tag{20}$$

and thus

$$\frac{1}{2}\frac{d|V_t|_H^2}{dt} \le C_B |V_t|_H^2 \left|X_t^{(2)}\right|_H.$$

The conclusion follows from Gronwall lemma.

The previous result is not stable in the limit of infinite dimensions. On the contrary, the assumption on B of the next lemma is stable in dimension 3.

Lemma 3.6. *Assume that*

$$\langle B(x,y), x\rangle_H \le C\,|x|_H^{1/2}\,\|x\|_V^{3/2}\,\|y\|_V$$

for every $x, y \in H$, where C is a universal constant. Then we have

$$|V_t|_H \le |V_0|_H\, e^{C\int_0^t \left\|X_s^{(2)}\right\|_V^4 ds}.$$

Proof. We restart from (20) and get now (we use Young inequality in the second step)

$$\begin{aligned}\frac{1}{2}\frac{d\,|V_t|_H^2}{dt} + \nu\,\|V_t\|_V^2 &\le C\,|V_t|_H^{1/2}\,\|V_t\|_V^{3/2}\left\|X_t^{(2)}\right\|_V \\ &\le \frac{1}{2}\,\|V_t\|_V^2 + C\,|V_t|_H^2\left\|X_t^{(2)}\right\|_V^4.\end{aligned}$$

Therefore

$$\frac{1}{2}\frac{d\,|V_t|_H^2}{dt} \le C\,|V_t|_H^2\left\|X_t^{(2)}\right\|_V^4$$

which implies the claim of the lemma, again by Gronwall lemma.

From the viewpoint of the limit to infinite dimensions for 3D fluids, the problem in the first lemma is the constant C_B. On the contrary, the problem in the second lemma is the term $\int_0^t \left\|X_s^{(2)}\right\|_V^4 ds$, on which we do not have bounds which are stable with the dimension.

To summarize, we have shown two simple computations which imply uniqueness for the finite dimensional problem but are useless for 3D fluids. There exist very many variants of these computations in different topologies, but the result, until now, is always the same: either the constant or some norm of the solutions blow-up in the case of the 3D Navier–Stokes equation.

Exercise 3.1. In the application to the 2-dimensional Navier–Stokes equations the continuity properties of B are stronger, due to the improvement coming from Sobolev embedding theorem. One has

$$\langle B(x,y), x\rangle_H \le C\,\|y\|_V\,|x|_H\,\|x\|_V$$

for every $x, y \in H$, where C is a universal constant. Prove that

$$|V_t|_H \le |V_0|_H\, e^{C\int_0^t \left\|X_s^{(2)}\right\|_V^2 ds}.$$

Deduce a pathwise uniqueness result.

Exercise 3.2. (from Schmalfuss [60]). Continue the 2-dimensional case but consider a multiplicative noise of the form

$$G(X_t)\,dW_t$$

in place of the additive noise $\sqrt{Q}dW_t$. Assume that G is a Lipschitz continuous mapping from H to the space of linear bounded operators in H. Under this more general condition, one can prove an existence result along the same lines developed above. However, the uniqueness is more difficult, since the equation for the difference of two solutions $V_t = X_t^{(1)} - X_t^{(2)}$ is still an Itô equation, and an estimate on $|V_t|^2$ cannot simply be obtained by a pathwise application of Gronwall lemma. On the other side, the inequality that one gets from Itô formula for $|V_t|^2$ cannot be closed at the level of mean values since it contains cubic terms. Solve the problem using Itô formula for

$$e^{-C\int_0^t \left\|X_s^{(2)}\right\|_V^2 ds} |V_t|^2 .$$

In another form, this trick comes out again in the paper of DaPrato and Debussche [23].

3.4 Existence and Uniqueness, Markov Property

In the next theorem we shall show that equation (4) has a unique strong solution $(X_t^x)_{t\geq 0}$ for every initial condition $x \in H$, that depends measurably on x. We then may define the operators $P_t : B_b(H) \to B_b(H)$ as

$$(P_t\varphi)(x) = E[\varphi(X_t^x)] .$$

Here $B_b(H)$ is the space of Borel bounded functions on H and $C_b(H)$ will be the space of continuous bounded ones. We prove also that (4) defines a Markov process in the sense that

$$E\left[\varphi\left(X_{t+s}^x\right)|\mathcal{F}_t\right] = (P_s\varphi)(X_t^x) \quad (P\text{-a.s.}).$$

for every $\varphi \in C_b(H)$, $t, s > 0$, $x \in H$. Taking the expectation in this identity one gets the semigroup property $P_{t+s} = P_t P_s$ on $C_b(H)$. We say that P_t is Feller if $P_t\varphi \in C_b(H)$ for every $\varphi \in C_b(H)$.

Theorem 3.1. *For every $\mathcal{F}_0$-measurable $X_0 : \Omega \to H$, there exists a unique continuous adapted solution $(X_t)_{t\geq 0}$ of equation (4) on $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P\right)$. If the initial conditions x^n converge to x in H, the corresponding solutions converge P-a.s., uniformly in time on bounded intervals. Equation (4) defines a Markov process with the Feller property.*

Proof. Uniqueness and continuous dependence have been proved above in lemma 3.5 (of course such a result is not stable in the limit of infinite dimensions). This implies also the Feller property. Let us divide the proof of existence and Markov property in several steps. They are classical and are given for completeness. Preliminary, we remark that the proof of existence can be performed either by means of classical probabilistic arguments or by a pathwise analysis, due to the additivity of the noise. Let us give the probabilistic proof.

Step 1. (existence for bounded X_0). It is sufficient to prove the existence on $[0,T]$. Assume that $|X_0|_H \leq C$ for some constant $C>0$. For any $n>C$, let $B_n(.) : H \to H$ be a Lipschitz continuous function such that $B_n(x) = B(x,x)$ for every $|x|_H \leq n$. Consider then the equation

$$dX_t^{(n)} = \left[-\nu A X_t^{(n)} - B_n\left(X_t^{(n)}, X_t^{(n)}\right)\right] dt + \sqrt{Q} dW_t$$

with initial condition X_0. It has globally Lipschitz coefficients, so there exists a unique continuous adapted solution $\left(X_t^{(n)}\right)_{t\geq 0}$. The proof of this classical result can be done by contraction principle in $L^2(\Omega; C([0,T];H))$. Let τ_n be defined as

$$\tau_n = \inf\left\{t \geq 0 : \left|X_t^{(n)}\right|_H = n\right\} \wedge T.$$

Up to τ_n the solution $X_t^{(n)}$ is also a solution of the original equation: it is sufficient to observe the integral form of the equations. Therefore, by lemma 3.2, applicable since $E|X_0|_H^2 < \infty$, we have

$$E\left(\sup_{t\in[0,T]}\left|X_{t\wedge\tau_n}^{(n)}\right|_H^2\right) \leq C_1\left(E|X_0|_H^2, TrQ, T\right).$$

In particular

$$E\left(1_{\{\tau_n<T\}}\left|X_{T\wedge\tau_n}^{(n)}\right|_H^2\right) \leq C_1\left(E|X_0|_H^2, TrQ, T\right)$$

which implies

$$P(\tau_n < T) \leq \frac{1}{n^2} C_1\left(E|X_0|_H^2, TrQ, T\right)$$

since $\left|X_{T\wedge\tau_n}^{(n)}\right|_H^2 = n^2$ on $\{\tau_n < T\}$. If $N > n$ then $\tau_N > \tau_n$ and

$$P\left(X_t^{(N)} = X_t^{(n)}, t \in [0,\tau_n]\right) = 1$$

Therefore, if $\tau_\infty := \sup_{n>C} \tau_n$, we may uniquely define a process $X_t^{(\infty)}$ for $t \in [0,\tau_\infty)$, equal to $X_t^{(n)}$ on $[0,\tau_n]$ for every n. Hence $X_t^{(\infty)}$ is a solution on $[0,\tau_\infty)$. But we have

$$P(\tau_\infty < T) \leq P(\tau_n < T) \leq \frac{C}{n^2}$$

for every n, hence $P(\tau_\infty < T) = 0$. Thus $X_t^{(\infty)}$ is a solution for $t \in [0, T-\varepsilon]$ for every small $\varepsilon > 0$. Since T is arbitrary, we have proved global existence. Denote by $(X_t^x)_{t\geq 0}$ the unique solution with initial condition $x \in H$.

Step 2. (existence for general X_0). Let $\Omega_n \in \mathcal{F}$ be defined as $\Omega_n = \left\{|X_0|_H^2 \leq n\right\}$. Define $X_0^{(n)}$ as X_0 on Ω_n, 0 otherwise. Let $\left(X_t^{(n)}\right)_{t\geq 0}$ be the unique solution of equation (4) with initial condition $X_0^{(n)}$. If $N > n$, then

$$P\left(\Omega_n \cap \left(X_t^{(N)} = X_t^{(n)} \text{ for every } t \geq 0\right)\right) = P\left(\Omega_n\right).$$

We may then uniquely define a process $X_t^{(\infty)}$ on $\Omega' = \cup_n \Omega_n$ as $X_t^{(\infty)} = X_t^{(n)}$ on Ω_n. Looking at the equation in integral form, in particular at its pathwise meaning, it is clear that $X_t^{(\infty)}$ solves the equation on Ω'. But $P\left(\Omega'\right) = 1$, hence we have proved the existence of a global solution.

Step 3. (Markov property). Given $x \in H$, $\varphi \in C_b\left(H\right)$, $t, s > 0$, we have to prove that

$$E\left[\varphi\left(X_{t+s}^x\right) Z\right] = E\left[\left(P_s\varphi\right)\left(X_t^x\right) Z\right]$$

for every bounded $\mathcal{F}_t$-measurable r.v. Z. By uniqueness

$$X_{t+s}^x = X_{t,t+s}^{X_t^x} \quad (P\text{-a.s.})$$

where $\left(X_{t_0,t}^\eta\right)_{t\geq t_0}$ denotes the unique solution on the time interval $[t_0, \infty)$, with the $\mathcal{F}_{t_0}$-measurable initial condition $X_{t_0,t_0}^\eta = \eta$. It is then sufficient to prove that

$$E\left[\varphi\left(X_{t,t+s}^\eta\right) Z\right] = E\left[\left(P_s\varphi\right)\left(\eta\right) Z\right]$$

for every H-valued $\mathcal{F}_t$-measurable r.v. η. By approximation (one has to use Lebesgue theorem and the fact that strong convergence of η_n in H implies that $\left(P_s\varphi\right)\left(\eta_n\right)$ converges P-a.s. to $\left(P_s\varphi\right)\left(\eta\right)$), it is sufficient to prove it for every r.v. η of the form $\eta = \sum_{i=1}^k \eta^{(i)} 1_{A^{(i)}}$ with $\eta^{(i)} \in H$ and $A^{(i)} \in \mathcal{F}_t$. By inspection (everything decomposes with respect to the partition $A^{(i)}$) one can see that it is sufficient to prove it for every deterministic element $\eta \in H$. Now the r.v. $X_{t,t+s}^\eta$ depends only on the increments of the Brownian motion between t and $t + s$, hence it is independent of $\mathcal{F}_t$. Therefore

$$E\left[\varphi\left(X_{t,t+s}^\eta\right) Z\right] = E\left[\varphi\left(X_{t,t+s}^\eta\right)\right] E\left[Z\right].$$

Since $X_{t,t+s}^\eta$ has the same law of X_s^η (by uniqueness), we have $E\left[\varphi\left(X_{t,t+s}^\eta\right)\right] = E\left[\varphi\left(X_s^\eta\right)\right]$ and thus

$$E\left[\varphi\left(X_{t,t+s}^\eta\right) Z\right] = \left(P_s\varphi\right)\left(\eta\right) E\left[Z\right] = E\left[\left(P_s\varphi\right)\left(\eta\right) Z\right].$$

The proof is complete.

3.5 Invariant Measures

Let P_t^* be the semigroup on the space of probability measures on H defined as

$$\left(P_t^*\mu\right)(f) = \mu\left(P_t f\right)$$

where we use the notation $\mu(f)$ for $\int_H f d\mu$. We have $P^*_{t+s} = P^*_t P^*_s$ for every $t, s \geq 0$ and P^*_0 is the identity. If $(X_t)_{t\geq 0}$ is a solution and ν_t denotes the law of X_t, then $P^*_{t-s}\nu_s = \nu_t$ for every $t \geq s \geq 0$.

We say that a probability measure μ is invariant if $P^*_t \mu = \mu$ for every $t \geq 0$. Equivalently, if

$$\mu(\varphi) = \mu(P_t \varphi)$$

for every $t \geq 0$ and $\varphi \in C_b(H)$.

We recall that, given a metric space (X, d) with its Borel σ-field $\mathcal{B}$, a set of probability measures Λ on $(X, \mathcal{B})$ is *tight* if the following condition holds: for every $\varepsilon > 0$ there is a compact set $K_\varepsilon \subset X$ such that $\mu(K_\varepsilon) > 1 - \varepsilon$ for every $\mu \in \Lambda$. Moreover, a set of probability measures Λ on $(X, \mathcal{B})$ is relatively compact if from every sequence $\{\mu_n\} \subset \Lambda$ one may extract a subsequence $\{\mu_{n_k}\}$ and find a probability measure μ on $(X, \mathcal{B})$ such that $\mu_{n_k} \to \mu$ weakly (by this we mean that $\mu_{n_k}(\varphi) \to \mu(\varphi)$ for every $\varphi \in C_b(H)$). If X is a Polish space, Prohorov theorem states that Λ is tight if and only if it is relatively compact. Notice that if X is compact, tightness is free and then also the relative compactness of Λ, but in our applications the metric space is H, so we need estimates to prove tightness.

Theorem 3.2. *There exists at least one invariant measure for (4), with the property*

$$\mu\left(\|.\|_V^2\right) \leq \frac{TrQ}{2\nu}. \tag{21}$$

Proof. **Step 1** (preparation). Following the general scheme attributed to Krylov and Bogoliubov, we consider a solution $(X_t)_{t\geq 0}$ with a suitable initial condition, say $X_0 = 0$, we denote the law of X_t by ν_t and introduce the time averages

$$\mu_T = \frac{1}{T}\int_0^T \nu_s ds = \frac{1}{T}\int_0^T P^*_s \nu_0 ds$$

or more explicitly $\mu_T(\varphi) = \frac{1}{T}\int_0^T \nu_s(\varphi) ds$ for every $\varphi \in C_b(H)$. The family of measures $\{\mu_T; T \geq 0\}$ is tight. Let us give two illuminating proofs of this fact.

Step 2 (first proof of tightness). We follow a clever argument of Chow and Khasminskii [20]. From the energy equality (8), taking into account the choice $X_0 = 0$, we know that

$$\frac{1}{T}\int_0^T E\|X_s\|_V^2 ds \leq \frac{TrQ}{2\nu}.$$

Notice that

$$\mu_T\left(\|.\|_V^2\right) = \frac{1}{T}\int_0^T \nu_s\left(\|.\|_V^2\right) ds = \frac{1}{T}\int_0^T E\|X_s\|_V^2 ds$$

(the first identity holds true for the test function $\|.\|_V^2 \wedge N$ by definition of μ_T, and then extends to the function $\|.\|_V^2$ by monotone convergence theorem). Hence

$$\mu_T\left(\|.\|_V^2\right) \leq \frac{TrQ}{2\nu} \tag{22}$$

and thus, by Chebyshev inequality,

$$\mu_T\left(\|x\|_V^2 \geq R^2\right) \leq R^{-2}\mu_T\left(\|.\|_V^2\right) \leq R^{-2}\frac{TrQ}{2\nu}.$$

This implies the tightness.

Step 3 (second proof of tightness). Equation (8), which reads

$$E\,|X_t|_H^2 + 2\nu\int_0^t E\,\|X_s\|_V^2\,ds = TrQ\;t$$

implies that $E\,|X_t|_H^2$ is differentiable and

$$\frac{dE\,|X_t|_H^2}{dt} + 2\nu E\,\|X_t\|_V^2 = TrQ.$$

Hence

$$\frac{dE\,|X_t|_H^2}{dt} \leq -2\nu\lambda E\,|X_t|_H^2 + TrQ.$$

This implies

$$E\,|X_t|_H^2 \leq e^{-2\nu\lambda t}E\,|X_0|_H^2 + \int_0^t e^{-2\nu\lambda(t-s)}TrQds \leq \frac{TrQ}{2\nu\lambda}.$$

The Gronwall-like inequality can be easily proved as Gronwall lemma, computing $\frac{d\left(e^{2\nu\lambda t}E|X_t|_H^2\right)}{dt}$ and integrating the result on $[0,t]$. From the previous inequality we have, as above,

$$\begin{aligned}\mu_T\left(|x|_H^2 \geq R^2\right) &= \frac{1}{T}\int_0^T \nu_s\left(|x|_H^2 \geq R^2\right)ds\\ &\leq \frac{1}{T}\int_0^T \nu_s\left(|.|_H^2\right)ds = \frac{1}{T}\int_0^T E\,|X_s|_H^2\,ds \leq \frac{TrQ}{2\nu\lambda}\end{aligned}$$

which yields the tightness. Notice that the result of this second method is weaker from the viewpoint of the topologies.

Step 4 (conclusion). From Prohorov theorem, there exists a sequence μ_{T_n} weakly convergent to a probability measure μ. Let us show that μ is invariant. We have, for every $f \in C_b(H)$, and using the fact that $P_t f \in C_b(H)$ by the Feller property,

$$\left(P_t^*\mu\right)(f) = \mu\left(P_t f\right) = \lim_{n\to\infty}\mu_{T_n}\left(P_t f\right) = \lim_{n\to\infty}\left(P_t^*\mu_{T_n}\right)(f)$$

and

$$P_t^* \mu_{T_n} = P_t^* \frac{1}{T_n} \int_0^{T_n} P_s^* \nu_0 ds = \frac{1}{T_n} \int_0^{T_n} P_{t+s}^* \nu_0 ds$$
$$= \frac{1}{T_n} \int_t^{t+T_n} P_\sigma^* \nu_0 d\sigma = \mu_{T_n}$$
$$- \frac{1}{T_n} \int_0^t P_\sigma^* \nu_0 d\sigma + \frac{1}{T_n} \int_{T_n}^{T_n+t} P_\sigma^* \nu_0 d\sigma.$$

The last two terms converge weakly to zero. This proves $P_t^* \mu = \mu$, so the existence of an invariant measure is assured.

Finally, from (22), we get

$$\mu_{T_n} \left(\|.\|_V^2 \wedge N \right) \leq \frac{TrQ}{2\nu}$$

for every $N, n > 0$, hence

$$\mu \left(\|.\|_V^2 \wedge N \right) \leq \frac{TrQ}{2\nu}$$

for every $N > 0$, and thus we have (21) by monotone convergence theorem. The proof is complete.

Remark 3.9. If Q is invertible, the invariant measure is unique and ergodic. We refer to specialized text for definitions and results.

Remark 3.10. In specific examples one can say more about ergodicity: it holds also for certain degenerate noises. Consider our main example 3.1 on the Galerkin approximation of the d-dimensional Navier–Stokes equations. Weinan E and Mattingly [27] in $d = 2$ and Romito [59] in $d = 3$ proved that ergodicity is true if the noise is active at least on a very small number of modes (like 4), properly displaced to "generate" all other modes through the action of the drift. Let us mention also the work of Mattingly and Hairer [53] on the true 2D Navier–Stokes equations (ergodicity with very degenerate noise) and the references therein.

In the previous theorem we have constructed an invariant measure with the property (21). For this purpose we had to start Krylov-Bogoliubov scheme from a good initial condition. In fact, this is not necessary: the invariance itself provides the mechanism to prove (21).

Theorem 3.3. *All invariant measures have the property (21). In fact they also satisfy*

$$\mu \left(\|.\|_V^2 \right) = \frac{TrQ}{2\nu}.$$

Proof. Let μ be an invariant measure. If $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P\right)$ is the filtered probability space where the Brownian motion is defined, consider the enlarged filtered probability space

$$\Omega' = \Omega \times H,\ \mathcal{F}' = \mathcal{F} \otimes \mathcal{B},\ \mathcal{F}'_t = \mathcal{F}_t \otimes \mathcal{B},\ P' = P \otimes \mu$$

with the new Brownian motion (W'_t) and the $\mathcal{F}'_0$-measurable r.v. X_0 defined as

$$W'_t(\omega, x) = W_t(\omega),\ X_0(\omega, x) = x.$$

The law of X_0 is μ. The unique solution (X_t) of (4) with initial condition X_0 is a stationary process, with the law of X_t equal to μ for every $t \geq 0$ (we do not give the details, the result is intuitively clear). Then we can apply corollary 3.1. Finally, from (8) we deduce that $\mu\left(\|.\|_V^2\right)$ is truly equal to $\frac{TrQ}{2\nu}$. The proof is complete.

Corollary 3.3. *Assume that A and Q commute and have eigenvalues λ_i and σ_i^2 respectively, with $\sum_i \sigma_i^2 \lambda_i < \infty$. Assume also that condition (19) holds true. Then all invariant measures μ have the property*

$$\mu\left[\frac{|Ax|_H^2}{\left(1+\|x\|_V^2\right)^2}\right] \leq C\frac{\sum_i \sigma_i^2}{2\nu^5} + \frac{1}{\nu}\left(1 + C\sum_i \sigma_i^2 \lambda_i\right)$$

$$\sqrt[3]{\nu}\mu\left[|Ax|_H^{2/3}\right] \leq C\left(1 + \frac{\sum_i \sigma_i^2}{2\nu^5}\right)\sum_i \sigma_i^2 \lambda_i$$

with universal constant $C > 0$.

3.6 Galerkin Stationary Measures for the 3D Equation

Our main concern are the stochastic Navier–Stokes equations (1) of Section 1. We work only with the Galerkin approximations and use the notations and definitions of Example 3.1 of Section 1, but restricted to real (not complex) spaces and operators. We assume for simplicity that the noise has the form

$$\sqrt{Q}W_t = \sum_{i=1}^{\infty} \sigma_i h_i(x)\,\beta_i(t)$$

where σ_i are real numbers, h_i are eigenfunctions of $\mathcal{A}$, β_i are independent Brownian motions. We do not define here the concept of solution to (1) (this will be done later on), but simply introduce a class of probability measures on $\mathcal{H}$ that we call *Galerkin stationary measures of equations* (1).

Let us say that a probability measure μ on $\mathcal{H}$ is a *cluster points* of Galerkin invariant measures if there exists a sequence $\{n_k\}$ diverging to infinity and for each n_k an invariant measure μ_{n_k} of the corresponding Galerkin approximation system, such that the sequence of measures $\{\mu_{n_k}\}$ weakly converges to μ on $\mathcal{H}$. Then we call Galerkin stationary measures of equations (1) every such cluster point. We denote by $\mathcal{P}_{NS}^{Galerkin}$ the set of all such probability measures on $\mathcal{H}$.

Theorem 3.4. *$P_{NS}^{Galerkin}$ is non empty. Every $\mu \in P_{NS}^{Galerkin}$ satisfies*

$$\mu\left(\|.\|_{\mathcal{V}}^2\right) \leq \frac{\sum_i \sigma_i^2}{2\nu}. \tag{23}$$

Proof. For every n, let μ_n be an invariant measure of the corresponding Galerkin approximation system. We have

$$\mu_n\left(\|.\|_{\mathcal{V}}^2\right) \leq \frac{\sum_i \sigma_i^2}{2\nu}$$

so the family $\{\mu_n\}$ of measures is bounded in probability in $\mathcal{V}$, and thus it is tight in $\mathcal{H}$ since the space $\mathcal{V}$ is compactly embedded into $\mathcal{H}$. By Prohorov theorem, there is a subsequence $\{\mu_{n_k}\}$ weakly convergent to some probability measure μ on $\mathcal{H}$. Thus $\mathcal{P}_{NS}^{Galerkin}$ is non empty.

From the previous uniform bound we also have

$$\mu_n\left(\sum_{i=1}^m \lambda_i \left|\langle\cdot, h_i\rangle\right|^2 \wedge N\right) \leq \frac{\sum_i \sigma_i^2}{2\nu}$$

for every $N > 0$ and integer $m > 0$, where we observe that

$$\|.\|_{\mathcal{V}}^2 = \sum_{i=1}^{\infty} \lambda_i \left|\langle\cdot, h_i\rangle\right|^2.$$

Now $\sum_{i=1}^m \lambda_i \left|\langle\cdot, h_i\rangle\right|^2 \wedge N \in C_b(\mathcal{H})$, hence we may take the limit as $n \to \infty$ and have

$$\mu\left(\sum_{i=1}^m \lambda_i \left|\langle\cdot, h_i\rangle\right|^2 \wedge N\right) \leq \frac{\sum_i \sigma_i^2}{2\nu}$$

which implies (23) by monotone converge theorem. In general, given $\mu \in P_{NS}^{Galerkin}$, by definition there is $\{\mu_{n_k}\}$ as above, so the previous argument applies, and (23) is proved for every $\mu \in P_{NS}^{Galerkin}$.

Remark 3.11. Unfortunately, even if for the finite dimensional approximations we have

$$\mu_n\left(\|.\|_{\mathcal{V}}^2\right) = \frac{\sum_i^{N_n} \sigma_i^2}{2\nu},$$

this equality does not pass to the limit because we cannot say that $\mu_n\left(\|.\|_{\mathcal{V}}^2\right)$ converges to $\mu\left(\|.\|_{\mathcal{V}}^2\right)$. We could have

$$\lim_{n\to\infty}\mu_n\left(\|.\|_{\mathcal{V}}^2\right)=\mu\left(\|.\|_{\mathcal{V}}^2\right)$$

(and then the equality above for μ) if

$$\begin{aligned}\left|\mu_n\left(\|.\|_{\mathcal{V}}^2\right)-\mu\left(\|.\|_{\mathcal{V}}^2\right)\right| &\le \left|\mu_n\left(\|.\|_{\mathcal{V}}^2\right)-\mu_n\left(\|.\|_{\mathcal{V}}^2\wedge N\right)\right|\\ &+\left|\mu_n\left(\|.\|_{\mathcal{V}}^2\wedge N\right)-\mu\left(\|.\|_{\mathcal{V}}^2\wedge N\right)\right|\\ &+\left|\mu\left(\|.\|_{\mathcal{V}}^2\wedge N\right)-\mu\left(\|.\|_{\mathcal{V}}^2\right)\right|\end{aligned}$$

can be made small for large n. The last term is small for large N. The second term is unclear in general, but under the assumption of the next theorem it is small since we may have weak convergence of μ_n to μ in $\mathcal{V}$. But the problem is to have the first term uniformly small in n, for large N. We have

$$\begin{aligned}\left|\mu_n\left(\|.\|_{\mathcal{V}}^2\right)-\mu_n\left(\|.\|_{\mathcal{V}}^2\wedge N\right)\right| &= \int_{\|.\|_{\mathcal{V}}^2>N}\|x\|_{\mathcal{V}}^2\,d\mu_n(x)\\ &\le \mu_n\left(\|.\|_{\mathcal{V}}^{2p}\right)^{1/p}\mu_n\left(\|.\|_{\mathcal{V}}^2>N\right)^{1/p'}\\ &\le \mu_n\left(\|.\|_{\mathcal{V}}^{2p}\right)^{1/p}\left(\frac{\mu_n\left(\|.\|_{\mathcal{V}}^2\right)}{N}\right)^{1/p'}\end{aligned}$$

and this would be small uniformly small in n, for large N, if $\mu_n\left(\|.\|_{\mathcal{V}}^{2p}\right)\le C$ for some $p>1$. But this is unknown. It can be proved in dimension 2.

We use now assumption (19), that holds true in the 3D case (lemma 2.2, plus the fact that the projection π_n is selfadjoint in $\mathcal{H}$ and commutes with $\mathcal{A}$).

Theorem 3.5. *If* $\sum_i\sigma_i^2\lambda_i<\infty$ *then* $\mu(D(\mathcal{A}))=1$ *for every* $\mu\in P_{NS}^{Galerkin}$ *and*

$$\sqrt[3]{\nu}\mu\left[|\mathcal{A}x|_{\mathcal{H}}^{2/3}\right]\le C\left(1+\frac{\sum_i\sigma_i^2}{2\nu^5}\right)\sum_i\sigma_i^2\lambda_i.$$

Proof. We have both $\sum_i\sigma_i^2\lambda_i<\infty$ and condition (19), so, given $\mu\in P_{NS}^{Galerkin}$ and a sequence $\{\mu_{n_k}\}$ converging to μ, by corollary 3.3 we have

$$\sqrt[3]{\nu}\mu_{n_k}\left[|\mathcal{A}x|_{\mathcal{H}}^{2/3}\right]\le C\left(1+\frac{\sum_i\sigma_i^2}{2\nu^5}\right)\sum_i\sigma_i^2\lambda_i$$

with a universal constant $C>0$. This easily implies the claim, with an argument already used in the previous proof. The proof is complete.

Let μ be a probability measure on $\mathcal{H}$. We say that it is *space homogeneous* if

$$\mu\left[f\left(u\left(\cdot-a\right)\right)\right]=\mu\left[f\left(u\right)\right] \tag{24}$$

for every $a \in \mathcal{T}_L$ and $f \in C_b\left(\mathcal{H}\right)$. We say it is *partial (or discrete) isotropic* if, for every rotation R that transforms the set of coordinate axes in itself, we have

$$\mu\left[f\left(u\left(R\cdot\right)\right)\right]=\mu\left[f\left(Ru(\cdot)\right)\right] \tag{25}$$

for all $f \in C_b\left(\mathcal{H}\right)$. This is the form of isotropy compatible with the symmetries of the torus. The same definitions apply to random fields, hence to $\sum_{i=1}^{\infty}\sigma_i h_i \beta_i\left(t\right)$ for given t. Notice that $\sum_{i=1}^{\infty}\sigma_i h_i \beta_i\left(t\right)$ is space homogeneous and partial isotropic for every $t \geq 0$ if and only if it is such for some t, being gaussian with covariance of the form Qt.

Theorem 3.6. *If $\sum_{i=1}^{\infty}\sigma_i h_i \beta_i\left(t\right)$ is space homogeneous and partial isotropic, then there exist $\mu \in P_{NS}^{Galerkin}$ that is space homogeneous and partial isotropic.*

Proof. There exist space homogeneous and partial isotropic invariant measures for the Galerkin approximations: it is sufficient to start the Krylov-Bogoliubov method from the initial condition equal to zero. Then their cluster points have the same property. The proof is complete.

The problem whether under the previous assumptions *all* elements of $P_{NS}^{Galerkin}$ are space homogeneous and partial isotropic, seems to be open (symmetry breaking).

4 Stochastic Navier–Stokes Equations in 3D

4.1 Concepts of Solution

Consider the abstract (formal) stochastic evolution equation (1) of Section 2 and its weak formulation over test functions (2) of that Section. From Lemma 2.1 we have

$$\begin{aligned}\int_0^t\left|\left\langle B\left(u_s,\varphi\right),u_s\right\rangle\right|ds &\leq \int_0^t C\left|u_s\right|_H^{1/2}\left\|u_s\right\|_V^{3/2}\left\|\varphi\right\|_V ds \\ &\leq C_\varphi \sup_{s\in[0,t]}\left|u_s\right|_H^{1/2}\int_0^t\left\|u_s\right\|_V^{3/2}ds\end{aligned} \tag{1}$$

hence the nonlinear term in (2) (Section 2) is well defined for functions u that live in $L^\infty\left(0,T;H\right)\cap L^2\left(0,T;V\right)$, $T>0$ (but many other spaces work as well, like $L^2\left(0,T;L^4\right)$).

As in the deterministic case, strong continuity of trajectories in H is an open problem. There will be strong continuity in weaker spaces, like $D(A)'$, and a uniform bound in H. Let H_σ be the space H with the weak topology. Since

$$C\left([0,T]\,;D(A)'\right)\cap L^\infty\left(0,T;H\right)\subset C\left([0,T]\,;H_\sigma\right)$$

(see lemma 4.6 below), the trajectories of the solutions will be at least weakly continuous in H. One could also prove strong continuity from the right at $t=0$ and for a.e. t, and in addition there is strong continuity in $\left[L^p\left(\mathcal{T}\right)\right]^3$ for $p<2$; we do not prove these results.

The following presentation is strongly inspired to [55].

Definition 4.1. *We call Brownian stochastic basis the object*

$$\left(W,\mathcal{F},\left(\mathcal{F}_t\right)_{t\geq 0},Q,\left(\beta_i\left(t\right)\right)_{t\geq 0,i\in\mathbb{N}}\right)$$

where $(W,\mathcal{F},Q)$ is a probability space, $\left(\mathcal{F}_t\right)_{t\geq 0}$ a filtration, $\left(\beta_i\left(t\right)\right)_{t\geq 0,i\in\mathbb{N}}$ a sequence of independent Brownian motions on $\left(W,\mathcal{F},\left(\mathcal{F}_t\right)_{t\geq 0},Q\right)$ (namely, the real valued processes β_i are independent, are adapted to $\left(\mathcal{F}_t\right)_{t\geq 0}$, are continuous and null at $t=0$, and have increments $\beta_i\left(t\right)-\beta_i\left(s\right)$ that are $N\left(0,t-s\right)$-distributed and independent of $\mathcal{F}_s$).

Definition 4.2 (strong solutions). *Let $\left(W,\mathcal{F},\left(\mathcal{F}_t\right)_{t\geq 0},Q,\left(\beta_i\left(t\right)\right)_{t\geq 0,i\in\mathbb{N}}\right)$ be a Brownian stochastic basis. Given $u_0:W\to H$, $\mathcal{F}_0$-measurable, we say that a $D(A)'$-valued process u on $(W,\mathcal{F},Q)$ is a strong solution of equation (1) with initial condition u_0 if:*

1. *u is a continuous adapted process in $D(A)'$ and*

$$u\left(.,\omega\right)\in L^\infty\left(0,T;H\right)\cap L^2\left(0,T;V\right)\quad Q\text{-a.s.}$$

for every $T>0$, and

2. *(2) is satisfied.*

Definition 4.3 (weak martingale solutions). *Given a probability measure μ_0 on H, a weak solution of equation (1) with initial law μ_0 consists of a Brownian stochastic basis $\left(W,\mathcal{F},\left(\mathcal{F}_t\right)_{t\geq 0},Q,\left(\beta_i\left(t\right)\right)_{t\geq 0,i\in\mathbb{N}}\right)$ and a $D(A)'$-valued process u on $(W,\mathcal{F},Q)$ such that*

[WM1] u is a continuous adapted process in $D(A)'$ and

$$u\left(.,\omega\right)\in L^\infty\left(0,T;H\right)\cap L^2\left(0,T;V\right)\quad Q\text{-a.s.}$$

for every $T>0$,

[WM2] (2) is satisfied

[WM3] $u_0:=u(0)$ has law μ_0.

Let us set

$$\Omega = C\left([0,\infty); D(A)'\right)$$

and denote by $(\xi_t)_{t\geq 0}$ the canonical process ($\xi_t(\omega) = \omega_t$), by F the Borel σ-algebra in Ω and by F_t the σ-algebra generated by the events ($\xi_s \in A$) with $s \in [0,t]$ and $A \in \mathcal{B}(D(A)')$.

Definition 4.4 (solution to the martingale problem). *Given a probability measure μ_0 on H, we say that a probability measure P on (Ω, F) is a solution of the martingale problem associated to equation (1) with initial law μ_0 if*

[MP1]for every $T > 0$

$$P\left(\sup_{t\in[0,T]} |\xi_t|_H + \int_0^T \|\xi_s\|_V^2 \, ds < \infty\right) = 1$$

[MP2] for every $\varphi \in \mathcal{D}^\infty$ the process M_t^φ defined P-a.s on (Ω, F) as

$$M_t^\varphi := \langle \xi_t, \varphi\rangle_H - \langle \xi_0, \varphi\rangle_H + \int_0^t \nu \langle \xi_s, A\varphi\rangle_H \, ds$$
$$- \int_0^t \langle B(\xi_s, \varphi), \xi_s\rangle_H \, ds$$

is square integrable and (M_t^φ, F_t, P) is a continuous martingale with quadratic variation

$$[M^\varphi]_t = \sum_{i=1}^\infty \sigma_i^2 \langle \varphi, h_i\rangle_H^2 \cdot t$$

[MP3] $\mu_0 = \pi_0 P$.

We have given the definition of strong solutions only for completeness, since unfortunately at present there is no result of existence of strong solutions for the 3D stochastic Navier–Stokes equation (except when identically $\sigma_i = 0$). In fact one can solve pathwise the equation with additive noise (see for instance [40]) and prove the existence of a measurable selection, but the existence of a *progressively* measurable selection remains an open problem. See also [57].

Therefore we concentrate on the other two notions. The term "weak martingale solution" has the following origin. In the theory of SDE's, weak solutions are those described by such a definition (let us say weak in the probabilistic sense). But in the theory of PDE's the term weak usually refers to some kind of distributional formulation (let us say weak in the deterministic sense). Here we have to mix-up both kind of weaknesses, and a way to remind that we mean weak also in the probabilistic sense is to add the qualification "weak martingale". No special martingale notion appear in the definition, but the next theorem of equivalence is a motivation for this choice of the name.

Theorem 4.1. *P is a solution of the martingale problem if and only if there exists a weak martingale solution with law P.*

Proof. **Step 1**. Let $\left(W, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, Q, (\beta_i(t))_{t\geq 0, i\in\mathbb{N}}\right)$ and u be the objects in the definition of weak martingale solution. Let $\widetilde{P}$ be the law of u on (Ω, F). Let us prove that $\widetilde{P}$ solves the martingale problem. The main point is to prove [MP2]. With the notation $u(\gamma)$ for the function $(u(t,\gamma))_{t\geq 0}$, we have

$$\begin{aligned} M_t^{\varphi}(u(\gamma)) &= \langle u(t,\gamma), \varphi\rangle_H - \langle u(0,\gamma), \varphi\rangle_H \\ &\quad + \int_0^t \nu \langle u(s,\gamma), A\varphi\rangle_H \, ds - \int_0^t \langle B(u(s,\gamma), \varphi), u(s,\gamma)\rangle_H \, ds \end{aligned}$$

so for Q-a.e. $\gamma \in W$

$$M_t^{\varphi}(u(\gamma)) = \sum_{i=1}^{\infty} \sigma_i \langle \varphi, h_i\rangle_H \beta_i(t,\gamma).$$

First, $M^{\varphi}(t)$ is square integrable: since $\widetilde{P}$ is the law of u, we have

$$E^{\widetilde{P}}\left[M^{\varphi}(t)^2\right] = E^P\left[M^{\varphi}(t, u(.))^2\right] = \sum_{i=1}^{\infty} \sigma_i^2 \langle \varphi, h_i\rangle_H^2.$$

The other assertions of [MP2] are a consequence of lemma (4.1).

Step 2. Let now P be a solution of the martingale problem. Due to the special shape of the quadratic variation of M_t^{φ}, by Levy martingale characterization of the Brownian motion it follows that M_t^{φ} is a Brownian motion. Furthermore, $\beta_i(t,\omega) := M^{h_i}(t,\omega)$ is a sequence of independent Brownian motions on (Ω, F, F_t, P) and

$$M_t^{\varphi}(\omega) = \sum_{i=1}^{\infty} \sigma_i \langle \varphi, h_i\rangle_{Hi}(t,\omega).$$

This immediately implies [WM2], from [MP2].

Remark 4.1. The Brownian motions $\beta_i(t,\gamma)$ depend on P. Thus this proof does not provide a space with a simultaneous solution for every initial condition.

Remark 4.2. For equations with non-constant diffusion term, step 2 requires a representation theorem for martingales.

Lemma 4.1. *Let $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P\right)$ and $\left(\Omega', \mathcal{F}', (\mathcal{F}'_t)_{t\geq 0}, P'\right)$ be two filtered probability spaces and $X : \Omega \to \Omega'$ be a measurable mapping such that $P' = XP$; and such that $Z' \circ X$ is $\mathcal{F}_t$-measurable for every $\mathcal{F}'_t$-measurable*

Z'. Let $(M'_t)_{t\geq 0}$ be a continuous adapted process on $\left(\Omega', \mathcal{F}', (\mathcal{F}'_t)_{t\geq 0}, P'\right)$ such that $M_t := M'_t \circ X$ is a martingale on $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P\right)$ and there is an increasing adapted process $(A_t)_{t\geq 0}$ on $\left(\Omega', \mathcal{F}', (\mathcal{F}'_t)_{t\geq 0}, P'\right)$ such that $A_t \circ X = [M]_t$. Then $(M'_t)_{t\geq 0}$ is a martingale on $\left(\Omega', \mathcal{F}', (\mathcal{F}'_t)_{t\geq 0}, P'\right)$, with quadratic variation $(A_t)_{t\geq 0}$.

The proof is left as an exercise.

4.2 Existence of Solutions to the Martingale Problem

The theorem of existence will be based on a classical Galerkin approximation scheme. For other purposes, like the existence of the so called suitable weak solutions (satisfying local energy inequalities) other approximations must be used, but we shall not take that direction (see Flandoli and Romito [38]).

Let H_n be the finite dimensional space spanned by the first N_n eigenvectors of A, with N_n increasing to infinity. We endow H_n with the inner product induced by $|.|_H$, and use the same notation. Let A_n be the restriction of A to H_n and $B_n(.,.) : H_n \times H_n \to H_n$ the continuous bilinear operator defined as

$$\langle B_n(u,v), w\rangle = \langle B(u,v), w\rangle$$

for every $u, v, w \in H_n$. We have also

$$B_n(u,v) = \pi_n B(u,v), \quad u, v \in H_n$$

where π_n is the orthogonal projection of H on H_n.

Consider the equation in H_n

$$dX_t^n + [\nu A_n X_t^n + B_n(X_t^n, X_t^n)]\,dt = \sum_{i=1}^{N_n} \sigma_i h_i d\beta_i. \tag{2}$$

The operators in this equation satisfy all the assumptions of the previous Section, so we may use all the results proved there. Of course we may take advantage only of those estimates having *universal* constants.

Theorem 4.2. *Assume $\sigma^2 := \sum_i \sigma_i^2 < \infty$. Let μ be a measure on H such that $m_2 := \int_H |x|_H^2 \mu(dx) < \infty$. Then there exists at least one solution to the martingale problem with initial condition μ.*

Proof. **Step 1** (a priori bounds on Galerkin approximations). Let

$$\left(W, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, Q, (\beta_i(t))_{t\geq 0, i\in\mathbb{N}}\right)$$

be a Brownian stochastic basis supporting also an $\mathcal{F}_0$-measurable r.v. $u_0 : W \to H$ with law μ (to construct such a basis it is sufficient to use product spaces, as we did in theorem 3.3). Let $X_0^n := \pi_n u_0$.

For every n, there exist a unique continuous adapted solution $(X_t^n)_{t\geq 0}$ of equation (2) in H_n, with initial condition X_0^n. Under the embedding $H_n \subset H$ we have that $(X_t^n)_{t\geq 0}$ is a continuous adapted process in H, so it defines a measure P_n on $C([0,\infty);H)$, and thus on (Ω, F). In Section 3 we have proved

$$P_n\left[\sup_{t\in[0,T]}|\xi_t|_H^2+\nu\int_0^T\|\xi_s\|_V^2\,ds\right]\leq C_1\left(m_2,\sigma^2,T\right)$$

We have used the fact that

$$Q\left[|X_0^n|_H^2\right]=\int_H|\pi_n x|_H^2\,\mu(dx)\leq m_2.$$

Moreover, in view of the time regularity, equation (2) has the form

$$X_t^n=X_0^n+J_t^n+\sum_{i=1}^{N_n}\sigma_i h_i d\beta_i$$

where

$$J_t^n=-\int_0^t\left[\nu A_n X_s^n+B_n\left(X_s^n,X_s^n\right)\right]ds$$

and we have, on one side,

$$Q\left\|\sum_{i=1}^{N_n}\sigma_n h_n\beta_\cdot\right\|_{W^{\alpha,p}(0,T;H)}^p<C$$

(C independent of n) for every $p>1$, $\alpha\in(0,1/2)$, $T>0$, from Corollary 4.2; on the other side, for J_t^n, chosen $\gamma\in(3/2,2)$, we have

$$\begin{aligned}&\|J_\cdot^n\|_{W^{1,2}(0,T;D(A^{-\gamma}))}^2\\ \leq C_\nu\int_0^T|A_nX_s^n|_{V'}^2\,ds+C\int_0^T&|B_n\left(X_s^n,X_s^n\right)|_{D(A^{-\gamma})}^2\,ds\\ \leq C_\nu\int_0^T\|X_s^n\|_V^2\,ds+C\sup_{s\in[0,T]}&|X_s^n|_H^2\int_0^T\|X_s^n\|_V^2\,ds\end{aligned}$$

since, for $x,\varphi\in\mathcal{D}^\infty$,

$$\begin{aligned}|B_n(x,x)|_{D(A^{-\gamma})}^2&=\sup_{|\varphi|_{D(A^\gamma)}\leq 1}\left|\langle B_n(x,x),\varphi\rangle_{D(A^{-\gamma}),D(A^\gamma)}\right|\\&=\sup_{|\varphi|_{D(A^\gamma)}\leq 1}|\langle B(x,x),\varphi\rangle|\leq C\,|x|_H^2\,\|x\|_V^2\end{aligned}$$

from the Sobolev embedding of $D(A^\gamma)$ in the continuous fields. Therefore

$$P_n\left[\|\xi\|_{W^{\alpha,2}(0,T;D(A^{-\gamma}))}\right]\leq C_4\left(\nu,m_2,\sigma^2,T\right)$$

for every $\alpha\in(0,1/2)$, $\gamma\in(3/2,2)$, $T>0$.

Step 2 (tightness). By Chebyshev inequality, given $\alpha \in (0,1/2)$, $\gamma \in (3/2,2)$, $T>0$, for every $\varepsilon>0$ there is a bounded set

$$B_\varepsilon \subset L^2(0,T;V) \cap W^{\alpha,2}\left(0,T;D\left(A^{-\gamma}\right)\right)$$

such that $P_n(B_\varepsilon) > 1-\varepsilon$ for every n. From theorem 4.6, there is a compact set

$$K_\varepsilon \subset L^2(0,T;H)$$

such that $P_n(K_\varepsilon) > 1-\varepsilon$ for every n. From the boundedness of the law of J^n in $W^{1,2}(0,T;D(A^{-\gamma}))$ and of the law of the Brownian motion in $W^{\alpha,p}(0,T;H)$ for every $p>1$ and $\alpha \in (0,1/2)$, we may apply lemma 4.3 and have a compact set

$$K'_\varepsilon \subset C([0,T];D(A)')$$

such that $P_n(K'_\varepsilon) > 1-\varepsilon$ for every n. Therefore the family of measures $\{P_n\}$ is tight in $L^2(0,T;H)$ and in $C([0,T];D(A)')$, with their Borel σ-fields. Hence there exists a probability measure P on

$$C([0,T];D(A)') \cap L^2(0,T;H)$$

that is the weak limit in such spaces of a subsequence $\{P_{n_k}\}$.

Step 3 (P is a solution to the martingale problem). From the uniform estimates on $\{P_{n_k}\}$ in $L^2(0,T;V)$ and $L^\infty(0,T;H)$ we may deduce that P gives probability one to each one of these spaces and has bounds in the mean similar to those uniform of P_{n_k}. The details of this fact are rather tedious so we give only a sample in the next section, see lemma 4.8. This way we have checked property [MP1] in the definition of solution to the martingale problem.

Concerning [MP3], we have

$$P_{n_k}(\varphi) \to P(\varphi)$$

for every $\varphi \in C_b(C([0,T];D(A)'))$, hence in particular $\pi_0 P_{n_k} \to \pi_0 P$ as probability measures on $D(A)'$. But $\pi_0 P_{n_k}$ is the law of $\pi_n u_0$, which converges to μ since $\pi_n u_0$ converges Q-a.s. to u_0. Hence $\pi_0 P$ is μ.

Finally, let us check property [MP2]. Given $\varphi \in \mathcal{D}^\infty$, we have to prove that for every $t>s\geq 0$ and every F_s-measurable bounded r.v. Z, we have

$$P\left[\left(M_t^\varphi\right)^2\right] < \infty$$
$$P\left[\left(M_t^\varphi - M_s^\varphi\right) Z\right] = 0$$
$$P\left[\left(\left(M_t^\varphi\right)^2 - \varsigma_t - \left(\left(M_s^\varphi\right)^2 - \varsigma_s\right)\right) Z\right] = 0$$

where $\varsigma_t := \sum_{i=1}^\infty \sigma_i^2 \langle \varphi, h_i \rangle_H^2 \cdot t$. For the measure P_{n_k} we know (by lemma 4.1) that $(M_t^{\varphi,n_k}, F_t, P_{n_k})$ is a square integrable martingale with quadratic variation

$$[M^{\varphi,n_k}]_t = \sum_{i=1}^{N_{n_k}} \sigma_i^2 \langle \varphi, h_i \rangle_H^2 \cdot t$$

where

$$M_t^{\varphi,n} := \langle \xi_t, \varphi \rangle_H - \langle \xi_0, \varphi \rangle_H + \int_0^t \nu \langle \xi_s, A\varphi \rangle_H ds - \int_0^t \langle B(\xi_s, \pi_n \varphi), \xi_s \rangle_H ds.$$

Thus $(M_t^{\varphi,n_k}, F_t, P_{n_k})$ is a Brownian motion and we have

$$\sup_k P_{n_k} \left[(M_t^{\varphi,n_k})^{2+\varepsilon} \right] < \infty$$
$$P_{n_k} \left[(M_t^{\varphi,n_k} - M_s^{\varphi,n_k}) Z \right] = 0$$
$$P_{n_k} \left[\left((M_t^{\varphi,n_k})^2 - \varsigma_t^{n_k} - \left((M_s^{\varphi,n_k})^2 - \varsigma_s^{n_k} \right) \right) Z \right] = 0$$

where $\varsigma_t^n := \sum_{i=1}^{N_n} \sigma_i^2 \langle \varphi, h_i \rangle_H^2 \cdot t$ and $\varepsilon > 0$. It is now sufficient to use lemma 4.2 below. The proof is complete.

Remark 4.3. In the case of noise depending on u, the passage to the limit (step 3, proof of [MP2]) is more involved and requires uniform estimates on p-moments of X_t^n, see [34].

Lemma 4.2. *On a Polish space X, if P_n converges weakly to P (in the sense of measures), $\varphi_n, \varphi : X \to \mathbb{R}$ are measurable and $\varphi_n(x_n) \to \varphi(x)$ for every $x \in X$ and any sequence $x_n \to x$, and*

$$P_n \left[|\varphi_n|^{1+\varepsilon} \right] \leq C$$

for some $\varepsilon, C > 0$, then $P[|\varphi|] < \infty$ and $P_n[\varphi_n] \to P[\varphi]$.

Proof. Let Y_n, Y be r.v. on a probability space (Ω, F, Q), with expectation E, with values in X, with laws P_n and P respectively, such that $Y_n \xrightarrow{X} Y$, Q-a.s. (Skorohod theorem). We have to prove that $E[|\varphi(Y)|] < \infty$ and $E[\varphi_n(Y_n)] \to E[\varphi(Y)]$. But we know that

$$E \left[|\varphi_n(Y_n)|^{1+\varepsilon} \right] \leq C$$

and $\varphi_n(Y_n) \to \varphi(Y)$, Q-a.s.; hence it is sufficient to apply Vitali convergence theorem.

For the definitions of space homogeneous and partial isotropic measure or random field, see section 3.6 above. The proof of the following facts is similar to the previous theorem and will be omitted. Under the assumptions of theorem 4.2 we have:

Theorem 4.3. *If μ and $\sum_{i=1}^{\infty}\sigma_i h_i \beta_i(1)$ are space homogeneous and partial isotropic, then there exists a solution P of the martingale problem with initial condition μ such that $\pi_t P$ is space homogeneous and partial isotropic for every $t \geq 0$.*

If $\sum_i \sigma_i^2 \lambda_i < \infty$ then there exists a solution P of the martingale problem with initial condition μ such that

$$P\left[\int_0^T \frac{|A\xi_s|_H^2}{\left(1+\|\xi_s\|_V^2\right)^2} ds\right] < \infty.$$

In particular, $P(\xi_t \in D(A)) = 1$ for a.e. $t \geq 0$.

We complete the section by stating a result proved in [34], proof that is a variant of the previous one and we do not repeat. By *stationary* martingale solution we mean a solution P of the martingale problem that is shift invariant (in time) on Ω.

Theorem 4.4. *There exists at least one stationary martingale solution P_{stat}, with the following properties:*

$$P_{stat}\left[\|\xi_t\|_V^2\right] \leq \frac{TrQ}{2\nu}, \quad P_{stat}\left[|\xi_t|_H^p\right] < \infty$$

for every $t \geq 0$ and $p \geq 2$.

Remark 4.4. In dimension $d = 2$ we have the identity

$$E^{P_{stat}}\|\xi_t\|_V^2 = \frac{TrQ}{2\nu}.$$

Theorem 4.5. *If $\sum_{i=1}^{\infty}\sigma_i h_i \beta_i(1)$ is space homogeneous and partial isotropic, there exists at least one stationary martingale solution P_{stat} such that $\pi_t P_{stat}$ is space homogeneous and partial isotropic for every $t \geq 0$.*

If $\sum_i \sigma_i^2 \lambda_i < \infty$ then there exists at least one stationary martingale solution P_{stat} such that for every $t \geq 0$ we have $P_{stat}(\xi_t \in D(A)) = 1$ and

$$P_{stat}\left[\frac{|A\xi_t|_H^2}{\left(1+\|\xi_t\|_V^2\right)^2}\right] < \infty.$$

In the usual context of Markov dynamics, one introduces the notion of invariant measure (see above in the finite dimensional case). This can be done in dimension 2, but in 3D we have the obstacle of the lack of Markov property (at least a priori; see the next sections). Nevertheless, there are several ways to introduce measures that may play the role of invariant measures. In order to stress the difference w.r.t. the Markov set-up, we shall call them stationary measures. Recall the definition of Galerkin stationary measure given above; call $P_{NS}^{Galerkin}$ the set of such measures; we have proved that it is non empty. Here, having the existence of stationary solutions, it is meaningful to consider the measures of the form $\pi_t P_{stat}$, where P_{stat} is a stationary solution of the martingale problem; call $P_{NS}^{stationary}$ the set of such measures. One can prove the following relation:

$$P_{NS}^{Galerkin} \subset P_{NS}^{stationary}$$

However, the other relations are less clear (opposite inclusion, relation to invariant measures for Markov selections, etc.).

4.3 Technical Complements

We collect here some technical facts used in the previous section.

Sobolev spaces of fractional order

Let E be a (separable) Banach space, $p > 1$, $\alpha \in (0,1)$, $T > 0$, $W^{\alpha,p}(0,T;E)$ the (Sobolev) space of all $u \in L^p(0,T;E)$ such that

$$[u]^p_{W^{\alpha,p}(0,T;E)} := \int_0^T \int_0^T \frac{|u(t)-u(s)|_E^p}{|t-s|^{1+\alpha p}} dt ds < \infty$$

endowed with the norm

$$\|u\|^p_{W^{\alpha,p}(0,T;E)} = \int_0^T |u(t)|_E^p \, dt + [u]^p_{W^{\alpha,p}(0,T;E)}.$$

One can show that $W^{\alpha,p}(0,T;E)$ is a (separable) Banach space. Moreover, if $\alpha p > 1$, $W^{\alpha,p}(0,T;E) \subset C^\gamma([0,T];E)$ for every $\gamma < \alpha p - 1$. We also have:

Lemma 4.3. *If $E \subset \widetilde{E}$ are two Banach spaces with compact embedding, $p > 1$ and $\alpha \in (0,1)$ satisfy $\alpha p > 1$, then $W^{\alpha,p}(0,T;E)$ is compactly embedded into $C\left([0,T];\widetilde{E}\right)$. Similarly, if $E_1,...,E_n$ are compactly embedded into $\widetilde{E}$ and $p_1,...,p_n > 1$, $\alpha_1,...,\alpha_n \in (0,1)$ satisfy $\alpha_i p_i > 1$ for every $i = 1,...,n$, then*

$$W^{\alpha_1,p_1}(0,T;E_1) + \cdots + W^{\alpha_n,p_n}(0,T;E_n)$$

is compactly embedded into $C\left([0,T];\widetilde{E}\right)$.

Theorem 4.6. *Let $E_0 \subset E \subset E_1$ be Banach spaces, E_0 and E_1 reflexive, E_0 compactly embedded in E, E continuously embedded into E_1. Given $p > 1$, $\alpha \in (0,1)$, $T > 0$, the space*

$$X := L^p(0,T;E_0) \cap W^{\alpha,p}(0,T;E_1)$$

is compactly embedded into $L^p(0,T;E)$.

Detailed proofs can be found in [34].

Gaussian measures in Hilbert spaces

Let X be the gaussian r.v. in H defined as

$$X = \sum_{i=1}^{\infty} \sigma_i h^{(i)} X_i$$

where $\left(h^{(i)}\right)$ is a c.o.s. in H, $\sum_{i=1}^{\infty} \sigma_i^2 < \infty$ and (X_i) is a sequence of independent standard Gaussian random variables on a probability space $(\Omega, \mathcal{F}, P)$ with expectation E. Then:

Lemma 4.4. *For all* $t \in [0, \inf_i \frac{1}{2\sigma_i})$ *we have*

$$E\left[e^{t|X|_H^2}\right] = \exp\left(-\frac{1}{2}\sum_{i=1}^{\infty} \log\left(1 - 2\sigma_i^2 t\right)\right).$$

Proof. We leave the proof as an exercise, based on the formula

$$\frac{1}{\sqrt{2\pi\sigma_i^2}} \int_{-\infty}^{+\infty} \exp\left(tx^2 - \frac{x^2}{2\sigma_i^2}\right) dx = \frac{1}{\sqrt{1 - 2\sigma_i^2 t}}.$$

Corollary 4.1. *For every* $p > 1$ *we have*

$$E\left[|X|_H^p\right] \leq C_p \left(\sum_{i=1}^{\infty} \sigma_i^2\right)^{p/2}.$$

Proof. If $p = 2m$ with a positive integer m, this follows from the lemma by differentiation. For $p \in (2m-1, 2m)$ we apply Hölder inequality:

$$E\left[|X|_H^p\right] \leq E\left[\left(|X|_H^p\right)^{\frac{2m}{p}}\right]^{\frac{p}{2m}} \leq C_{2m} \left(\sum_{i=1}^{\infty} \sigma_i^2\right)^{p/2}.$$

Remark 4.5. Applied to Gaussian martingales, this is the upper bound in Burkholder-Davies-Gundy (BDG) inequality. In fact, if we want to deal with non additive noise, we have to replace the present arguments with BDG inequality.

Corollary 4.2. *If* $B(t)$ *is a Brownian motion in* H *given by*

$$B(t) = \sum_{i=1}^{\infty} \sigma_i h^{(i)} \beta^{(i)}(t)$$

with $\sum_{i=1}^{\infty}\sigma_i^2<\infty$ *and* $\left(\beta^{(i)}(t)\right)$ *a sequence of independent standard Brownian motions, then for every* $p>1$, $\alpha\in(0,1/2)$, $T>0$,

$$E\left\|B\right\|^p_{W^{\alpha,p}(0,T;H)}<C\left(p,\alpha,T\right)\left(\sum_{i=1}^{\infty}\sigma_i^2\right)^{p/2}.$$

Proof. From the corollary above we have

$$\begin{aligned}&E\int_0^T\int_0^T\frac{\left|B(t)-B(s)\right|_H^p}{\left|t-s\right|^{1+\alpha p}}dtds\\&\leq\int_0^T\int_0^T\frac{C_p\left(\sum_{i=1}^{\infty}\sigma_i^2\right)^{p/2}(t-s)^{p/2}}{\left|t-s\right|^{1+\alpha p}}dtds\\&=C_p\left(\sum_{i=1}^{\infty}\sigma_i^2\right)^{p/2}\int_0^T\int_0^T\frac{1}{\left|t-s\right|^{1+\left(\alpha-\frac{1}{2}\right)p}}dtds.\end{aligned}$$

The integral is finite since $1+\left(\alpha-\frac{1}{2}\right)p<1$. The proof is complete.

Remark 4.6. If we would not know yet that $B(t)$ has a.s. continuous trajectories, we could deduce it from the previous result.

Remarks on $\Omega=C\left([0,\infty);D(A)'\right)$

The following and other similar results are used several times throughout this Section, often without mention. We discuss the following ones as a sample. The general idea of the following results is that apparently stronger topologies in $D(A)'$ and Ω define measurable sets and functions. First, H is a Borel set in $D(A)'$. Indeed, there is a c.o.s. $\{e_n\}$ in H made of elements of $D(A)$, such that, for an element $x\in D(A)'$, we have $x\in H$ if and only if $\sum\langle x,e_n\rangle^2<\infty$ (since $e_n\in D(A)$, $\langle x,e_n\rangle$ is a priori well defined for $x\in D(A)'$, so $\sum\langle x,e_n\rangle^2$ either converges or diverges to $+\infty$). Similarly, the (possibly infinite) function $x\mapsto|x|_H$ is measurable on $D(A)'$.

Lemma 4.5. $C\left([0,\infty);H\right)$ *is a Borel set in* Ω.

Proof. Fix a dense countable set D in $[0,\infty)$. For an $\omega\in\Omega$ we have $\omega\in C\left([0,\infty);H\right)$ if and only if it is uniformly continuous on every bounded subset of D: for every$N>0$, $n>0$ there is $m>0$ such that $t,s\in D\cap[0,N]$, $|t-s|<1/m$ implies $|\omega(t)-\omega(s)|_H<1/n$. This condition is expressed by means of countably many operations (recall also that $x\mapsto|x|_H$ is measurable on $D(A)'$).

Lemma 4.6. *Let $B([0,\infty);H)$ be the set of H-valued functions ω, bounded on every bounded set, i.e. such that for every $T>0$*

$$\sup_{t\in[0,T]} |\omega(t)|_H < \infty.$$

Then

$$B([0,\infty);H)\cap \Omega = C\left([0,\infty);H_\sigma\right)\cap \Omega.$$

Moreover

$$B([0,\infty);H)\cap \Omega \in F$$

and the (possibly infinite) function

$$f(\omega) = \sup_{t\in[0,T]} \sum_{n=1}^{\infty} \langle \omega(t), e_n\rangle^2$$

is measurable, for every $T>0$.

Proof. If $\omega \in B([0,\infty);H)\cap\Omega$ and $\varphi\in H$, then, given$T>0$, $t_0\in[0,T]$ and $\varepsilon>0$, let $\varphi'\in D(A)$ be such that $|\varphi-\varphi'|_H\le\varepsilon$, and take $\delta>0$ such that $|\omega(t)-\omega(t_0)|_{D(A)'}\le\varepsilon$ for $|t-t_0|\le\delta$, $t,t_0\in[0,T]$. We have

$$\begin{aligned}|\langle \omega(t)-\omega(t_0),\varphi\rangle| &\le |\langle \omega(t)-\omega(t_0),\varphi-\varphi'\rangle| \\ &\quad + |\langle \omega(t)-\omega(t_0),\varphi'\rangle| \\ &\le \varepsilon C + \varepsilon C.\end{aligned}$$

Viceversa, if $\omega\in C\left([0,\infty);H_\sigma\right)\cap\Omega$, then, for every $T>0$ and $\varphi\in H$,

$$\sup_{t\in[0,T]} |\langle \omega(t),\varphi\rangle| \le \infty.$$

So $(\omega(t))$ is a family of functionals on H that are pointwise equibounded. By Banach-Steinhaus theorem, they are equibounded in the operator norm, namely

$$\sup_{t\in[0,T]} |\omega(t)|_H^2 < \infty.$$

Finally, about the measurability, consider the sets

$$A_{N,R} = \left\{ \omega\in\Omega : \sup_{t\in[0,T]} \sum_{n=1}^{N} \langle \omega(t), e_n\rangle^2 \le R \right\}.$$

They are measurable and

$$\bigcup_{R>0}\bigcap_{N>0} A_{N,R} = B([0,\infty);H)\cap\Omega.$$

The proof is complete.

Lemma 4.7. *Let $P \in \Pr(\Omega)$ be such that*

$$P\left(C\left([0,\infty);H_\sigma\right)\cap\Omega\right)=1.$$

Then, given $t \geq 0$, the mapping $\omega \mapsto \omega(t)$, a priori F_t-measurable with values in $D(A)'$, has a P-modification on F_t that is F_t-measurable with values in $(H, \mathcal{B}(H))$.

Lemma 4.8. *Let $P_n \in \Pr(\Omega)$, weakly convergent to P. Assume that for every $T > 0$ there is a constant $C_T > 0$ such that*

$$P_n\left[\sup_{t\in[0,T]}|\omega(t)|_H^2\right]\leq C_T.$$

Then $P\left(B([0,\infty);H)\cap\Omega\right)=1$ and

$$P\left[\sup_{t\in[0,T]}|\omega(t)|_H^2\right]\leq C_T.$$

Proof. Given $R, N > 0$, the functional

$$f_{N,R}(\omega)=\sup_{t\in[0,T]}\sum_{n=1}^{N}\langle\omega(t),e_n\rangle^2\wedge R$$

belong to $C_b(H)$, so

$$\lim_{n\to\infty}P_n\left(f_{N,R}\right)=P\left(f_{N,R}\right).$$

Moreover,

$$f_{N,R}(\omega)\leq\sup_{t\in[0,T]}|\omega(t)|_H^2$$

so $P_n(f_{N,R}) \leq C$ and thus $P(f_{N,R}) \leq C$. By monotone convergence we get the result.

4.4 An Abstract Markov Selection Result

The topics of this and the following sections are quite technical and a complete treatment of them would exceed the reasonable size of this note. Therefore we limit the discussion to the main ideas. Details of this section can be found in [39].

Let $\mathcal{V} \subset \mathcal{H} \subset \mathcal{V}'$ be a Gelfand triple of separable Hilbert spaces with continuous dense injections. In our application $\mathcal{V}$ will be $D(A)$. Denote by Ω the space $C([0,\infty);\mathcal{V}')$, with Borel σ-field $\mathcal{B}$, and for every $t \geq 0$ we set $\Omega^t := C([t,\infty);\mathcal{V}')$, with its Borel σ-field $\mathcal{B}^t$; clearly Ω^t is isomorphic to Ω by the natural map $\Phi_t : \Omega \to \Omega^t$, $(\Phi_t\omega)(t+s) = \omega(s)$ for every $s \geq 0$. Denote also by $(F_t)_{t\geq 0}$ the canonical filtration on Ω.

Given P on Ω, given $t>0$, there is an F_t-measurable (P-unique) function $\omega \mapsto P_\omega^{F_t}$, from Ω to $\Pr(\Omega^t)$, such that

$$P\left(A^{ext}\cap F\right) = \int_F P_\omega^{F_t}(A)\, dP(\omega)$$

for every $F \in F_t$ and $A \in \mathcal{B}^t$, where $A^{ext} \in \mathcal{B}$ is the set $\left\{\omega|_{[t,\infty)} \in A\right\}$. Moreover,

$$P_\omega^{F_t}(\xi_t = \omega(t)) = 1$$

for P-a.e. $\omega \in \Omega$. The existence of such function $P_\omega^{F_t}$ comes from the existence of a regular conditional probability distribution, which exists since Ω is Polish and F_t is countably generated. We also have:

Lemma 4.9. *Given P on Ω and an F_t-measurable function $\omega \mapsto Q_\omega$, from Ω to $\Pr(\Omega^t)$, such that*

$$Q_\omega(\xi_t = \omega(t)) = 1$$

for every $\omega \in \Omega$, there is a (unique) measure P^Q on Ω such that

$$P^Q(F) = P(F) \text{ for every } A \in F_t$$

$$\left(P^Q\right)_\omega^{F_t} = Q_\omega$$

for P^Q-a.e. $\omega \in \Omega$.

Details can be found in [62]. The idea is to define

$$P^Q(F \times A) = \int_F Q_\omega(A)\, dP(\omega)$$

and verify all the assertions.

Definition 4.5. *Let $\{P^x\}_{x\in H} \subset \Pr(\Omega)$ be a family of measures such that*

$$P^x(\Omega \cap C([0,\infty);\mathcal{H}_\sigma)) = 1.$$

We say it is Markov if for every $t \geq 0$

$$(P^x)_\omega^{F_t} = \Phi_t P^{\omega(t)} \text{ for } P^x\text{-a.e. } \omega \in \Omega.$$

A priori $\omega \in \Omega$, so $P^{\omega(t)}$ could be not-well-defined, but we require the identity only for P^x-a.e. ω, and we know that P^x is supported by $\mathcal{H}$ valued functions. So the previous definition is meaningful.

Definition 4.6. *Let $\{C^x \subset \Pr(\Omega)\,; x \in \mathcal{H}\}$ be a collection of families of measures such that $P(\Omega \cap C([0,\infty);\mathcal{H}_\sigma)) = 1$ for every $P \in C^x$. We say it is pre-Markov if for every $t \geq 0$ the following two assertions hold:*

i) for every $P \in C^x$,

$$(P)_\omega^{F_t} \in \Phi_t C^{\omega(t)} \text{ for } P\text{-a.e. } \omega \in \Omega$$

ii) for every $P \in C^x$, and every F_t-measurable function $\omega \mapsto Q_\omega$, from Ω to $\Pr(\Omega^t)$*, such that $Q_\omega \in \Phi_t C^{\omega(t)}$ for every $\omega \in \Omega$,*

$$P^Q \in C^x.$$

Remark 4.7. If a family of singletons is pre-Markov, then it is Markov.

The set $\Pr(\Omega)$ with the weak converge is Polish. Denote by $Comp(\Pr(\Omega))$ the family of all compact sets in $\Pr(\Omega)$. It is a metric space and we can talk about measurability of functions from a measurable space to $Comp(\Pr(\Omega))$ (see [62] for details).

Remark 4.8. To understand the following proof it may be useful to recall the following well-known principle in the calculus of variations: if the functional to be maximized is "local", then every segment of a global maximizer is a maximizer of the corresponding segmented functional. To be more specific, assume $f^*(t)$ maximizes the functional $J_0(f) := \int_0^T \varphi(f(t))dt$ (under suitable assumptions on φ) with the constraint $f(0) = x^0$. Then, given $s \in (0,T)$, the segment $f^*|_{[s,T]}$ maximizes the functional $J_s(f) := \int_s^T \varphi(f(t))dt$ with the constraint $f(s) = f^*(s)$. Indeed, if this would not be true, if g is a function on $[s,T]$ with $g(s) = f^*(s)$ and $J_s(g) > J_s(f^*)$, then the function

$$\widetilde{f} = \begin{cases} f^* \text{ on } [0,s] \\ g \text{ on } [s,T] \end{cases}$$

has the property $J_0\left(\widetilde{f}\right) > J_0(f^*)$, contradicting the assumption that f^* was optimal for J_0.

Theorem 4.7. *Let $\{C^x \subset \Pr(\Omega); x \in \mathcal{H}\}$ be a pre-Markov family such that*

$$P(\Omega \cap C([0,\infty); \mathcal{H}_\sigma)) = 1$$

for every $P \in C^x$. If C^x is a convex compact set in $\Pr(\Omega)$ for every $x \in \mathcal{H}$ and $x \mapsto C^x$ is measurable, then there exist a Markov selection.

Proof. We essentially repeat the proof of [62], with minor topological remarks due to the infinite dimensions and different notations.

Step 1 (reduction of C^x by local functionals; preparation). Given a measurable pre-Markov family $\{C^x \subset \Pr(\Omega); x \in \mathcal{H}\}$, given $\lambda > 0$, let us define the operator $R_\lambda^+ : C_b(\mathcal{V}') \to C_b(\mathcal{V}')$ as

$$\left(R_\lambda^+ \varphi\right)(x) = \sup_{P \in C^x} J_{\varphi,\lambda}(P)$$

where, for any $\varphi \in C_b(\mathcal{V}')$ and $\lambda > 0$, the functional $J_{\varphi,\lambda}$ on $\Pr(\Omega)$ is defined as

$$J_{\varphi,\lambda}(P) = P\left[\int_0^\infty e^{-\lambda t}\varphi(\xi_t)\,dt\right].$$

The notation R_λ^+ is inspired by the particular case when C^x is a singleton and thus we have a Markov process $\{P^x \in \Pr(\Omega); x \in \mathcal{H}\}$; if it is sufficiently regular and L is its infinitesimal generator, then

$$R_\lambda^+ = (\lambda - L)^{-1}$$

(a rigorous formulation of this sentence requires specification of function spaces and regularities that are not of interest here).

Since the function

$$\omega \mapsto \int_0^\infty e^{-\lambda t}\varphi(\xi_t(\omega))\,dt$$

is bounded and continuous on Ω, $J_{\varphi,\lambda}$ is continuous on $\Pr(\Omega)$. Therefore, given $x \in \mathcal{H}$, on the compact set C^x there is at least one maximizing element for $J_{\varphi,\lambda}$. Denote by $C^x_{\varphi,\lambda}$ the set of all such maximizing elements; thus

$$\left(R_\lambda^+\varphi\right)(x) = J_{\varphi,\lambda}\left(C^x_{\varphi,\lambda}\right).$$

Let us show that the family

$$\left\{C^x_{\varphi,\lambda} \subset \Pr(\Omega); x \in \mathcal{H}\right\}$$

is pre-Markov and has all the same properties of $\{C^x \subset \Pr(\Omega); x \in \mathcal{H}\}$. Clearly

$$P\left(\Omega \cap C([0,\infty);\mathcal{H}_\sigma)\right) = 1$$

for every $P \in C^x_{\varphi,\lambda}$. The set $C^x_{\varphi,\lambda}$ is compact (it is the set of maximizing elements of a continuous mapping on a compact set). The mapping $x \mapsto C^x_{\varphi,\lambda}$ is measurable since the two mappings $x \mapsto C^x$ and $C^x \mapsto C^x_{\varphi,\lambda}$ are measurable (the last assertion comes from [62], lemma 12.1.7). Finally, $C^x_{\varphi,\lambda}$ is convex: given $P^i \in C^x_{\varphi,\lambda}$ and $\alpha_i \geq 0$, $i = 1,2$, such that $\alpha_1 + \alpha_2 = 1$, setting $P = \alpha_1 P^1 + \alpha_2 P^2$, we have

$$J_{\varphi,\lambda}(P) = \alpha_1 J_{\varphi,\lambda}\left(P^1\right) + \alpha_2 J_{\varphi,\lambda}\left(P^2\right)$$

which implies that $P \in C^x_{\varphi,\lambda}$. Let us prove it is pre-Markov.

Step 2 (pre-Markov property, part 1). First, let us prove that for every $P \in C^x_{\varphi,\lambda}$,

$$P\left[\omega \in \Omega : (P)^{F_t}_\omega \in \Phi_t C^{\omega(t)}_{\varphi,\lambda}\right] = 1. \tag{3}$$

As a preliminary remark, notice that $\omega \mapsto (P)^{F_t}_\omega$ is F_t-measurable with values in $\Pr(\Omega^t)$, and up to a P-modification $\omega \mapsto \Phi_t C^{\omega(t)}_{\varphi,\lambda}$ is F_t-measurable with values in compact sets in $\Pr(\Omega^t)$, because $\omega \mapsto \omega(t)$ is F_t-measurable with values in $\mathcal{H}$ and $x \mapsto \Phi_t C^x_{\varphi,\lambda}$ is measurable from $\mathcal{B}(\mathcal{H})$ to compact sets in $\Pr(\Omega^t)$. Therefore, by [62], lemma 12.1.9, the set

$$\left[\omega \in \Omega : (P)^{F_t}_\omega \in \Phi_t C^{\omega(t)}_{\varphi,\lambda}\right]$$

belongs to F_t. If (3) is not true, there is $A \in F_t$ such that $P(A) > 0$ and $(P)^{F_t}_\omega \notin \Phi_t C^{\omega(t)}_{\varphi,\lambda}$ for every $\omega \in A$, namely

$$J_{\varphi,\lambda}\left(\Phi_t^{-1}(P)^{F_t}_\omega\right) < \max_{C^{\omega(t)}} J_{\varphi,\lambda}$$

for every $\omega \in A$.

Choose an F_t-measurable selection from $\Omega \ni \omega \mapsto \Phi_t C^{\omega(t)}_{\varphi,\lambda}$, call it Q_ω, define the F_t-measurable mapping

$$R_\omega = \begin{cases} Q_\omega & \text{if } \omega \notin A \\ (P)^{F_t}_\omega & \text{if } \omega \in A \end{cases}$$

with values in $\Pr(\Omega^t)$, and define the probability measure P^R. We have

$$J_{\varphi,\lambda}\left(\Phi_t^{-1}(P)^{F_t}_\omega\right) < J_{\varphi,\lambda}\left(\Phi_t^{-1} Q_\omega\right) \tag{4}$$

for every $\omega \in A$.

We show now that $J_{\varphi,\lambda}\left(P^R\right) > J_{\varphi,\lambda}(P)$, which is a contradiction since P is a maximizer; the proof of (3) will be then complete, by contradiction. We have

$$\begin{aligned} J_{\varphi,\lambda}\left(P^R\right) &= P^R\left[P^R\left[\left.\int_0^\infty e^{-\lambda s}\varphi(\xi_s)\,ds\right| F_t\right]\right] \\ &= P^R\left[\int_0^t e^{-\lambda s}\varphi(\xi_s)\,ds\right] + e^{-\lambda t}P^R\left[P^R\left[\left.\int_t^\infty e^{-\lambda(s-t)}\varphi(\xi_s)\,ds\right| F_t\right]\right] \\ &= P\left[\int_0^t e^{-\lambda s}\varphi(\xi_s)\,ds\right] + e^{-\lambda t}P\left[P^R\left[\left.\int_t^\infty e^{-\lambda(s-t)}\varphi(\xi_s)\,ds\right| F_t\right]\right] \end{aligned}$$

and

$$\begin{aligned} & P^R\left[\left.\int_t^\infty e^{-\lambda(s-t)}\varphi(\xi_s)\,ds\right| F_t\right](\omega) \\ &= \begin{cases} Q_\omega\left[\int_t^\infty e^{-\lambda(s-t)}\varphi(\xi_s)\,ds\right] & \text{if } \omega \notin A \\ (P)^{F_t}_\omega\left[\int_t^\infty e^{-\lambda(s-t)}\varphi(\xi_s)\,ds\right] & \text{if } \omega \in A \end{cases} \\ &= \begin{cases} \Phi_t^{-1}Q_\omega\left[\int_0^\infty e^{-\lambda s}\varphi(\xi_s)\,ds\right] & \text{if } \omega \notin A \\ \Phi_t^{-1}(P)^{F_t}_\omega\left[\int_0^\infty e^{-\lambda s}\varphi(\xi_s)\,ds\right] & \text{if } \omega \in A \end{cases} \\ &= \begin{cases} J_{\varphi,\lambda}\left(\Phi_t^{-1}Q_\omega\right) & \text{if } \omega \notin A \\ J_{\varphi,\lambda}\left(\Phi_t^{-1}(P)^{F_t}_\omega\right) & \text{if } \omega \in A \end{cases} \end{aligned}$$

so, by (4),

$$
\begin{aligned}
P\left[P^R\left[\int_t^\infty e^{-\lambda(s-t)}\varphi(\xi_s)\,ds\bigg|\,F_t\right]\right] &> P\left[J_{\varphi,\lambda}\left(\Phi_t^{-1}(P)_\omega^{F_t}\right)\right] \\
&= P\left[(P)_\omega^{F_t}\left[\int_t^\infty e^{-\lambda(s-t)}\varphi(\xi_s)\,ds\right]\right] \\
&= P\left[P\left[\int_t^\infty e^{-\lambda(s-t)}\varphi(\xi_s)\,ds\bigg|\,F_t\right]\right] \\
&= P\left[\int_t^\infty e^{-\lambda(s-t)}\varphi(\xi_s)\,ds\right].
\end{aligned}
$$

Therefore

$$
\begin{aligned}
J_{\varphi,\lambda}\left(P^R\right) &> P\left[\int_0^t e^{-\lambda s}\varphi(\xi_s)\,ds\right] + e^{-\lambda t}P\left[\int_t^\infty e^{-\lambda(s-t)}\varphi(\xi_s)\,ds\right] \\
&= J_{\varphi,\lambda}(P).
\end{aligned}
$$

The proof of the first part of the pre-Markov property is complete.

Step 3 (pre-Markov property, part 2). Let us prove that for every $P \in C_{\varphi,\lambda}^x$, and every F_t-measurable function $\omega \mapsto Q_\omega$, from Ω to $\Pr(\Omega^t)$, such that $Q_\omega \in \Phi_t C_{\varphi,\lambda}^{\omega(t)}$ for every $\omega \in \Omega$,

$$
P^Q \in C_{\varphi,\lambda}^x.
$$

We have, similarly to some of the above arguments,

$$
\begin{aligned}
J_{\varphi,\lambda}\left(P^Q\right) &= P\left[\int_0^t e^{-\lambda s}\varphi(\xi_s)\,ds\right] + e^{-\lambda t}P\left[P^Q\left[\int_t^\infty e^{-\lambda(s-t)}\varphi(\xi_s)\,ds\bigg|\,F_t\right]\right] \\
&= P\left[\int_0^t e^{-\lambda s}\varphi(\xi_s)\,ds\right] + e^{-\lambda t}P\left[Q_\omega\int_t^\infty e^{-\lambda(s-t)}\varphi(\xi_s)\,ds\right] \\
&\geq P\left[\int_0^t e^{-\lambda s}\varphi(\xi_s)\,ds\right] + e^{-\lambda t}P\left[(P)_\omega^{F_t}\int_t^\infty e^{-\lambda(s-t)}\varphi(\xi_s)\,ds\right]
\end{aligned}
$$

since Q_ω is a maximizer,

$$
\begin{aligned}
&= P\left[\int_0^t e^{-\lambda s}\varphi(\xi_s)\,ds\right] + e^{-\lambda t}P\left[P\left[\int_t^\infty e^{-\lambda(s-t)}\varphi(\xi_s)\,ds\bigg|\,F_t\right]\right] \\
&= J_{\varphi,\lambda}(P)
\end{aligned}
$$

hence P^Q is a maximizer.

Step 4 (iterative reduction to singletons) Let $\{\psi_j\}$ and $\{\theta_k\}$ be dense subsets of $(0,\infty)$ and $C_b(\mathcal{V}')$ respectively. Let $\{\varphi_n,\lambda_n\}$ be an enumeration of $\{\psi_j,\theta_k\}_{j,k}$. Given $x \in H$, let $C_{\varphi_1,\lambda_1}^x$ be the set of maximizers of J_{φ_1,λ_1} over C^x, $C_{\varphi_2,\lambda_2}^x$ be the set of maximizers of J_{φ_2,λ_2} over $C_{\varphi_1,\lambda_1}^x$, and so on. The sets

$C^x_{\varphi_n,\lambda_n}$ are a decreasing sequence of compact sets, hence they have non empty compact intersection, that we denote by $\widetilde{C}^x$.

The family $\left\{\widetilde{C}^x\right\}_{x\in H}$ is pre-Markov: it is the intersection of a sequence of pre-Markov families (it is easy to check that the pre-Markov property is preserved by countable intersection).

Let us prove that $\widetilde{C}^x$ is a singleton; this will imply that the family $\left\{\widetilde{C}^x\right\}_{x\in H}$ is Markov. If $P,Q\in\widetilde{C}^x$, then, for every n, $P,Q\in C^x_{\varphi_n,\lambda_n}$, hence

$$J_{\varphi_n,\lambda_n}(P)=J_{\varphi_n,\lambda_n}(Q)$$

This means

$$\int_0^\infty e^{-\theta_k t}P\left[\psi_j(\xi_t)\right]dt=\int_0^\infty e^{-\theta_k t}Q\left[\psi_j(\xi_t)\right]dt$$

for every j,k, and since $t\mapsto P\left[\psi_j(\xi_t)\right]$ and $t\mapsto Q\left[\psi_j(\xi_t)\right]$ are continuous, from the uniqueness of the Laplace transform we have

$$P\left[\psi_j(\xi_t)\right]=Q\left[\psi_j(\xi_t)\right]$$

for every t and j. Hence

$$P\left[\varphi(\xi_t)\right]=Q\left[\varphi(\xi_t)\right]$$

for every t and $\varphi\in C_b(V')$.

Step 5 (conclusion). Let us summarize what we know: that for every $x\in\mathcal{H}$, for every $P,Q\in\widetilde{C}^x$, for every t and $\varphi\in C_b(\mathcal{V}')$ we have $P\left[\varphi(\xi_t)\right]=Q\left[\varphi(\xi_t)\right]$. We have to prove the following statement: given $x\in\mathcal{H}$ and $P,Q\in\widetilde{C}^x$, for every n, every $0\le t_1<...<t_n$ and every $\varphi_1,...,\varphi_n\in C_b(\mathcal{V}')$,

$$P\left[\varphi_1(\xi_{t_1})...\varphi_n(\xi_{t_n})\right]=Q\left[\varphi_1(\xi_{t_1})...\varphi_n(\xi_{t_n})\right].$$

We prove it by induction. It is true for $n=1$. Assume it is true for n. Denote by $M_{t_1.....t_n}$ the σ-field generated by $\xi_{t_1},...,\xi_{t_n}$. We have

$$\begin{aligned}&P\left[\varphi_1(\xi_{t_1})...\varphi_n(\xi_{t_n})\varphi_{n+1}\left(\xi_{t_{n+1}}\right)\right]\\&=P\left[\varphi_1(\xi_{t_1})...\varphi_n(\xi_{t_n})P\left[\varphi_{n+1}\left(\xi_{t_{n+1}}\right)|M_{t_1.....t_n}\right]\right]\end{aligned}$$

so if we prove that

$$P\left[\varphi_{n+1}\left(\xi_{t_{n+1}}\right)|M_{t_1.....t_n}\right]=Q\left[\varphi_{n+1}\left(\xi_{t_{n+1}}\right)|M_{t_1.....t_n}\right],\quad P-a.s.$$

the proof will be complete, by the induction hypothesis (notice we cannot take simply F_{t_n} in place of $M_{t_1.....t_n}$ because we have to apply here the induction hypothesis). With other notations, we have to prove that

$$P^{M_{t_1.....t_n}}_\omega\left[\varphi\left(\xi_{t_{n+1}}\right)\right]=Q^{M_{t_1.....t_n}}_\omega\left[\varphi\left(\xi_{t_{n+1}}\right)\right],\quad P-a.s.$$

for every $\varphi \in C_b(\mathcal{V}')$. If we had F_{t_n} here in place of $M_{t_1,\ldots,t_n}$, the proof would be complete by the main assumption, since a.s. we have $P_\omega^{F_{t_n}}, Q_\omega^{F_{t_n}} \in \Phi_{t_n}\widetilde{C}^{\omega(t_n)}$. Let us use this fact.

We know that the family $\left\{\widetilde{C}^x\right\}_{x\in H}$ is pre-Markov, so there are sets N_P, $N_Q \in F_{t_n}$, with $P(N_P) = 0$ and $Q(N_Q) = 0$, such that $P_\omega^{F_{t_n}} \in \Phi_{t_n}\widetilde{C}^{\omega(t_n)}$ for every $\omega \notin N_P$ and $Q_\omega^{F_{t_n}} \in \Phi_{t_n}\widetilde{C}^{\omega(t_n)}$ for every $\omega \notin N_Q$. Therefore

$$P_\omega^{F_{t_n}}\left[\varphi\left(\xi_{t_{n+1}}\right)\right] = Q_\omega^{F_{t_n}}\left[\varphi\left(\xi_{t_{n+1}}\right)\right]$$

for every $\omega \notin N_P \cup N_Q$.

We have

$$P_\omega^{M_{t_1,\ldots,t_n}}(.) = \int P_{\omega'}^{F_{t_n}}(.)\, P_\omega^{M_{t_1,\ldots,t_n}}(d\omega')$$

and $P_\omega^{M_{t_1,\ldots,t_n}}(\xi(t_n) = \omega(t_n)) = 1$, so the integral is a convex combination of elements of $\Phi_{t_n}\widetilde{C}^{\omega(t_n)}$. By the convexity property of $\Phi_{t_n}\widetilde{C}^{\omega(t_n)}$ we get $P_\omega^{M_{t_1,\ldots,t_n}} \in \Phi_{t_n}\widetilde{C}^{\omega(t_n)}$, and similarly for $Q_\omega^{M_{t_1,\ldots,t_n}}$. This implies $P = Q$ and the proof is complete.

Remark 4.9. As in [62], one can easily show that there exists a unique Markov selection if and only if for every $x \in \mathcal{H}$ the set C^x is a singleton.

Remark 4.10. With less easy notations one can prove the strong Markov property, under a pre-strong Markov property for C^x, see [62], [39].

Remark 4.11. A Markov selection allows us to define a Markov semigroup on $B_b(H)$:

$$(P_t\varphi)(x) = P^x[\varphi(\xi_t)].$$

The semigroup property is a consequence of the Markov property:

$$\begin{aligned}(P_{t+s}\varphi)(x) &= P^x[\varphi(\xi_{t+s})] = P^x[P^x[\varphi(\xi_{t+s})|F_s]] \\ &= P^x\left[P^{\xi_s}[\varphi(\xi_t)]\right] = (P_sP_t\varphi)(x).\end{aligned}$$

Remark 4.12. Let C^x be a pre-Markov family with the associated operator R_λ^+ defined in the previous proof. Let P_t be the semigroup generated by one of the Markov selections and let R_λ be the operator associated to it by

$$R_\lambda\varphi = \int_0^\infty e^{-\lambda t}(P_t\varphi)(x)\,dt.$$

If (λ_1, φ_1) is the first pair used in the selection procedure of P_t, then we have

$$R_{\lambda_1}\varphi_1 = R_{\lambda_1}^+\varphi_1.$$

In the particular case when C^x is the singleton P^x, we have $R_{\lambda_1} = R_{\lambda_1}^+$.

Remark 4.13. In our applications, from Itô formula we have

$$P^x\left[e^{-\lambda T}\theta\left(\xi_T\right)\right]=\theta\left(x\right)+\int_0^T e^{-\lambda t}P^x\left[\left(\left(L_0-\lambda\right)\theta\right)\left(\xi_t\right)\right]dt$$

for every θ of the form

$$\theta\left(x\right)=\psi\left(\left\langle v_1,x\right\rangle_{V,V'},...\left\langle v_n,x\right\rangle_{V,V'}\right)$$

with $v_i\in V$ and $\psi\in C_b^2\left(\mathbb{R}^n\right)$; denote this class of functions by $\mathcal{F}C_b^2$. Here L_0 is defined as

$$\left(L_0\theta\right)\left(x\right)=\frac{1}{2}Tr\left[QD^2\theta\left(x\right)\right]-\left\langle D\theta\left(x\right),Ax+B\left(x,x\right)\right\rangle_{V,V'}$$

for $\theta\in\mathcal{F}C_b^2$ and $x\in V$. As $T\to\infty$ we get

$$\theta\left(x\right)=\int_0^\infty e^{-\lambda t}\left(P_t\left(\lambda-L_0\right)\theta\right)\left(x\right)dt.$$

Then $\lambda-L_0$ is injective,

$$\left|\theta\right|_\infty\leq\frac{1}{\lambda}\left|\left(\lambda-L_0\right)\theta\right|_\infty\quad\text{for every }\theta\in\mathcal{F}C_b^2$$

and for every

$$\varphi\in\mathcal{E}_\lambda:=Range\left(\left(\lambda-L_0\right)\mathcal{F}C_b^2\right)$$

we have

$$\left(\left(\lambda-L_0\right)^{-1}\varphi\right)\left(x\right)=\int_0^\infty e^{-\lambda t}\left(P_t\varphi\right)\left(x\right)dt=R_\lambda\varphi\left(x\right).$$

Moreover, while R_λ depends on the Markov process, L_0 does not. Therefore

$$\left(P_t\varphi\right)\left(x\right)$$

is independent of the Markov selection for every $\varphi\in\cap_{\lambda>0}\mathcal{E}_\lambda$. If $\cap_{\lambda>0}\mathcal{E}_\lambda$ would be a separating class, then we have uniqueness of the Markov selection and also of the martingale solutions. This is one of the several ways to see that density properties of the range of $\lambda-L_0$ over $\mathcal{F}C_b^2$ are related to uniqueness. See [58] for an example of rigorous use of this argument.

4.5 Markov Selection for the 3D Stochastic NSE's

Let us go back to equation (1). Verifying that martingale solutions of (1) satisfy the assumptions of the abstract theorem 4.7 is a very difficult task. On one side we need a definition of martingale solution that is stable by disintegration and recollection. On the other side, we have to prove compactness of

C^x, namely as a first step its tightness. For the tightness we need quantitative bounds on elements $P \in C^x$ and 3D Navier–Stokes equations have the unpleasant feature that it is not possible to perform computations on weak solutions, so we cannot prove bounds from the definition given in a previous section. The usual trick in similar problems is to include the bounds in the definition itself: on one side one can prove the existence of martingale solutions satisfying such bounds, on the other the bounds hold true by definition. But here comes the conflict with the first requirement, that solutions should be stable by disintegration and recollection. Indeed, quantitative properties on mean values are not stable by disintegration.

Therefore the idea is to include in the definition of martingale solution not the final mean energy inequality, but an instrument that implies it and is stable by disintegration and recollection. The main instrument that is stable is the concept of martingale. All properties that we express in terms of martingales, or sub or super - martingales, will be stable. This is the reason why we include in the definition of martingale solution a special *super-martingale* property that implies the mean energy inequality.

Unfortunately, even if the idea is clear, the details are still hard since we need energy inequalities over generic intervals $[s,t]$, not only $[0,t]$, since we have to disintegrate at time s and have the same property after time s. Here a detail emerges, namely that for 3D Navier–Stokes equations there is a problem to prove energy inequalities on $[s,t]$ for every s, while one can prove them for *almost every* s with respect to the Lebesgue measure.

Because of these details, the topic is very technical and we address to [39]. In this section we content ourselves with a *conditional* result. We introduce two concepts: enriched martingale problem and a.s. enriched martingale problem; in the definition of the first one we include a super-martingale property; in the definition of the second one we include an a.s. martingale property. Then we prove the existence of at least one solution to the a.s. enriched martingale problem. On the other side, *assuming the existence* of at least one solution to the enriched martingale problem, we prove the existence of a Markov selection.

In this section

$$\Omega = C\left([0,\infty); D(A)'\right).$$

Recall that $(\theta_t, F_t, P)_{t\geq 0}$ is a super-martingale if $P[\theta_t] < \infty$ for every $t \geq 0$ and

$$P[\theta_t 1_A] \leq P[\theta_s 1_A]$$

for every $t \geq s \geq 0$ and every $A \in F_s$.

We say that $(\theta_t, F_t, P)_{t\geq 0}$ is an *almost sure* (a.s.) *super-martingale* if $P[\theta_t] < \infty$ for every $t \geq 0$ and there exists a full Lebesgue measure set $S \subset [0,\infty)$ (namely a set $S \subset [0,\infty)$, with Lebesgue measure of $[0,\infty) \setminus S$ equal to zero), with $0 \in S$, such that

$$P[\theta_t 1_A] \leq P[\theta_s 1_A]$$

holds for every $s \in S$, every $t \geq s$, and every $A \in F_s$.

Definition 4.7 (solution of the enriched martingale problem). *Given a probability measure μ_0 on H, we say that a probability measure P on (Ω, F) is a solution of the enriched martingale problem associated to equation (1) with initial law μ_0 if*

[MP1]

$$P\left(\sup_{t\in[0,T]} |\xi_t|_H^2 + \int_0^T \|\xi_s\|_V^2\, ds < \infty\right) = 1$$

[MP2] for every $\varphi \in \mathcal{D}^\infty$, the process M_t^φ defined P-a.s. on (Ω, F) as

$$M_t^\varphi := \langle \xi_t, \varphi\rangle_H - \langle \xi_0, \varphi\rangle_H + \int_0^t \nu \langle \xi_s, A\varphi\rangle_H\, ds$$
$$- \int_0^t \langle B(\xi_s, \varphi), \xi_s\rangle_H\, ds;$$

is P-square integrable and (M_t^φ, F_t, P) is a continuous martingale with quadratic variation

$$[M^\varphi]_t = \sum_{i=1}^\infty \sigma_i^2 \langle \varphi, h_i\rangle_H^2 \cdot t$$

[MP3] the process N_t defined P-a.s. on (Ω, F) as

$$N_t := |\xi_t|_H^2 + 2\nu \int_0^t \|\xi_s\|_V^2\, ds - |\xi_0|_H^2 - \sum_{i=1}^\infty \sigma_i^2\, t$$

is P-integrable and (N_t, F_t, P) is a super-martingale

[MP4] more generally, for every integer $n > 0$ the process $N_t^{(2n)}$ defined P-a.s. on (Ω, F) as

$$N_t^{(2n)} := |\xi_t|_H^{2n} + 2n\nu \int_0^t |\xi_s|_H^{2n-2} \|\xi_s\|_V^2\, ds$$
$$- |\xi_0|_H^{2n} - n(2n-1) \sum_{i=1}^\infty \sigma_i^2 \int_0^t |\xi_s|_H^{2n-2}\, ds$$

is P-integrable and (N_t, F_t, P) is a super-martingale

[MP5] $\mu_0 = \pi_0 P$.

Definition 4.8 (solution of the a.s. enriched martingale problem). *The definition of solution of the **a.s.** enriched martingale problem associated to equation (1) with initial law μ_0 is the same as the previous one, with the only difference that $N_t^{(2n)}$ are a.s. super-martingales, for $n \geq 1$.*

Remark 4.14. In the deterministic case, a basic concept is the validity of the energy inequality; the analog in the stochastic case is the super-martingale property [MP3]. It seems that the decreasing process of the Doob-Meyer decomposition of N_t is the extra dissipation process which could exist in 3D fluids (it is an open problem whether it is zero or not).

Similarly to Theorem 4.2 we have:

Theorem 4.8. *Given $x \in H$, there exists at least one solution to the a.s. enriched martingale problem with initial condition x.*

Proof. **Step 1**. The construction of P is the same done in Theorem 4.2.

Step 2. Let us check that P fulfills [MP3]. The property

$$P\left[|N_t|\right] < \infty$$

is a consequence of the estimate

$$P_{n_k}\left[\sup_{t\in[0,T]} |\xi_t|_H^2 + \nu \int_0^T \|\xi_s\|_V^2 \, ds\right] \le C_1\left(m_2, \sigma^2, T\right)$$

proved in Section 3, that passes to the limit to P. We have to prove that there exists a full Lebesgue measure set $S \subset [0, \infty)$ with $0 \in S$, such that

$$P\left[(N_t - N_s)\, 1_A\right] \le 0$$

holds for every $s \in S$, every $t \ge s$, and every $A \in F_s$. Namely, we need to have

$$P\left[\left(|\xi_t|_H^2 + 2\nu \int_s^t \|\xi_r\|_V^2 \, dr - |\xi_s|_H^2 - \sum_{i=1}^{\infty} \sigma_i^2 \; (t-s)\right) 1_A\right] \le 0. \tag{5}$$

From the results of Section 3 we have

$$P_{n_k}\left[\left(|\xi_t|_H^2 + 2\nu \int_s^t \|\xi_r\|_V^2 \, dr - |\xi_s|_H^2 - \sum_{i=1}^{N_{n_k}} \sigma_i^2 \; (t-s)\right) 1_A\right] = 0.$$

Let us argue as in the proof of lemma 4.2. Let Y_{n_k}, Y be r.v. on a probability space $(\Sigma, \mathcal{G}, Q)$, with expectation E, with values in

$$X = C\left([0,T]\,; D(A)'\right) \cap L^2\left(0, T; H\right)$$

with laws P_{n_k} and P respectively, such that $Y_{n_k} \xrightarrow{X} Y$, Q-a.s. Let us prove that for every $t \ge s \ge 0$ there is a subsequence Y'_{n_k} of Y_{n_k} such that

$$E\left[1_A \int_s^t \|Y(r)\|_V^2 \, dr\right] \le \liminf_{k\to\infty} E\left[1_A \int_s^t \left\|Y'_{n_k}(r)\right\|_V^2 \, dr\right]. \tag{6}$$

We know that

$$E\left[\int_s^t \|Y_{n_k}(r)\|_V^2\, dr\right] = P_{n_k}\left[\int_s^t \|\xi_r\|_V^2\, dr\right] \le C$$

uniformly in k, hence there is a subsequence Y'_{n_k} of Y_{n_k} that converges weakly to some $\widetilde{Y}$ in $L^2([s,t]\times\Sigma;V)$; relaxing to weaker common topologies we identify $Y=\widetilde{Y}$; then (6) holds true.

Let us prove that there is a subsequence Y''_{n_k} of Y'_{n_k} such that

$$E\left[\left|Y''_{n_k}(t)\right|_H^2\right] \to E\left[|Y(t)|_H^2\right] \quad \text{for a.e. } t\ge 0. \tag{7}$$

It follows from

$$E\left[\left|Y''_{n_k}(t)-Y(t)\right|_H^2\right] \quad \text{for a.e. } t\ge 0.$$

There exists such Y''_{n_k} since

$$E\int_0^T \left|Y'_{n_k}(t)-Y(t)\right|_H^2 dt \to 0 \tag{8}$$

and this convergence to zero is true since $\int_0^T |Y_{n_k}(t)-Y(t)|_H^2\, dt$ converges to zero Q-a.s. and we have uniform Q-integrability by the estimates ($p^*>2$)

$$E\sup_{t\in[0,T]} |Y_{n_k}(t)|_H^{p^*} < C_3\left(p, E|x|_H^p, TrQ, T\right)$$

proved in Section 3. Properties (6) and (7) imply (5) for a.e. t and $s,t\ge s\ge 0$. Given such an s, the extension to every $t>s$ comes from weak continuity in H of trajectories and Fatou lemma.

Step 3. Let us check that P fulfills [MP4]. Set $p=2n$. The proof that $P\left[\left|N_t^{(p)}\right|\right]<\infty$ is like in step 2 and to prove that $P\left[(N_t-N_s)1_A\right]\le 0$ the only novelty is to show that there is a subsequence Y'_{n_k} of, say, Y_{n_k} such that

$$E\left[\int_s^t |Y(r)|_H^{p-2}\, dr\right] = \lim_{k\to\infty} E\left[\int_s^t \left|Y'_{n_k}(r)\right|_H^{p-2} dr\right].$$

From (8) (for Y_{n_k}) there exists Y'_{n_k} which converges to Y a.s. in (t,σ) ($\sigma\in\Sigma$). Thus it is sufficient to apply Vitali convergence theorem; the uniform integrability is guaranteed always by the p^*-estimates of Section 3, where p^* is arbitrary since x is deterministic. The proof is complete.

Let us now prove a conditional result of Markov selection (see [39] fo a non conditional result). Let C^x be the collection of all solutions of the enriched martingale problem. Assume $C^x\ne\emptyset$ for every $x\in H$. Let us prove that $\{C^x\}_{x\in H}$ is pre-Markov and satisfies the assumptions of theorem 4.7, namely that:

Claim. The family $\{C^x \subset \Pr(\Omega); x \in H\}$ has the following properties:

Lemma 4.10. *i) for every* $P \in C^x$,

$$P_\omega^{F_t} \in \Phi_t C^{\omega(t)} \text{ for } P\text{-a.e. } \omega \in \Omega$$

ii) for every $P \in C^x$, *and every* F_t*-measurable function* $\omega \mapsto Q_\omega$, *from* Ω *to* $\Pr(\Omega^t)$, *such that* $Q_\omega \in \Phi_t C^{\omega(t)}$ *for every* $\omega \in \Omega$,

$$P^Q \in C^x$$

iii) C^x *is a convex compact set in* $\Pr(\Omega)$ *for every* $x \in H$ *and* $x \mapsto C^x$ *is measurable.*

As a consequence, under the conditional assumption $C^x \neq \emptyset$, there exists a Markov selection from the family of all solutions of the martingale problem associated to equation (1).

The proof of the claim is rather lengthy and we only describe the idea (see [39] for the details). The difficult property in (iii) is the compactness, that can be proved as follows. Let $\{P_m^x\} \subset C^x$ be given. From the supermartingale properties we have

$$P_m^x\left[|\xi_t|_H^2 + 2\nu \int_0^t \|\xi_s\|_V^2 \, ds - |x|_H^2 - \sum_{i=1}^{\infty} \sigma_i^2 \, t\right] \leq 0$$

$$\begin{aligned} P_m^x\Bigg[&|\xi_t|_H^{2n} + 2n\nu \int_0^t |\xi_s|_H^{2n-2} \|\xi_s\|_V^2 \, ds \\ &- |x|_H^{2n} - n(2n-1) \sum_{i=1}^{\infty} \sigma_i^2 \int_0^t |\xi_s|_H^{2n-2} \, ds\Bigg] \leq 0 \end{aligned}$$

and in addition we may use Doob's maximal inequality to estimate the supremum in time (recall that given a supermartingale X_t on a discrete set of times $t = 0, ..., T$ one has

$$P\left(\sup_{t \leq T} X_t \geq \lambda\right) \leq \frac{1}{\lambda}\left(E[X_0] + E\left[X_T^-\right]\right)$$

for every $\lambda > 0$). From these estimates one has bounds, uniform in m, similar to those of Galerkin approximations, and then the proof of the existence of a subsequence converging to some $P^x \in C^x$ is similar to the case treated above.

As to points (i) and (ii) of the lemma, we use the fact that the martingale and super-martingale properties are stable under disintegration and recombination; the same is true for properties having probability zero or one. Let us state the theoretical results in this direction that we have to use and omit some of the proofs, that can be found in [39] (some of the results are taken from [62], Thm. 1.2.10). This is the machinery to complete the proof.

The proof of the following lemma is easy.

Lemma 4.11. *Given $P \in \Pr(\Omega)$, a σ-field $G \subset \mathcal{B}$, a set $A \in \mathcal{B}$ and a measurable mapping $\omega \mapsto Q_\omega$ from (Ω, G) to $\Pr(\Omega)$, defined P^Q as*

$$P^Q = \int_\Omega Q_\omega dP(\omega)$$

the following statements are equivalent:
i) $P^Q(A) = 1$
ii) there is a P-null set $N \in G$ such that, for all $\omega \notin N$

$$Q_\omega(A) = 1.$$

The proof of the following two lemmas is less elementary; the first one is very similar to [62], Thm. 1.2.10; for the second one see [39].

Lemma 4.12. *Given $P \in \Pr(\Omega)$, two continuous adapted processes $\theta, \zeta : [0, \infty) \times \Omega \to \mathbb{R}$ and $t_0 \geq 0$, the following conditions are equivalent:*
i) $(\theta_t, F_t, P)_{t \geq t_0}$ is a P-square integrable martingale with quadratic variation $(\zeta_t)_{t \geq t_0}$
ii) there is a P-null set $N \in F_{t_0}$ such that, for all $\omega \notin N$, $\left(\theta_t, F_t, P_\omega^{F_{t_0}}\right)_{t \geq t_0}$ is a $P_\omega^{F_{t_0}}$-square integrable martingale with quadratic variation $(\zeta_t)_{t \geq t_0}$; and $P\left[P_\cdot^{F_{t_0}}[\zeta_t]\right] < \infty$ for every $t \geq t_0$.

Lemma 4.13. *Let $\alpha, \beta : [0, \infty) \times \Omega \to \mathbb{R}_+$ be two adapted processes, β being non decreasing, and let*

$$\theta = \alpha - \beta.$$

Assume θ is left lower semicontinuous. Given $P \in \Pr(\Omega)$ and $t_0 \geq 0$, the following conditions are equivalent:
i) $(\theta_t, F_t, P)_{t \geq t_0}$ is a super-martingale, $P[\alpha_t] < \infty$ and $P[\beta_t] < \infty$ for every $t \geq t_0$;
ii) there is a P-null set $N \in F_{t_0}$ such that, for all $\omega \notin N$, $\left(\theta_t, F_t, P_\omega^{F_{t_0}}\right)_{t \geq t_0}$ is a super-martingale, $P_\omega^{F_{t_0}}[\alpha_t] < \infty$ and $P_\omega^{F_{t_0}}[\beta_t] < \infty$ for every $t \geq t_0$; and $P\left[P_\cdot^{F_{t_0}}[\beta_t]\right] < \infty$ for every $t \geq t_0$.

4.6 Continuity in u_0 of Markov Solutions

Although the uniqueness of solutions to (1) is still an open problem (as in the deterministic case), striking results in the direction of the well posedness have been proved by Da Prato and Debussche [23]. Under proper assumptions on non degeneracy of the noise, they have proved the existence of a selection that depends continuously on the initial conditions (in the sense that the associated Markov semigroup is Strong Feller). They have also provided a direct solution of the Kolmogorov equation and certain gradient estimates that could be helpful in relation with the range problem of remark 4.13 and thus the uniqueness problem.

In this section we revisit their approach and prove that *every Markov* process associated to equation (1) has a Strong Feller like property of continuous dependence on initial conditions. We give here only a few details, and address the reader to [33].

Recall that a Markov operator P_t is Strong Feller if it maps bounded Borel functions in bounded continuous functions; here we always talk about a Strong Feller like property since it will turn out that P_t maps bounded Borel functions in continuous but possibly unbounded functions; and moreover the topology of continuity is that of $D(A)$.

It is difficult to figure out how non-uniqueness could be compatible with such a result if we compare this situation with that of measurable selections, as described by the following simple remark.

Remark 4.15. Let X, Y be two metric spaces, with Borel σ-fields $\mathcal{B}(X)$ and $\mathcal{B}(Y)$, and let Φ be a measurable multivalued mapping from X to Y. Assume that X has no isolated points. If every measurable selection from Φ is continuous, then Φ is univalued. Indeed, let φ be a measurable selection. Given $(x_0, y_0) \in X \times Y$ such that $y_0 \in \Phi(x_0)$, the function $\widetilde{\varphi} : X \to Y$ equal to φ on $X \setminus \{x_0\}$ and to y_0 at x_0, is a measurable selection too, hence it is continuous, hence

$$y_0 = \lim_{x \to x_0} \widetilde{\varphi}(x) = \lim_{x \to x_0} \varphi(x) = \varphi(x_0).$$

This means that Φ is univalued.

However, the situation with Markov selections is not so simple: the Markov structure is much more demanding than measurability only, to the extent that different Strong Feller Markov selections may exists, as in the example of exercise 6.7.7 of Stroock-Varadhan [62]. Thus the uniqueness problem remains open.

Example 4.1. We briefly recall the following example from [62]. Consider the equation on the real line

$$dX_t = \left(|X_t|^{1/4} \wedge 1 \right) dW_t, \quad X_0 = x. \tag{9}$$

Existence of solutions is not a problem. Until $X_t \neq 0$, there is also uniqueness. If $x = 0$, then $X_t \equiv 0$ is a solution; one can embed it into a Markov process: given $x \neq 0$, when the solution from x meets zero we glue it with the zero solution. Let us give another solution (of the martingale problem) from zero and another Markov process. Let $(B_t)_{t \geq 0}$ be a Brownian motion and let $(\tau_t^x(\omega))_{t \geq 0}$ be a solution of the equation

$$\frac{d\tau_t^x(\omega)}{dt} = \left(\left| x + B_{\tau_t^x(\omega)}(\omega) \right|^{1/4} \wedge 1 \right), \quad \tau_0^x = 0.$$

Then $X_t^x := x + B_{\tau_t^x}$ is a solution of (9) and is a Markov process. One can choose $\left(\tau_t^0(\omega)\right)_{t\geq 0}$ different from zero and have a process different from the first one described above. One can also show that they are Feller. Moreover, there are other Markov selections, described in [62].

Nevertheless, the fact that every Markov process is Strong Feller may have interesting consequences. In the following two remarks we describe informally two consequences.

Remark 4.16. If the equation is well posed for **one** initial condition, and the noise allows us to prove irreducibility, then the equation is well posed for **every** initial condition. The scheme of the proof is first to show, by irreducibility and the Markov property, that there is a dense set of initial conditions for which the equation is well posed. Such dense set then belongs to every Markov process; by the Strong Feller property, a priori different Markov processes will coincide on every initial conditions. Notice that in the deterministic case the well-posedness is known for sufficiently small and regular initial conditions. But the proofs of such results do not extend to the case of additive white noise, since preservation of smallness is impossible in such a case. This is an interesting dichotomy between the deterministic and the stochastic case.

Remark 4.17. Under the assumptions that produce the Strong Feller property and irreducibility, we have the following result: for every initial condition $x \in H$, for every solution P^x of the martingale problem from x (at least for those that are members of some Markov selection), at **every** time $t \geq 0$ we have

$$P^x\left(\xi_t \in D(A)\right) = 1. \tag{10}$$

Related easier and well-known results are: i) just under the assumption $\sum_{i=1}^{\infty} \lambda_i \sigma_i^2 < \infty$, without any other condition of nondegeneracy, (10) is true for a.e. t; ii) (10) is known at every time t for stationary solutions. To pass to the case of every solution P^x and every t we take any Markov selection $\left(\widetilde{P}^x\right)$, we prove it has an invariant measure μ with full support, from (ii) we deduce that given t, for μ-a.e. $x \in D(A)$ we have $\widetilde{P}^x\left(\xi_t \in D(A)\right) = 1$. Then we use Strong Feller to extend to every $x \in D(A)$. Finally we use (i) to extend to every $x \in H$. This proves the claim for every solution $\widetilde{P}^x$ being a member of a Markov selection. The result should be extendible to every solution P^x by the reconstruction theorem of [62], but the details should be investigated. Finally, notice that this property looks related to the fact, proved in [23], that the Kolmogorov equation is solvable for initial conditions defined only on $D(A)$.

To shorten a few details, we directly assume that Q has the form

$$Q = A^{-\alpha}, \quad 5/2 < \alpha < 3.$$

The assumption $\alpha > 5/2$ implies that $A^{\frac{1}{2}+\varsigma}\sqrt{Q}$ is Hilbert-Schmidt in H (the embedding of $H^{3/2+\varepsilon}$ into L^2 is Hilbert-Schmidt, in 3D) for some $\varsigma > 0$, so that the noise lives in $D\left(A^{\frac{1}{2}+\varsigma}\right)$. This will imply $z \in C\left([0,\infty); D(A)\right)$ and other regularity properties. The assumption $\alpha < 3$, on the contrary, allows us to deal with $Q^{-1/2}D_x u_t^x$.

The first ingredient is the bunch of regular paths that every weak solution has for a positive local (random) time, when the initial condition is regular. Following [23] we work with $x \in D(A)$ but other choices seem possible (like $x \in V$). To introduce and analyze this bunch of regular paths, that we call the *regular plume*, we study pathwise equation (1). Notice that such a pathwise analysis is possible, for a given x, also for the solutions of the martingale problem that are probability measures in path space, since the theorem of equivalence describes them as the law of a pathwise solution on some Brownian filtered space (which may depend on x).

Consider the deterministic equation

$$u(t) + \int_0^t \left(Au\left(s\right) + B\left(u,u\right)\right) ds = x + \omega\left(t\right)$$

(interpreted in weak form over test functions $\varphi \in \mathcal{D}^\infty$) and the corresponding Galerkin approximation (an equation in H_n)

$$u_n(t) + \int_0^t \left(Au_n\left(s\right) + \pi_n B\left(u_n,u_n\right)\right) ds = \pi_n x + \pi_n \omega\left(t\right)$$

when $\omega \in \cap_{\alpha\in(0,1/2)} C^\alpha \left([0,\infty); D(A^{\frac{1}{2}+\varsigma})\right)$, $\varsigma > 0$ given in the assumptions on Q. Consider also the auxiliary Stokes equations

$$z(t) + \int_0^t Az\left(s\right) ds = \omega\left(t\right)$$

having the unique mild solution

$$z(t) = e^{-tA}\omega\left(t\right) - \int_0^t Ae^{-(t-s)A}\left(\omega\left(s\right) - \omega\left(t\right)\right) ds.$$

From elementary arguments based on the analytic estimates $\left|A^\alpha e^{-tA}\right| \leq \frac{C_{\alpha,T}}{t^\alpha}$ for $t \in (0,T)$, we have (see for instance [32] for details)

$$z \in C\left([0,\infty); D(A^{1+\varsigma-\varepsilon})\right)$$

for every $\varepsilon > 0$. In particular,

$$z \in C\left([0,\infty); D(A)\right).$$

Lemma 4.14. *Given $x \in D(A)$, there exists $t_0 > 0$ and a unique solution $u \in C([0,t_0); D(A))$; moreover, there is at least one weak solution*

$$u \in C([0,\infty; H_\sigma) \cap L^2_{loc}([0,\infty); V) \cap L^{2/3}_{loc}([0,\infty); D(A)).$$

Local uniqueness, on $[0,t_0)$, holds in the weak class too, thus any such weak solution coincides on $[0,t_0)$ with the unique $u \in C([0,t_0); D(A))$. Finally, given $T > 0$, if for a weak solution we have

$$\int_0^T |Au(t)|^2 \, dt < \infty$$

then there exists a unique solution $u \in C([0,T]; D(A))$.

Proof. The proof of this result is standard, so we omit the details; let us simply show, formally, that an a priori estimate in $C([0,t_0]; D(A))$ can be proved locally, and it holds true on an interval $[0,T]$ if $\int_0^T |Au(t)|^2 \, dt < \infty$. The new variable $v = u - z$ satisfies

$$v(t) + \int_0^t (Av(s) + B(u,u)) \, ds = x$$

hence

$$\frac{dv}{dt} + Av + B(u,u) = 0$$

$$\begin{aligned} \frac{d|Av(t)|^p}{dt} &= p|Av|^{p-2} \left\langle Av, A\frac{dv}{dt} \right\rangle \\ &= -p|Av|^{p-2} \langle Av, AAv + AB(u,u) \rangle \end{aligned}$$

and therefore

$$\frac{d|Av(t)|^p}{dt} + p|Av|^{p-2} \|Av\|_V^2 = -p|Av|^{p-2} \langle Av, AB(u,u) \rangle$$

$$\leq p|Av|^{p-2} \|Av\|_V \left| A^{1/2} B(u,u) \right| \leq Cp|Av|^{p-2} \|Av\|_V |Au|^2$$

from lemma 2.3; this implies

$$\frac{d|Av(t)|^p}{dt} \leq C|Av|^{p-2} |Au|^4. \tag{11}$$

Hence in particular

$$\frac{d|Av(t)|^2}{dt} \leq C|Av|^2 |Av|^2 + C|Az|^4.$$

It is not difficult to deduce the results from this estimate.

Given an initial condition $x \in D(A)$ and a corresponding weak solution u^x define

$$\tau^x = \infty \text{ if } \int_0^T |Au^x(t)|^2 \, dt < \infty \text{ for every } T \geq 0, \text{ otherwise}$$
$$\tau^x = \inf \left\{ T \geq 0 : \int_0^T |Au^x(t)|^2 \, dt = \infty \right\}$$

and notice that $\tau^x > 0$ (for $x \in D(A)$) because of the aforementioned results. A priori this definition depends on the weak solution, because of lack of global uniqueness.

Lemma 4.15. *The definition of τ^x is independent of the weak solution. It depends only on $u|_{[0,\tau^x)}$, that is unique and continuous in $D(A)$. Moreover, if $\tau^x < \infty$, then $\int_0^{\tau^x} |Au^x(t)|^2 \, dt = \infty$. Finally, τ^x coincides with $\widetilde{\tau}^x$ defined as $\widetilde{\tau}^x = \infty$ if u^x is locally bounded around t in $D(A)$ for every $t \geq 0$, otherwise*

$$\widetilde{\tau}^x = \inf \{ t \geq 0 : u^x \text{ locally bounded around } t \text{ in } D(A) \} .$$

Proof. Recall that $\int_0^T |Au^x(t)|^2 \, dt < \infty$ implies u^x regular and unique on $[0,T]$. Denote by τ_1^x, τ_2^x, the times associated to two weak solutions u_1^x and u_2^x. If $\tau_1^x = \infty$ then u_1^x is globally unique, hence $u_1^x \equiv u_2^x$ and $\tau_1^x = \tau_2^x$. Therefore (by symmetry) it is sufficient to consider the case $\tau_1^x, \tau_2^x < \infty$. In such a case $\int_0^T |Au_1^x(t)|^2 \, dt < \infty$ for every $T < \tau_1^x$, hence u_1^x is regular and unique on $[0,\tau^x)$, thus $u_2^x \equiv u_1^x$ on $[0,\tau^x)$. This implies $\int_0^T |Au_2^x(t)|^2 \, dt < \infty$ for every $T < \tau_1^x$, hence $\tau_2^x \geq \tau_1^x$. Reversing the role of τ_1^x and τ_2^x we prove the converse inequality, thus $\tau_1^x = \tau_2^x$.

If $\int_0^{\tau_1^x} |Au_1^x(t)|^2 \, dt$ would be finite, then u_1^x would be regular on $[0,\tau^x]$, in particular $u_1^x(\tau^x) \in D(A)$, hence it could be prolonged as a continuous function in $D(A)$ on some interval $[0,\tau^x+\varepsilon]$, $\varepsilon > 0$, contradicting the fact that $\int_0^{\tau_1^x+\varepsilon} |Au_1^x(t)|^2 \, dt = \infty$. Therefore $\int_0^{\tau_1^x} |Au_1^x(t)|^2 \, dt = \infty$.

Finally, $\tau^x \leq \widetilde{\tau}^x$ because $\int_0^T |Au^x(t)|^2 \, dt < \infty$ implies $u \in C([0,T];D(A))$. Viceversa, if $T < \widetilde{\tau}^x$, then (by a covering argument) u is bounded on $[0,T]$ with values in $D(A)$, hence $\int_0^T |Au^x(t)|^2 \, dt < \infty$; therefore $T < \tau^x$, and thus $\widetilde{\tau}^x \leq \tau^x$. The proof is complete.

Lemma 4.16. *For $t < \tau^x$, we have*

$$u_n \to u \text{ in } C([0,t];D(A))$$

(hence in particular $\int_0^t |Au_n(r)|^2 \, dr \to \int_0^t |Au(r)|^2 \, dr$) while for $t \geq \tau^x$ we have

$$\int_0^t |Au_n(r)|^2 \, dr \to \infty.$$

Proof. **Step 1**. On u_n we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left|A\left(u_n-z_n\right)\right|^2+\left|A^{3/2}\left(u_n-z_n\right)\right|^2 \\
&=-\left\langle A^2\left(u_n-z_n\right),\pi_n B\left(u_n,u_n\right)\right\rangle \leq C\left|A^{3/2}\left(u_n-z_n\right)\right|\left|Au_n\right|^2 \\
&\leq \left|A^{3/2}\left(u_n-z_n\right)\right|^2+C\left|A\left(u_n-z_n\right)\right|^4+C\left|Az_n\right|^4
\end{aligned}$$

Hence, given $R,T>0$, there is $\triangle t>0$ such that for every $t_0\in[0,T]$

$$|Au_n(t_0)|\leq R \Rightarrow \sup_{t\in[t_0,t_0+\triangle t]}|Au_n(t)|\leq 2R.$$

Step 2. With the notations

$$\begin{aligned}
v_n\left(t\right)&=u_n(t)-u(t)\\
\widetilde{v}_n\left(t\right)&=\left(u_n(t)-u(t)\right)-\left(\pi_n-I\right)z\left(t\right)
\end{aligned}$$

we have

$$\widetilde{v}_n\left(t\right)+\int_0^t A\widetilde{v}_n\left(t\right)ds+\int_0^t\left[\pi_n B\left(u_n,u_n\right)-B\left(u,u\right)\right]ds=x_n-x.$$

Formally (in some intermediate computations we use $\|Az\left(.\right)\|_V$ that we do not know to be finite; but in the final inequality (12) it disappears; then the rigorous proof can be done by an approximation, see [33] for details)

$$\frac{1}{2}\frac{d}{dt}\left|A\widetilde{v}_n\right|^2+\left|A^{3/2}\widetilde{v}_n\right|^2=-\left\langle A^2\widetilde{v}_n,\pi_n B\left(u_n,u_n\right)-B\left(u,u\right)\right\rangle.$$

We have

$$\pi_n B\left(u_n,u_n\right)-B\left(u,u\right)=\left(\pi_n-I\right)B\left(u,u\right)+\pi_n\left[B\left(u_n-u,u_n\right)+B\left(u,u_n-u\right)\right]$$

Therefore

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left|A\widetilde{v}_n\right|^2+\left|A^{3/2}\widetilde{v}_n\right|^2\leq\left|\left\langle A\widetilde{v}_n,\left(\pi_n-I\right)AB\left(u,u\right)\right\rangle\right| \\
&+\left|\left\langle A\widetilde{v}_n,\pi_n AB\left(u_n-u,u_n\right)\right\rangle\right|+\left|\left\langle A\widetilde{v}_n,\pi_n AB\left(u,u_n-u\right)\right\rangle\right| \\
&\leq\left|A^{3/2}\widetilde{v}_n\right|^2+f_n+C\left|Av_n\right|^2\left(\left|Au_n\right|^2+\left|Au\right|^2\right)
\end{aligned}$$

and thus

$$\frac{d}{dt}\left|A\widetilde{v}_n\right|^2\leq\widetilde{f}_n+C\left|A\widetilde{v}_n\right|^2\left(\left|Au_n\right|^2+\left|Au\right|^2\right)$$

or in integral form

$$|A\widetilde{v}_n(t)|^2 \leq |A\widetilde{v}_n(t_0)|^2 + \int_{t_0}^{t} C\,|A\widetilde{v}_n|^2 \left(|Au_n|^2 + |Au|^2\right) ds + \int_{t_0}^{t} \widetilde{f}_n ds \tag{12}$$

where

$$f_n = \left|(\pi_n - I)\,A^{1/2}B(u,u)\right|^2,$$
$$\widetilde{f}_n = f_n + C\,|A(\pi_n - I)\,z(t)|^2 \left(|Au_n|^2 + |Au|^2\right).$$

On any interval $[t_0, t_0 + \triangle t]$ we have

$$|A\widetilde{v}_n(t)|^2 \leq \left(|A\widetilde{v}_n(t_0)|^2 + \int_{t_0}^{t_0+\triangle t} \widetilde{f}_n ds\right) + \int_{t_0}^{t} C\,|A\widetilde{v}_n|^2 \left(|Au_n|^2 + |Au|^2\right) ds$$

and we notice that $\int_{t_0}^{t_0+\triangle t} \widetilde{f}_n ds \to 0$ by Lebesgue theorem, if on that interval we can invoke the result of step 1.

Step 3. Given $u_0 \in D(A)$, $t_1 < \tau^{u_0}$, we have

$$\int_0^{t_1} |Au(r)|^2 dr < \infty, \quad u \in C([0, t_1]; D(A)).$$

Let

$$R := 1 + \sup_{t \in [0,t_1]} |Au(t)|$$

and $\triangle t$ be given by step 1. We can apply the result of step 1 for every n, since $|A\pi_n u_0| \leq R$.

By Gronwall lemma, for $t_0 = 0$, we have

$$|A\widetilde{v}_n(t)|^2 \leq e^{\int_0^t C\left(|Au_n|^2 + |Au|^2\right) ds} \left(|A\widetilde{v}_n(0)|^2 + \int_0^{\triangle t} \widetilde{f}_n ds\right)$$

and therefore

$$\sup_{[0,\triangle t]} |Av_n(t)| \to 0.$$

We have established the result of the first part of the lemma on $[0, \triangle t]$. Moreover, since $|Au(\triangle t)| \leq R - 1$, eventually in n we have $|Au_n(\triangle t)| \leq R$. Hence we can apply the result of step 1 eventually in n on the interval $[\triangle t, 2\triangle t]$. By Gronwall lemma for $t_0 = \triangle t$, we have

$$|A\widetilde{v}_n(t)|^2 \leq e^{\int_{\triangle t}^t C\left(|Au_n|^2 + |Au|^2\right) ds} \left(|A\widetilde{v}_n(\triangle t)|^2 + \int_{\triangle t}^{2\triangle t} \widetilde{f}_n ds\right)$$

and therefore

$$\sup_{[\triangle t, 2\triangle t]} |Av_n(t)| \to 0.$$

In a finite number of steps we prove the first claim of the lemma.

Step 4. It is sufficient to consider the case $\tau^x < \infty$ and prove that

$$\int_0^{\tau^x} |Au_n(r)|^2 \, dr \to \infty.$$

By contradiction, assume there is a constant $C > 0$ and a subsequence (u_{n_k}) such that $\int_0^{\tau^x} |Au_{n_k}(r)|^2 \, dr \leq C$ for every k. In addition to the global usual estimates, this implies that there exists a further subsequence $\left(u'_{n_k}\right)$ and an element $u' \in L^2([0, \tau^x]; D(A))$ (beyond the usual regularities) such that $u'_{n_k} \to u'$ strongly in $L^2([0, \tau^x]; H)$, weakly in $L^2([0, \tau^x]; D(A))$, etc. Then it is possible to prove that u is a weak solution on $[0, \tau^x]$. On $[0, \tau^x)$ it must coincide with u. Hence $u \in L^2([0, \tau^x]; D(A))$, which contradicts the definition of τ^x. The proof is complete.

Let us now apply the previous results to the stochastic case. Given $\omega \in \Omega$, let $\tau(\omega) \in [0, \infty]$ be defined as

$$\tau(\omega) = \infty \text{ if } \int_0^T |A\omega(t)|^2 \, dt < \infty \text{ for every } T \geq 0, \text{ otherwise}$$
$$\tau(\omega) = \inf \left\{ T \geq 0 : \int_0^T |A\omega(t)|^2 \, dt = \infty \right\}$$

Definition 4.9 ((of regular plume)). *Given a Brownian stochastic basis*

$$\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, Q, \left(\beta^{(i)}(t) \right)_{t \geq 0, i \in \mathbb{N}} \right),$$

given $x \in D(A)$, equation (1) can be uniquely solved pathwise on $[0, \tau)$, giving rise to a locally defined continuous process in $D(A)$. Its value, in H, at time τ is uniquely prescribed by weak continuity in H of any weak solution. The process $(u^x_{t \wedge \tau})_{t \geq 0}$ so defined will be called the regular plume from x associated to equation (1).

Definition 4.10 (of regularized semigroup). *Given a Brownian stochastic basis, with expectation E, and the regular plume $(u^x_{t \wedge \tau})_{t \geq 0}$ from any $x \in D(A)$, the regularized semigroup associated to equation (1) is defined as*

$$\begin{aligned}(S_t\varphi)(x) &= \int_{\{t < \tau^x\}} e^{-K \int_0^t |Au_r^x|^2 dr} \varphi(u_t^x) \, dQ \\ &= E\left[e^{-K \int_0^t |Au_r^x|^2 dr} \varphi(u_t^x) 1_{t < \tau^x} \right]\end{aligned}$$

(with the understanding that $e^{-\infty} = 0$*) for every* $t \geq 0$*,* $x \in D(A)$*,* $\varphi \in B_b(D(A))$*. Here* K *is any positive constant, so we should write* $S_t^{(K)}$*, but we shall omit the superscript. Given any Markov selection* $\{P^x\}_{x \in H}$*, we also have*

$$(S_t \varphi)(x) = P^x \left[e^{-K \int_0^t |A\xi_r|^2 dr} \varphi(\xi_t) 1_{t<\tau} \right].$$

Remark 4.18. Here and below we use the notation $E[X 1_A]$ to denote $\int_A X dQ$; so X may be infinite or even not well defined on $\Omega \setminus A$.

Lemma 4.17. *Given* $p > 0$*, if* K *is sufficiently large,* $(S_t \varphi)(x)$ *is also well defined for every* $t \geq 0$*,* $x \in D(A)$ *and measurable* $\varphi : D(A) \to \mathbb{R}$*, such that*

$$|\varphi(x)| \leq C(1 + |Ax|^p)$$

for some $C > 0$*,* $p > 0$*. In such a case we have*

$$|(S_t \varphi)(x)| \leq C'(1 + |Ax|^p).$$

Proof. From (11) we have ($v_t = u_t^x - z_t$)

$$\begin{aligned} \frac{d|Av|^p}{dt} &\leq Cp|Av|^{p-2}|Au|^2 \left(|Av|^2 + |Az|^2\right) \\ &\leq Cp|Av|^p|Au|^2 + Cp|Av|^{p-2}|Au|^2|Az|^2 \\ &\leq C'p|Av|^p|Au|^2 + C'p|Au|^2|Az|^p. \end{aligned}$$

Hence, for $K = C'p$,

$$\begin{aligned} \left(e^{-K\int_0^t |Au|^2 dr}|Av|^p\right)' &\leq K e^{-K\int_0^t |Au|^2 dr}|Au|^2|Az|^p \\ &\leq -C_z^{(p)}\left(e^{-K\int_0^t |Au|^2 dr}\right)' \end{aligned}$$

where $C_z^{(p)} = \sup_{t\in[0,T]}|Az(t)|^p$, which implies

$$e^{-K\int_0^t |Au|^2 dr}|Av(t)|^p \leq |Ax|^p + C_z^{(p)}$$

hence

$$e^{-K\int_0^t |Au|^2 dr}|Au(t)|^p \leq p\left(|Ax|^p + 2C_z^{(p)}\right) \tag{13}$$

which finally implies the result since $C_z^{(p)}$ has all finite moments.

Given a Brownian stochastic basis $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, Q, \left(\beta^{(i)}(t)\right)_{t\geq 0, i\in\mathbb{N}}\right)$ (with expectation E) on which we have constructed the regular plume, let $(u_t^{x,n})$ be the unique adapted continuous solution of the Galerkin approximation. Define the semigroup S_t^n on $B_b(D(A))$ as

$$(S_t^n \varphi)(x) = E\left[e^{-K\int_0^t |Au_r^{x,n}|^2 dr}\varphi(u_t^{x,n})\right].$$

Lemma 4.18. *If K is sufficiently large, for every continuous $\varphi : D(A) \to \mathbb{R}$, such that*

$$|\varphi(x)| \leq C_\varphi (1 + |Ax|)^k$$

for some $C > 0$, $k \geq 0$, we have

$$(S_t^n \varphi)(x) \to (S_t \varphi)(x)$$

for every $t \geq 0$, $x \in D(A)$.

Proof. If $k = 0$ (φ bounded), it is sufficient to use lemma 4.16 to check that, given x, t, Q-a.s., we have

$$e^{-K \int_0^t |Au_r^{x,n}|^2 dr} \varphi(u_t^{x,n}) \to e^{-K \int_0^t |Au_r^x|^2 dr} \varphi(u_t^x) 1_{t<\tau^x} \tag{14}$$

as $n \to \infty$ (as explained above, the understanding of $e^{-K \int_0^t |Au_r^x|^2 dr} \varphi(u_t^x) 1_{t<\tau^x}$ is that it is zero for $t \geq \tau^x$, even if $\varphi(u_t^x)$ is not well defined) and then apply Lebesgue theorem. For a general k, (14) is still true for the same reason for $t < \tau^x$, while it requires more care for $t \geq \tau^x$. Indeed, for $t \geq \tau^x$, we know that $e^{-K \int_0^t |Au_r^{x,n}|^2 dr} \to 0$, but we need a control on the possible rate of explosion of $\varphi(u_t^{x,n})$. But, as in the previous proof, we have

$$e^{-K \int_0^t |Au_t^{x,n}|^2 dr} |Au_t^{x,n}|^k \leq C_k \left(|Ax|^k + C_z + C_z^{k/2} \right)$$

with the same constants. Choose $K' \geq 2K$, where K is the one of the latter estimate. Then

$$\begin{aligned} &e^{-K' \int_0^t |Au_r^{x,n}|^2 dr} |\varphi(u_t^{x,n})| \\ &\leq e^{-K \int_0^t |Au_r^{x,n}|^2 dr} C_\varphi \left(1 + C_k \left(|Ax|^k + C_z + C_z^{k/2} \right) \right) \end{aligned}$$

which goes to zero Q-a.s., as $n \to \infty$, Hence (14) is true also for $t \geq \tau^x$, with Q-probability one, with the constant K' at the exponent. From the same estimate we see that

$$e^{-K' \int_0^t |Au_r^{x,n}|^2 dr} |\varphi(u_t^{x,n})| \leq C_\varphi \left(1 + C_k \left(|Ax|^k + C_z + C_z^{k/2} \right) \right)$$

for every t, hence we can apply again Lebesgue theorem. The proof is complete.

Let us now prove a Strong Feller property for the regularized semigroup.

Lemma 4.19. *Given $k \geq 0$, if K is sufficiently large, for every measurable $\varphi : D(A) \to \mathbb{R}$, such that*

$$|\varphi(x)| \leq C_\varphi (1 + |Ax|)^k$$

for some $C > 0$, $k \geq 0$, we have

$$\begin{aligned} &|(S_t \varphi)(x) - (S_t \varphi)(y)| \\ &\leq \left[c \cdot C_\varphi \left(t^{\varepsilon - 1} + 1 \right) (|Ax| + |Ay| + 1)^k \right] \cdot |A(x - y)| \end{aligned}$$

for every $t > 0$, $x, y \in D(A)$ and for some $\varepsilon > 0$.

Proof. **Step 1**. From [23] we know that (lemma 4.1)

$$|D_h S_t^n \varphi(x)| \leq c \cdot C_\varphi \left(t^{\varepsilon-1}+1\right)|Ah|\left(|Ax|+1\right)^k \tag{15}$$

for a certain $\varepsilon > 0$. We sketch the proof below. Here by D_h we denote the derivative in the direction h. Hence, for $x, y \in D(A)$,

$$\begin{aligned}&|(S_t^n\varphi)(x) - (S_t^n\varphi)(y)| \\ &\leq \left[c \cdot C_\varphi \left(t^{\varepsilon-1}+1\right)\left(|Ax|+|Ay|+1\right)^k\right] \cdot |A(x-y)|.\end{aligned}$$

Up to mollification of φ, we may apply the previous lemma and get the result.

Step 2. To have an idea of the role of the regularization given by the potential and of the assumption $\alpha < 3$, let us sketch the proof of (15). Following [23], from a variant of Bismut-Elworthy-Li formula we have

$$D_h(S_t^n\varphi)(x) = \frac{1}{t}(I_1 + I_2)$$

$$I_1 = E\left[e^{-K\int_0^t |Au_r^{x,n}|^2 dr}\varphi\left(u_t^{x,n}\right)\int_0^t \left\langle Q^{-1/2}D_h u_s^{x,n}, dW_s\right\rangle\right]$$

$$I_2 = E\int_0^t S_{t-s}^n \varphi\left(u_s^{x,n}\right) D_h\left[e^{-K\int_0^s |Au_r^{x,n}|^2 dr}\right] ds.$$

Let us treat only the (most difficult) term I_1. We have ($u_t = u_t^{x,n}$ for brevity)

$$I_1 \leq E\left[e^{-K\int_0^t |Au_r|^2 dr} C_\varphi^2 \left(1+|Au_t|\right)^{2k}\right]^{1/2} E\left[\varsigma_t^2\right]^{1/2}$$

with

$$\varsigma_t := e^{-\frac{K}{2}\int_0^t |Au_r|^2 dr}\int_0^t \left\langle Q^{-1/2}D_h u_s, dW_s\right\rangle.$$

The first factor can be treated by the analog of (13) for $u_t^{x,n}$. As to the second factor,

$$\begin{aligned}d\varsigma_t^2 &= -K|Au_t|^2 \varsigma_t^2 dt + 2\varsigma_t e^{-\frac{K}{2}\int_0^t |Au_r|^2 dr}\left\langle Q^{-1/2}D_h u_t, dW_t\right\rangle \\ &\quad + e^{-\frac{K}{2}\int_0^t |Au_r|^2 dr}\left|Q^{-1/2}D_h u_t\right|^2 dt\end{aligned}$$

$$E\left[\varsigma_t^2\right] \leq E\int_0^t e^{-\frac{K}{2}\int_0^s |Au_r|^2 dr}\left|A^{\alpha/2}D_h u_s\right|^2 ds.$$

We have thus to analyze the regularity of $\eta_t^{h,x,n} = D_h u_t^{x,n}$.

Step 3. We have ($\eta_t = \eta_t^{h,x,n}$ for brevity), for every $\beta \geq 0$

$$\frac{d\eta_t}{dt} + A\eta_t + \pi_n B(\eta_t, u_t) + \pi_n B(u_t, \eta_t) = 0, \quad \eta_0 = \pi_n h.$$

$$\frac{d\left|A^{\beta}\eta_t\right|^2}{dt}+2\left|A^{\beta+\frac{1}{2}}\eta_t\right|^2\leq 2\left|\left\langle A^{2\beta}\eta_t,B\left(\eta_t,u_t\right)+B\left(u_t,\eta_t\right)\right\rangle\right|$$

and from lemma 2.3, for $\beta\in(1/2,1)$,

$$\begin{aligned}\frac{d\left|A^{\beta}\eta_t\right|^2}{dt}+\left|A^{\beta+\frac{1}{2}}\eta_t\right|^2&\leq\left|A^{\beta-\frac{1}{2}}\left[B\left(\eta_t,u_t\right)+B\left(u_t,\eta_t\right)\right]\right|\\&\leq C\left|Au_t\right|^2\left|A^{\beta}\eta_t\right|^2\end{aligned}$$

which implies, for sufficiently large K,

$$\int_0^t e^{-\frac{K}{2}\int_0^s\left|Au_r\right|^2dr}\left|A^{\beta+\frac{1}{2}}\eta_s\right|^2ds\leq\left|A^{\beta}h\right|^2.$$

For $\beta=\frac{\alpha-1}{2}$ we get the estimate $E\left[\varsigma_t^2\right]\leq\left|A^{\beta}h\right|^2$. This is the essence of the proof of (15).

The previous result has consequences on the Markov processes associated to equation (1) by means of the following variation of constant formula. Given a Markov selection $\{P^x\}_{x\in H}$ we associate to it the Markov semigroup P_t on $B_b\left(H\right)$ defined as

$$\left(P_t\varphi\right)(x)=P^x\left[\varphi\left(\xi_t\right)\right]$$

A priori, this semigroup depends on the selection, but it satisfies the same relation w.r.t. S_t.

Lemma 4.20. *Let $K>0$ be large enough. For every*

$$x\in D(A)\ \textit{and}\ \varphi\in B_b\left(H\right)$$

we have

$$P^x\left[\varphi\left(\xi_t\right)\right]=\left(S_t\varphi\right)(x)+\int_0^t\left(S_s\left(K\left|A\cdot\right|^2\left(P_{t-s}\varphi\right)(\cdot)\right)\right)(x)\,ds.$$

Proof. **Step 1**. We use the convention $e^{-\infty}=0$; $0\cdot\infty$ is not necessarily defined but $\int_a^b f\left(t\right)dt$ is well defined as soon as f is well defined a.s., and integrable.

Let $x\in D(A)$ and u_t^x be a weak solution of the deterministic equation (usual assumptions on ω). The main result of step 1 is to show that

$$e^{-K\int_0^t\left|Au_r^x\right|^2dr}-1=-\int_0^t e^{-K\int_0^s\left|Au_r^x\right|^2dr}K\left|Au_s^x\right|^2ds$$

for every $t\geq 0$. For $t<\tau^x$ this is obvious. The proof for $t\geq\tau^x$ seems to be non trivial. We have $\int_0^t\left|Au_r^x\right|^2dr=\infty$, then $e^{-K\int_0^t\left|Au_r^x\right|^2dr}-1=-1$; about the integral, the integrand for $s\in[0,\tau^x)$ is obviously defined and finite, while for $s\in[\tau^x,t]$ we have $e^{-K\int_0^s\left|Au_r^x\right|^2dr}=0$, $\left|Au_s^x\right|^2$ is finite a.s., hence

$e^{-K\int_0^s |Au_r^x|^2 dr} |Au_s^x|^2$ is well defined and equal to zero a.s.; in conclusion, for $t \geq \tau^x$ we have

$$\int_0^t e^{-K\int_0^s |Au_r^x|^2 dr} |Au_s^x|^2 \, ds = \int_0^{\tau^x} e^{-K\int_0^s |Au_r^x|^2 dr} |Au_s^x|^2 \, ds.$$

By (13) this integral is finite and

$$\begin{aligned}\int_0^{\tau^x} e^{-K\int_0^s |Au_r^x|^2 dr} K |Au_s^x|^2 \, ds &= \lim_{\eta\uparrow\tau^x} \int_0^{\eta} e^{-K\int_0^s |Au_r^x|^2 dr} K |Au_s^x|^2 \, ds \\ &= \lim_{\eta\uparrow\tau^x} \left(e^{-K\int_0^\eta |Au_r^x|^2 dr} - 1 \right).\end{aligned}$$

We know that $\int_0^{\tau^x} |Au_r^x|^2 \, dr = \infty$. By monotone convergence theorem,

$$\lim_{\eta\uparrow\tau^x} \int_0^{\eta} |Au_r^x|^2 \, dr = \lim_{\eta\uparrow\tau^x} \int_0^{\tau^x} |Au_r^x|^2 \, 1_{r\leq\eta} dr = \int_0^{\tau^x} |Au_r^x|^2 \, dr = \infty$$

hence $\lim_{\eta\uparrow\tau^x} \left(e^{-K\int_0^\eta |Au_r^x|^2 dr} - 1 \right) = -1$, and the identity is proved also for $t \geq \tau^x$.

Step 2. Take now a non negative $\varphi \in B_b(H)$ (by linearity, this is sufficient), $x \in D(A)$ and u_t^x be a weak martingale solution of the stochastic equation having law P^x (the element of the Markov selection under investigation). Let

$$\left(\Omega^x, \mathcal{F}^x, (\mathcal{F}_t^x)_{t\geq 0}, Q^x, \left(\beta_x^{(i)}(t) \right)_{t\geq 0, i\in\mathbb{N}} \right)$$

be the Brownian stochastic basis in the definition of the weak martingale solution u_t^x. From the identity of the previous step, Q^x-a.s. we have

$$\begin{aligned}\varphi(u_t^x) &= e^{-K\int_0^t |Au_r^x|^2 dr} 1_{t<\tau^x} \varphi(u_t^x) \\ &\quad + \int_0^t e^{-K\int_0^s |Au_r^x|^2 dr} K |Au_s^x|^2 \varphi(u_t^x) \, ds.\end{aligned}$$

The first two terms are clearly Q^x-integrable and equal to $(P_t\varphi)(x)$ and $(S_t\varphi)(x)$ resp., thus also the third one is Q^x-integrable and we have

$$\begin{aligned}(P_t\varphi)(x) &= (S_t\varphi)(x) + \\ &\quad Q^x \left[\int_0^t e^{-K\int_0^s |Au_r^x|^2 dr} K |Au_s^x|^2 \varphi(u_t^x) \, ds \right].\end{aligned}$$

The last term is, by monotone convergence, the limit as $N \to \infty$ of

$$Q^x \left[\int_0^t e^{-K\int_0^s |Au_r^x|^2 dr} K \left(|Au_s^x|^2 \wedge N \right) \varphi(u_t^x) \, ds \right]$$

which in turns is equal to

$$\int_0^t Q^x \left[e^{-K\int_0^s |Au_r^x|^2 dr} K \left(|Au_s^x|^2 \wedge N \right) \varphi\left(u_t^x\right) \right] ds$$
$$= \int_0^t P^x \left[e^{-K\int_0^s |A\xi_r|^2 dr} K \left(|A\xi_s|^2 \wedge N \right) \varphi\left(\xi_t\right) \right] ds$$
$$= \int_0^t P^x \left[e^{-K\int_0^s |A\xi_r|^2 dr} K \left(|A\xi_s|^2 \wedge N \right) P^x \left[\varphi\left(\xi_t\right) |F_s\right] \right] ds$$

The Markov property gives us (this is a crucial point)

$$P^x \left[\varphi\left(\xi_t\right) |F_s\right] = \left(P_{t-s}\varphi\right)\left(\xi_s\right).$$

Therefore the previous integral is equal to

$$\int_0^t P^x \left[e^{-K\int_0^s |A\xi_r|^2 dr} K \left(|A\xi_s|^2 \wedge N \right) \left(P_{t-s}\varphi\right)\left(\xi_s\right) \right] ds$$
$$= \int_0^t P^x \left[e^{-K\int_0^s |A\xi_r|^2 dr} 1_{s<\tau} K \left(|A\xi_s|^2 \wedge N \right) \left(P_{t-s}\varphi\right)\left(\xi_s\right) \right] ds$$

that converges, by monotone convergence, as $N \to \infty$, to

$$\int_0^t P^x \left[e^{-K\int_0^s |A\xi_r|^2 dr} 1_{s<\tau} K \left|A\xi_s\right|^2 \left(P_{t-s}\varphi\right)\left(\xi_s\right) \right] ds$$
$$= \int_0^t \left(S_s \left(K \left|A\cdot\right|^2 \left(P_{t-s}\varphi\right)(\cdot) \right) \right)(x)\, ds.$$

The proof is complete.

We can now prove the main result of this section.

Theorem 4.9. *Given a Markov selection $\{P^x\}_{x\in H}$, for every $\varphi \in B_b(H)$ and $x, y \in D(A)$ we have*

$$\left|P^x\left[\varphi\left(\xi_t\right)\right] - P^y\left[\varphi\left(\xi_t\right)\right]\right|$$
$$\leq c\left[t^{\varepsilon-1} + 1 + t^{\varepsilon}\left(|Ax| + |Ay| + 1\right)^2\right] \left|A\left(x-y\right)\right|.$$

Proof. From the variation of constant formula and lemma 4.19 we have

$$\left|P^x\left[\varphi\left(\xi_t\right)\right] - P^y\left[\varphi\left(\xi_t\right)\right]\right| \leq \left|\left(S_t\varphi\right)(x) - \left(S_t\varphi\right)(y)\right|$$
$$+ \int_0^t \left| \left(S_s \left(K \left|A\cdot\right|^2 P_{t-s} \right) \right)(x) - \left(S_s \left(K \left|A\cdot\right|^2 P_{t-s} \right) \right)(y) \right| ds$$
$$\leq \left[c\left(t^{\varepsilon-1} + 1\right) \right] \left|A\left(x-y\right)\right|$$
$$+ \left[\int_0^t \left(s^{\varepsilon-1} + 1\right) ds \cdot c\left(|Ax| + |Ay| + 1\right)^2 \right] \left|A\left(x-y\right)\right|$$
$$\leq c\left[t^{\varepsilon-1} + 1 + t^{\varepsilon}\left(|Ax| + |Ay| + 1\right)^2\right] \left|A\left(x-y\right)\right|.$$

The proof is complete.

5 Some Topics on Turbulence

5.1 Introduction and a Few Keywords

We shall mainly refer to the ideas related to Kolmogorov and Obukhov theory developed around 1941 (shortly denoted by K41 theory, see [46]). It is an example of *phenomenology of turbulence*. Following Frisch [42], by this we mean that we create in ourselves a *mental image* of what a turbulent fluid could be, with the help of intuitive *geometric structures* that usually are called *eddies* (or *vortex filaments* when they have strongly elongated shapes, or *vortex pancakes* when they are more surface like, etc.). A typical intuition we have about them is that they *rotate* (in a complex way, not as rigid bodies), with a *typical velocity* U of rotation. We also idealize their shape and associate a *size* l to them (a *typical length scale* of the eddy, something like its diameter). When the structure is more filament (or pancake) like, there could be more than one typical length scales involved, but for the time being let us discuss the case of structures with only one length scale l.

A usual idealization is to think that l takes the values 2^{-n} for positive integers n (more physically we should say $l = l_0 2^{-n}$ where l_0 is a measure of the length of the whole space occupied by the fluid). We apply the intuitive correspondence between length l and *wave number* k saying that a structure of size l has wave number $k = l^{-1}$. Hence we have the different wave numbers $k = 2^n$ (or better $k = k_0 2^n$).

We may think that the various geometric structures *interact*. There are very many interactions that could take place. For the time being, let us forget the interaction between structures of the same size. Let us concentrate our attention on the *interaction between different scales*. Its description is not simple at all and to some extent not completely understood. Let us idealize it by thinking that structures of size $l = l_0 2^{-n}$ produce, by *instability*, smaller structures of size $l = l_0 2^{-n'}$ for some $n' > n$ (for instance, when an instability of the so called Kelvin-Helmoltz type occurs, a new vortex structure originates, and it has a smaller size than the typical size of the original part of fluid where the instability took place). This is the so called *direct cascade*. There could be an *inverse cascade*, in which several small structures merge to form a larger structure (Onsager provided a statistical mechanics explanation of this fact); this seems to be a relevant effect mostly for 2D fluids, so we do not discuss it.

Let us mention the fact that a relevant part of the cascade seems to be due to *vortex stretching*: an eddy undergoes geometric transformations that make it longer in a direction and thinner along the orthogonal plane, so from a blob like shape it gets a filament like shape. This would oblige us to introduce the multiple length scales of filaments now, so for simplicity we do not discuss this mechanism for a while.

The mental image proposed by Kolmogorov is roughly the following one. The turbulent fluid that we observe is entirely composed of eddies; for each eddy size $l = l_0 2^{-n}$, there are eddies of that size that *fill in the whole space* occupied by the fluid. A cascade of energy takes place from length l to length $l/2$ for every l: eddies of size l produce eddies of size $l/2$ by instability and in such a transformation the larger eddies transfer part of their energy to the smaller ones. *Energy is injected* by some mechanism at the largest scales (some solid object which moves in the fluid, some external force). Such *energy is transferred* from scale to scale, from the larger to the smaller scales. There is a length scale η, that we shall call *Kolmogorov dissipation scale*, such that eddies of scale l of the order of η or smaller dissipate very fast their energy (into heat), so the cascade mechanism described above stops. The rate of energy dissipated by such small eddies will be equal to the rate of energy injected at large scales, otherwise the system would be not in a stationary regime.

The viscosity of the fluid does not play a major role on structures with l essentially larger than η, while it is a basic ingredient at scales $l \sim \eta$ or smaller. If the viscosity is decreased, then the scale η becomes smaller, but always positive. If we keep unchanged the external forces acting at large scales, so the rate of energy injection remains constant when we send ν to zero, to preserve a stationary regime the rate of energy dissipation will remain also constant (in spite of the fact that, as we shall see, it is the product of ν which goes to zero, and the mean square gradient of the velocity).

If there is hope to discover universal statistical laws which hold for every turbulent fluid (independently of geometry of the region, particular features of the mechanism of injection of energy etc.), it is reasonable to expect that they will hold at quite small scales and when the viscosity is very small. For this reason the previous comment on the limit as $\nu \to 0$ is relevant: this is the "regime" where one looks for universal statistical laws. We need a mathematical model which incorporate the idea that when $\nu \to 0$ the energy injection and the energy dissipation remain constant. The model proposed above of the stochastic equation

$$dX_t = \left[-\nu A X_t - B\left(X_t, X_t\right)\right] dt + \sqrt{Q} dW_t$$

is in this direction; for the finite dimensional approximations we have the identity

$$\nu E \left\|X_t\right\|_V^2 = \frac{TrQ}{2}$$

for stationary solutions which states precisely that energy injection and the energy dissipation remain constant in the limit $\nu \to 0$. Unfortunately, for the 3D infinite dimensional model the equality above is an open problem, but perhaps this is just a technical issue related to our present poor understanding of the well posedness.

With these intuitive ideas in mind, Kolmogorov analyzed a specific statistical index of the turbulent fluid, the so called structure function of order 2, and proposed a formula for it. This formula is still now one of the few quantitative predictions that are close to experimental results (up to some approximation).

5.2 K41 Scaling Law: Heuristics and Unclear Issues

Let $u(t,x)$ be the velocity field of the fluid. Statistical quantities like the mean value or the covariance of this field are not expected to have universal behavior (their size, for instance, depends too much on the particular conditions of the fluid), except for some qualitative feature: we assume $x \mapsto u(t,x)$ to be *space homogeneous and isotropic*.

The velocity displacements $u(t,x+ry)-u(t,x)$ on the contrary could have universal statistical properties when r is small. So let us look at their second moment $E\left[|u(t,x+ry)-u(t,x)|^2\right]$. We assume to work on time-stationary fields, so this mean value is independent of t; we shall drop t and write $E\left[|u(x+ry)-u(x)|^2\right]$. Since we assume space homogeneity, this mean value will not depend on x; by isotropy, it will not depend on the particular direction y either. So take a unitary coordinate vector e and consider the quantity

$$S_2(r) = E\left[|u(re)-u(0)|^2\right]$$

(assume 0 is a point of the domain where the fluid lives). This is called *second-order structure function*. The structure function of order p is simply the same expression with the p power; we do not discuss it at the beginning.

Kolmogorov and Obukhov conjectured that $S_2(r)$ could have a universal behavior in r, for small r and small viscosity. Let us describe the argument by *dimensional analysis* that Kolmogorov proposed to obtain a formula for $S_2(r)$.

Let us denote by $[L]$ the dimension of a length, $[T]$ for time. Velocity has dimension $[L][T]^{-1}$, hence $S_2(r)$ has dimension $[L]^2[T]^{-2}$.

Let us assume that, when $\nu \to 0$, for small r the function $S_2(r)$ depends only on r and the rate of energy dissipation ε, as a power law. The latter is defined as

$$\varepsilon = \nu E\left[|Du(0)|^2\right]$$

(we advertise that this discussion is still heuristic; precise definitions will be given below for random fields on the torus). So the assumption is that

$$S_2(r) = C r^{\alpha} \varepsilon^{\beta}$$

for some adimensional constant C and some exponents α and β.

The dimension of ε is a little tricky to determine. A simple way is to think that the energy dissipated is dimensionally as the time derivative of the kinetic energy (this is clear from the Navier–Stokes equations). The kinetic energy $\frac{1}{2}E\left[|u(0)|^2\right]$ has dimension $[L]^2[T]^{-2}$, so its time derivative has dimension $[L]^2[T]^{-3}$. Thus ε has dimension $[L]^2[T]^{-3}$.

Using finally the fact that r has obviously dimension $[L]$, from the power assumption above we must have

$$
\begin{aligned}
[L]^2 [T]^{-2} &= [L]^\alpha \left([L]^2 [T]^{-3} \right)^\beta \\
&= [L]^{\alpha+2\beta} [T]^{-3\beta} .
\end{aligned}
$$

The only solution is $\beta = 2/3$ and $\alpha = 2 - 2\beta = 2/3$. Hence

$$
S_2 (r) = C r^{2/3} \varepsilon^{2/3}. \tag{1}
$$

In particular, the behavior in r is like $r^{2/3}$, or

$$
\lim_{r \to 0} \frac{\log S_2 (r)}{\log r} = \frac{2}{3}.
$$

It is common to observe in (sophisticated enough) experiments that the log-log plot of $S_2 (r)$ has a plateau of approximate slope $\frac{2}{3}$. As we said, this is still now one of the two best statistical laws compared to experiments (the other one is concerned with boundary layers).

We have just remarked that in experiments a slope close to $\frac{2}{3}$ is observed along a plateau of the curve, but not along the whole curve. So the previous argument has to be made a little more precise about the range of r where it is expected to be true. Going back to the mental image described in the previous section, we have seen that we may expect an energy cascade up to some dissipation scale η only; and the universal behavior that we try to describe with the structure function $S_2 (r)$ refers to scales r in such a range where the cascade takes place, called *inertial range*. So the prescription of K41 theory is that the law (1) holds true in a range $r \in [\eta, r_0]$. We have now to determine η.

Let us find η again by a dimensional argument. Assume that η depends only on ε and the viscosity ν, as a power law. [Notice that these assumptions are a very strong idealization but they are reasonable: the rate of energy injection or dissipation ε has to play a role in the quantitative laws; for $S_2 (r)$ there should be also an obvious dependence on r, while for η there should be a dependence on ν, as already remarked in the previous section.]

So we assume

$$
\eta = C \nu^\alpha \varepsilon^\beta .
$$

The dimension of η is $[L]$ and the dimension of ε is $[L]^2 [T]^{-3}$. The dimension θ of ν could be found from the relation $\varepsilon = \nu E \left[|Du (0)|^2 \right]$: Du has dimension $[T]^{-1}$, so from $[L]^2 [T]^{-3} = \theta [T]^{-2}$ we deduce $\theta = [L]^2 [T]^{-1}$. Therefore the power relation imposes

$$
\begin{aligned}
[L] &= \left([L]^2 [T]^{-1} \right)^\alpha \left([L]^2 [T]^{-3} \right)^\beta \\
&= [L]^{2\alpha+2\beta} [T]^{-\alpha-3\beta} .
\end{aligned}
$$

The only solution is $\alpha = 3/4$ and $\beta = -1/4$. Thus the law for η is

$$\eta = C\nu^{3/4}\varepsilon^{-1/4}.$$

There are more refined arguments which support the power laws given here, but all of them are in any case based on unproved assumptions, never deduced from the Navier–Stokes equations.

In the sequel we give a rigorous definition of the K41 scaling law and prove some necessary and some sufficient conditions for it, with the hope to throw some light on this problem. The presentation is based on the work by Flandoli, Gubinelli, Hairer and Romito [37], but, to simplify, we avoid anomalous exponents and restrict ourselves to K41 theory. It is necessary to say that we do not believe K41 is exactly true for the Navier–Stokes equations. Neverthless, understanding necessary and/or sufficient conditions for K41 may help to start a rigorous investigation of such scaling laws.

The arguments just presented rely on some assumptions, namely the dependence of $S_2(r)$ and η only on certain variables in the form of a power law, that are unjustified. They may look natural:

- it is clear that $S_2(r)$ should depend on r and that η should depend on ν (we have $\lim_{\nu\to 0}\eta = 0$);
- it can be intuitively clear that both of them should depend on ε: by analogy with queuing theory, in the stationary regime, independently of the complexity of the queuing network, the rate of input is equal to the rate of output, and such rate is a basic parameter that affects several main quantities of the system;
- but it is not clear why no other quantity should be involved.

The agreement of K41 prediction with experimental results is good but not perfect. The same dimensional argument described above may be applied to 2D fluids (thin layers of fluids), where on the contrary the experiments do not confirm the K41 prediction. Essentially the explanation has something to do with the additional conservation law for 2D fluids, that is the conservation of enstrophy (for zero viscosity).

Another failure of the scaling argument above is when it is applied to the *structure function of order* p: here it is better to work with the *longitudinal* structure function to appreciate also the case of odd numbers (especially $p = 3$)

$$S_p^l(r) = E\left[\langle u(re) - u(0), e\rangle^p\right].$$

For even p the behavior in r is expected to be the same as that of

$$S_p(r) = E\left[\|u(re) - u(0)\|^p\right].$$

If we believe that the assumptions of K41 theory apply to $S_p^l(r)$ as well, by the same dimensional analysis we would find

$$S_p^l(r) \sim r^{p/3}.$$

On the contrary, if we "define" the numbers ζ_p as those for which

$$S_p^l(r) \sim r^{\zeta_p}$$

there is experimental evidence that the function

$$p \mapsto \zeta_p$$

is strictly concave for large p's, not equal to the line $p \mapsto p/3$. For $p = 3$ it seems both from the experimental viewpoint and from some heuristics that the correct value is really $\zeta_3 = 3/3 = 1$. For $p = 2$ the experiments give values ζ_2 very close to $2/3$, but possibly slightly larger. But for large p the experimental values of ζ_p, although not coinciding from one experiment to the other, is definitely smaller that $p/3$.

5.3 Definitions and Examples

Given the unitary torus $\mathcal{T} = [0,1]^d$, $d = 2,3$, recall the definitions of H and $D(A)$. We denote by $\mathcal{P}$ the class of all probability measures μ on H (with the Borel σ-algebra) such that $\mu(D(A)) = 1$, μ is space homogeneous and partial isotropic, and

$$\mu\left[\int_{\mathcal{T}} \|Du(x)\|^2 \, dx\right] < \infty.$$

Since $\mathbb{H}^2(\mathcal{T}) \subset \mathbb{C}(\mathcal{T})$ by Sobolev embedding theorem, the elements of $D(A)$ are continuous (have a continuous element in their equivalence class). Consequently, given $x_0 \in \mathcal{T}$, the mapping $u \mapsto u(x_0)$ is well defined on $D(A)$, with values in $\mathbb{R}^d$. In particular, any expression of the form

$$\mu\left[f\left(u(x_1), ..., u(x_n)\right)\right]$$

is well defined for given $x_1, ..., x_n \in \mathcal{T}$, given $\mu \in \mathcal{P}$, and suitable $f : \mathbb{R}^{nd} \to \mathbb{R}$ (for instance measurable non negative). It will follow that $S_2^\mu(r)$ is well defined (possibly infinite) for every $\mu \in \mathcal{P}$. On the contrary we cannot evaluate pointwise Du and D^2u, but we may use the quantities

$$\mu\left[\int_{\mathcal{T}} \|Du(x)\|^2 \, dx\right], \quad \mu\left[\int_{\mathcal{T}} \left\|D^2u(x)\right\|^2 \, dx\right]$$

the first of which is finite by assumption for $\mu \in \mathcal{P}$, while the second one is either finite or equal to $+\infty$.

For every $\mu \in \mathcal{P}$ we introduce the *second order structure function*

$$S_2^\mu(r) = \mu\left[\left\|u(r \cdot e) - u(0)\right\|^2\right] \tag{2}$$

for some coordinate unitary vector e, with $r > 0$ (the results proved below extend to the so called longitudinal structure function; we consider (2) to fix the ideas). The symmetries in $\mathcal{P}$ imply that $S_2^{\mu}(r)$ is independent of the coordinate unitary vector e, and the velocity difference could be taken at any other point x: $u(x + r \cdot e) - u(x)$ gives us the same result.

We are going to define K41 scaling law for a set $\mathcal{M} \subset \mathcal{P} \times \mathbb{R}_+$. The reason is that equation (1) may have (a priori) more than one stationary measure for any given ν and in certain claims it seems easier to consider a set of measures for a given ν. Given $\nu > 0$ we use the notation $\mathcal{M}_\nu$ for the set section $\{\mu \in \mathcal{P} : (\mu, \nu) \in \mathcal{M}\}$.

Given $(\mu, \nu) \in \mathcal{P} \times \mathbb{R}_+$, we define the *mean energy dissipation rate* as

$$\varepsilon = \varepsilon(\mu, \nu) := \nu \cdot \mu \left[\int_{\mathcal{T}} \|Du(x)\|^2 \, dx \right].$$

In the sequel, to simplify the exposition, we impose on families $\mathcal{M} \subset \mathcal{P} \times \mathbb{R}_+$ the following condition (*constant mean energy dissipation rate as the viscosity goes to zero*):

$$\varepsilon(\mu, \nu) = \varepsilon_0 \text{ for every } (\mu, \nu) \in \mathcal{M}. \tag{3}$$

This is true if we consider the finite dimensional models of Section 3 (with the identification of $\varepsilon(\mu, \nu)$ with $\nu \cdot \mu \left(\|\cdot\|_V^2 \right)$). It remains true for the stochastic Navier–Stokes equation (1), if the dimension is $d = 2$. Unfortunately, in 3D, it is an open problem, as illustrated in the section on Galerkin stationary measures. So, in a sense, we impose here an assumption that we do not know whether it is satisfied by our main example, the 3D case. We do this for simplicity of exposition: in [37] the assumption is partially removed.

Given $(\mu, \nu) \in \mathcal{P} \times \mathbb{R}_+$, we also define the quantity

$$\eta = \eta(\mu, \nu) := \nu^{3/4} \varepsilon(\mu, \nu)^{-1/4}.$$

Under the assumption (3) we simply have

$$\eta(\mu, \nu) = \nu^{3/4} \eta_0$$

where, for shortness, we have used the symbol $\eta_0 = \varepsilon_0{}^{-1/4}$.

Let us come to the definition of K41 scaling law. See [37] for a more general version, including also a correction to the 2/3 exponent.

Definition 5.1. *We say that a scaling law of K41 type holds true for a set $\mathcal{M} \subset \mathcal{P} \times \mathbb{R}_+$ if there exist $\nu_0 > 0$, $C > c > 0$, $r_0 > 0$ such that the bound*

$$c \cdot r^{2/3} \le S_2^{\mu}(r) \le C \cdot r^{2/3}$$

holds for every pair $(\mu, \nu) \in \mathcal{M}$ and every r such that $\nu \in (0, \nu_0]$ and $\eta(\mu, \nu) < r < r_0$, namely

$$\nu^{3/4} \eta_0 < r < r_0.$$

This is the mathematical formulation of K41 theory that we analyze. We shall prove some necessary conditions and some sufficient conditions for it. Let us insist on the fact that we do not claim that K41 is true. Presumably a version with an exponent larger than 2/3 exponent is true. The reason to state a definition is to attempt a rigorus investigation of this scaling property.

Before going into some rigorous results about this definition, let us ask ourselves a few preliminary apparently easy questions: can we give examples of functions $f(\nu, r)$ (we have the association in mind $f(\nu, r) = S_2^{\mu(\nu)}(r)$) such that

$$c \cdot r^{2/3} \leq f(\nu, r) \leq C \cdot r^{2/3} \text{ for } \nu^{3/4} < r < 1 \text{ and small } \nu?$$

It is important to realize that our usual way of thinking in mathematics is about limit properties. The previous property is not a limit one, but it is a property in an intermediate range, with some kind of uniformity as a parameter goes to a limit ($\nu \to 0$).

The easiest way to answer the previous question is by the example

$$f(\nu, r) = r^{2/3} \text{ for } \textit{all } (r, \nu).$$

But such an example cannot be related to our models. Indeed, we shall see below that, due to the property $\mu(D(A)) = 1$, we must have a regular behavior in $r \to 0$, for every given ν:

$$f(\nu, r) \sim r^2 \text{ as } r \to 0, \text{ for every } \nu.$$

More precisely, we shall see that essentially we have

$$f(\nu, r) = \frac{r^2}{\nu} \text{ for sufficiently small } r.$$

So let us refine our question and ask whether we may:

- find examples of functions $f(\nu, r)$ such that

  $$C_1 \cdot r^{2/3} \leq f(\nu, r) \leq C_2 \cdot r^{2/3} \tag{4}$$

  for $C_3 \nu^{3/4} < r < 1$ and

  $$C_4 \cdot r^2 \leq f(\nu, r) \leq C_5 \cdot r^2 \tag{5}$$

  for $r < C_6 \nu^{3/4}$.

  This is not easy (unless we artificially define piecewise $f(\nu, r)$).

Example 5.1 (Negative example). Let us preliminary understand better the function

$$f_0(\nu, r) := \frac{r^2}{\nu}$$

which certainly satisfies the second part of the requirement. We have

$$f_0(\nu, r) = r^{2/3} \left(r\nu^{-3/4}\right)^{4/3}$$

so we have

$$C_1 \cdot r^{2/3} \leq f(\nu, r) \leq C_2 \cdot r^{2/3} \text{ for } C_3\nu^{3/4} < r < C_4\nu^{3/4}$$

for suitable constants. Apparently we get a property very close to the required one, but it holds true only on an interval of r whose boundary values are infinitesimal of the same order. This is a basic point: to have a meaningful definition of a scaling law we must ask its validity on an interval whose boundary points diverge one w.r.t. the other. What is meaningless in the previous $2/3$ property on $C_3\nu^{3/4} < r < C_3'\nu^{3/4}$ is simply that a similar property holds true replacing $2/3$ with any other exponent $\alpha \in (0, 2)$. Indeed we have

$$f_0(\nu, r) = r^{\alpha} \left(r\nu^{-\frac{1}{2-\alpha}}\right)^{2-\alpha}$$

hence

$$C_1' \cdot r^{\alpha} \leq f_0(\nu, r) \leq C_2' \cdot r^{\alpha} \text{ for } C_3'\nu^{\frac{1}{2-\alpha}} < r < C_4'\nu^{\frac{1}{2-\alpha}}.$$

Summarizing, $f_0(\nu, r) = \frac{r^2}{\nu}$ clearly does not have any interesting non-integer scaling law, as we see from its definition, and it shows in addition that in the definition of a scaling law it is necessary to impose that the range of r has boundary points that diverge one from each other.

Example 5.2 (Positive example). This example arose in the computations made of a random vortex model of Flandoli and Gubinelli, see [36]. Consider the function

$$f(\nu, r) = \int_{\eta}^{1} l^{2/3} \left(\frac{l \wedge r}{l}\right)^2 \frac{dl}{l}$$

with $\eta = \nu^{3/4}$. We have

$$r \leq \eta \Rightarrow f(\nu, r) = \int_{\eta}^{1} l^{2/3} \left(\frac{r}{l}\right)^2 \frac{dl}{l} = \frac{3}{4} r^2 \left[\nu^{-1} - 1\right]$$

which gives us (5). On the other hand,

$$\begin{aligned} r \in [\eta, 1] \Rightarrow f(\nu, r) &= \int_{\eta}^{r} l^{2/3} \frac{dl}{l} + \int_{r}^{1} l^{2/3} \left(\frac{r}{l}\right)^2 \frac{dl}{l} \\ &= \frac{9}{4} r^{2/3} - \frac{3}{2} \nu^{1/2} - \frac{3}{4} r^2 \end{aligned}$$

which is bounded above and below by the order $r^{2/3}$ since $r \in \left[\nu^{3/4}, 1\right]$ ($\nu^{1/2} \leq r^{2/3}$). This implies (4).

5.4 Brownian Eddies and Random Vortex Filaments

Before entering into some rigorous results around K41 scaling law for the stochastic Navier–Stokes equations, let us get more intuition from a phenomenological model. This model is a priori given, in the sense that it does not come from the Navier–Stokes equations, but it is nevertheless defined in rigorous mathematical terms and it is an attempt to describe some of the numerical observations of vortex filaments obtained in the last 15 years. The main source of motivation has been the book of Chorin [19], where discrete vortex filaments based on paths of self-avoiding walk are investigated. Other related works are [10], [37].

Let $(W_t)_{t\geq 0}$ be a 3D Brownian motion. Let $\rho : \mathbb{R}^3 \to \mathbb{R}$ be the function $\rho(x) = \exp\left(-\|x\|^2\right)$ (there is a lot of freedom in the choice of ρ, this is just a convenient example). Given $\ell > 0$, rescale ρ as

$$\rho_\ell(x) = \rho\left(\frac{x}{\ell}\right) = \exp\left(-\frac{\|x\|^2}{\ell^2}\right).$$

Let $K_\ell(x) : \mathbb{R}^3 \to \mathbb{R}^3$ be the field

$$K_\ell(x) = \frac{1}{4\pi}\int_{\mathbb{R}^3} \rho_\ell(y)\frac{x-y}{|x-y|^3}dy.$$

Remark 5.1. If $\gamma : \mathbb{R} \to \mathbb{R}^3$ is the curve $\gamma(t) = (0,0,t)$, then the vector field (interpret it as a velocity field)

$$u_{Burgers}(x) := \int_{-\infty}^{+\infty} K_\ell(x-\gamma(t)) \wedge \dot{\gamma}(t)\, dt$$

is called a Burgers vortex, and is easily seen to be a rotating field around the z-axis, with some decay at infinity. It is also given by the Biot-Savart law with respect to the "vorticity" field $\xi(x)$:

$$u_{Burgers}(x) = \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{x-y}{|x-y|^3} \wedge \xi_{Burgers}(y)\, dy$$

$$\xi_{Burgers}(y) := \int_{-\infty}^{+\infty} \rho_\ell(y-\gamma(t)) \wedge \dot{\gamma}(t)\, dt.$$

The number ℓ is a measure of the "cross section" of the vortex "tube".

We repeat the mathematics of the previous example but starting from the Brownian motion $(W_t)_{t\geq 0}$ in place of the line γ.

Definition 5.2. *Let us call Brownian eddy at scale $l > 0$ the following random field $(W)_\ell^\perp(x)$:*

$$(W)_\ell^\perp(x) := \frac{1}{\ell^2}\int_0^{\ell^2} K_\ell(x-W_t) \wedge dW_t \qquad x \in \mathbb{R}^3.$$

Remark 5.2. We use the notation $(.)_{.}^{\perp}$ because the field $(W)_{\ell}^{\perp}(x)$ is somewhat orthogonal to the trajectory of the Brownian motion.

Remark 5.3. Let us explain the rigorous use of the notation. Given a stochastic process $(X) = (X_t)_{t\geq 0}$, if the stochastic integral is well define we may introduce the associated random field

$$(X)_{\ell}^{\perp}(x) := \frac{1}{\ell^2}\int_0^{\ell^2} K_{\ell}(x - X_t) \wedge dX_t \qquad x \in \mathbb{R}^3.$$

Such an integral is well defined for instance for every semimartingale (X). The notation $(.)_{\ell}^{\perp}$ denotes a mapping from processes to random fields; for instance, if we write

$$(X_0 + W)_{\ell}^{\perp}(x)$$

where X_0 is a 3D random variable, we understand the vector field associated to the process $X_0 + W$, that is a Brownian motion starting from position X_0.

Remark 5.4. It may help the intuition to figure out the shape of a Brownian eddy. The Brownian motion has sometimes long excursions: along them a Brownian eddy looks like an irregular Burgers vortex. On the contrary, most often the trajectory of a Brownian motion is very much folded around itself: in such a case the Brownian eddy is more blob-like. The number ℓ is a measure of the size, and also of the smoothness. Notice finally that the typical displacement of a Brownian motion in a time ℓ^2 is of the order ℓ, that is the size of the kernel K_{ℓ}: therefore for most of the trajectories of the Brownian motion the associated eddy is as long as large, eddy-like more than filament-like.

Remark 5.5. One can verify by stochastic analysis that the field $(W)_{\ell}^{\perp}(x)$ is very regular (it has C^{∞}-realizations) and

$$\mathsf{div}\,(W)_{\ell}^{\perp}(x) = 0.$$

The reason why this field turns out to be interesting is the following scaling property.

Lemma 5.1. *For every $\lambda, \ell > 0$,*

$$(W)_{\ell}^{\perp}(\lambda x) \stackrel{\mathcal{L}}{=} (W)_{\ell/\lambda}^{\perp}(x)$$

the equality in law being at the level of random fields.

Proof.

$$\begin{aligned}(W)_{\ell}^{\perp}(\lambda x) &= \frac{1}{\ell^2}\int_0^{\ell^2} K_{\ell}\left(\lambda\left(x - \frac{W_t}{\lambda}\right)\right) \wedge \lambda d\left(\frac{W_t}{\lambda}\right) \\ &= \frac{1}{(\ell/\lambda)^2}\int_0^{\ell^2} K_{\ell/\lambda}\left(x - \frac{W_t}{\lambda}\right) \wedge d\left(\frac{W_t}{\lambda}\right)\end{aligned}$$

since

$$K_\ell(\lambda x) = \lambda K_{\ell/\lambda}(x).$$

The processes $\left(\frac{W_t}{\lambda}\right)$ and $\left(W_{t/\lambda^2}\right)$ have the same law, hence

$$(W)_\ell^\perp(\lambda x) \stackrel{\mathcal{L}}{=} \frac{1}{(\ell/\lambda)^2}\int_0^{\ell^2} K_{\ell/\lambda}\left(x - W_{t/\lambda^2}\right)\wedge d\left(W_{t/\lambda^2}\right)$$

where it is not difficult to see that the equality in law is at the level of random fields (namely jointly in different locations x). Finally, simple arguments on time change in stochastic integrals show that

$$(W)_\ell^\perp(\lambda x) \stackrel{\mathcal{L}}{=} \frac{1}{(\ell/\lambda)^2}\int_0^{(\ell/\lambda)^2} K_{\ell/\lambda}(x - W_t)\wedge d(W_t).$$

The proof is complete.

Remark 5.6. In particular,

$$(W)_\ell^\perp(\ell x) \stackrel{\mathcal{L}}{=} (W)_1^\perp(x).$$

This says that the velocities we observe in $(W)_\ell^\perp$ are the same as those of $(W)_1^\perp$. The energy will be much smaller: the "support" of an ℓ-eddy is roughly of order ℓ^3, hence its kinetic energy is roughly of order ℓ^3 times the kinetic energy of $(W)_1^\perp$.

Remark 5.7. The analogous definition of fractional Brownian eddy at scale $\ell > 0$ and Hurst parameter $H \in (0,1)$ would be

$$\left(W^H\right)_\ell^\perp(x) = \frac{1}{\ell^2}\int_0^{\ell^{1/H}} K_\ell\left(x - W_t^H\right)\wedge dW_t^H \qquad x \in \mathbb{R}^3$$

whenever the integral is well defined, where $\left(W_t^H\right)$ is a fractional Brownian motion in $\mathbb{R}^3$ with Hurst parameter $H \in (0,1)$. With the same proof one can show that

$$\left(W^H\right)_\ell^\perp(\lambda x) \stackrel{\mathcal{L}}{=} \left(W^H\right)_{\ell/\lambda}^\perp(x).$$

Indeed, the only difference in the proof is that now the processes $\left(\frac{W_t^H}{\lambda}\right)$ and $\left(W_{t/\lambda^{1/H}}\right)$ have the same law.

With the same proof of the previous lemma we have:

Lemma 5.2. *Given a non anticipating 3D r.v. X_0, for every $\lambda, \ell > 0$,*

$$(X_0 + W)_\ell^\perp(\lambda x) \stackrel{\mathcal{L}}{=} \left(\frac{X_0}{\lambda} + W\right)_{\ell/\lambda}^\perp(x)$$

the equality in law being at the level of random fields.

With the help of the previous objects we may define more complex random fields. Let $\left\{X_0^{(n)}\right\}_{n\in\mathbb{N}}$ be a sequence of 3D i.i.d. random variables, $\left\{W^{(n)}\right\}_{n\in\mathbb{N}}$ be a sequence of 3D independent Brownian motions, $\left\{\ell_{(n)}\right\}_{n\in\mathbb{N}}$ be a sequence of positive i.i.d. random variables, with all these objects independent one of each others. Then define *formally* the series

$$u(x) = \sum_{n=1}^{\infty} \ell_{(n)}^{1/3} \left(X_0^{(n)} + W^{(n)}\right)_{\ell_{(n)}}^{\perp} (x) \, .$$

Assume *formally* that

- $X_0^{(n)}$ are uniformly distributed in $\mathbb{R}^3$
- $\ell_{(n)}$ are distributed according to $\frac{d\ell}{\ell^4}$ on $(0,\infty)$.

Exercise 5.1. Understand intuitively that the natural distribution to have space filling of eddies of every size is $\frac{d\ell}{\ell^4}$ and not $\frac{d\ell}{\ell^3}$. The "invariance" below of the law of $\left(X_0^{(n)}, \ell_{(n)}\right)$ by homotheties is a technical explanation.

These sentences are not rigorous as they stand since they refer to measures which are only σ-finite, but they may be made rigorous by using Poisson point processes.

The intuitive geometric idea about $u(x)$ is that at every (small) interval of scales $[\ell, \ell + \triangle\ell]$ we see the space filled in of vortex eddies of size in $[\ell, \ell + \triangle\ell]$. And $u(x)$ is the velocity field associated to such a fluid composed of many eddies of every size.

Let us show *formally* that

$$u(\lambda x) \overset{\mathcal{L}}{=} \lambda^{1/3} u(x). \tag{6}$$

By the lemma above we have

$$\begin{aligned} u(\lambda x) &\overset{\mathcal{L}}{=} \sum_{n=1}^{\infty} \ell_{(n)}^{1/3} \left(\frac{X_0^{(n)}}{\lambda} + W^{(n)}\right)_{\ell_{(n)}/\lambda}^{\perp} (x) \\ &= \lambda^{1/3} \sum_{n=1}^{\infty} \left(\frac{\ell_{(n)}}{\lambda}\right)^{1/3} \left(\frac{X_0^{(n)}}{\lambda} + W^{(n)}\right)_{\ell_{(n)}/\lambda}^{\perp} (x) \\ &\overset{\mathcal{L}}{=} \lambda^{1/3} \sum_{n=1}^{\infty} \ell_{(n)}^{1/3} \left(X_0^{(n)} + W^{(n)}\right)_{\ell_{(n)}}^{\perp} (x) = \lambda^{1/3} u(x) \end{aligned}$$

where we have used the formal fact that the joint law of $\left(X_0^{(n)}, \ell_{(n)}\right)$ is invariant by homotheties:

$$E\left[\varphi\left(\frac{X_0^{(n)}}{\lambda},\frac{\ell_{(n)}}{\lambda}\right)\right]=\int_0^\infty\int_{\mathbb{R}^3}\varphi\left(\frac{x}{\lambda},\frac{\ell}{\lambda}\right)dx\frac{d\ell}{\ell^4}$$

$$\overset{x'=\frac{x}{\lambda}}{=}\int_0^\infty\lambda^3\int_{\mathbb{R}^3}\varphi\left(x',\frac{\ell}{\lambda}\right)dx'\frac{d\ell}{\ell^4}=\int_{\mathbb{R}^3}\int_0^\infty\varphi\left(x',\frac{\ell}{\lambda}\right)\frac{d\left(\ell/\lambda\right)}{\left(\ell/\lambda\right)^4}dx'$$
$$\overset{\ell'=\frac{\ell}{\lambda}}{=}\int_{\mathbb{R}^3}\int_0^\infty\varphi\left(x',\ell'\right)\frac{d\ell'}{\left(\ell'\right)^4}dx'=E\left[\varphi\left(X_0^{(n)},\ell_{(n)}\right)\right].$$

Unfortunately all these computations are not rigorous since the series defining $u(x)$ does not converge! That something is wrong can be immediately guessed from the fact that (6) implies that either $u(x)$ is identically zero, or that in some sense it is identically infinite. Indeed as $\lambda\to 0$, if we accept continuity, we get $u(0)\overset{\mathcal{L}}{=}0\cdot u(0)$; and on the other side $u(x)$ should be a space homogeneous random field, so $u(x)\overset{\mathcal{L}}{=}0\cdot u(x)$ at every x. Even without continuity, any meaning of stationarity implies that $u(x)$ and $u(\lambda x)$ should have certain equal quantities, and this is compatible only with the multiplier $\lambda=1$. In addition, certainly it is not reasonable to believe that the field u written above is identically zero, so we have to conclude that in a sense it is infinite everywhere. We do not make this argument rigorous, since the final result is a negative one and there is no intuitive hope that the behavior is better than the one just described.

However, having a random field with the self-similarity property (6) would give us an example of random field with a K41-type property over an infinite range of r:

$$\begin{aligned}S_2(r)&=E\left[\|u(r\cdot e)-u(r\cdot 0)\|^2\right]\\&=E\left[\left\|r^{1/3}u(e)-r^{1/3}u(0)\right\|^2\right]=r^{1/3}S_2(1).\end{aligned}$$

If $S_2(1)$ where different from zero and finite (but it is infinite) we would get K41, even without limitations on r.

All of this is formal but very instructive. One should come back to this example after having learned more about the scaling transformations that we shall perform on the stochastic Navier–Stokes equations.

In fact the only problem with the previous objects is that $\ell_{(n)}$ have distributions that extend to infinite too much, so that there are arbitrarily large and intense eddies. It is sufficient to cut-off ℓ and we get a rigorous example of random field that has the K41 scaling law. But it is not exactly self-similar: only "at small distances" (see Kupiainen [50]).

Theorem 5.1. *Given a positive real number $\ell_{\max}$, consider the σ-finite measure $\frac{d\ell}{\ell^4}1_{l\in(0,\ell_{\max}]}$. Assume that the r.v. $\ell_{(n)}$ are distributed according to this measure. Then the random field $u(x)$ above is well defined, it has all finite* *moments, it is space homogeneous and isotropic, and its second order structure function* $S_2(r)$ *is bounded above and below (uniformly in* $r \in (0,1)$*, say) by the function*

$$f(r) = \int_0^1 \ell^{2/3} \left(\frac{\ell \wedge r}{\ell} \right)^2 \frac{d\ell}{\ell}$$

and therefore by $r^{2/3}$*.*

The proof is very technical (based on Burkholder-Davis-Gundy inequality, strong Markov property, arguments of potential theory similar to those of the theory of the Brownian sausage) and may be found in [36]. The meaning of the r.v.'s distributed according to only σ-finite measures is rigorously given in [36] by means of Poisson point processes, as we have already said; this is a quite technical issue so we do not give the details here.

The following modification of the previous theorem, again proved in [36], which includes a cut-off at viscous scales, may also be of interest.

Theorem 5.2. *For every* $\nu \in (0, \ell_{\max})$ *consider the* σ*-finite measure* $\frac{d\ell}{\ell^4} 1_{l \in (\eta, \ell_{\max}]}$ *where*

$$\eta = \eta(\nu) = \nu^{3/4}.$$

Assume that the r.v. $\ell_{(n)}$ *are distributed according to this measure. Then the same conclusions of the previous theorem hold but with the function*

$$f(\nu, r) = \int_\eta^1 \ell^{2/3} \left(\frac{\ell \wedge r}{\ell} \right)^2 \frac{d\ell}{\ell}$$

and therefore satisfies K41 scaling law (see section 5.2).

Is it the idealization proposed by this Brownian eddies model close to reality? We do not know the answer, simply it looks like the mental image described by Kolmogorov [46]. Let us only mention that other models give the same result. We may introduce more filament-like random vortices of the form

$$u_{\text{single}}^{(X_0, \ell, T, U)}(x) = \frac{U}{\ell^2} \int_0^T K_\ell (x - X_0 - W_t) \wedge dW_t.$$

If T is larger than ℓ^2 the displacement of the Brownian trajectory is typically longer than the cross-section ℓ. With these fields as building objects one may still construct random field with K41 law but not eddy-like. Notice that numerical simulations in the last 15 years have often shown that a turbulent fluid is rich of elongated vortex filaments (but their relevance for the statistics is not proved).

In favor of the previous model of Brownian eddies we may quote the local self-similarity, which seems to be one of the observed features of turbulent fluids and is also related to the scaling properties of the stochastic Navier–Stokes equations, as we shall see. In this respect, if we believe that more filament like objects are also present in the fluid, they could constitute a secondary object, but maybe important to explain certain intermittent features and corrections to K41.

5.5 Necessary Conditions for K41

Let us leave random vortex filaments and go back to the rigorous analysis of K41 property. We give some general results and then their application to the stochastic Navier–Stokes equations.

The first results of this subsection apply to suitable families of probability measures, without any use of the Navier–Stokes equations. They will be applied to stochastic Navier–Stokes equations at the end of the section.

Given a measure $\mu \in \mathcal{P}$, $\mu \neq \delta_0$, we introduce the number $\theta = \theta(\mu)$ defined by the identity

$$\theta^2 = \frac{\mu\left[\int_{[0,1]^d} \|Du(x)\|^2 \, dx\right]}{\mu\left[\int_{[0,1]^d} \|D^2u(x)\|^2 \, dx\right]} \tag{7}$$

with the understanding that $\theta = 0$ when $\mu\left[\int_{[0,1]^d} \left\|D^2u(x)\right\|^2 dx\right] = \infty$ and $\theta = 1$ when $\mu = \delta_0$. θ has the dimension of a length and we interpret it as *an estimate of the length scale where dissipation is more relevant.* Indeed, very roughly, from

$$\frac{\int_{\mathcal{T}} \left\|D^2u(x)\right\|^2 dx}{\int_{\mathcal{T}} \|Du(x)\|^2 \, dx} \sim \frac{\sum |k|^2 \left(|k|^2 |\widehat{u}(k)|^2\right)}{\sum |k|^2 |\widehat{u}(k)|^2}$$

we see that $\theta(\mu)^{-2}$ has the meaning of typical square wave length of dissipation (looking at $|k|^2 |\widehat{u}(k)|^2$ as a sort of distribution in wave space of the dissipation).

Lemma 5.3. *For every $\mu \in \mathcal{P}$ such that $\theta(\mu) > 0$ we have*

$$\frac{1}{4d} \cdot r^2 \leq \frac{S_2^\mu(r)}{\mu\left[\int_{\mathcal{T}} \|Du(x)\|^2 \, dx\right]} \leq r^2 \text{ for every } r \in (0, \frac{\theta(\mu)}{4d}]. \tag{8}$$

Proof. We have to use Taylor formula, but the measures μ are concentrated a priori only on $W^{2,2}$ vector fields. For sake of brevity, we give the proof under the additional assumption that

$$\mu\left(D(A) \cap C^2(\mathcal{T})\right) = 1$$

for all the measures μ involved. In [37] one may found the proof in the general case, performed by mollification.

By space homogeneity of μ

$$\mu\left[\|u(re)-u(0)\|^2\right] \le r^2 \int_0^1 \mu\left[\|Du(\sigma e)\|^2\right] d\sigma$$
$$= r^2 \mu\left[\|Du\|^2\right]$$

and thus the right-hand inequality of (8) is proved for every $r > 0$.

On the other side, for smooth vector fields we have

$$u(re) - u(0) = Du(0)\, re + r^2 \int_0^1 D^2 u(\sigma e)(e,e)\, d\sigma$$

and thus

$$\mu\left[\|Du \cdot re\|^2\right] \le 2\mu\left[\|u(re)-u(0)\|^2\right]$$
$$+ 2\mu\left[\left\|r^2 \int_0^1 D^2 u(\sigma e)(e,e)\, d\sigma\right\|^2\right].$$

Again from space homogeneity of μ,

$$\mu\left[\left\|r^2 \int_0^1 D^2 u(\sigma e)(e,e)\, d\sigma\right\|^2\right] \le r^4 \mu\left[\left\|D^2 u\right\|^2\right]$$

and from discrete isotropy we have (see the appendix of [37])

$$\mu\left[\|Du \cdot e\|^2\right] = \frac{1}{d}\mu\left[\|Du\|^2\right].$$

Therefore

$$\mu\left[\|u(re)-u(0)\|^2\right] \ge \frac{r^2}{2d}\mu\left[\|Du\|^2\right] - r^4 \mu\left[\left\|D^2 u\right\|^2\right].$$

Therefore, by definition of $\theta(\mu)$,

$$S_2(r) \ge \left(\frac{1}{2d} - \frac{r^2}{\theta(\mu)}\right)\mu\left[\|Du\|^2\right] \cdot r^2.$$

This implies the left-hand inequality of (8) for $r \in (0, \frac{\theta(\mu)}{4d}]$. The proof is complete.

Theorem 5.3. *Let $\mathcal{M} \subset \mathcal{P} \times \mathbb{R}_+$ be a set with the following scaling property: there is a function $\widetilde{\eta} : \mathcal{M} \to \mathbb{R}_+$ (the length scale of the scaling property), with*

$$\lim_{\nu \to 0} \sup_{\mu \in \mathcal{M}_\nu} \widetilde{\eta}(\mu, \nu) = 0,$$

a scaling exponent $\alpha \in (0,2)$ *and constants* $C_2 \geq C_1 > 0$, $\nu_0 > 0$, $r_0 > 0$ *such that*

$$C_1 \cdot r^\alpha \leq S_2^\mu (r) \leq C_2 \cdot r^\alpha \quad \text{for } r \in [\widetilde{\eta}(\mu,\nu), r_0] \tag{9}$$

for every $\nu \in (0,\nu_0)$ *and every* $\mu \in \mathcal{M}_\nu$. *Let* $\theta(\mu)$ *be the dissipation length scale defined above.*

Then the two length scales $\theta(\mu)$ *and* $\widetilde{\eta}(\mu,\nu)$ *are related by the following property: there exist* $C > 0$, $\nu_1 > 0$ *such that*

$$\theta(\mu) \leq C \cdot \widetilde{\eta}(\mu,\nu) \tag{10}$$

for every $\nu \in (0,\nu_1)$ *and every* $\mu \in \mathcal{M}_\nu$.

For the proof we address to [37]; we do not repeat it here since it is intuitively rather clear that (8) is in contradiction with (9) if the ranges of r where the two properties hold overlap, so we need the bound (10).

Remark 5.8. The divergence of the range of r's in the definition (9) of a scaling law is essential to have a non trivial definition. If, on the contrary, we simply ask that the scaling law holds on a bounded interval $r \in [C_3\eta_\nu, C_4\eta_\nu]$, we have a definition without real interest, as it is explained by the example of section 5.1.

Let us finally state two general consequences of the previous theorem, that we shall apply to stochastic Navier–Stokes equations.

Corollary 5.1. *Given a family* $\mathcal{M} \subset \mathcal{P} \times \mathbb{R}_+$, *if*

$$\inf_{(\mu,\nu)\in\mathcal{M}} \theta(\mu) > 0$$

then no scaling law in the sense of the previous theorem may hold true.

Corollary 5.2. *Let* $\mathcal{M} \subset \mathcal{P} \times \mathbb{R}_+$ *be a family with the K41 scaling law, in the sense of definition 5.1. Then there exists* $\nu_0 > 0$ *and* $C > 0$ *such that*

$$\mu\left[\int_{\mathcal{T}} \left\|D^2 u(x)\right\|^2 dx\right] \geq C\varepsilon_0^{3/2} \cdot \nu^{-5/2}$$

for every $\nu \in (0,\nu_0)$ *and every* $\mu \in \mathcal{M}_\nu$.

Proof. From (10), the definition of $\eta(\mu,\nu)$ and the definition of $\theta^2(\mu)$ we have

$$\frac{\mu\left[\int_{\mathcal{T}} \|Du(x)\|^2 dx\right]}{\mu\left[\int_{\mathcal{T}} \|D^2 u(x)\|^2 dx\right]} \leq C\nu^{3/2}\eta_0^2$$

Thus, from the definition of ε_0,

$$\frac{\varepsilon_0}{\mu\left[\int_{\mathcal{T}} \|D^2 u(x)\|^2 dx\right]} \leq C\nu^{5/2}\eta_0^2.$$

This implies the claim of the Corollary. The proof is complete.

Remark 5.9. Dimensional analysis says that ν has dimension $[L]^2 [T]^{-1}$, ε has dimension $[L]^2 [T]^{-3}$, so $\varepsilon_0^{3/2} \cdot \nu^{-5/2}$ has dimension $[L]^{-2} [T]^{-2}$, the correct dimension of $\mu \left[\int_{\mathcal{T}} \left\| D^2 u(x) \right\|^2 dx \right]$.

Remark 5.10. The previous and next corollaries are based only on the scaling exponents of $\eta(\mu, \nu)$, not on the exponent $2/3$ in definition 5.1. Therefore, any other $\alpha \in (0, 2)$ in place of $2/3$ would give us the same result.

Let us first apply the general results to disprove K41 in the 2D case. The following result is well known.

Lemma 5.4. *Let μ be an invariant measure of (1) ($d = 2$) such that*

$$\mu \left[\int_{\mathcal{T}} \| Du(x) \|^2 dx \right] < \infty.$$

Then $\mu \in \mathcal{P}_0$ and

$$\nu \cdot \mu \left[\int_{\mathcal{T}} \| Du(x) \|^2 dx \right] = \frac{1}{2} \sum_{i=1}^{\infty} \sigma_i^2$$

$$\nu \cdot \mu \left[\int_{\mathcal{T}} \| D \mathsf{curl} u(x) \|^2 dx \right] = \frac{1}{2} \sum_{i=1}^{\infty} \sigma_i^2 \lambda_i.$$

Since $\int_{\mathcal{T}} \left\| D^2 u \right\|^2 dx = \int_{\mathcal{T}} \| D \mathsf{curl} u \|^2 dx$, we readily have:

Corollary 5.3. *In 2D, there exists a positive constant θ_0, independent of ν, such that*

$$\theta(\mu) = \theta_0$$

for every invariant measure $\mu \in \mathcal{P}$ of (1). Hence a family of invariant measures $\mathcal{M} \subset \mathcal{P} \times \mathbb{R}_+$ of (1) cannot have any scaling law (in the sense of (9).

Remark 5.11. Under our assumptions on the noise, invariant measures of (1) that belong to $\mathcal{P}$ certainly exist. In principle there could be others not in $\mathcal{P}$, but this cannot happen in all those cases when uniqueness of the invariant measure is known (see [53] and the references therein).

Remark 5.12. Consider equation (1) without the nonlinear term (called Stokes equations):

$$du(t) + \nu A u(t) dt = \sum_{i=1}^{\infty} \sigma_i h_i d\beta_i(t).$$

in dimension $d = 2, 3$. Let $\mathcal{M} \subset \mathcal{P} \times \mathbb{R}_+$ be a family of invariant measures for it. Then the same results of the previous theorem hold true. The proof is the same. Alternatively, one may work componentwise in h_i and prove easily the claims.

Let us treat now the 3D case. Recall the concept of *Galerkin stationary measures* introduced at the end of Section 3 and the notations $\mathcal{P}_{NS}^{Galerkin}(\nu)$ for the set of all such measures and $\mathcal{S}^n(\nu)$ for the invariant measures of the approximating Galerkin system (we underline here the dependence on ν).

Given $u \in V$, let S_u be the tensor with $L^2(\mathcal{T})$ components

$$S_u = \frac{1}{2}\left(Du + Du^T\right)$$

(called stress tensor). The scalar field

$$S_u(x)\,\mathsf{curl}u(x) \cdot \mathsf{curl}u(x)$$

describes the *stretching of the vorticity field*. If we set $\xi = \mathsf{curl}u$, then formally we have

$$\frac{\partial \xi}{\partial t} + (u \cdot \nabla)\,\xi = \nu \triangle \xi + S_u \xi + \sum_{i=1}^{\infty} \sigma_i\,(\mathsf{curl}h_i)\,\dot{\beta}_i(t)\,.$$

A *formal* application of Itô formula yields the inequality

$$\nu \cdot \mu \int_{\mathcal{T}} \|D\mathsf{curl}u\|^2\,dx \le \mu \int_{\mathcal{T}} S_u \mathsf{curl}u \cdot \mathsf{curl}u\,dx + \frac{1}{2}\sum_{i=1}^{\infty} \sigma_i^2 \lambda_i. \tag{11}$$

for $\mu \in \mathcal{P}_{NS}^{Galerkin}(\nu)$ (in fact formally the identity). Along with the general results proved above we would get

$$\mu\left[\int_{\mathcal{T}} S_u(x)\,\mathsf{curl}u(x) \cdot \mathsf{curl}u(x)\,dx\right] \ge C\varepsilon_0^{3/2}\nu^{-3/2}. \tag{12}$$

This would be the final result of this section, having an interesting physical interpretation. However, we cannot prove this result in this form, without further assumptions. We give, without proof (see [37]), two results around (12): theorem 5.4 reformulates it for the coarse graining scheme given by Galerkin approximations; theorem 5.4 expresses the most natural statement directly for $\mu \in \mathcal{P}_{NS}^{Galerkin}(\nu)$ but it requires an additional unproved regularity assumption.

Theorem 5.4. *Let $\mathcal{M} \subset \mathcal{P} \times \mathbb{R}_+$, with $\mathcal{M}_\nu \subset \mathcal{P}_{NS}^{Galerkin}(\nu)$, be a family with the K41 scaling law. Then there exists $\nu_0 > 0$ and $C > 0$ such that*

$$\liminf_{k\to\infty} \mu_{n_k}\left[\int_{\mathcal{T}} S_u \mathsf{curl}u \cdot \mathsf{curl}u dx\right] \ge C\varepsilon_0^{3/2}\nu^{-3/2}$$

for every $\nu \in (0, \nu_0)$, every $\mu \in \mathcal{M}_\nu$ and every sequence $\mu_{n_k} \in \mathcal{S}^{k_n}(\nu)$ such that μ_{k_n} converges to μ on H.

Lemma 5.5. *If $\mu \in \mathcal{P}_{NS}^{Galerkin}(\nu)$ is the weak limit of a sequence $\mu_{n_k} \in \mathcal{S}^{k_n}(\nu)$ such that $\mu_{n_k}\left[\|\cdot\|_V^{2+\varepsilon}\right] \leq C$ for some $\varepsilon, C > 0$, then*

$$\nu \cdot \mu \left[\int_{\mathcal{T}} \|Du(x)\|^2 \, dx\right] = \frac{1}{2}\sum_{i=1}^{\infty} \sigma_i^2.$$

If in addition $\mu_{n_k}\left[\|\cdot\|_V^{3+\varepsilon}\right] \leq C$ then (11) holds true.

Corollary 5.4. *Let $\mathcal{M} \subset \mathcal{P} \times \mathbb{R}_+$, with $\mathcal{M}_\nu \subset \mathcal{P}_{NS}^{Galerkin}(\nu)$, be a family with the K41 scaling law. Assume that every μ in $\mathcal{M}$ is the weak limit of a sequence $\mu_{n_k} \in \mathcal{S}^{k_n}(\nu)$ such that*

$$\mu_{n_k}\left[\|\cdot\|_V^{3+\varepsilon}\right] \leq C$$

for some $\varepsilon, C > 0$. Then there exists $\nu_0 > 0$ and $C > 0$ such that (12) holds for every $\nu \in (0, \nu_0)$ and every $\mu \in \mathcal{M}_\nu$.

Remark 5.13. If K41 scaling law holds then vortex stretching must be intense. Heuristically, no geometrical depletion of such stretching may occur (in contrast to the 2D case where the stretching term is zero because $\mathsf{curl} u(x)$ is aligned with the eigenvector of eigenvalue zero of $S_u(x)$): indeed, if we extrapolate the behavior $E\left[|Du|^2\right] \sim \frac{1}{\nu}$ as $Du \sim \frac{1}{\sqrt{\nu}}$, $\mathsf{curl} u \sim \frac{1}{\sqrt{\nu}}$, then we get $E\left[S_u \mathsf{curl} u \cdot \mathsf{curl} u\right] \sim \frac{1}{\nu\sqrt{\nu}}$ if there is no help from the geometry. Another way to explain this idea is the following sort of generalized Hölder inequality (for the proof, see [37]).

Corollary 5.5. *Let $\mathcal{M} \subset \mathcal{P} \times \mathbb{R}_+$, with $\mathcal{M}_\nu \subset \mathcal{P}_{NS}^{Galerkin}(\nu)$, be a family with the K41 scaling law, fulfilling the assumptions of theorem 5.4. Then there exists $\nu_0 > 0$ and $C > 0$ such that*

$$\left(\mu \int_{\mathcal{T}} \|Du\|^2 \, dx\right)^{1/2} \leq C\left(\mu\left[\int_{\mathcal{T}} \|S_u \mathsf{curl} u \cdot \mathsf{curl} u\|^2 \, dx\right]\right)^{1/3}$$

for every $\nu \in (0, \nu_0)$ and every $\mu \in \mathcal{M}_\nu$.

5.6 A Condition Equivalent to K41

We continue with the notations and concepts just introduced in the last section on the 3D case. Let $u(t,x)$ be a solution of equation (1) *on the unitary torus* ($L = 1$). We analyze the K41 property for it. Given $L > 0$, consider the new fields (see Kupiainen [50])

$$u_L(t,x) = L^{1/3} u(L^{-2/3}t, L^{-1}x)$$

and $p_L(t,x) = L^{2/3} p\left(L^{-2/3}t, L^{-1}x\right)$. To help the intuition, think that L is large so we blow-up the solution u. Formally, these fields satisfy the equations on the torus of size L

$$\frac{\partial u_L}{\partial t} + (u_L \cdot \nabla) u_L + \nabla p_L = \nu_L \triangle u_L + \sum_{i=1}^{\infty} \sigma_i h_i \left(\frac{x}{L}\right) \overset{\cdot}{\beta}{}_i^L (t) \tag{13}$$

where h_i were the eigenfunctions of the Stokes operator on the unitary torus, $\beta_i^L (t)$ are the independent Brownian motions

$$\beta_i^L (t) = L^{1/3} \beta_i \left(L^{-2/3} t\right)$$

and

$$\nu_L = \nu L^{4/3}.$$

The heuristic proof of this fact is a simple exercise: all terms $\frac{\partial u_L}{\partial t}$, $(u_L \cdot \nabla) u_L$, etc. are equal to $L^{-1/3}$ times the analogous terms $\frac{\partial u}{\partial t}$, $(u \cdot \nabla) u$, etc., and formally

$$\overset{\cdot}{\beta}{}_i^L (t) = L^{1/3} \dot{\beta}_i \left(L^{-2/3} t\right) L^{-2/3} = L^{-1/3} \dot{\beta}_i \left(L^{-2/3} t\right).$$

The same computation can be performed for the more general transformation

$$u_{(\lambda,\alpha)} (t, x) = \lambda^\alpha u \left(\lambda^{\alpha+1} t, \lambda x\right), \quad p_{(\lambda,\alpha)} (t, x) = \lambda^{2\alpha} p \left(\lambda^{\alpha+1} t, \lambda x\right)$$

but the previous choice of exponents is the only one such that the energy input per unit time and space is the same for every L, or λ (no coefficient depending on the scale parameter appears in front of the noise). Heuristically, if we believe in a uniform (not spiky, not intermittent) cascade picture of the energy (without essential inverse cascade), this invariance of the energy input should imply that the small scale properties of (1) (on the unitary torus) and (13) are the same, namely that they are invariant under this transformation; so we should expect that in the stationary regime $u_L (x)$ and $u (x)$ have approximatively the same law. But this would imply that $L^{1/3} u(L^{-1} x)$ and $u (x)$ have approximatively the same law, namely $u (Lx) \sim L^{1/3} u(x)$. Such scaling property would imply K41.

Let us stress again that not only we cannot prove claims like $u(Lx) \sim L^{1/3} u(x)$ but we do not believe they are exactly true. Presumably the correct result is closer to $u(Lx) \sim L^{1/3+k} u(x)$ for some $k > 0$.

Let us denote by $\mathcal{P}_{NS}^{Galerkin} (\nu)$ the family of Galerkin stationary measures for (1) on the unitary torus. Similarly, given L and a number $\tilde{\nu}$ (not necessarily equal to $\nu L^{4/3}$), let us denote by $\mathcal{P}_{NS}^{Galerkin} (\tilde{\nu}, L)$ the family of Galerkin stationary measures for equation (13) on the torus of size L, where we replace the symbol ν_L by $\tilde{\nu}$.

Let us denote by $\mathcal{P}_{NS}^{Galerkin} \times \mathbb{R}_+$ the set of *all pairs* (μ, ν) such that $\mu \in \mathcal{P}_{NS}^{Galerkin} (\nu)$. Similarly, let us denote by $\widetilde{\mathcal{P}}_{NS}^{Galerkin} \times \mathbb{R}_+^2$ the set of *all triples* $(\widetilde{\mu}, \tilde{\nu}, L)$ such that $\widetilde{\mu} \in \widetilde{\mathcal{P}}_{NS}^{Galerkin} (\tilde{\nu}, L)$.

Let $\mathcal{P}$ be the set of measures of the previous sections relative to the unitary torus. Let $\widetilde{\mathcal{P}}_L$ for the set of probability measures analogous to $\mathcal{P}$, but on the torus $[0,L]^3$. Denote by $\widetilde{\mathcal{P}} \times \mathbb{R}_+^2$ the set of all triples $(\widetilde{\mu}, \tilde{\nu}, L)$ such that $(\tilde{\nu}, L) \in \mathbb{R}_+^2$ and $\widetilde{\mu} \in \widetilde{\mathcal{P}}_L$. In the next definition and later on we use the notation $\widetilde{\mu}\left[\|u(e)-u(0)\|^2\right]$ when $\widetilde{\mu} \in \widetilde{\mathcal{P}}_L$ (and other similar mean values): this means

$$\widetilde{\mu}\left[\|u(e)-u(0)\|^2\right] = \int_{H_L} \|u(e)-u(0)\|^2 \, d\widetilde{\mu}(u)$$

where H_L is the usual space H but on the torus $[0,L]^3$.

The following condition seems interesting since it looks rather qualitative, in contrast to the definition of the K41 law, and shows that the exponent 2/3 arises from the scaling properties of the stochastic Navier–Stokes equations. Also the exponent in the range of r's arises spontaneously from this transformation.

Condition. *A subset $\tilde{\mathcal{M}} \subset \widetilde{\mathcal{P}} \times \mathbb{R}_+^2$ is said to satisfy Condition A if there exist $\tilde{\nu}_0 > 0$, $L_0 > 0$, $C > c > 0$ such that*

$$c \leq \widetilde{\mu}\left[\|u(e)-u(0)\|^2\right] \leq C \tag{14}$$

for every $(\widetilde{\mu}, \tilde{\nu}, L) \in \tilde{\mathcal{M}}$ with $\tilde{\nu} \leq \tilde{\nu}_0$, $L \geq L_0$.

Theorem 5.5. *The set $\widetilde{\mathcal{P}}_{NS}^{Galerkin} \times \mathbb{R}_+^2$ satisfies Condition A if and only if the set $\mathcal{P}_{NS}^{Galerkin} \times \mathbb{R}_+$ has a scaling law of K41 type, in the sense of Definition 5.1.*

Proof. **Step 1** (preparation). The proof is simple but notationally non trivial. The statement of K41 property involves two parameters, (ν, r), subject to the following constraints:

$$\nu \leq \nu_0 \,, \qquad r \in [\eta_0 \nu^{3/4}, r_0].$$

Hence we deal with the region

$$K_{\nu_0, r_0} = \left\{(\nu, r) \in \mathbb{R}_+^2 : \nu < \nu_0 \,, r \in [\left(\eta_0 \nu^{3/4}, r_0\right)\right\}$$

It is not restrictive to assume $r_0 = \nu_0^{3/4}\eta_0$, so the region K_{ν_0, r_0} looks like a right-angled triangle with a round hypotenuse (we suggest the reader to draw a picture of this set in the plane $\nu - r$).

Condition A involves two other parameters, $(\tilde{\nu}, L)$, subject to the constraint

$$\tilde{\nu} \leq \tilde{\nu}_0, \qquad L \geq L_0.$$

Hence, in condition A, we deal with the region

$$D_{\tilde{\nu}_0, L_0} = \left\{(\tilde{\nu}, L) \in \mathbb{R}_+^2 : \tilde{\nu} < \tilde{\nu}_0, L > L_0\right\}.$$

Such a region is a vertical semi-strip open upwards. Let us introduce the transformation $f : \mathbb{R}_+^2 \to \mathbb{R}_+^2$ defined as

$$f(\nu, r) = \left(\nu r^{-4/3}, r^{-1}\right)$$

which is invertible, with inverse given by

$$f^{-1}(\tilde{\nu}, L) = \left(\tilde{\nu} L^{-4/3}, L^{-1}\right).$$

We have

$$f(K_{\nu_0,r_0}) = D_{\tilde{\nu}_0,L_0}$$

if $\tilde{\nu}_0 = \eta_0^{-4/3}$, $L_0 = r_0^{-1}$. The piece of the curve $r = \eta_0 \nu^{3/4}$ pertaining to the boundary of K_{ν_0,r_0} is mapped into the vertical half-line $\tilde{\nu} = \tilde{\nu}_0, L > L_0$ of the boundary of $D_{\tilde{\nu}_0,L_0}$. The horizontal boundary segment of K_{ν_0,r_0} is mapped into the horizontal boundary segment of $D_{\tilde{\nu}_0,L_0}$. The vertical boundary segment of K_{ν_0,r_0} is mapped into the vertical half-line $\tilde{\nu} = \eta_0^{-4/3}, L > L_0$ of $D_{\tilde{\nu}_0,L_0}$.

Given any $L > 0$, let us also consider the mapping $S_L : \mathcal{H}_L \to \mathcal{H}$ defined by (see the scaling transformation above)

$$u(x) = L^{-1/3} \widetilde{u}(Lx).$$

It is possible to prove rigorously (see [37]) that

$$S_L\left(\widetilde{\mathcal{P}}_{NS}^{Galerkin}(\tilde{\nu}, L)\right) = \mathcal{P}_{NS}^{Galerkin}(\nu)$$
$$\text{for every } (\nu, \tilde{\nu}, L) \in \mathbb{R}_+^3 \text{ such that } \nu = \tilde{\nu} L^{-4/3}.$$

The heuristic has been given above before the theorem.

Step 2 (Condition A implies K41). Given $\tilde{\nu}_0, L_0$ in the definition of Condition A, choose ν_0, r_0 such that $f(K_{\nu_0,r_0}) \subset D_{\tilde{\nu}_0,L_0}$. Given $(\nu, r) \in K_{\nu_0,r_0}$ and $\mu \in \mathcal{P}_{NS}^{Galerkin}(\nu)$, let $(\tilde{\nu}, L) = f(\nu, r)$, so that $\nu = \tilde{\nu} L^{-4/3}$, and denote by $\widetilde{\mu}$ the measure in $\widetilde{\mathcal{P}}_{NS}^{Galerkin}(\tilde{\nu}, L)$ such that $\mu = S_{r^{-1}} \widetilde{\mu}$. We have

$$\begin{aligned}
S_2^{\mu}(r) &= \int_H \|u(re) - u(0)\|^2 \, d\mu(u) \\
&= \int_H \|u(re) - u(r0)\|^2 \, d(S_{r^{-1}} \widetilde{\mu})(u) \\
&= \int_{H_{r^{-1}}} \left\| r^{1/3} \widetilde{u}(e) - r^{1/3} \widetilde{u}(0) \right\|^2 d\widetilde{\mu}(\tilde{u}) \\
&= r^{2/3} \int_{H_{r^{-1}}} \|\widetilde{u}(e) - \widetilde{u}(0)\|^2 \, d\widetilde{\mu}(\tilde{u}).
\end{aligned}$$

By Condition A, $\int_{H_{r^{-1}}} \|\widetilde{u}(e) - \widetilde{u}(0)\|^2 \, d\widetilde{\mu}(\tilde{u})$ is bounded between two constants, hence we get K41.

Step 3 (K41 implies Condition A). The proof proceeds like step 2 but in the opposite direction and is left to the reader.

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