Chapter II
Intersection and Diametric Problems

In this chapter we first introduce the problem of finding the maximal cardinality of a system (or family) of subsets (in particular from $\binom{[n]}{k}$), such that any two subsets from the system intersect in not less than t elements. We call such a system of subsets t-intersecting family. We also consider the diametric problem in two different spaces. The diametric problem in one of the spaces is closely connected with the intersection problem. This connection is based on a technique that was invented by Ahlswede and Khachatrian. One can understand how it works by following the proof of the Complete Intersection Theorem, which we introduce later. This technique the reader first meets in the proof of Lemma 5. It is quite different from induction or other methods known before. In some sense it is a combination of shifting, with proving of necessity of the symmetry of the family under permutations of a sufficiently large number of components. The whole method becomes clear when the reader goes through the proofs and finds out that this method allows to solve several problems that had been considered hopeless for solving before. The reward for the efforts of the reader going along the lines of rather long proofs will be the satisfaction he attains at the end.

Lecture 1 The Complete Intersection Theorem

We turn to the problem of finding the maximal cardinality of a t-intersecting family of k-subsets (subsets of cardinality k of ground set $[n]$). It is easy to see that when $n \le 2k-t$, then the whole family $\binom{[n]}{k}$ of k-subsets is t-intersecting. We thus consider the case when $n > 2k-t$.

We come to necessary considerations and definitions. A system of sets $\mathcal{A} \subset 2^{[n]}$ is called t-intersecting if for arbitrary $A_1,\ A_2 \in \mathcal{A},\ \ |A_1 \cap A_2| \ge t$. Denote by $\mathcal{I}(n,t)$ the set of unrestricted t-intersecting systems and

$$\mathcal{I}(n,k,t) = \left\{ \mathcal{A} \in \mathcal{I}(n,t) : \mathcal{A} \subset \binom{[n]}{k} \right\}.$$

Our main goal is to determine the value

$$M(n,k,t) = \max_{\mathcal{A} \in \mathcal{I}(n,k,t)} |\mathcal{A}|.$$

We denote

$$\mathcal{F}(i) = \left\{ F \in \binom{[n]}{k} : |F \cap [t+2i]| \geq t+i \right\}, 0 \leq i \leq k-t.$$

In words: $\mathcal{F}(i)$ is the family of all k-element subsets of $[n]$ containing at least $t+i$ elements in the first $[t+2i]$ positions. Obviously, $\mathcal{F}(i)$ is a t-intersecting system: all pairs of k-element subsets from $\mathcal{F}(i)$ intersect in at least t elements already in the first $[t+2i]$ positions.

Theorem 3 (Complete Intersection Theorem (Ahlswede and Khachatrian 1997))

(i) *For* $n = 2k,\ t = 1$

$$M(n,k,1) = \binom{n-1}{k-1}.$$

For $1 \leq t \leq k \leq n,\ n > 2k-t,$

(ii) *If for some* $r \in \{0,1,2\dots\}$

$$(k-t+1)\left(2+\frac{t-1}{r+1}\right) < n < (k-t+1)\left(2+\frac{t-1}{r}\right), \tag{1}$$

then we have

$$M(n,k,t) = |\mathcal{F}(r)|.$$

Here we set $\frac{t-1}{r} = \infty$ *if* $r = 0$.

(iii) *If for some* $r \in \{0,1,2\dots\}$ *and* $t > 1$

$$(k-t+1)\left(2+\frac{t-1}{r+1}\right) = n, \tag{2}$$

then

$$M(n,k,t) = |\mathcal{F}(r)| = |\mathcal{F}(r+1)|.$$

Moreover, all optimal systems are known. In case (i) one must choose for each $A \in 2^{[n]}$ one set from $A, \bar{A}$. In the other cases, up to permutations on $[n]$, there is uniqueness in case (ii) and there are two systems in case (iii).

From this theorem it follows that if $n \geq (k-t+1)(t+1)$ then

$$M(n,k,t) = \binom{n-t}{k-t}. \tag{3}$$

In other words in this case the set

$$\mathcal{A}(n,k,t)=\left\{A\in\binom{[n]}{k}:\ [1,t]\subset A\right\}$$

is a maximal intersecting set.

Most famous in this subject is the $4m$-conjecture, which was stated more than 70 years ago (see [E87]) and says that

$$M(4m,2m,2)=\frac{\binom{4m}{2m}-\binom{2m}{m}^2}{2},$$

which is based on the construction

$$\{F\in\binom{[4m]}{2m}:|F\cap[1,2m]|\geq m+1\}.$$

It is the first case ($t=2$) for which relation (3), which is based on the "naive" construction $\mathcal{A}=\left\{A\in\binom{[n]}{k}:[1,t]\subset A\right\}$, is not optimal. Indeed, simple calculations give for $m=1$ still $M(4,2,2)=1=\binom{4-2}{2-2}$; however, for $m=2$ $M(8,4,2)=17>15=\binom{6}{2}$.

The $4m$-conjecture, which was mentioned in [E90] as the last open problem from [EKR61], attracted the attention of many mathematicians for a long time (see [DF83] and [CF92] for an upper bound). It also made it into the book [CG98].

Proof of Case (i). Obviously $M(2k,k,1)\geq\binom{2k-1}{k-1}$ and since with every A its complement cannot be in an intersecting family,

$$M(2k,k,1)\leq\frac{1}{2}\binom{2k}{k}=\frac{1}{2}\left(\binom{2k-1}{k}+\binom{2k-1}{k-1}\right)=\binom{2k-1}{k-1}.$$

Now we turn to other cases of this theorem. For every $1\leq i<j\leq n$ define on $2^{[n]}$ the left shifting operator L_{ij} by the equation

$$L_{ij}(A)=\begin{cases}\{i\}\cup(A\setminus\{j\}), & \text{if } i\notin A,\ j\in A,\\ A, & \text{otherwise.}\end{cases}$$

Also for every $1\leq i<j\leq n$ and set system $\mathcal{A}\in 2^{[n]}$ define an operator on $\mathcal{A}$

$$\mathcal{L}_{ij}(A,\mathcal{A})=\begin{cases}L_{ij}(A), & \text{if } L_{ij}(A)\notin\mathcal{A},\\ A, & \text{otherwise.}\end{cases}$$

Also set

$$\mathcal{L}_{ij}(\mathcal{A})=\{\mathcal{L}_{ij}(A,\mathcal{A}),\ A\in\mathcal{A}\}.$$

We say that the set system $\mathcal{A}\subset 2^{[n]}$ is left-compressed if $\mathcal{A}=\mathcal{L}_{ij}(\mathcal{A})$ for all $1\leq i<j\leq n$. Let $L\mathcal{I}$ be the set of all left-compressed systems belonging to $\mathcal{I}$. It is easy to see (Exercise 3) that

$$M(n,k,t)=\max_{\mathcal{A}\in L\mathcal{I}(n,k,t)}|\mathcal{A}|. \tag{4}$$

For arbitrary $A \in 2^{[n]}$ we denote by $A_{i,j},\ 1 \le i,j \le n$ the set obtained from A by exchanging coordinates i,j. For a set system $\mathcal{A} \subset 2^{[n]}$ and $1 \le i,j \le n$ we denote by $\mathcal{A}_{i,j}$ the set system obtained from $\mathcal{A}$ by exchanging the coordinates i,j in every $A \in \mathcal{A}$. Suppose that $\mathcal{A} \in L\mathcal{I}(n,k,t)$ and $\mathcal{A}$ is not right-compressed. Let $\ell < n$ be the largest integer such that $\mathcal{A}$ is invariant under exchange operations in $[0,\ell]$, that is,

$$\mathcal{A} = \mathcal{A}_{i,j}, 1 \le i,j \le \ell, \text{ but } \mathcal{A} \ne \mathcal{A}_{i,\ell+1} \text{ for some } 1 \le i \le \ell. \tag{5}$$

We set

$$\mathcal{A}' = \{A \in \mathcal{A} : A_{i,\ell+1} \notin \mathcal{A} \text{ for some } 1 \le i \le \ell\}. \tag{6}$$

Lemma 4 *The following relations are valid (Exercise 4):*

(i) $\ell+1 \notin A$ *for all* $A \in \mathcal{A}'$.
(ii) *Let* $A \in \mathcal{A}'$ *and* $j \in A,\ 1 \le j \le \ell$, *then* $A_{j,\ell+1} \notin \mathcal{A}$.
(iii) *Let* $A \in \mathcal{A}'$, $A = B \cup C$, *where* $B = A \cap [\ell],\ C = A \cap [\ell+1,n]$, *then* $B' \cup C \in \mathcal{A}'$ *for every* $B' \subset [1,\ell]$ *with* $|B'| = |B|$.
(iv) *Let* $A \in \mathcal{A}'$ *and* $D \in \mathcal{A} \setminus \mathcal{A}'$, *then*

$$|A_{i,\ell+1} \cap D| \ge t$$

for all $1 \le i \le \ell$.
(v) *Let* $A_1, A_2 \in \mathcal{A}'$, $B_i = A_i \cap [\ell]$; $i = 1,2$ *and suppose that* $|B_1| + |B_2| \ne \ell + t$, *then* $|A_1 \cap A_2| \ge t+1$.

Proposition 1 *Let* $\mathcal{B} \subset 2^{[n]}$ *be a set system, such that* $\mathcal{B} = \bar{\mathcal{B}} = \{\bar{B} : B \in \mathcal{B}\}$ *and* $\bar{B} = [n] \setminus B$. *Then every maximal intersecting* $\mathcal{B}' \subset \mathcal{B}$ *has cardinality* $|\mathcal{B}|/2$ *(Exercise 4).*

We will need the following key lemma.

Lemma 5 *Let* $\mathcal{A} \subset L\mathcal{I}(n,k,t)$, $|\mathcal{A}| = M(n,k,t)$, $n > 2k-t$ *and*

$$n < (k-t+1)\left(2 + \frac{t-1}{r}\right). \tag{7}$$

Then $\mathcal{A}_{i,j} = \mathcal{A}$ *for all* $1 \le i,j \le t+2r$.

Proof. We can assume that $t \ge 2$, because in the case $t = 1$ inequalities (7) and $n > 2k-t$ are incomparable ($r \ne 0$). We suppose that the statement of the lemma is not valid and come to a contradiction. Let $\ell < t+2r$ be such that $\mathcal{A}_{i,j} = \mathcal{A},\ 1 \le i,\ j \le \ell$, but $\mathcal{A}' = \{A \in \mathcal{A} : A_{i,\ell+1} \notin \mathcal{A} \text{ for some } 1 \le i \le \ell\} \ne \emptyset$. We will prove that in this case under the assumption (7) there exists a $\mathcal{B} \in \mathcal{I}(n,k,t)$ such that $|\mathcal{B}| > |\mathcal{A}|$, a contradiction.

Let

$$\mathcal{A}' = \bigcup_{i=1}^{\ell} \mathcal{A}(i),\ \mathcal{A}(i) = \{A \in \mathcal{A}' : |A \cap [1,\ell] = i\}.$$

From Lemma 4 it follows that $\mathcal{A}(i) = \emptyset$ when $1 \le i < t$. We will prove that all set systems $\mathcal{A}(i)$ are empty. Suppose that $\mathcal{A}(i) \neq \emptyset$ for some $i:\ t \le i \le \ell$. From (iii) of Lemma 4 it follows that

$$|\mathcal{A}(i)| = \binom{\ell}{i} |\mathcal{A}^*(i)|, \tag{8}$$

where

$$\mathcal{A}^*(i) = \{A \cap [\ell+2, n] :\ A \in \mathcal{A}(i)\}. \tag{9}$$

From (i) of Lemma 4 it follows that $\ell+1 \notin A$ for all $A \in \mathcal{A}'$. Also note that when $n = \ell+1$, we have $\mathcal{A}^*(i) = \{\emptyset\}$ and hence $|\mathcal{A}^*(i)| = 1$. Let

$$\mathcal{B}(i) = \{B : |B \cap [1,\ell]| = i-1,\ \ell+1 \in B,\ B \cap [\ell+2, n] \in \mathcal{A}^*(i)\}.$$

Then by (ii) of Lemma 4

$$|\mathcal{B}(i)| = \binom{\ell}{i-1} |\mathcal{A}^*(i)|,\ \mathcal{B}(i) \cap \mathcal{A} = \emptyset. \tag{10}$$

Similar to (8) and (10) we have

$$|\mathcal{A}(\ell+t-i)| = \binom{\ell}{\ell+t-i} |\mathcal{A}^*(\ell+t-i)|, \tag{11}$$

$$|\mathcal{B}(\ell+t-i)| = \binom{\ell}{\ell+t-i-1} |\mathcal{A}^*(\ell+t-i)|. \tag{12}$$

Next we consider two subcases: **1.** $i \neq \ell+t-i$ and **2.** $i = \ell+t-i$.

Subcase 1. From (v) of Lemma 4 it follows:

For $B \in \mathcal{B}(i),\ A \in \mathcal{A}(j)$ with $i+j \neq \ell+t$ we have $|B \cap A| \ge t$. Thus using this inequality, from (iv) of Lemma 4 we obtain

$$\mathcal{H}_1 = ((\mathcal{A} \setminus \mathcal{A}(\ell+t-i)) \cap \mathcal{B}(i)) \in \mathcal{I}(n,k,t),$$
$$\mathcal{H}_2 = ((\mathcal{A} \setminus \mathcal{A}(i)) \cap \mathcal{B}(\ell+t-i)) \in \mathcal{I}(n,k,t).$$

Next we show that in this case

$$\max\{|\mathcal{H}_1|, |\mathcal{H}_2|\} > |\mathcal{A}| = M(n,k,t), \tag{13}$$

which will be a contradiction. If the opposite to (13) is true, then from (8), (10)–(12) the inequalities

$$\binom{\ell}{i-1} |\mathcal{A}^*(i)| \le \binom{\ell}{\ell+t-i} |\mathcal{A}^*(\ell+t-i)|, \tag{14}$$
$$\binom{\ell}{\ell+t-i-1} |\mathcal{A}^*(\ell+t-i)| \le \binom{\ell}{i} |\mathcal{A}^*(i)|$$

follow. As $\mathcal{A}(i) \neq \emptyset$, from the first inequality in (14) it follows that $\mathcal{A}(\ell+t-i) \neq \emptyset$.

However, (14) yields the inequality

$$i(\ell+t-i) \le (\ell-i+1)(i+1-t),$$

which could not be true, because $t \ge 2$, and hence

$$i > i+1-t,\ \ell+t-i > \ell-i+1.$$

Thus $\mathcal{A}(i) = \emptyset$ for all $i \ne \ell+t-i$.

Subcase 2. Here we have $2|(\ell+t)$ and hence $\ell+2 \le n$. Therefore, if $\mathcal{A}\left(\frac{\ell+t}{2}\right) \ne \emptyset$, then $\mathcal{A}^*\left(\frac{\ell+t}{2}\right) \ne \emptyset$.

We have

$$\left|\mathcal{A}\left(\frac{\ell+t}{2}\right)\right| = \binom{\ell}{\frac{\ell+t}{2}}\left|\mathcal{A}^*\left(\frac{\ell+t}{2}\right)\right|$$

and any $A \in \mathcal{A}\left(\frac{\ell+t}{2}\right)$ can be written as $A = B \cup C$ with

$$B = (A \cap [1,\ell]) \in \binom{[\ell]}{\frac{\ell+t}{2}},\ C = (S \cap [\ell+2,n]) \in \mathcal{A}^*\left(\frac{\ell+t}{2}\right),$$

where $|C| = k - \frac{\ell+t}{2}$ since $\ell+1 \notin A$.

Using the pigeon-hole principle we establish the existence of an element in $[\ell+2,n]$ and $\mathcal{D} \subset \mathcal{A}^*\left(\frac{\ell+t}{2}\right)$ such that $d \in D$ for all $D \in \mathcal{D}$ and

$$|\mathcal{D}| \ge \left|\mathcal{A}^*\left(\frac{\ell+t}{2}\right)\right| \frac{k-\frac{\ell+t}{2}}{n-\ell-1}. \tag{15}$$

We set

$$\mathcal{A}_1\left(\frac{\ell+t}{2}\right) = \left\{A \in \mathcal{A}\left(\frac{\ell+t}{2}\right) : (A \cap [\ell+2,n]) \in \mathcal{D}\right\},$$
$$\mathcal{A}_2\left(\frac{\ell+t}{2}\right) = \mathcal{A}\left(\frac{\ell+t}{2}\right) \setminus \mathcal{A}_1\left(\frac{\ell+t}{2}\right).$$

Then

$$\mathcal{A}\left(\frac{\ell+t}{2}\right) = \mathcal{A}_1\left(\frac{\ell+t}{2}\right) \cup \mathcal{A}_2\left(\frac{\ell+t}{2}\right).$$

Also set

$$\mathcal{H} = \left(\mathcal{A} \setminus \mathcal{A}_2\left(\frac{\ell+t}{2}\right)\right) \cup \mathcal{G},$$

where

$$\mathcal{G} = \left\{B \in \binom{[n]}{k} : (B \cap [1,\ell]) \in \binom{[\ell]}{\frac{\ell+t}{2}-1},\ \ell+1 \in B, B \cap [\ell+2,n] \in \mathcal{D}\right\}.$$

From (ii) of Lemma 4 follows that $\mathcal{G} \cap \mathcal{A} = \emptyset$. Also note that $\mathcal{H} \in \mathcal{I}(n,k,t)$.

Next we show that under the conditions (1) and

$$\ell < t+2r,\ 2|(\ell+t) \tag{16}$$

the inequality

$$|\mathcal{H}| > |\mathcal{A}| \tag{17}$$

is valid, which will be a contradiction to the assumption about maximality of $\mathcal{A}$.

Note that from inequalities (16) the inequality

$$\ell \le t+2r-2 \tag{18}$$

follows. Since

$$|\mathcal{G}| = \binom{\ell}{\frac{\ell+t}{2}-1}|\mathcal{D}|,\ \left|\mathcal{A}_2\left(\frac{\ell+t}{2}\right)\right| = \binom{\ell}{\frac{\ell+t}{2}}\left(\left|\mathcal{A}^*\left(\frac{\ell+t}{2}\right)\right| - |\mathcal{D}|\right),$$

from (17) we get

$$\binom{\ell}{\frac{\ell+t}{2}-1}|\mathcal{D}| > \binom{\ell}{\frac{\ell+t}{2}}\left(\left|\mathcal{A}^*\left(\frac{\ell+t}{2}\right)\right| - |\mathcal{D}|\right)$$

or

$$\binom{\ell+1}{\frac{\ell+t}{2}}|\mathcal{D}| > \binom{\ell}{\frac{\ell+t}{2}}\left|\mathcal{A}^*\left(\frac{\ell+t}{2}\right)\right|.$$

From (15) it follows that for the validity of the last inequality it is sufficient to set

$$\binom{\ell+1}{\frac{\ell+t}{2}}\frac{k-\frac{\ell+t}{2}}{n-\ell-1} > \binom{\ell}{\frac{\ell+t}{2}}. \tag{19}$$

Inequality (19) is equivalent to $(k-t+1)\left(2+\frac{2(t-1)}{\ell-t+2}\right) > n$. The validity of the last inequality follows from (18) and assumption (7):

$$(k-t+1)\left(2+\frac{2(t-1)}{\ell-t+2}\right) \ge (k-t+1)\left(2+\frac{t-1}{r}\right) > n.$$

This proves Lemma 5. □

Proof of Theorem 3.
Case (ii): Suppose first that

$$(k-t+1)\left(2+\frac{t-1}{r+1}\right) < n < (k-t+1)\left(2+\frac{t-1}{r}\right) \tag{20}$$

and $\mathcal{A} \in L\mathcal{I}(n,k,t)$, $|\mathcal{A}| = M(n,k,t)$. From Lemma 5 it follows that $\mathcal{A}$ is invariant under the permutations of any positions in $[1,t+2r]$, hence $k \ge t+r$. Also it is easy to see that $\bar{\mathcal{A}}$ is right-compressed, $\bar{\mathcal{A}} \in \mathcal{I}(n,n-k,n-2k+t)$, and

$|\mathcal{A}| = |\bar{\mathcal{A}}| = M(n,k,t) = M(n,n-k,n-2k+t)$. From (20) the inequalities

$$(k'-t'+1)\left(2+\frac{t'-1}{r'+1}\right) < n < (k'-t'+1)\left(2+\frac{t'-1}{r'}\right)$$

follow, where $k' = n-k$, $t' = n-2k+t$, and $r' = k-t-r$ ($\frac{t'-1}{r'} = \infty$ when $r' = 0$).

Now it is easy to see that Lemma 5 can be formulated for right-compressed sets with obvious changes in the proof. This proof shows that $\bar{\mathcal{A}}$ is invariant under the permutations of the positions in $[n-t'-2r'+1,n] = [t+2r+1,n]$. Hence such invariance is also valid for $\mathcal{A}$ and $[t+2r+1,n]$. Since $\mathcal{A}$ is left-compressed and $n > 2k-t$, we have

$$|A_1 \cap A_2 \cap [1,t+2r]| \geq t,\ A_1,A_2 \in \mathcal{A}. \tag{21}$$

But $\mathcal{A}$ is invariant under permutations of the positions from $[1,t+2r]$. Thus the unique maximal set $\mathcal{A} \in L\mathcal{I}(n,k,t)$ is $\mathcal{A} = \mathcal{F}(r)$.

Case (iii): $n = (k-t+1)\left(2+\frac{t-1}{r+1}\right)$. Similar to the previous case we consider the complement set of $\mathcal{A}$ and using the same approach with one exception $n = 2,\ k,t = 1$, we derive an inequality similar to (21):

$$|A_1 \cap A_2 \cap [1,t+2r+2]| \geq t,\ A_1,A_2 \in \mathcal{A}. \tag{22}$$

Then (22) and Lemma 5 deliver two optimal sets: either $\mathcal{A} = \mathcal{F}(r)$ or $\mathcal{A} = \mathcal{F}(r+1)$ and

$$|\mathcal{A}| = |\mathcal{F}(r)| = |\mathcal{F}(r+1)|.$$

The answer in the case $n = 2,\ k,t = 0$ is obvious. The theorem is proved. Thus the problem of finding a maximal t-intersecting family is completely solved.

Now we turn our attention to the uniqueness of the optimal families. In case (i) one gets the optimal families by choosing from every set $\{A,\bar{A}\}$, $A \in \mathcal{A}$, exactly one element. Up to permutations there is exactly one optimal family in case (ii) and there are exactly two cases in case (iii). This we prove next.

We will need the following:

Lemma 6 *Suppose $\mathcal{A} \in \mathcal{I}(n,k,t)$ and $\mathcal{A}$ gets transformed by left shifting operations into the set $\mathcal{F}(r)$ for some $0 \leq r \leq (n-t)/2$. Then necessarily $\mathcal{A}$ is obtained from $\mathcal{F}(r)$ by permutations of the elements, provided that*

$$\begin{aligned} &n \geq 2k-t+2, \text{ for } t \geq 2,\\ &n = 2k-t+1, \text{ for } t \geq 2 \text{ and } k = t+r \text{ or } k = t+r+1,\\ &n \geq 2k+1, \text{ for } t = 1 \text{ and } r = 0 \text{ or } r = 1. \end{aligned}$$

Proof. W.l.o.g. we assume that

$$\mathcal{L}_{ij}(\mathcal{A}) = \mathcal{F}(r). \tag{23}$$

It is clear that if $i,j \in [1,t+2r]$ or $i,j \notin [1,t+2r]$, then $\mathcal{A} = \mathcal{F}(r)$.

Suppose now that $i = t + 2r$ and $j = n$. Let

$$\mathcal{A}_1 = \{A \in \mathcal{A} : \ j \in A, \ i \notin A, \ ((A \setminus \{j\}) \cup \{i\}) \notin \mathcal{A}\},$$
$$\mathcal{A}_2 = \{A \in \mathcal{A} : \ j \notin A, \ i \in A, \ ((A \setminus \{i\}) \cup \{j\}) \notin \mathcal{A}\}.$$

Clearly, if $\mathcal{A}_1 = \emptyset$, then $\mathcal{A} = \mathcal{F}(r)$ and if $\mathcal{A}_2 = \emptyset$, then $\mathcal{A}$ is obtained from $\mathcal{F}(r)$ by exchanging the coordinates $i = t + 2r$ and $j = n$. Suppose now that $\mathcal{A}_1, \mathcal{A}_2 \neq \emptyset$ and let us show that in this case $\mathcal{A} \notin \mathcal{I}(n,k,t)$. Consider

$$\mathcal{H} = \left\{ H \in \binom{[n] \setminus \{i,j\}}{k-1} : \ |H \cap [1, t+2r-1]| = t + r - 1 \right\}.$$

Observe that from (23) it follows that, for any $B \in \mathcal{A}_1 \cup \mathcal{A}_2$, $|B \cap [1, t+2r-1]| = t + r - 1$ holds. Moreover, from the same assumption (23) we have the following: for every $H \in \mathcal{H}$ either $H \cup \{j\} \in \mathcal{A}_1$ or $H \cup \{i\} \in \mathcal{A}_2$.

Now we form a graph $G = (V,E)$ as follows:

$$V = \mathcal{H}, \ e(H_1,H_2) \in E \text{ iff } \left| H_1 \cap H_2 \right| = t - 1.$$

One can easily verify that graph G is connected iff the conditions of the lemma hold. Hence under these conditions, if $\mathcal{A}_1 \neq \emptyset$ and $\mathcal{A}_2 \neq \emptyset$, then there exist $B_1 \in \mathcal{A}_1$ and $B_2 \in \mathcal{A}_2$ with $|B_1 \cap B_2| = t - 1$, which contradicts $\mathcal{A} \in \mathcal{I}(n,k,t)$. □

Now we are ready to prove the uniqueness of the optimal set system in the Complete Intersection Theorem. Let $n > 2k - t$, $\mathcal{A} \in \mathcal{I}(n,k,t)$, and $|\mathcal{A}| = M(n,k,t)$, and after finitely many left shifting operations let $\mathcal{A}$ be transformed to the left-compressed set system $\mathcal{A}' \in L\mathcal{I}(n,k,t)$, $|\mathcal{A}'| = M(n,k,t)$. We know that $\mathcal{A}' = \mathcal{F}(r)$ for some $r \in \mathbb{N} \cup 0$, where r is defined by the conditions of the theorem. It can be easily verified that these r's satisfy the conditions of the lemma and hence $\mathcal{A}$ is obtained from $\mathcal{F}(r)$ by permutations of the elements. ■

Now we consider the case when there is no restriction on the cardinality of a set from the t-intersecting family. This case turns out to be much simpler than the previous one. Denote

$$M(n,t) = \max_{\mathcal{A} \in \mathcal{I}(n,t)} |\mathcal{A}|,$$
$$\mathcal{K}(n,t) = \left\{ A \in 2^{[n]} : |A| \geq \frac{n+t}{2} \right\} = \bigcup_{i=\frac{n+t}{2}}^{n} \binom{[n]}{i}, \ if \ 2|(n+t).$$

Theorem 4 (Unrestricted Intersection Theorem (Katona 1964)) *The following identities hold:*

$$M(n,t) = \begin{cases} |\mathcal{K}(n,t)|, & 2|(n+t), \\ 2|\mathcal{K}(n-1,t)|, & 2 \not| (n+t). \end{cases} \tag{24}$$

Moreover, in the case $2|(n+t)$, $t>1$ *the optimal family is unique, while in the case* $2 \not|(n+t)$, $t>1$ *it is unique up to permutations of the ground set* $[n]$.

Proof. We will give the simple proof of this theorem, which was presented in [AK05]. It uses only shifting and induction. Consider only the case $2|(n+t)$, the case $2 \not|(n+t)$ has a similar proof. For $t=1$ and $t=n$ the theorem is obviously true $M(n,1)=2^{n-1}$, because if $A \in \mathcal{A}$, then $[n] \setminus A \notin \mathcal{A}$). We can assume, that $\mathcal{A} \in L\mathcal{I}(n,t)$. Let

$$\begin{aligned}
\mathcal{A}_1 &= \{A \in \mathcal{A}:\ 1 \in A\},\\
\mathcal{A}_0 &= \mathcal{A} \setminus \mathcal{A}_1,\\
\mathcal{A}_j^* &= \{A \cap [2,n]:\ A \in \mathcal{A}_j\},\ j=0,1,
\end{aligned}$$

Then $\mathcal{A}_1^* \in \mathcal{I}(n-1,t-1)$, $\mathcal{A}_0^* \in \mathcal{I}(n-1,t+1)$.

We have by induction

$$\begin{aligned}
|\mathcal{A}| = |\mathcal{A}_0^*| + |\mathcal{A}_1^*| &\le \sum_{i=\frac{n+t}{2}-1}^{n-1} \binom{n-1}{i} + \sum_{i=\frac{n+t}{2}}^{n-1} \binom{n-1}{i}\\
&= \sum_{i=\frac{n+t}{2}}^{n} \binom{n}{i}.
\end{aligned}$$

The uniqueness of the family $\mathcal{A}$ for $t>1$ also follows using induction. For $t=1$ it is delegated to Exercise 5. ■

Lecture 2 The Diametric Problem for Vertices in the Hamming Metric

Next we consider the diametric problem in the Hamming space $\mathcal{H}_q^n$, which is the space of n-tuples with elements from $[0,q-1]$ endowed with the Hamming metric $d_H(a^n,b^n) = n - \sum_{i=1}^n \delta_{a_i,b_i}$. As we will see, the solution of this problem is closely related to the t-intersection problem. The diametric problem is in some sense similar to the intersection problem. In the case, when all n-tuples in the family have exactly w nonzero symbols, these two problems coincide: if two n-tuples from the family intersect in t positions, then the distance between them is $2(w-t)$. Next we consider the nonrestrictive diametric problem: we find the maximal cardinality of a family of n-tuples with prescribed diameter of this family. Note that in the binary case $q=2$ this problem was solved a long time ago (formula (5) below). For $a^n, b^n \in \mathcal{H}_q^n$ denote

$$int(a^n,b^n) = \Big|\{j : a_j = b_j\}\Big|.$$

We call $\mathcal{A} \subset \mathcal{H}_q^n$ a $t-\mathcal{H}_q^n$ intersecting family if for all $a^n, b^n \in \mathcal{A}$,

$$int(a^n,b^n) \ge t.$$

Let $I_q(n,t)$ denote the set of all such families. Since

$$d_H(a^n, b^n) = n - int(a^n, b^n),$$

we have that the diameter of a $t - \mathcal{H}_q^n$ intersecting family $\mathcal{A}$ is not greater than $n-t$. Hence the problem of finding a maximal $t - \mathcal{H}_q^n$ intersecting family is equivalent to the problem of finding a maximal family with given diameter. Also note that the notions of $t - \mathcal{H}_2^n$ intersecting family and t-intersecting family in $2^{[n]}$ are different.

We are interested in finding a formula for the volume of a maximal $t - \mathcal{H}_q^n$ intersecting family

$$N_q(n,t) = \max_{\mathcal{A} \in I_q(n,t)} |\mathcal{A}|. \tag{1}$$

Let

$$B(a^n) = \{j : a_j = q-1\}, \tag{2}$$

$$\mathcal{K}(i) = \left\{a^n \in \mathcal{H}_q^n : |B(a^n) \cap [1, t+2i]| \geq t+i\right\}.$$

Clearly, $\mathcal{K}(i) \in I_q(n,t),\ i \in \{0,1,\ldots,(n-t)/2\}$. Obviously $\mathcal{K}(i)$ has diameter $n-t$. Indeed any two n-tuples already in the first $t+2i$ positions intersect in t positions. The next theorem gives the complete solution of the diametric problem.

Theorem 5 (Ahlswede and Khachatrian 1998) *For $q \geq 2,\ t > 1$ or $q = 2,\ t = 1$ let $r \in \{0,1,2\ldots\}$ be the largest integer such that*

$$t + 2r < \min\left\{n+1, t+2\frac{t-1}{q-2}\right\}. \tag{3}$$

Then $N_q(n,t) = |\mathcal{K}(r)|$. We set here $(t-1)/(q-2) = \infty$ if $q = 2$. Also we have

$$N_q(n,1) = |\mathcal{K}(0)| = q^{n-1}. \tag{4}$$

Uniqueness properties are delegated to the Exercises. In the case $q = 2$, this theorem was proved by Kleitman [K66a] and we write the explicit solution in that case

$$N_2(n,t) = \begin{cases} \sum_{i=0}^{\frac{n-t}{2}} \binom{n}{i}, & 2|(n-t), \\ 2\sum_{i=0}^{\frac{n-t-1}{2}} \binom{n-1}{i}, & 2 \not| (n-t). \end{cases} \tag{5}$$

This equality can be proved using the same arguments as in the proof of Theorem 4 (Exercise 7). We will use this equality later when we demonstrate the solution of the diametric problem in the Taxi metric. For $q > 2$ and small values of n, Frankl and Füredi [FF80] proved $N_q(n,t) = q^{n-t}$ iff $t \leq q-1$ or $t = n, n-1$. A generalization to $q \geq 2$ was proved in [M82].

Exercise 8 asks the reader to prove equality (4) directly. A natural candidate for the solution of the diametric problem in the case of arbitrary t is

$$N_q(n,t) = q^{n-t}. \tag{6}$$

The maximal family $\mathcal{B}_q(n,t)$ in this case can be chosen to be

$$\mathcal{B}_q(n,t) = \{B = (q_1,\dots,q_n) \in \mathcal{H}_q^n : (q_1,\dots,q_t) = (a_1,\dots,a_t)\}$$

for some $(a_1,\dots,a_t),\ a_i \in [q]$. However, as it follows from Theorem 5, this is not true in the general case. Before in [FF80] it was proved that this is true when $t \geq 15$ and $n \leq t+1$ or $q \geq t+1$.

Also it is interesting to mention one more particular case when

$$n \leq t + 1 + \log t / \log(q-1).$$

In this case

$$N_q(n,t) = \left|\mathcal{K}\left(\left\lfloor \frac{n-t}{2} \right\rfloor\right)\right|.$$

Proof of the Theorem. One can see that the definitions of the families $\mathcal{F}(i)$ and $\mathcal{K}(i)$ are quite similar. This gives the hint that the proofs of this theorem and Theorem 3 should also have common features. The reader will find in the proof of Theorem 5 a lot of technique from the proof of the Complete Intersection Theorem. Note that in the case $t = 1, q > 2$ inequality (3) is not satisfied for $r = 0,1,\dots$. It can be easily seen by following the beginning of the next proof that in this case $N_q(n,1) = |\mathcal{K}(0)| = q^{n-1}$ (see also Exercise 7). The uniqueness of the optimal configuration in this case up to permutations of the components and elements of the alphabet first was proved in [L79a]. In the case $t = 1,\ q = 2$ there are many possibilities of the choice of the optimal configuration (see Exercise 6). Exercises 7 and 9 ask to establish the uniqueness of the optimal configuration in other cases.

Now we turn to the proof of the theorem.

For $\mathcal{A} \subset \mathcal{H}_q^n,\ a^n \in \mathcal{A}$, and $j \in \{1,2,\dots,n\},\ i \in \{0,1,\dots,q-1\}$ we define

$$T_{ji}(a^n) = \begin{cases} b^n = (a_1,a_2,\dots,a_{j-1},q-1,a_{j+1},\dots,a_n), & b^n \notin \mathcal{A} \text{ and } a_j = i, \\ a^n, & \text{otherwise.} \end{cases}$$

Also we put

$$T_{ji}(\mathcal{A}) = \left\{T_{ji}(a^n) : a^n \in \mathcal{A}\right\}.$$

We say that the set $\mathcal{A} \subset \mathcal{H}_q^n$ is canonical if

$$T_{ji}(\mathcal{A}) = \mathcal{A}$$

for all $j = 1,2,\dots,n$ and $i = 0,1,\dots,q-1$. It is easy to see that by a finite number of operations T_{ji} every set $\mathcal{A}$ becomes canonical. Also, each transformation T_{ji} keeps the cardinality and the $t-\mathcal{H}_q^n$-intersection property unchanged, that is, $|T_{ji}(\mathcal{A})| = |\mathcal{A}|$ and $\mathcal{A} \in I_q(n,t) \Rightarrow T_{ji}(\mathcal{A}) \in I_q(n,t)$.

Hence

$$N_q(n,t) = \max_{\mathcal{A} \in CI_q(n,t)} |\mathcal{A}|, \tag{7}$$

where $CI_q(n,t) \subset I_q(n,t)$ is the set of canonical families in $I_q(n,t)$.

With each system $\mathcal{A} \in CI_q(n,t)$ we associate the "image" $\mathcal{B}(\mathcal{A}) = \{B(a^n) :\ a^n \in \mathcal{A}\}$, where $B(a^n)$ is defined in (2). It is not difficult to see that if $\mathcal{A} \in CI_q(n,t)$, then

$$\mathcal{B}(\mathcal{A}) \in \mathcal{I}(n,t). \tag{8}$$

■

Directly from the definition follows (Exercise 10)

Proposition 2 *Let $\mathcal{A} \in CI_q(n,t)$ be maximal: $|\mathcal{A}| = N_q(n,t)$ and let $\mathcal{B}(\mathcal{A})$ be the image of $\mathcal{A}$. Then*

(i) *$\mathcal{B}(\mathcal{A})$ is an upset.*

(ii)

$$|\mathcal{A}| = \sum_{B \in \mathcal{B}(\mathcal{A})} (q-1)^{n-|B|} = \sum_{i=0}^{n} g(i)(q-1)^{n-i},$$

where

$$g(i) = \left| \mathcal{B}(\mathcal{A}) \cap \binom{[n]}{i} \right|.$$

Denote by $LCI_q(n,t) \subset CI_q(n,t)$ the set of all systems $\mathcal{A}$ from $CI_q(n,t)$ with $\mathcal{B}(\mathcal{A}) \in L\mathcal{I}(n,t)$. From the definitions it follows that

$$N_q(n,t) = \max_{\mathcal{A} \in LCI_q(n,t)} |\mathcal{A}|. \tag{9}$$

For $E \in 2^{[n]}$ denote

$$\mathcal{V}(E) = \left\{ a^n \in \mathcal{H}_q^n : B(a^n) \in \mathcal{U}(E) \right\}. \tag{10}$$

Here $\mathcal{U}(E)$ is the upset with one minimal set E. Obviously

$$|\mathcal{V}(E)| = q^{n-|E|}. \tag{11}$$

For $\mathcal{E} \subset 2^{[n]}$ we put

$$\mathcal{V}(\mathcal{E}) = \bigcup_{E \in \mathcal{E}} \mathcal{V}(E).$$

We call $\mathcal{A}$ a q-upset, if

$$\mathcal{A} = \mathcal{V}(\mathcal{B}(\mathcal{A})).$$

For $E = \{e_1, e_2, \ldots, e_{|E|}\} \in 2^{[n]},\ e_1 < e_2 < \ldots < e_{|E|}$ we set $s^+(E) = e_{|E|}$ and for $\mathcal{E} \subset 2^{[n]}$ we set

$$s^+(\mathcal{E}) = \max_{E \in \mathcal{E}} s^+(E).$$

The next results follow from the definitions.

Proposition 3 *Let $\mathcal{A} \in LCI_q(n,t)$ be a q-upset and $\mathcal{B}(\mathcal{A})$ be the image of $\mathcal{A}$. Let $\mathcal{M}(\mathcal{A})$ be the set of minimal elements of $\mathcal{B}(\mathcal{A})$ (in the sense of set inclusion). Then*

$\mathcal{A}$ is a disjoint union

$$\mathcal{A} = \bigcup_{E \in \mathcal{M}(\mathcal{A})} D(E),$$

where

$$D(E) = \left\{ a^n = (a_1, a_2, \dots, a_n) \in \mathcal{H}_q^n : B(a^n) \cap [1, s^+(E)] = E \right\}. \tag{12}$$

Proposition 4 *Let $\mathcal{A} \in LCI_q(n,t)$ be a q-upset. For $E \in \mathcal{M}(\mathcal{A})$ such that $s^+(E) = s^+(\mathcal{M}(\mathcal{A}))$, denote*

$$\mathcal{A}_E = \mathcal{V}(E) \setminus \mathcal{V}(\mathcal{M}(\mathcal{A}) \setminus E).$$

This is the set of elements from $\mathcal{A}$ which are generated only by E.

Then

$$\mathcal{A}_E = D(E)$$

and

$$|\mathcal{A}_E| = (q-1)^{s^+(E)-|E|} q^{n-s^+(E)}. \tag{13}$$

Proposition 5 *Let $\mathcal{A} \in LCI_q(n,t)$ be a q-upset and let $E_1, E_2 \in \mathcal{M}(\mathcal{A})$ have the properties $i \notin E_1 \cup E_2,\ j \in E_1 \cap E_2$ for some $i, j \in [n],\ i < j$. Then*

$$|E_1 \cap E_2| \geq t+1.$$

We need the following key result.

Lemma 7 *For $q > 2$ and $\mathcal{A} \in LCI_q(n,t)$ with $|\mathcal{A}| = N_q(n,t)$ for some $r \in \{0,1,2,\dots\}$ we have*

$$s^+(\mathcal{M}(\mathcal{A})) = t + 2r \leq t + \frac{2(t-1)}{q-2}. \tag{14}$$

If $(t-1)/(q-2)$ is a positive integer, then there exists an $\mathcal{A}' \in LCI_q(n,t)$ with $|\mathcal{A}'| = N_q(n,t)$ such that for some $r' \in \{0,1,2,\dots\}$

$$s^+(\mathcal{M}(\mathcal{A}')) = t + 2r' < t + \frac{2(t-1)}{q-2}. \tag{15}$$

Proof. First we prove (14). Suppose the opposite is true:

$$s^+(\mathcal{M}(\mathcal{A})) = \ell > t + \frac{2(t-1)}{q-2} \tag{16}$$

or $2 \not| (\ell - t)$ and

$$\ell \leq t + \frac{2(t-1)}{q-2}. \tag{17}$$

Let us show that in this case there exists $\mathcal{A}' \in I_q(n,t)$ such that $|\mathcal{A}'| > |\mathcal{A}|$. The proof of this fact is quite similar to the proof of the t-intersection theorem and we frequently refer to it. Consider the partition

$$\mathcal{M}(\mathcal{A}) = \mathcal{M}_0(\mathcal{A}) \cup \mathcal{M}_1(\mathcal{A}),$$

where

$$\mathcal{M}_0(\mathcal{A}) = \{E \in \mathcal{M}(\mathcal{A}) : s^+(E) = s^+(\mathcal{M}(\mathcal{A})) = \ell\}$$

and

$$\mathcal{M}_1(\mathcal{A}) = \mathcal{M}(\mathcal{A}) \setminus \mathcal{M}_0(\mathcal{A}).$$

Note that for $E_1 \in \mathcal{M}_0(\mathcal{A})$ and $E_2 \in \mathcal{M}_1(\mathcal{A})$ we have

$$|(E_1 \setminus \{\ell\}) \cap E_2| \geq t.$$

Similar to Lemma 4 (v), using Proposition 5, it can be proved (Exercise 11) that if $E_1, E_2 \in \mathcal{M}_0(\mathcal{A})$ and $|E_1 \cap E_2| = t$, then

$$|E_1| + |E_2| = \ell + t. \tag{18}$$

Like in the proof of the t-intersection theorem, we consider the partition

$$\mathcal{M}_0(\mathcal{A}) = \bigcup_i \mathcal{R}(i),$$

where $\mathcal{R}(i) = \mathcal{M}_0(\mathcal{A}) \cap \binom{[n]}{i}$ and

$$\mathcal{R}'(i) = \{E \subset [1, \ell - 1] : E \cup \{\ell\} \in \mathcal{R}(i)\}.$$

Now we prove that all $\mathcal{R}(i)$ are empty. Suppose that for some i, $\mathcal{R}(i) \neq \emptyset$. Note that from (18) it follows that if $E_1' \in \mathcal{R}'(i)$, $E_2' \in \mathcal{R}'(j)$, and $i + j \neq \ell + t$, then

$$|E_1' \cap E_2'| \geq t. \tag{19}$$

As before, we consider two cases: **a**. $i \neq (\ell + t)/2$ and **b**. $i = (\ell + t)/2$.

Case a. According to (19) the two sets

$$\begin{aligned} \mathcal{F}_1 &= \mathcal{M}_1(\mathcal{A}) \cup (\mathcal{M}_0(\mathcal{A}) \setminus (\mathcal{R}(i) \cup \mathcal{R}(\ell + t - i))) \cup \mathcal{R}'(i), \\ \mathcal{F}_2 &= \mathcal{M}_1(\mathcal{A}) \cup (\mathcal{M}_0(\mathcal{A}) \setminus (\mathcal{R}(i) \cup \mathcal{R}(\ell + t - i))) \cup \mathcal{R}'(\ell + t - i), \end{aligned}$$

have the property $\mathcal{F}_1,\ \mathcal{F}_2 \in \mathcal{I}(n,t)$ and hence $\mathcal{A}_i = \mathcal{V}(\mathcal{F}_i) \in I_q(n,t),\ i = 1,2$. We show that under the assumption $\mathcal{R}(i) \neq \emptyset$ we have

$$\max\{|\mathcal{A}_1|, |\mathcal{A}_2|\} > |\mathcal{A}|, \tag{20}$$

which will be a contradiction to the maximality of $\mathcal{A}$.

From the definitions of $\mathcal{F}_1$ and $\mathcal{R}(i)$ it follows that

$$\mathcal{A} \setminus \mathcal{A}_1 = \bigcup_{E \in \mathcal{R}(\ell + t - i)} D(E),$$

and from Proposition 4 we have

$$|\mathcal{A} \setminus \mathcal{A}_1| = |\mathcal{R}(\ell+t-i)|(q-1)^{i-t} q^{n-\ell}. \tag{21}$$

Now we estimate the value $|\mathcal{A}_1 \setminus \mathcal{A}|$. Let $E_1 \in \mathcal{R}'(i)$. Then, denote

$$D'(E_1) = \left\{ a^n \in \mathcal{H}_q^n : B(a^n) \cap [\ell] = E_1 \right\}. \tag{22}$$

We have

$$D'(E_1) \in \mathcal{A}_1 \setminus \mathcal{A}. \tag{23}$$

Since

$$|D'(E_1)| = (q-1)^{\ell-i+1} q^{n-\ell}$$

and

$$D'(E_1) \cap D'(E_2) = \emptyset,\ E_1, E_2 \in \mathcal{R}(i),\ E_1 \neq E_2$$

we obtain

$$|\mathcal{A}_1 \setminus \mathcal{A}| \geq |\mathcal{R}(i)|(q-1)^{\ell-i+1} q^{n-\ell}. \tag{24}$$

In a similar way we show that

$$|\mathcal{A} \setminus \mathcal{A}_2| = |\mathcal{R}(i)|(q-1)^{\ell-i} q^{n-\ell}, \tag{25}$$

$$|\mathcal{A}_2 \setminus \mathcal{A}| \geq |\mathcal{R}(\ell+t-i)|(q-1)^{i-t+1} q^{n-\ell}. \tag{26}$$

It is left to the reader to show that even more, (24) and (26) are equalities!

From (21), (24), (25), and (26) follows that if (20) is not true, then

$$|\mathcal{R}(i)|(q-1)^{\ell-i+1} \leq |\mathcal{R}(\ell+t-i)|(q-1)^{i-t},$$
$$|\mathcal{R}(\ell+t-i)|(q-1)^{i-t+1} \leq |\mathcal{R}(i)|(q-1)^{\ell-i}.$$

If $\mathcal{R}(i) \neq \emptyset$ and $q > 2$, these inequalities are inconsistent. This implies that $\mathcal{R}(i) = \emptyset$ for all $i \neq (\ell+t)/2$. In particular, we prove that if $\mathcal{R}(i) \neq \emptyset$, then $2|(\ell+t)$ and $i = \frac{\ell+t}{2}$.

Case b. By the pigeon-hole principle there exists an $i \in [1, \ell-1]$ and a $\mathcal{G} \subset \mathcal{R}'\left(\frac{t+\ell}{2}\right)$ such that $i \notin E$ for all $E \in \mathcal{G}$ and

$$|\mathcal{G}| \geq \frac{\ell-t}{2(\ell-1)} \left| \mathcal{R}'\left(\frac{\ell+t}{2}\right) \right|. \tag{27}$$

As $|E_1 \cap E_2| \geq t,\ E_1, E_2 \in \mathcal{G}$, and $\mathcal{R}(i) = \emptyset,\ i \neq (\ell+t)/2$, we have

$$\mathcal{F}' = \left(\mathcal{M}(\mathcal{A}) \setminus \mathcal{R}\left(\frac{\ell+t}{2}\right) \right) \cup \mathcal{G} \in \mathcal{I}(n,t).$$

Thus

$$\mathcal{V}(\mathcal{F}') \in I_q(n,t).$$

Next we show that under the condition (14),

$$|\mathcal{V}(\mathcal{F}')| > |\mathcal{A}|, \tag{28}$$

which is a contradiction to the maximality of $\mathcal{A}$. Consider the partition

$$\mathcal{A} = \mathcal{V}(\mathcal{M}(\mathcal{A})) = D_1 \cup D_2,$$

where

$$D_1 = \mathcal{V}\left(\mathcal{M}(\mathcal{A}) \setminus \mathcal{R}\left(\frac{\ell+t}{2}\right)\right),$$
$$D_2 = \mathcal{V}\left(\mathcal{R}\left(\frac{\ell+t}{2}\right)\right) \setminus \mathcal{V}\left(\mathcal{M}(\mathcal{A}) \setminus \mathcal{R}\left(\frac{\ell+t}{2}\right)\right),$$

and

$$\mathcal{V}(\mathcal{F}') = D_1 \cup D_3,$$

where

$$D_3 = \mathcal{V}(\mathcal{G}) \setminus \mathcal{V}\left(\mathcal{M}(\mathcal{A}) \setminus \mathcal{R}\left(\frac{\ell+t}{2}\right)\right).$$

Inequality (28) is equivalent to the inequality

$$|D_3| > |D_2|. \tag{29}$$

From Proposition 4 we have

$$|D_2| = \left|\mathcal{R}\left(\frac{\ell+t}{2}\right)\right|(q-1)^{(\ell-t)/2} q^{n-\ell}. \tag{30}$$

Let $E \in \mathcal{G}$, $E \subset [\ell-1]$, and $|E| = (\ell+t)/2 - 1$. Denote

$$\mathcal{C}(E) = \left\{a^n \in \mathcal{H}_q^n : B(a^n) \cap [\ell-1] = E\right\}.$$

Then $\mathcal{C}(E) \subset D_3$ and we have the partition

$$D_3 = \bigcup_{E \in \mathcal{G}} \mathcal{C}(E)$$

and hence

$$|D_3| = |\mathcal{G}|(q-1)^{(\ell-t)/2} q^{n-\ell+1}. \tag{31}$$

Using inequality (27), from (29), (30), and (31) we get that the following inequality is sufficient for (28) to hold:

$$\frac{\ell-t}{2(\ell-1)}\left|\mathcal{R}\left(\frac{\ell+t}{2}\right)\right|(q-1)^{(\ell-t)/2} q^{n-\ell+1}$$
$$> \left|\mathcal{R}\left(\frac{\ell+t}{2}\right)\right|(q-1)^{(\ell-t)/2} q^{n-\ell}.$$

From inequality (16) it follows that the last inequality is true ($\mathcal{R}\left(\frac{\ell+t}{2}\right) \neq \emptyset$). Hence assumption (16) is false and the first part of Lemma 7 is proved.

To prove the second part of the lemma, suppose that $(t-1)/(q-2)$ is a positive integer and

$$s^+(\mathcal{M}(\mathcal{A})) = \ell = t + 2\frac{t-1}{q-2}. \tag{32}$$

We have already proved that for all $E \in \mathcal{M}(\mathcal{A})$ with $s^+(E) = \ell$ we have $|E| = (\ell+t)/2$. One can repeat the proof of *Case b* and show that instead of (28) under assumption (32) we have the inequality $|\mathcal{V}(\mathcal{F}')| \geq |\mathcal{A}|$. This completes the proof of the lemma. □

In the proof of Theorem 5 we use a lemma that allows to reduce the problem to another one, which we have already solved with Theorem 3. Let $S \subset 2^{2^{[m]}}$ and

$$H(S, \beta_t, \ldots, \beta_m) = \max_{\mathcal{L}\in S} \sum_{i=t}^{m} |\mathcal{L}(i)|\beta_i, \tag{33}$$

where $\mathcal{L}(i) = \mathcal{L} \cap \binom{[m]}{i},\ t \leq m,\ \mathcal{L} \subset 2^{[m]}$, and $\beta_t, \beta_{t+1}, \ldots, \beta_m \in \mathbb{R}_+$. Suppose that for some $S \subset 2^{2^{[m]}}$ there is an $\mathcal{L}^* \in S$ such that for some $r \in \{1, 2, \ldots\}$, $\mathcal{L}^*(i) = \emptyset$ for $t \leq i < t+r$ and $|\mathcal{L}^*(i)| \geq |\mathcal{L}(i)|$ for $t+r \leq i \leq m$ and all $\mathcal{L} \in S$.

Lemma 8 *Let* $\beta_t, \beta_{t+1}, \ldots, \beta_m \in \mathbb{R}_+$ *and*

$$\mathcal{L}^* = \arg\max_{\mathcal{L}\in S} \sum_{i=t}^{m} |\mathcal{L}(i)|\beta_i$$

have the properties described above. Then, for any $\gamma_t, \ldots, \gamma_m \in \mathbb{R}_+$ *such that*

$$\frac{\beta_i}{\beta_{i+1}} \geq \frac{\gamma_i}{\gamma_{i+1}},\ i = t, \ldots, m-1, \tag{34}$$

it holds

$$\mathcal{L}^* = \arg\max_{\mathcal{L}\in S} \sum_{i=t}^{m} |\mathcal{L}(i)|\gamma_i.$$

Proof. W.l.o.g. we can assume that $\beta_m = \gamma_m = 1$. We introduce the numbers $\beta_t, \ldots, \beta_m$ and $\gamma_t, \ldots, \gamma_m$ in the form

$$\begin{aligned}
\beta_m &= 1, & \gamma_m &= 1;\\
\beta_{m-1} &= \delta_{m-1}, & \gamma_{m-1} &= \varepsilon_{m-1};\\
\beta_{m-2} &= \delta_{m-1}\delta_{m-2}, & \gamma_{m-2} &= \varepsilon_{m-1}\varepsilon_{m-2},\\
&\vdots & &\vdots\\
\beta_i &= \delta_{m-1}\delta_{m-2}\ldots\delta_i, & \gamma_i &= \varepsilon_{m-1}\varepsilon_{m-2}\ldots\varepsilon_i;\\
&\vdots & &\vdots\\
\beta_t &= \delta_{m-1}\delta_{m-2}\ldots\delta_t, & \gamma_t &= \varepsilon_{m-1}\varepsilon_{m-2}\ldots\varepsilon_t.
\end{aligned}$$

We have

$$\delta_i \geq \varepsilon_i,\ i = 1, \ldots, m-1. \tag{35}$$

Let $\ell \in \{1, 2, \ldots\}$ be the largest integer such that $\delta_i = \varepsilon_i,\ i \geq m - \ell + 1$.

Introduce the positive numbers $\beta'_t, \ldots, \beta'_m$ satisfying $\beta'_m = \beta_m, \ldots, \beta'_{m-\ell+1} = \beta_{m-\ell+1}$ and $\beta'_i = \beta_i \varepsilon_{m-\ell}/\delta_{m-\ell},\ t \leq i \leq m - \ell$.

If $m - \ell + 1 \leq t + r$, then

$$\sum_{i=t}^{m} |\mathcal{L}^*(i)|\beta'_i = \sum_{i=t}^{m} |\mathcal{L}^*(i)|\beta_i \geq \sum_{i=t}^{m} |\mathcal{L}(i)|\beta_i \geq \sum_{i=t}^{m} |\mathcal{L}(i)|\beta'_i.$$

If $m - \ell + 1 > t + r$, then the inequality

$$\sum_{i=1}^{m} |\mathcal{L}^*(i)|\beta'_i \geq \sum_{i=1}^{m} |\mathcal{L}(i)|\beta'_i \tag{36}$$

is equivalent to

$$\begin{aligned} &\sum_{i=m-\ell+1}^{m} |\mathcal{L}^*(i)|\beta_i + \sum_{i=t+r}^{m-\ell} |\mathcal{L}^*(i)| \frac{\beta_i \varepsilon_{m-\ell}}{\delta_{n-\ell}} \\ \geq &\sum_{i=m-\ell+1}^{m} |\mathcal{L}(i)|\beta_i + \sum_{i=t}^{m-\ell} |\mathcal{L}(i)| \frac{\beta_i \varepsilon_{m-\ell}}{\delta_{m-\ell}}, \end{aligned}$$

or

$$\begin{aligned} &(\delta_{m-\ell} - \varepsilon_{m-\ell}) \sum_{i=m-\ell+1}^{m} (|\mathcal{L}^*(i)| - |\mathcal{L}(i)|)\beta_i \\ &+ \varepsilon_{m-\ell} \left(\sum_{i=t}^{m} |\mathcal{L}^*(i)|\beta_i - \sum_{i=t}^{m} |\mathcal{L}(i)|\beta_i \right) \geq 0. \end{aligned}$$

The last inequality is true since $\delta_{m-\ell} > \varepsilon_{m-\ell}$ and $|\mathcal{L}^*(i)| \geq |\mathcal{L}(i)|$ for $i \geq m - \ell + 1 > t + r$. Continuing this transformation we obtain step by step the coefficients $\gamma_t, \ldots, \gamma_m$ and this proves Lemma 8. $\square$

Now we are ready to prove Theorem 5. It is convenient now to denote the Hamming ball in the space of binary sequences of length $t + 2r$, which has radius r and is centered at $[t + 2r]$ by

$$\mathcal{D}(r,t) = \left\{ D \in 2^{[t+2r]} : |D| \geq t + r \right\}.$$

Let

$$\mathcal{D}(i) = \mathcal{D}(r,t) \cap \binom{[t+2r]}{i}.$$

We have $|\mathcal{D}(i)| = 0,\ i < t+r$, and $|\mathcal{D}(i)| = \binom{t+2r}{i}, i \geq t+r$. Also note that $\mathcal{D}(r,t) \in \mathcal{I}(t+2r,t)$. It is easy to show that the following relations are valid:

$$\begin{aligned}
|\mathcal{F}(r)| &= \sum_{j=0}^{r} \binom{2r+t}{t+r+j}\binom{n-2r-t}{k-t-r-j} \\
&= \sum_{i=0}^{t+2r} |\mathcal{D}(i)| \binom{n-2r-t}{k-i}, \\
|\mathcal{K}(r)| &= \sum_{j=0}^{r} \binom{2r+t}{t+r+j} (q-1)^{r-j} q^{n-2r-t} \\
&= q^{n-2r-t} \sum_{j=0}^{t+2r} |\mathcal{D}(i)| (q-1)^{2r+t-i}.
\end{aligned}$$

We can reformulate Theorem 3 as follows. Let for some $r = \{0,1,2,\dots\}$

$$(k-t+1)\left(2+\frac{t-1}{r+1}\right) < m_0 < (k-t+1)\left(2+\frac{t-1}{r}\right), \tag{37}$$

and for $i \geq t$

$$\gamma_i = \binom{m_0-2r-t}{k-i}. \tag{38}$$

Then

$$\mathcal{D}(r,t) = \arg \max_{\mathcal{M} \in \mathcal{I}(2r+t,t)} \sum_{i=t}^{t+2r} |\mathcal{M}(i)| \gamma_i,$$

where $\mathcal{M}(i) = \mathcal{M} \cap \binom{[t+2r]}{i}$.

When $q \geq 2,\ t > 1$ or $q = 2,\ t = 1$, let us choose $r \in \{0,1,2,\dots\}$ such that

$$t+2r < \min\left\{n+1, t+\frac{2(t-1)}{q-2}\right\} \tag{39}$$

and

$$\begin{aligned}
N_q(n,t) &= \max_{\mathcal{M} \in \mathcal{I}(t+2r,t)} \sum_{i=t}^{t+2r} |\mathcal{M}(i)| (q-1)^{t+2r-i} q^{n-t-2r} \\
&= q^{n-t-2r} \max_{\mathcal{M} \in \mathcal{I}(t+2r,t)} \sum_{i=t}^{t+2r} |\mathcal{M}(i)| (q-1)^{t+2r-i}.
\end{aligned} \tag{40}$$

The possibility of such a choice follows from Lemma 7. Next we apply Lemma 8 for $m = t+2r$, $S = \mathcal{I}(t+2r,t) \subset 2^{2^{[t+2r]}}$, $\gamma_i = \binom{m_0-2r-t}{k-i}$, $i = t, t+1, \dots, t+2r$, where m_0 satisfies (37) and $\beta_i = (q-1)^{t+2r-i}$. Also, we take $\mathcal{L}^* = \mathcal{D}(r,t) \in \mathcal{I}(1+2r,t)$. It is easy to see that $\mathcal{D}(r,t)$ enjoys the properties from Lemma 8 for the set $\mathcal{L}^*$.

Now it remains to make a proper choice of the parameters k and m_0. We will show for given r satisfying (39), the existence of m_0 from the interval (37) with condition

$$\frac{\gamma_i}{\gamma_{i+1}} \geq q-1 = \frac{\beta_i}{\beta_{i+1}},\ i = t, \dots, t+2r-1 \tag{41}$$

from Lemma 8 holding. Therefore

$$k \geq t+2r, \tag{42}$$
$$m_0 \geq q(k-t)+t+2r-1. \tag{43}$$

It remains to prove that there exists $k \in \{1,2,\dots\}$ such that the system

$$\begin{cases} (k-t+1)\left(2+\frac{t-1}{r+1}\right) < m_0 < (k-t+1)\left(2+\frac{t-1}{r}\right), \\ q(k-t)+2r+t-1 \leq m_0 \end{cases} \tag{44}$$

has a solution $m_0 \in \{1,2,\dots\}$ and

$$k \geq t+2r,\ r < \frac{t-1}{q-2}. \tag{45}$$

We rewrite the system (44) in a way to get the following conditions on k:

$$\begin{cases} \frac{rm_0}{2r+t-1}+t-1 < k < \frac{(r+1)m_0}{2r+t+1}+t-1, \\ k \leq \frac{m_0}{q} - \frac{2r+t-1}{q}+t \end{cases}. \tag{46}$$

To be able to choose an integer k satisfying the first inequality, it is enough to satisfy the inequality

$$\frac{rm_0}{2r+t-1}+t-1 < \frac{(r+1)m_0}{2r+t+1}+t-2,$$

or

$$m_0 > \frac{(2r+t+1)(2r+t-1)}{t-1}. \tag{47}$$

Consider now the second inequality from (46). For this we impose the condition

$$\frac{rm_0}{2r+t-1}+t-1 < \frac{m_0}{q} - \frac{2r+t-1}{q}+t-1$$

or, since $r < (t-1)/(q-2)$, we have

$$m_0 > \frac{(2r+t-1)^2}{2r+t-qr-1}. \tag{48}$$

We also impose the condition

$$2r+t < \frac{m_0}{2r+t-1}+t-1,$$

or

$$m_0 > \frac{(2r+1)(2r+t-1)}{r}. \tag{49}$$

Finally, we choose m_0 that satisfies (47)–(49) and take k to be the smallest integer such that $k > rm_0/(2r+t-1)+t-1$. For such a choice of m_0, k inequalities (41) hold and hence we can apply Lemma 8. Thus we get

$$N_q(n,t) = |\mathcal{K}(r)| = \alpha^{n-2r-t} \sum_{i=0}^{t+2r} |\mathcal{D}(i)|(q-1)^{2r+t-i}. \tag{50}$$

It is easy to show that the maximum of the RHS of (50) is achieved when r is the maximal number that satisfies (39). This completes the proof of Theorem 5 in the case $q \geq 2,\ t > 1$ and $q = 2,\ t = 1$. When $q > 2, t = 1$ we derive $r = 0$ from Lemma 7 and the theorem follows trivially. ■

Lecture 3 The Diametric Problem for Vertices in the Taxi Metric

Now we turn to a problem that has considerably different methods of proof. However, there are several connections with the previous material. First of all we once more deal with the diametric problem, but in the Taxi metric (definitions will come next). In the case of binary n-tuples this metric coincides with the Hamming metric and Kleitman's result (5) gives the solution for both metrics.

Consider the diametric problem in a space, which is a direct product of paths. This problem is in some sense easier than its q-ary Hamming space counterpart (the direct product of the complete graphs of given size) and has been solved before the latter one. The metric in the space, which is a direct product of paths, is called the Taxi metric. In other words, we consider the space $\mathcal{T}^n$ of sequences $x^n = (x_1, \dots, x_n)$ with components $x_i \in \mathcal{X}_i$, where the nodes $\mathcal{X}_i = \left\{x_1, \dots, x_{|\mathcal{X}_i|}\right\}$ are nodes of the path $x_1 - x_2 - \dots - x_{\mathcal{X}_i}$, $|x_i - x_j| = |i-j|$ and the distance between n-tuples is

$$\Delta(x^n, y^n) = \sum_{i=1}^{n} |x_i - y_i|.$$

In the case $|\mathcal{X}_i| > 2$, the structure and the solution of the diametric problem becomes much more difficult in comparison with the binary case. Next we come to the formulation of the problem and the results. For any subset $A \subset \mathcal{X}^n$, the diameter $D(A)$ and the radius $R(A)$ are defined as usual:

$$\begin{aligned} D(A) &= \max_{x^n, y^n \in A} \Delta(x^n, y^n), \\ R(A) &= \min_{x^n \in A} \max_{y^n \in A} \Delta(x^n, y^n). \end{aligned}$$

We are interested in determining the quantity

$$C(d,n) = \max\{|A| :\ D(A) \leq d\}.$$

We show how to completely solve this problem in some important cases, namely when all $|\mathcal{X}_i|$ are odd or all $|\mathcal{X}_i|$ are even. In the solution of the diametric problem when all $|\mathcal{X}_i|$ are odd and all $|\mathcal{X}_i|$ are even, quite different approaches are used. But note that in both cases the maximal set of diameter d is the ball of radius $d/2$. It is interesting that the center of the ball in the case of all even $|\mathcal{X}_i|$ is not a point in $\prod_{i=1}^n \mathcal{X}_i$ but some point with coordinates in the intervals $[\min_{x\in\mathcal{X}_i} x, \max_{x\in\mathcal{X}_i} x]$.

Note the important (probably the main) conclusion here that in all cases the maximal set is a ball in L^1-metric of radius $d/2$ with some specified center, which can vary in different cases.

We start with the case when all $|\mathcal{X}_i|$ are odd. For convenience we write the alphabets in the form

$$\mathcal{X}_i = \{-q_i, \dots, -1, 0, 1, \dots, q_i\},\ |\mathcal{X}_i| = 2q_i + 1,$$

denote $q^n = (q_1, \dots, q_n)$, and define for convenience the q^n-space by $\mathcal{B} = \mathcal{X}^n = \mathcal{X}_1 \times \cdots \times \mathcal{X}_n$. Let

$$B(0^n, r) = \{x^n \in \mathcal{B} :\ ||x^n|| \leq r\}$$

be the Taxi ball of radius r with the center in the origin (here $||x^n|| = \sum_i |x_i|$). Denote $N(r,n) = |B(0^n, r)|$. The next two theorems give the solution of the diametric problem in Taxi metric when all $|\mathcal{X}_i|$ are odd. The first theorem gives the solution for even diameter, and the next one for odd diameter of the set.

Theorem 6 (Ahlswede, Cai, and Zhang 1992) *$C(2r,n) = N(r,n)$, if all $|\mathcal{X}_i|$ are odd.*

Proof. We define the order $<_c$ on $\mathcal{X}_i$ by arranging its elements in the form $0, 1, -1, \dots$ and the order $\leq_c$ on $\mathcal{B}$ by setting $x^n \leq y^n$ iff $x_i \leq_c y_i$ for $i = 1, \dots, n$. By means of this order we introduce the "pushing to the center operator" P as follows: for any set $A \subset \mathcal{B}$ and any $x_j^n = (x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n) \in \prod_{1\leq i\neq j\leq n} \mathcal{X}_i$ we set

$$A(x_j^n) = \{(z_1, \dots, z_n) \in A :\ z_i = x_i \ for\ i \neq j\},$$

let $P_j A(x_j^n) = \{(x_1, \dots, x_{j-1}, x, x_{j+1}, \dots, x_n) :\ x$ be one of the $|A(x_j^n)|$ c-smallest elements in $\mathcal{X}_j\}$ and also let $P_j(A) = \bigcup_{x^n} P_j A(x_j^n)$.

If $P_j A = A$ for all j, then we say that A is a c-downset. It is easy to verify that every $A \subset \mathcal{B}$ can be pushed into a c-downset A' such that

$$|A| = |A'|,$$
$$D(A) \geq D(A').$$

One easily verifies the fact (I) that $||x^n|| - ||y^n|| = 0 \pmod 2$ implies $\Delta(x^n, y^n) = 0 \pmod 2$.

We proceed with the proof of the theorem by induction on n. The case $n = 1$ being trivial, let now $q^n = q_1 q^{n-1}$ and let $A \subset \mathcal{B}$ satisfy $D(A) \leq 2r$. We can assume that A is a c-downset. Therefore, we have for $u >_c v$

$$A_u \subset A_v$$

if $A_u = \{x^{n-1} :\ ux^{n-1} \in A\}$, and for every nonnegative integer $\theta \le q_1$ we have $A_{-\theta} \subset A_\theta$. Consider now the sets

$$A_\theta^0 = \{x^{n-1} :\ ||x^{n-1}|| \text{ is odd, } x^{n-1} \in A_\theta \setminus A_{-\theta}\},$$
$$A_\theta^e = \{x^{n-1} :\ ||x^{n-1}|| \text{ is even, } x^{n-1} \in A_\theta \setminus A_{-\theta}\}$$

and define

$$A_{-\theta}^* = A_{-\theta} \cup A_\theta^0,\ A_\theta^* = A_\theta \setminus A_\theta^0 = A_{-\theta} \cup A_\theta^e.$$

We then have

$$D(A_{-\theta}^*) = \max\{D(A_{-\theta}), D(A_\theta^0), D(A_{-\theta}, A_\theta^0)\}, \tag{1}$$

where we define

$$D(U,V) = \max_{u \in U,\ v \in V} \Delta(u,v).$$

We shall show next that

$$D(A_{-\theta}^*) \le 2(r-\theta). \tag{2}$$

For this, notice that for $a^{n-1}, b^{n-1} \in A_{-\theta} \subset A_\theta$ and $x^{n-1}, y^{n-1} \in A_\theta^0$ the following sequences are in the set A :

$$(-\theta)a^{n-1},\ (-\theta)b^{n-1},\ \theta a^{n-1},\ \theta b^{n-1},\ \theta x^{n-1},\ \theta y^{n-1},\ (-\theta+1)x^{n-1},\ (-\theta+1)y^{n-1}.$$

From the fact $D(A) \le 2r$ we obtain therefore the inequalities

$$\begin{aligned} \Delta(a^{n-1}, b^{n-1}),\quad \Delta(a^{n-1}, x^{n-1}) &\le 2(r-\theta), \\ \Delta(x^{n-1}, y^{n-1}) &\le 2(r-\theta)+1. \end{aligned} \tag{3}$$

However, since $||x^{n-1}||$ and $||y^{n-1}||$ are odd, by (I), $\Delta(x^{n-1}, y^{n-1})$ must be even. This shows that actually

$$\Delta(x^{n-1}, y^{n-1}) \le 2(r-\theta).$$

This inequality together with (1) and (3) implies (2).

Similarly one can prove that

$$D(A_\theta^*) \le 2(r-\theta).$$

By the induction hypothesis we conclude our proof with

$$|A| = \sum_{u=-q_1}^{q_1} |A_u| = \sum_{u=-q_1}^{q_1} |A_u^*| \le \sum_{u=-q_1}^{q_1} N(r-|u|, q^{n-1}) = N(r, q^n).$$

■

We address now the case of an odd diameter. Again we present a complete solution for spaces with odd $|\mathcal{X}_i|$.

For this, we introduce the ball $S^*(r,n)$ in L^1-metric with the center in $(1/2,0,\ldots,0)$. For $d=2r+1$ and $q^n=q^{n-1}q_n$ with $q_1 \geq q_i,\ i=2,\ldots,n$ we set

$$S^*(r,n)=\{x^n:\ x_1 \leq 0, ||x^n|| \leq r,\ or\ x_1>0,\ ||x^n|| \leq r+1\}.$$

Clearly

$$D(S^*(r,n))=d.$$

Theorem 7 (Ahlswede, Cai, and Zhang 1992) *If we assume w.l.o.g. $q_1 \geq q_i$ for $i=2,\ldots,n$, then we have $C(2r+1,n)=|S^*(r,n)|$ for $d=2r+1$, when all $|\mathcal{X}_i|$ are odd.*

For $a^n,b^n \in \mathcal{B}$ denote

$$\overline{\Delta}(a^n,b^n)=\max\{\Delta(a'^n,b'^n):\ a'^n \leq_c a^n,\ b'^n \leq_c b^n\}.$$

We introduce a metric $\Delta^*:\mathcal{B}\times\mathcal{B}\to\mathbb{R}_+$ by

$$\Delta^*(a^n,b^n)=\begin{cases}\overline{\Delta}(a^n,b^n), & a^n \neq b^n,\\ 0, & a^n=b^n,\end{cases}$$

and the diameter

$$D^*(A)=\max\{\Delta^*(a^n,b^n):\ a^n,b^n \in A\}.$$

The following result can easily be verified.

Proposition 6 *(i)* $\Delta^*(a^n,b^n)=||a^n||+||b^n||-|\{i:\ a_i>0,\ b_i>0\}|$ *if* $a^n \neq b^n$,

(ii) Δ^* *is a metric,*

(iii) $D^*(M_c(A))=D(A)$ *for a c-downset* $A\subset\mathcal{B}$, *where* $M_c(A)$ *is the set of c-maximal elements in* A.

We assume that $q_1 \geq \ldots \geq q_n$. The operator below is based on the mapping φ : $\mathcal{X}_{n-1}\times\mathcal{X}_n \to \mathcal{X}_{n-1}\times\mathcal{X}_n$ defined by

$$\varphi(x,y)=\begin{cases}(-x,-y), & x<0,y>0\\ (-x+1,-y), & x>0,y>0\\ (y,0), & x=0,y>0\\ (x,y), & \text{otherwise.}\end{cases}$$

We will use this function to define for any $A\subset\mathcal{B}$ a mapping $\phi:\ A\to\mathcal{B}$ by

$$\phi(a^n)=\begin{cases}a^n, & \text{if } a_n>0,\ a^{n-2}\varphi(a_{n-1},a_n)\in A,\\ a^{n-2}\varphi(a_{n-1},a_n), & \text{otherwise.}\end{cases}$$

We also write $\phi(A)=\{\phi(a^n):\ a^n \in A\}$.

For any set $B\subset\mathcal{B}$ we introduce the associated c-downset $\mathcal{D}_c(B)=\{x^n:\ \exists b^n \in B$ such that $x^n \leq_c b^n\}$. Now we define an operator Q by putting

$$Q(A)=\mathcal{D}_c(\phi(A)).$$

Clearly

$$|Q(A)| \geq |\phi(A)| = |A|.$$

We summarize some properties that follow immediately from the definitions.

Proposition 7 *For any set $A \subset \mathcal{B}$*

(i) $M_c(Q(A)) = M_c(\phi(A)) \subset \phi(A) \subset Q(A)$,
(ii) $a^{n-2}a_{n-1}a_n \in \phi(A)$ *implies* $a^{n-2}\varphi(a_{n-1},a_n) \in \phi(A)$.

We need the following:

Lemma 9 *For a c-downset A, $D(Q(A)) \leq D(A)$.*

Proof. By (iii) in Proposition 6,

$$D(Q(A)) = D^*(M_c(Q(A)) = D^*(M_c(\phi(A))) \leq D^*(\phi(A))$$

and, since A is a c-downset, also

$$D(A) = D^*(A).$$

It suffices therefore to show that $D^*(\phi(A)) \leq D^*(A)$ or that

$$\Delta^*(\phi(a^n),\phi(b^n)) \leq D^*(A) \tag{4}$$

for all $a^n, b^n \in A$. In the case $\phi(a^n) = a^n,\ \phi(b^n) = b^n$, which includes the case $a_n \leq 0,\ b_n \leq 0$, this is of course true.

In the case $a_n \leq 0,\ b_n > 0$ we notice that ϕ does not increase $||\cdot||$ and only in the case when $b_{n-1} > 0$, ϕ may decrease $|\{i:\ a_i > 0,\ b_i > 0\}|$, but by at most 1. Furthermore, in the case $b_{n-1} > 0,\ b_n > 0$ we have $||\phi(b^n)|| = ||b^n|| - 1$. Therefore, by (i) in Proposition 6, we obtain

$$\Delta^*(\phi(a^n),\phi(b^n)) \leq \Delta^*(a^n,b^n)$$

and thus (4).

The case $a_n > 0, b_n \leq 0$ being symmetrically the same, we are left with the case $a_n > 0,\ b_n > 0$, and (again by symmetry) $\phi(b^n) \neq b^n$. We divide this into two subcases:

1. $\phi(a^n) \neq a^n$. We establish (4) by proving $\Delta^*(\phi(a^n),\phi(b^n)) = \Delta^*(a^n,b^n)$. To prove it one should verify that $\Delta^*(a^n,b^n) - \overline{\Delta}(a^{n-2},b^{n-2})$ and $\Delta^*(\phi(a^n),\phi(b^n)) - \overline{\Delta}(a^{n-2},b^{n-2})$ are equal.

2. $\phi(a^n) = a^n$. Here necessarily $\tilde{a}^n = a^{n-2}\varphi(a_{n-1},a_n) \in A$. We can easily prove that $\Delta^*(\phi(a^n),\phi(b^n)) = \Delta^*(\tilde{a}^n,b^n)$ by verifying the validity of the equality $\Delta^*(\tilde{a}^n,b^n) - \overline{\Delta}(a^{n-2},b^{n-2}) = \Delta^*(\phi(a^n,b^n)) - \overline{\Delta}(a^{n-2},b^{n-2})$. □

Now we are able to prove Theorem 7. As before, we proceed by induction on n. The case $n = 1$ is clear. By Proposition 7 and Lemma 9 we can assume that A is a c-downset with the property

$$a^n = a^{n-2}a_{n-1}a_n \in M_c(A), \tag{5}$$

which implies

$$a^{n-2}\varphi(a_{n-1},a_n) \in A.$$

Let $A_x = \{x^{n-1} :\ x^{n-1}x \in A\}$ and consider for $\theta > 0$ the sets

$$\begin{aligned} A_\theta^+ &= \{x^{n-2}x_{n-1} \in A_\theta \setminus A_{-\theta} :\ x_{n-1} > 0\} \\ A_\theta^- &= \{x^{n-2}x_{n-1} \in A_\theta \setminus A_{-\theta} :\ x_{n-1} \le 0\} \\ A_\theta^* &= A_\theta \setminus A_\theta^- = A_{-\theta} \cup A_\theta^+ \\ A_{-\theta}^* &= A_{-\theta} \cup A_\theta^-,\ A_0^* = A_0. \end{aligned}$$

Since A is a c-downset, we have $A_\theta \supset A_{-\theta}$. Therefore, for $a^{n-1}, b^{n-1} \in A_{-\theta} \subset A_\theta$ and $x^{n-1} \in A_\theta$ we also have $a^{n-1}(-\theta),\ b^{n-1}\theta,\ x^{n-1}\theta \in A$ and thus

$$\Delta(a^{n-1},b^{n-1}),\ \Delta(a^{n-1},x^{n-1}) \le d - 2\theta$$

and

$$D(A_{-\theta}), D(A_{-\theta},A_\theta^-), D(A_{-\theta},A_{-\theta}^+) \le d - 2\theta. \tag{6}$$

Now we are going to prove that also

$$D(A_\theta^-) = D^*(M_c(A_\theta^-)) \le d - 2\theta \tag{7}$$

$$D(A_\theta^+) = D^*(M_c(A_\theta^+)) \le d - 2\theta. \tag{8}$$

Suppose (7) is not true, then for some $a^{n-1}, b^{n-1} \in M_c(A_\theta^-)$

$$\Delta^*(a^{n-1},b^{n-1}) > d - 2\theta. \tag{9}$$

Since $a^{n-1} \notin A_{-\theta}$ and $a^{n-1}\theta \in M_c(A)$, we have $a^{n-2}\varphi(a_{n-1},\theta) \in A$ by (5). Moreover, since $a_{n-1} \le 0$ and $\theta > 0$, by our definitions

$$\varphi(a_{n-1},\theta) = \begin{cases} (-a_{n-1},-\theta), & a_{n-1} < 0, \\ (\theta,0), & a_{n-1} = 0. \end{cases}$$

Thus, noticing that $\theta > 0$ and $b_{n-1} \le 0$, we can conclude that

$$\begin{aligned} d \ge D(A) &\ge \Delta^*(a^{n-2}\varphi(a_{n-1},\theta), b^{n-1}\theta) = \overline{\Delta}(a^{n-2},b^{n-2}) \\ &+ |a_{n-1}| + |b_{n-1}| + 2\theta = \Delta^*(a^{n-1},b^{n-1}) + 2\theta > d. \end{aligned}$$

This contradiction proves (7).

Now suppose that (8) is not true, that is, for some $a^{n-1}, b^{n-1} \in M_c(A_\theta^+)$ (9) holds. By the reasoning given before $a^{n-2}\varphi(a_{n-1},\theta) \in A$. Now $\varphi(a_{n-1},\theta) = (-(a_{n-1}-1),-\theta)$, because $a_{n-1} > 0$ and $\theta > 0$ in this case. We arrive again at a contradiction

$$\begin{aligned} d \ge D(A) &\ge \Delta^*(a^{n-2}\varphi(a_{n-1},\theta), b^{n-1}\theta) \\ &= \overline{\Delta}(a^{n-2},b^{n-2}) + \overline{\Delta}(\varphi(a_{n-1},\theta), b_{n-1}\theta) \\ &= \overline{\Delta}(a^{n-2},b^{n-2}) + |a_{n-1}| + |b_{n-1}| + 2\theta - 1 \\ &= \Delta^*(a^{n-1},b^{n-1}) + 2\theta > d. \end{aligned}$$

So (8) holds. From (6), (7), and (8) we conclude that

$$D(A_\ell^*) \le d - 2|\ell|$$

for all ℓ and by the induction hypothesis

$$|A_\ell^*| \le |S^*(r - |\ell|, n)|.$$

Note that $A_\ell^* = \emptyset$ when $|\ell| > r$, and $S^*(z,n) = \emptyset$ when $z < 0$. Therefore,

$$|A| \le \sum_{\ell=-q_n}^{q_n} |S^*(r - |\ell|, n)| = |S^*(r,n)|.$$

This completes the proof of Theorem 7. ■

Now we consider the diametric problem in the case, when all $|\mathcal{X}_i|$ are even. As we have already mentioned, the proof that some ball of radius $d/2$ is a maximal set of diameter d in this case is quite different.

First of all, we prove that in this case a maximal set of diameter $d = b(\mathcal{B}) - 1$, where $b(\mathcal{B}) = \sum_{i=1}^n q_i$, contains half of the points from $\mathcal{B}$. Moreover, such a set can be chosen to be a ball of radius $(b(\mathcal{B}) - 1)/2$ with the center depending on the parity of $b(\mathcal{B}) + n$.

Let L^1 be the space of n-tuples of reals with the L^1-metric

$$\Delta(x^n, y^n) = \sum_{i=1}^n |x_i - y_i|,\ x^n = (x_1, \dots, x_n),\ y^n = (y_1, \dots, y_n) \in L^1.$$

We consider the set $\mathcal{B}$ imbedded into the space L^1 in such a way that the ith coordinate of $\mathcal{B}$ takes the values from $\mathcal{X}_i = \left\{-q_i + \frac{1}{2}, \dots, -\frac{1}{2}, \frac{1}{2}, \dots, q_i - \frac{1}{2}\right\}$. Next we show that a set of diameter $b(\mathcal{B})$ cannot contain more than $|\mathcal{B}|/2$ points from $\mathcal{B}$. Indeed, consider the set $\mathcal{B}_j \subset \mathcal{B}$, which belongs to some orthant of L^1 (by orthant we mean a set with a prescribed sign of each component), and couple it with the set $\tilde{\mathcal{B}}_j : \tilde{\mathcal{B}}_j = -\mathcal{B}_j$. To every point $x^n = (x_1, \dots, x_n) \in \mathcal{B}_j$ there is a corresponding unique point $\tilde{x}^n = (\tilde{x}_1, \dots, \tilde{x}_n) \in \tilde{\mathcal{B}},\ \tilde{x}_i = -q_i + x_i$. It is easy to see that this correspondence is a bijection and

$$\Delta(x^n, \tilde{x}^n) = b(\mathcal{B}).$$

Thus only one of the points from a pair $(x^n, \tilde{x}^n)$ can be in a set of diameter $b(\mathcal{B}) - 1$, and a set of diameter $b(\mathcal{B}) - 1$ contains not more than half of the points from $\mathcal{B}$.

The next lemma solves the problem of representing a maximal set of diameter $d = b(\mathcal{B}) - 1$ as a ball of radius $d/2$.

Lemma 10 *Let $d = b(\mathcal{B}) - 1$. If $b(\mathcal{B}) + n$ is odd, then the ball $B\left(0^n, \frac{d}{2}\right)$, $0^n = (0, \dots, 0) \in L^1$ contains half of the points from $\mathcal{B}$. In the case when $b(\mathcal{B})$ is odd, the ball $B\left(z^n, \frac{d}{2}\right)$, $z^n = (1/2, \dots, 1/2)$ also contains half of the points from $\mathcal{B}$ and in the case of even $b(\mathcal{B})$ the same assertion is true with $z^n = (0, 1/2, \dots, 1/2)$.*

If $b(\mathcal{B})+n$ is even, the ball $B\left(z^n,\frac{d}{2}\right)$, $z^n=(1/2,0,\ldots,0)$ contains half of the points from $\mathcal{B}$. If $b(\mathcal{B})$ is even, the ball $B\left(z^n,\frac{d}{2}\right)$, $z^n=(0,1/2,\ldots,1/2)$ also contains half of the points from $\mathcal{B}$. Here in the case of odd $b(\mathcal{B})$ the same assertion is true with $z^n=(1/2,\ldots,1/2)$.

Proof. Suppose that $b(\mathcal{B})+n$ is odd. We call the point $0^n=(0,\ldots,0)\in L^1$ the center of $\mathcal{B}$. Let us show that each orthant intersects the ball $B\left(0^n,\frac{b-1}{2}\right)$ in exactly half of the points. W.l.o.g. we consider the orthant $\mathcal{B}_+$ with all-positive coordinates. Again we consider coupling, now the points being from $\mathcal{B}_+$. To each point $x^n=(x_1,\ldots,x_n)\in\mathcal{B}_+$ we assign in a one-to-one manner the point $\bar{x}^n=(\bar{x}_1,\ldots,\bar{x}_n)\in\mathcal{B}_+$ with $\bar{x}_i=q_i-x_i$. Next we show that the ball $B\left(0^n,\frac{b(\mathcal{B})-1}{2}\right)$ contains at least (actually exactly) one point from each pair $(x,\bar{x})$.

Indeed, if

$$
\begin{aligned}
||x^n|| &> \frac{b(\mathcal{B})-1}{2}, \\
||\bar{x}^n|| &> \frac{b(\mathcal{B})-1}{2},
\end{aligned}
\tag{10}
$$

or

$$\frac{b(\mathcal{B})+1}{2}>\sum_{i=1}^{n}x_i>\frac{b(\mathcal{B})-1}{2}.$$

The only possibility for these inequalities to be valid is

$$\sum_{i=1}^{n}x_i=\frac{b(\mathcal{B})}{2}.$$

For some positive integer y_i, $x_i=y_i-\frac{1}{2}=\frac{2y_i-1}{2}$; thus

$$\sum_{i=1}^{n}(2y_i-1)=b(\mathcal{B})$$

or

$$2\sum_{i=1}^{n}y_i=n+b(\mathcal{B}). \tag{11}$$

We see that the RHS of (11) is odd and the LHS is even, leading to a contradiction. Thus, when $b(\mathcal{B})+n$ is odd, the ball $B\left(0^n,\frac{d}{2}\right)$ is maximal and contains exactly half of the points from $\mathcal{B}_+$ and hence, by symmetry, also from $\mathcal{B}$.

Using the same method, it is easy to check that if $2|(b(\mathcal{B})-1)$, then the ball $B\left(z^n,\frac{b(\mathcal{B})-1}{2}\right)$, $z^n=(1/2,\ldots,1/2)$ is also a maximal set of diameter $b(\mathcal{B})-1$ and contains half of $\mathcal{B}$. To prove this we consider the pairing $(x^n,\bar{x}^n)$, $x^n=(x_1,\ldots,x_n)$, $\bar{x}^n=(\bar{x}_1,\ldots,\bar{x}_n)\in\mathcal{B}$ with

$$\bar{x}_i=\begin{cases} q_i+x_i, & x_i<1/2, \\ -q_i+x_i, & x_i\geq 1/2. \end{cases} \tag{12}$$

As before, the correspondence $x^n \leftrightarrow \bar{x}^n$ is a bijection and the relations, similar to (10), look as follows:

$$||x^n - z^n|| = \sum_{i=1}^{n} \left| x_i - \frac{1}{2} \right| > \frac{b(\mathcal{B})-1}{2},$$
$$||\bar{x}^n - z^n|| = \sum_{x_i<1/2} \left| q_i + x_i - \frac{1}{2} \right| + \sum_{x_i \ge 1/2} \left| -q_i + x_i - \frac{1}{2} \right| > \frac{b(\mathcal{B})-1}{2}$$

or

$$\sum_{i=1}^{n} |x_i| + \frac{\alpha-\beta}{2} > \frac{b(\mathcal{B})-1}{2},$$
$$b(\mathcal{B}) - \sum_{i=1}^{n} |x_i| - \frac{\alpha-\beta}{2} > \frac{b(\mathcal{B})-1}{2},$$

where

$$\alpha = |\{i:\ x_i < 1/2\}|,\ \beta = |\{i:\ x_i \ge 1/2\}|.$$

Hence we have

$$\frac{b(\mathcal{B})+1}{2} > \sum_{i=1}^{n} |x_i| + \frac{\alpha-\beta}{2} > \frac{b(\mathcal{B})-1}{2}$$

or

$$2\sum_{i=1}^{n} |x_i| + \alpha - \beta = b(\mathcal{B}).$$

It is easy to see that the LHS of this equality is even, while $b(\mathcal{B})$ is odd, again a contradiction.

If $b(\mathcal{B})$ is even, we prove that the subset $\mathcal{B}' \subset \mathcal{B}$ with the first coordinate being positive intersects $B\left(z^n, \frac{d}{2}\right)$, $z^n = (0, 1/2, \ldots, 1/2)$ in $|\mathcal{B}'|/2$ points, from which by symmetry follows that the ball $B\left(z^n, \frac{d}{2}\right)$ contains half of the points from $\mathcal{B}$. Consider now the coupling defined by the transformation (12) in all but one coordinate (for example, when $i = 2, \ldots, n$) and set

$$\bar{x}_1 = q_1 - x_1.$$

As before, we impose the conditions

$$||x^n - z^n|| = \sum_{i=2}^{n} \left| x_i - \frac{1}{2} \right| + x_1 > \frac{b(\mathcal{B})-1}{2},$$
$$||\bar{x}^n - z^n|| = \sum_{x_i<1/2} \left| q_i + x_i - \frac{1}{2} \right| + \sum_{i>1,\ x_i \ge 1/2} \left| -q_i + x_i - \frac{1}{2} \right| + q_1 - x_1$$
$$> \frac{b(\mathcal{B})-1}{2}$$

or

$$\sum_{i=1}^{n} |x_i| + \frac{\alpha - \beta_1}{2} > \frac{b(\mathcal{B}) - 1}{2},$$
$$b(\mathcal{B}) - \sum_{i=1}^{n} |x_i| - \frac{\alpha - \gamma}{2} > \frac{b(\mathcal{B}) - 1}{2},$$

where

$$\gamma = \#\{i > 1 : \ x_i \geq 1/2\}.$$

Hence we have

$$\frac{b(\mathcal{B}) + 1}{2} > \sum_{i=1}^{n} |x_i| + \frac{\alpha - \gamma}{2} > \frac{b(\mathcal{B}) - 1}{2}$$

or

$$2\sum_{i=1}^{n} |x_i| + \alpha - \gamma = b(\mathcal{B}).$$

But $\alpha - \gamma$ is odd and $b(\mathcal{B})$ is even, a contradiction.

We are done with the case of odd $b(\mathcal{B}) + n$. The case of even $b(\mathcal{B}) + n$ can be settled analogously and we leave it to the reader. □

Now we are ready to prove the theorem, which says that a ball of radius $d/2$ with center in some specified point in L^1 is a maximal set of diameter d in $\mathcal{B}$.

Theorem 8 *Let us assume that all $|\mathcal{X}_i|$ are even, then there is a ball (in Taxi metric) of radius $d/2$, which is a maximal set of diameter d in $\mathcal{B}$. The center of the ball can be chosen to be $z^n = (1/2, \ldots, 1/2)$ if d is even and $d < b(\mathcal{B})$ or $z^n = (0, 1/2, \ldots, 1/2)$ if d is odd and $d < b(\mathcal{B})$.*

If $d \geq b(\mathcal{B})$, then we can choose $z^n = (0, \ldots, 0)$ if $d - n$ is even and $z^n = (1/2, 0, \ldots, 0)$ if $d - n$ is odd.

Proof. To prove the theorem we use the result (5) of Lecture 2, which solves the problem in the case when all q_i are equal to 1 (the binary $\mathcal{B}$). It is easy to check that the solution of the problem in the binary case is consistent with the general case, formulated in the theorem.

As in the proof for odd values of $|\mathcal{X}_i|$ we can assume that the maximal set is p-compressed according to the p-order on each $\mathcal{X}_i: \ -q_i + 1/2 >_p q_i - 1/2 >_p -q_i + 3/2 >_p q_i - 3/2 >_p \ldots >_p -1/2 >_p 1/2$.

Suppose first that d is odd and $d < b(\mathcal{B})$. Fix some coordinate i with $q_i > 1$. Let $\mathcal{B}_1$ be the set obtained from $\mathcal{B}$ by deleting the extremal points from the set $\mathcal{X}_i$, that is, the points $q_i - 1/2, \ -q_i + 1/2$, and $\mathcal{B}_2 = \mathcal{B} \setminus \mathcal{B}_1$. We shift the ith coordinate of $\mathcal{B}_2$ to zero in the following way: $q_i - 1/2 \to 1/2; \ -q_i + 1/2 \to -1/2$. Let S be the maximal p-compressed set in $\mathcal{B}$ and $S_j = S \cap \mathcal{B}_j, \ j = 1, 2$ (we also assume that the ith coordinate of S_2 is shifted simultaneously with the ith coordinate of $\mathcal{B}_2$). Let $z^n = (0, 1/2, \ldots, 1/2)$. If $d_j = d(S_j)$, then

$$d_1 \leq d,$$
$$b(\mathcal{B}_1) = b(\mathcal{B}) - 1,$$

$$d_2 \leq d - 2q_i + 2,$$
$$b(\mathcal{B}_2) = b(\mathcal{B}) - q_i + 1.$$

Here the inequality for d_2 is valid, because the set S is p-compressed.

We deduce that if $d < b(\mathcal{B})$, then $d_1 \leq b(\mathcal{B}_1)$ and $d_2 \leq b(\mathcal{B}_2) - 1$. Assume at first that $d_1 < b(\mathcal{B})$. Note that the RHS of the restrictions for d_j from (13) have the same parities as d. For the set S_2 we can use induction and choose the ball $B_2 = B\left(z^n, \frac{d-2q_i+2}{2}\right)$ in $\mathcal{B}_2$ as a maximal set of diameter $d - 2q_i + 2$. If $d < b(\mathcal{B}_1) - 1$, then we choose the ball $B_1 = B\left(z^n, \frac{d}{2}\right)$ in $\mathcal{B}_1$ as a maximal set of diameter d. Then the ball $B_1 \cup B_2 = B\left(z^n, \frac{d}{2}\right)$ gives a maximal set of diameter d in $\mathcal{B}$. If $d_1 = d = b(\mathcal{B}_1) = b(\mathcal{B}) - 1$, then we can apply Lemma 10 for even $b(\mathcal{B})$ to justify the statement of the theorem. Also in some step it can happen that $\mathcal{B}_1$ and/or $\mathcal{B}_2$ become binary and in this case we use (5) of Lecture 2 and choose the center of the maximal ball in the binary space as needed for induction (make the necessary considerations in this case!).

Since the RHS of restrictions for d_j in (13) has the same parities as d, the proof of the theorem in the case of even $d < b(\mathcal{B}) - 1$ is similar to the case of odd d and we leave it to the reader.

Now consider the case when $d \geq b(\mathcal{B})$. Here we make another splitting of the set $\mathcal{B}$. Again we choose i such that $q_i > 1$ and choose $\mathcal{B}_3 \subset \mathcal{B}$ to be the set of all n-tuples $x^n = (x_1, \ldots, x_n)$ from $\mathcal{B}$ with $x_1 = \pm 1/2$ and $\mathcal{B}_4 = \mathcal{B} \setminus \mathcal{B}_3$. Again we shift the ith coordinate of $\mathcal{B}_4$ by making the transformation

$$x_i \to \begin{cases} x_i + 1, \, x_i < 0, \\ x_i - 1, \, x_i > 0. \end{cases}$$

With the same notation as before we have

$$d_3 \geq \min\{d, t\},$$
$$b(\mathcal{B}_3) = b(\mathcal{B}) - q_i + 1,$$
$$d_4 = d - 2,$$
$$b(\mathcal{B}_4) = b(\mathcal{B}) - 1,$$

where $t = d(\mathcal{B}_3)$. Hence if $d \leq t$, then in any case $d_3 > b(\mathcal{B}_3)$ and if $d_4 \geq b(\mathcal{B}_4)$, then (as the restrictions on d_j have the same parity as d) we can deduce that the cardinality of S_3 or S_4 is upper-bounded by the cardinality of the balls $B\left(z^n, \frac{d}{2}\right)$ in $\mathcal{B}_3$ or $B\left(z^n, \frac{d-2}{2}\right)$ in $\mathcal{B}_4$, respectively, where the common center z^n depends on the parity of d (or, for fixed n, the parity of $d - n$). Thus the cardinality of S does not exceed the cardinality of the ball $B\left(z^n, \frac{d}{2}\right)$ in $\mathcal{B}$.

If $d > t$, then we can choose the center z^n of the ball $B\left(z^n, \frac{d-2}{2}\right)$ in $\mathcal{B}_4$ as the maximal set of diameter $d - 2$ in $\mathcal{B}_4$ and we have $\mathcal{B}_3 \subset B\left(z^n, \frac{d}{2}\right)$ for the ball in $\mathcal{B}$.

In the case $d_4 = b(\mathcal{B}_4) - 1$, we use Lemma 10 in the same way as before.

Again it is possible that on some step $\mathcal{B}_j$ becomes binary. In that case we use (5) of Lecture 2 and a consistent choice of the ball center z^n. Check that such a choice always exists. ■

Theorems 6, 7, and 8 completely solve the problem of determining maximal sets of a given diameter in the Taxi metric, when all components $\mathcal{X}_i$ have even or odd lengths.

Lecture 4 The Diametric Problem for Edges in Hamming Metric

Theorem 5 deals with the vertex-diametric problem: we find the maximal cardinality of a set with given diameter. It was started in [AK00b] to consider the situation where one wants to find a set with given diameter that has maximal number of edges. For a given set $\mathcal{A} \subset \{0,1\}^n$ (in this problem we consider only the binary case, and the general case of an arbitrary alphabet is not solved) the edge set is defined as

$$\mathcal{E}(\mathcal{A}) = \{(a^n, b^n) : \ a^n, b^n \in \mathcal{A}, \ d_H(a^n, b^n) = 1\}.$$

We denote $D(n,d) = I_2(n, n-d)$. The edge-diametric problem is to find the value

$$E(n,d) = \max_{\mathcal{A} \in D(n,d)} |\mathcal{E}(\mathcal{A})|.$$

Theorem 9 gives the complete solution of this problem. As in the case of the vertex diametric problem, while following the proof of this theorem the reader will see that some parts of it use technique from the proof of the Complete Intersection Theorem.

Let

$$\begin{aligned} \mathcal{W}(n) &= \{(a_1, \dots, a_n) \in \{0,1\}^n : \ a_1 = 1\}, \\ \mathcal{G}(r) &= \left\{ A \in 2^{[n]} : \ \left| A \cap [1, t+2r] \right| \geq t + r \right\}, \ t = n - d. \end{aligned}$$

Note that $\mathcal{G}(r)$ is the Cartesian product of the Hamming ball on the length $2t + r$ with radius r and center in $[1, 2t + r]$ and the whole space $2^{[n-t-2r]}$ on the rest length $n - t - 2r$.

Theorem 9 (Ahlswede and Khachatrian 2000) *Let $t = n - d$. The following relation is valid:*

$$E(n,d) = \begin{cases} |\mathcal{E}(\mathcal{W}(n))|, & if \ d = n-1, \\ \left| \mathcal{E}\left(\mathcal{G}\left(\frac{d}{2}\right)\right) \right|, & if \ d \leq n-2, \ 2|d, \\ \left| \mathcal{E}\left(\mathcal{G}\left(\frac{d-1}{2}\right)\right) \right|, & if \ d \leq n-2, \ 2 \not| d. \end{cases}$$

For $A \in \mathcal{A}$ we denote $T_j(A) = T_{j0}(A)$. It is easy to see that in addition to the mentioned properties, T_j satisfies the relation

$$|\mathcal{E}(T_j(\mathcal{A}))| \geq |\mathcal{E}(\mathcal{A})|.$$

Let $UD(n,d)$ be the set of all upsets in $D(n,d)$. We have

$$E(n,d) = \max_{\mathcal{A} \in UD(n,d)} |\mathcal{E}(\mathcal{A})|. \tag{1}$$

On the other hand, if $\mathcal{A} \subset 2^{[n]}$ is an upset and has diameter d, then any $A_1, A_2 \in \mathcal{A}$ have at least $(n-d)$ componentwise common 1's.

Hence

$$E(n,d) = \max_{\mathcal{A} \in UI(n,n-d)} |\mathcal{E}(\mathcal{A})|, \tag{2}$$

where $UI(n,n-d)$ denotes the set of all $(n-d)$-intersecting systems, which are also upsets.

Also, it is easy to see that

$$E(n,d) = \max_{\mathcal{A} \in LUI(n,n-d)} |\mathcal{E}(\mathcal{A})|, \tag{3}$$

where $LUI(n,n-d)$ is the set of all left-compressed sets from $UI(n,n-d)$. We define the sets $\mathcal{A}_{i,j}$, $\mathcal{A}'$, etc. in the same way as in Lemma 4. Then all statements of the lemma are still valid in our case. In addition, it is easy to see that the following items (vi) and (vii) are also true.

(vi) Let $A \in \mathcal{A}'$. Then for any $B' \subset [1,\ell]$ with $|B'| < |B|$ and $C' \subseteq C$ we have

$$B' \cup C' \notin \mathcal{A}.$$

(vii) Let $A \in \mathcal{A}'$. It can be shown that for any $C' \subset C$, $B \cup C' \in \mathcal{A}$ implies $B \cup C' \in \mathcal{A}'$.

We will need two more results. The verification of Proposition 8 is left to the reader.

Proposition 8 *Let $\mathcal{A} \subset 2^{[n]}$ be an upset. Then*

$$|\mathcal{E}(\mathcal{A})| = \sum_{A \in \mathcal{A}} (n - |A|).$$

Proposition 9 *The following relation holds:*

$$\max_{\mathcal{A} \subset 2^{[n]},\ |\mathcal{A}| = 2^{n-1}} |\mathcal{E}(\mathcal{A})| = |\mathcal{E}(\mathcal{W}(n))|.$$

The proof of Proposition 9 is given at the end of the lecture. We start with the following lemma.

Lemma 11 *Let $\mathcal{S} \subset 2^{[m]}$ have the following properties:*

(i) $\mathcal{S}$ is complement closed, that is, from $A \in \mathcal{S}$ follows that $\bar{A} \in \mathcal{S}$,
(ii) $\mathcal{S}$ is convex, that is, from $A, C \in \mathcal{S}$ and $A \subset B \subset C$ follows that $B \in \mathcal{S}$.

Then there exists an $\mathcal{S}' \subset \mathcal{S}$ such that $\mathcal{S}' \in I(m)$ and

$$\sum_{A \in \mathcal{S}'} (m - |A|) \geq \frac{m-1}{2m} \sum_{A \in \mathcal{S}} (m - |A|) = \frac{m-1}{4} |\mathcal{S}|. \tag{4}$$

Moreover, if $\mathcal{S} \neq 2^{[m]}$, then there exists an $\mathcal{S}' \subset \mathcal{S}$, $\mathcal{S}' \in I(m)$ for which strict inequality in (4) holds.

Proof. First we notice that the identity in (4) follows from property (i). In the case $\mathcal{S} = 2^{[m]}$, by taking $\mathcal{S}' = \{A \in 2^{[m]} : 1 \in A\}$, we have $\mathcal{S}' \in I(m)$, $|\mathcal{S}'| = \frac{|\mathcal{S}|}{2} = 2^{m-1}$, and easily get (4) in this case.

Let now $\mathcal{S} \neq 2^{[m]}$, let $B \in \mathcal{S}$ be any element with minimal cardinality, and let $i \in B$. We consider the following partition of $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3 \cup \mathcal{S}_4$, where

$$\begin{aligned}
\mathcal{S}_1 &= \{A \in \mathcal{S} : i \in A, (A \setminus \{i\}) \in \mathcal{S}\}, \ \mathcal{S}_2 = \{A \in \mathcal{S} : i \notin A, A \cup \{i\} \in \mathcal{S}\}, \\
\mathcal{S}_3 &= \{A \in \mathcal{S} : i \in A, (A \setminus \{i\}) \notin \mathcal{S}\}, \ \mathcal{S}_4 = \{A \in \mathcal{S} : i \notin A, A \cup \{i\} \notin \mathcal{S}\}.
\end{aligned}$$

It is easily seen that

$$\bar{\mathcal{S}}_1 = \mathcal{S}_2, \ \bar{\mathcal{S}}_3 = \mathcal{S}_4.$$

Hence $|\mathcal{S}_1| = |\mathcal{S}_2|$ and $|\mathcal{S}_3| = |\mathcal{S}_4|$. Also $S_3 \neq \emptyset$, since $i \in B \in \mathcal{S}$ and B has minimal cardinality. Also, for every $A \in \mathcal{S}_4$ and $A' \in \mathcal{S} \setminus \mathcal{S}_3$, $A \cap A' \neq \emptyset$ holds. Hence $(\mathcal{S}_1 \cup \mathcal{S}_4), (\mathcal{S}_1 \cup \mathcal{S}_3) \in I(m)$.

We have

$$\sum_{A \in \mathcal{S}_3 \cup \mathcal{S}_4} (m - |A|) = m \frac{|\mathcal{S}_3| + |\mathcal{S}_4|}{2}.$$

Consequently,

$$\max \left\{ \sum_{A \in \mathcal{S}_3} (m - |A|), \sum_{A \in \mathcal{S}_4} (m - |A|) \right\} \geq m \frac{|\mathcal{S}_3| + |\mathcal{S}_4|}{4}. \tag{5}$$

On the other hand, by construction of $\mathcal{S}_1, \mathcal{S}_2$ and the property $\bar{\mathcal{S}}_1 = \mathcal{S}_2$, we have

$$m \frac{|\mathcal{S}_1| + |\mathcal{S}_2|}{2} = \sum_{A \in \mathcal{S}_1} (m - |A|) + \sum_{A \in \mathcal{S}_2} (m - |A|) = 2 \sum_{A \in \mathcal{S}_1} (m - |A|) + \frac{|\mathcal{S}_1| + |\mathcal{S}_2|}{2}.$$

Hence

$$\sum_{A \in \mathcal{S}_1} (m - |A|) = \frac{m-1}{4} (|\mathcal{S}_1| + |\mathcal{S}_2|). \tag{6}$$

Therefore, from (5) and (6) we get

$$\max\left\{\sum_{A\in\mathcal{S}_1\cup\mathcal{S}_3}(m-|A|),\sum_{A\in\mathcal{S}_1\cup\mathcal{S}_4}(m-|A|)\right\}\geq\frac{m}{4}(|\mathcal{S}_3|+|\mathcal{S}_4|)$$
$$+\ \frac{m-1}{4}(|\mathcal{S}_1|+|\mathcal{S}_2|)\geq\frac{m-1}{4}(|\mathcal{S}_1|+|\mathcal{S}_2|+|\mathcal{S}_3|+|\mathcal{S}_4|)=\frac{m-1}{4}|\mathcal{S}|.$$

□

Corollary 1 *Let $\mathcal{S}\subset 2^{[m]}$ be defined as in the previous lemma and let (4) hold for $\mathcal{S}'\subset\mathcal{S},\ \mathcal{S}'\in I(m),\ |\mathcal{S}'|=\frac{|\mathcal{S}|}{2}$. Then for any $c\in\mathbb{R}$*

$$\sum_{A\in\mathcal{S}'}(m-|A|+c)\geq\frac{m+2c-1}{2(m+2c)}\sum_{A\in\mathcal{S}}(m-|A|+c). \tag{7}$$

Proof. We just notice that (7) follows from (4) and the identities

$$\frac{m+2c-1}{2(m+2c)}\sum_{A\in\mathcal{S}}(m-|A|+c)=\frac{m+2c-1}{2(m+2c)}\left(\frac{m}{2}|\mathcal{S}|+c|\mathcal{S}|\right)$$
$$=\frac{m-1}{4}|\mathcal{S}|+\frac{c}{2}|\mathcal{S}|,$$
$$\sum_{A\in\mathcal{S}'}(m-|A|+c)=\sum_{A\in\mathcal{S}'}(m-|A|)+\frac{c}{2}|\mathcal{S}|.$$

□

Now let $\mathcal{A}\in D(n,d)$ and $|\mathcal{E}(\mathcal{A})|=E(n,d)$. We can assume that $\mathcal{A}\in LUI(n,n-d)$. The next lemma plays the central role in the proof of Theorem 9.

Lemma 12 *Let $\mathcal{A}$ be the set that was described just above. Then, necessarily, $\mathcal{A}$ is invariant under exchange operations in*

(i) $[1,n]$, *if* $2|d$ *and* $d\leq n-3$
(ii) $[1,n-2]$, *if* $2|d$ *and* $d=n-2$
(iii) $[1,n-1]$, *if* $2\not|d$ *and* $d\leq n-2$.

Proof. The proof of this lemma is quite similar to the proof of Lemma 5. Let ℓ be the largest integer such that $\mathcal{A}_{i,j}=\mathcal{A}$ for all $1\leq i,j\leq\ell$. Assume the opposite to the statement of the lemma is true:

$$\ell<n_1, \tag{8}$$

where $n_1\in\{n-2,n-1,n\}$ depends on the case. We are going to show that, under the assumption (8), there exists a $\mathcal{B}\in I(n,n-d)$ with $|\mathcal{E}(\mathcal{B})|>|\mathcal{E}(\mathcal{A})|$, which is a contradiction. As in the proof of Lemma 5, we start with the partition $\mathcal{A}'=\bigcup_{i=1}^{\ell}\mathcal{A}(i)$. From Lemma 4 it follows that $\mathcal{A}(i)=\emptyset$ for all $1\leq i<n-d=t$. We will show that all $\mathcal{A}(i)$s are empty. Suppose that $\mathcal{A}(i)\neq\emptyset$ for some $i,\ t\leq i\leq\ell$. From Lemma 4 we know that

$$|\mathcal{A}(i)|=\binom{\ell}{i}|\mathcal{A}^*(i)|. \tag{9}$$

Note that in the case $n = \ell + 1$ we have $\mathcal{A}^*(i) = \emptyset$ and $|\mathcal{A}^*(i)| = 1$. Now as before we consider the sets $\mathcal{B}(i)$. From Lemma 4 it follows that for $B \in \mathcal{B}(i), A \in \mathcal{A}(j)$ with $i + j \neq \ell + t,\ |A \cap B| \geq t$ holds. Hence, using this and (iv) of Lemma 4, we have

$$\mathcal{H}_1 = ((\mathcal{A} \setminus \mathcal{A}(\ell+t-i)) \cup \mathcal{B}(i)) \in I(n, n-d),$$

and

$$\mathcal{H}_2 = ((\mathcal{A} \setminus \mathcal{A}(i)) \cup \mathcal{B}(\ell+t-i)) \in I(n, n-d).$$

Let us show that

$$\max\{|\mathcal{E}(\mathcal{H}_1)|, |\mathcal{E}(\mathcal{H}_2)|\} > |\mathcal{E}(\mathcal{A})| = E(n,d), \tag{10}$$

which will be a contradiction.

From the additional (vi) and (vii) of Lemma 4 one can easily show that the sets $\mathcal{H}_1$, $\mathcal{H}_2$, $(\mathcal{A} \setminus \mathcal{A}(j))$ are upsets. Therefore, using Proposition 8, we have

$$\begin{aligned}
|\mathcal{E}(\mathcal{A})| &= |\mathcal{E}(\mathcal{A} \setminus \mathcal{A}(\ell+t-i)| + \sum_{A \in \mathcal{A}(\ell+t-i)} (n - |A|) \\
&= |\mathcal{E}(\mathcal{A} \setminus \mathcal{A}(i))| + \sum_{A \in \mathcal{A}(i)} (n - |A|), \\
|\mathcal{E}(\mathcal{H}_1)| &= |\mathcal{E}(\mathcal{A} \setminus \mathcal{A}(\ell+t-i))| + \sum_{A \in \mathcal{B}(i)} (n - |A|) \qquad (11) \\
|\mathcal{E}(\mathcal{H}_2)| &= |\mathcal{E}(\mathcal{A} \setminus \mathcal{A}(i))| + \sum_{A \in \mathcal{B}(\ell+t-i)} (n - |A|).
\end{aligned}$$

Hence the negation of (10) is

$$\begin{aligned}
\sum_{A \in \mathcal{A}(\ell+t-i)} (n - |A|) &\geq \sum_{A \in \mathcal{B}(i)} (n - |A|), \qquad (12) \\
\sum_{A \in \mathcal{A}(i)} (n - |A|) &\geq \sum_{A \in \mathcal{B}(\ell+t-i)} (n - |A|).
\end{aligned}$$

Since we have assumed $\mathcal{A}(i) \neq \emptyset$, then clearly $\mathcal{A}(\ell + t - i) \neq \emptyset$ as well, because otherwise the first inequality of (12) would be false.

Using properties of the sets $\mathcal{A}(i), \mathcal{B}(i)$ we can write (12) in the form

$$\begin{aligned}
&\binom{\ell}{\ell+t-i} \sum_{C \in \mathcal{A}^*(\ell+t-i)} (n - \ell - t + i - |C|) \qquad (13) \\
&\geq \binom{\ell}{i-1} \sum_{D \in \mathcal{A}^*(i)} (n - i - |D|), \\
&\binom{\ell}{i} \sum_{D \in \mathcal{A}^*(i)} (n - i - |D|) \\
&\geq \binom{\ell}{\ell+t-i-1} \sum_{C \in \mathcal{A}^*(\ell+t-i)} (n - \ell - t + i - |C|).
\end{aligned}$$

However, (13) implies

$$(\ell-i+1)(i+1-t) \geq (\ell+t-i)i,$$

which is false, because $t \geq 2$ and, consequently, $i > i+1-t,\ \ell+t-i > \ell-i+1$.

Hence $\mathcal{A}(i) = \emptyset$ for all $i \neq \ell+t-i$.

Let now $i = \frac{\ell+t}{2}$. Here necessarily $2|(\ell+t)$ and therefore by assumption (8) we have in Lemma 12 $\ell \leq n-2$ in the case (i), $\ell \leq n-4$ in the case (ii), and $\ell \leq n-3$ in the case (iii). Let us call these conditions "conditions C."

Now we consider any element $A' = B' \cup C'$, where $B' \in \binom{[\ell]}{\frac{\ell+t}{2}}$, $C \subset C' \subset [\ell+2,n]$, and $C \in \mathcal{A}^*\left(\frac{\ell+t}{2}\right)$. Of course $A' \in \mathcal{A}$, since $\mathcal{A}$ is an upset and $(B' \cup C) \in \mathcal{A}' \subset \mathcal{A}$, $(B' \cup C) \subset (B' \cup C')$. It is also clear by the definition that, if $A' \in \mathcal{A}'$, then $A' \in \mathcal{A}\left(\frac{\ell+t}{2}\right)$. Using Lemma 4 we can say more: $A' = B' \cup C' \in \mathcal{A}\left(\frac{\ell+t}{2}\right)$ iff there is a $C'' \in \mathcal{A}^*\left(\frac{\ell+t}{2}\right)$ with $C'' \cap C' = \emptyset$, and hence with every $C \in \mathcal{A}^*\left(\frac{\ell+t}{2}\right)$ we have also $\bar{C} = ([\ell+2,n] \setminus C) \in \mathcal{A}^*\left(\frac{\ell+t}{2}\right)$. Moreover, it is easily seen that $\mathcal{A}^*\left(\frac{\ell+t}{2}\right)$ is a convex set. Therefore, $\mathcal{A}^*\left(\frac{\ell+t}{2}\right)$ has the properties described in Lemma 11 and we can apply this lemma and the corollary to get an intersecting set $\mathcal{A}_1^*\left(\frac{\ell+t}{2}\right) \subset \mathcal{A}^*\left(\frac{\ell+t}{2}\right)$ for which (7) holds:

$$\sum_{D \in \mathcal{A}_1^*\left(\frac{\ell+t}{2}\right)} (m-|D|+c) \geq \frac{m+2c-1}{2(m+2c)} \sum_{D \in \mathcal{A}^*\left(\frac{\ell+t}{2}\right)} (m-|D|+c) \tag{14}$$

for $m = n-\ell-1$ and any constant c.

Now denote

$$\mathcal{B}_1 = \left\{ B : |B \cap [1,\ell]| = \frac{\ell+t}{2} - 1,\ \ell+1 \in B,\ (B \cap [\ell+2,n]) \in \mathcal{A}_1^*\left(\frac{\ell+t}{2}\right) \right\}$$

$$\mathcal{A}_1\left(\frac{\ell+t}{2}\right) = \left\{ A \in \mathcal{A}\left(\frac{\ell+t}{2}\right) :\ (A \cap [\ell+2,n]) \in \mathcal{A}_1^*\left(\frac{\ell+t}{2}\right) \right\} \tag{15}$$

and consider the following competitor of the set $\mathcal{A}$:

$$\mathcal{H}_3 = \left(\left(\mathcal{A} \setminus \mathcal{A}\left(\frac{\ell+t}{2}\right) \right) \cup \mathcal{A}_1\left(\frac{\ell+t}{2}\right) \cup \mathcal{B}_1 \right).$$

It is easily seen that $\mathcal{H}_3 \in I(n, n-d)$.

We are going to show that

$$|\mathcal{E}(\mathcal{H}_3)| > |\mathcal{E}(\mathcal{A})|, \tag{16}$$

which will be a contradiction.

It is easily verified that both $\mathcal{H}_3$ and $\mathcal{A} \setminus \mathcal{A}\left(\frac{\ell+t}{2}\right)$ are upsets. Therefore, by Proposition 8 we can write

$$|\mathcal{E}(\mathcal{A})| = \left|\mathcal{E}\left(\mathcal{A}\setminus\mathcal{A}\left(\frac{\ell+t}{2}\right)\right)\right| + \sum_{A\in\mathcal{A}\left(\frac{\ell+t}{2}\right)} (n-|A|),$$

$$|\mathcal{E}(\mathcal{H}_3)| = \left|\mathcal{E}\left(\mathcal{A}\setminus\mathcal{A}\left(\frac{\ell+t}{2}\right)\right)\right| + \sum_{A\in\mathcal{A}_1\left(\frac{\ell+t}{2}\right)\cup\mathcal{B}_1} (n-|A|).$$

Hence the negation of (16) is

$$\sum_{A\in\mathcal{A}\left(\frac{\ell+t}{2}\right)} (n-|A|) \geq \sum_{A\in\mathcal{A}_1\left(\frac{\ell+t}{2}\right)\cup\mathcal{B}_1} (n-|A|),$$

which can be written in the form

$$\begin{aligned} &\tbinom{\ell}{\frac{\ell+t}{2}} \textstyle\sum_{D\in\mathcal{A}^*\left(\frac{\ell+t}{2}\right)} (m+c-|D|) \\ \geq &\left(\tbinom{\ell}{\frac{\ell+t}{2}} + \tbinom{\ell}{\frac{\ell+t}{2}-1}\right) \textstyle\sum_{D\in\mathcal{A}_1^*\left(\frac{\ell+t}{2}\right)} (m+c-|D|) \\ = &\tbinom{\ell+1}{\frac{\ell+t}{2}} \textstyle\sum_{d\in\mathcal{A}_1^*\left(\frac{\ell+t}{2}\right)} (m+c-|D|), \end{aligned}$$

$m = n-\ell-1,\ c = \frac{\ell-t+2}{2}$. This is equivalent to

$$\frac{\ell-t+2}{2(\ell+1)} \sum_{D\in\mathcal{A}^*\left(\frac{\ell+t}{2}\right)} (m+c-|D|) \geq \sum_{D\in\mathcal{A}_1^*\left(\frac{\ell+t}{2}\right)} (m+c-|D|). \tag{17}$$

However, (14) for $m = n-\ell-1,\ c = \frac{\ell-t+2}{2}$, and (17) imply

$$\frac{n-t}{n-t+1} \leq \frac{\ell-t+2}{\ell+1}, \tag{18}$$

which is false, since $t \geq 2$ and conditions C can be checked to hold. $\square$

Now we are ready to make the final step in the proof of the theorem. Let $\mathcal{A} \in D(n,d)$ be a set with $|\mathcal{E}(\mathcal{A})| = E(n,d)$. We can assume that $\mathcal{A} \in LUI(n,n-d)$.

In the case $d = n-1$ we just notice that any maximal set $\mathcal{B} \in D(n,n-1)$ has cardinality $|\mathcal{B}| = 2^{n-1}$. Now the equality $E(n,n-1) = |\mathcal{E}(\mathcal{H})|$ immediately follows from Proposition 9.

In the case $2|d,\ d \leq n-3$ we get from Lemma 12 $|A| \geq n-\frac{d}{2}$ for all $A \in \mathcal{A}$, since $\mathcal{A}$ is invariant in $[n]$ and at the same time $\mathcal{A} \in I(n,n-d)$. This implies $\mathcal{A} \subset \mathcal{G}\left(\frac{d}{2}\right) \in D(n,d)$, and by maximality of $\mathcal{A}$ we get

$$\mathcal{A} = \mathcal{G}\left(\frac{d}{2}\right).$$

Now we consider the case $2|d,\ d=n-2$. Looking at the proof of Lemma 12, (ii), we see that in (18) for $t=n-d=2,\ \ell=n-2$ we have equality, which means it can be slightly changed to

(ii*) If $2|d$ and $d=n-2$, then there exists an optimal set that is invariant in $[1,n]$. Therefore, in this case again, we have

$$E(n,d)=\left|\mathcal{G}\left(\mathcal{K}\left(\frac{d}{2}\right)\right)\right|.$$

We verify (for $2|d,\ d=n-2$) that

$$\left|\mathcal{E}\left(\mathcal{G}\left(\frac{d}{2}-1\right)\right)\right|=\left|\mathcal{E}\left(\mathcal{G}\left(\frac{d}{2}\right)\right)\right|$$

and hence $\mathcal{G}\left(\frac{d}{2}-1\right)$ is the second optimal configuration in this case.

Finally, the case $2\not|d,\ d\leq n-2$ follows from Lemma 12, (iii), by similar arguments. ■

Proof of Proposition 9. First we will make some definitions. A k-subcube of the n-cube is a set of all vertices, which have the same components in some set of $n-k$ positions.

A shadow of a k-subcube is obtained by changing one of the $n-k$ fixed positions. Each k-subcube has $n-k$ shadows.

The following algorithm will number ℓ vertices of the n-cube so that the configuration of these ℓ vertices gives a maximal number of connections: assign one to an arbitrary vertex; having assigned $1,\dots,\ell-1$, assign ℓ to an unnumbered vertex (not necessarily unique), which has the most numbered nearest neighbors. We will prove by induction on ℓ and n.

But first we find out which configurations the algorithm delivers. The answer follows from the fact that whenever $\ell=2^k$, a k-subcube is numbered. This is trivial for $k=0$. Assume that we have numbered 2^{k-1} vertices of an n-cube and that by the inductive hypothesis they form a $(k-1)$-subcube. This cube will have $n-(k-1)$ disjoint shadows. When the $(2^{k-1}+1)$th vertex is numbered, it will be in any of the shadows. The next $2^{k-1}-1$ numbers will also fall in this shadow. Since no shadow of that shadow intersects any other shadow, there will always be unnumbered vertices in the first shadow, which will have two or more numbered nearest neighbors. Thus, it is inductively apparent that, for any ℓ, the construction gives a series of cubes, corresponding to the ones in the binary expansion of ℓ, each shadow of every larger cube.

Now we perform induction. It is obviously true for $\ell=1$. Suppose it is true for $1,\dots,\ell-1$ and suppose that we have a maximally connected configuration of ℓ vertices of an n-cube. The n-cube may be divided into two $(n-1)$-subcubes in n ways. Choose one of them. Suppose we have a numbered vertices contained in one of the halves and $b\leq a$ in the other one. If $b=0$, induction on n completes the proof. If $b>0$, then the number of connections is maximized by having a maximally connected configuration in each half and b connections between them. By the hypothesis, the maximal configurations for a and b would be built of cubes, so that the smaller one will fit into the shadow of the larger one and so make b connections. This then is the case. Now suppose that 2^k is the largest power of two equal to or less than ℓ. Then if $2^k \leq a$, we have, by the induction hypothesis, a k-subcube in the larger configuration. If $2^k > a$, then 2^{k-1} is the largest power of two equal to or less than both a and b. In this case both a and b configurations contain $(k-1)$-subcubes, and since they are the largest such, each must lie in the other's shadow. In either case we have a k-subcube in maximal configuration. At last we must show that the remaining $\ell - 2^k$ vertices lie in a single shadow of the k-subcube. If not, c vertices lie in one shadow and d lie in another $(c+d \leq \ell - 2^k)$. Let 2^j be the smallest power of two equal to or greater than $c+d$. There can be no connections between the c and d configurations, so that the inductive hypothesis tells us that they are series of subcubes, each in the shadows of all larger ones. Look at a j-subcube which contains the c configurations and lies entirely within the shadow. Note that the complement of the c configurations in that j-subcube is also of the maximally connected type, so that the d configurations could be placed into it without changing its number of connections. But since $c+d > 2^{j-1}$, placing them both in the same j-cube would produce at least one more connection, contradicting our assumption that the configuration was maximally connected with $c, d > 0$.

At last note that the natural numbering of the n-cube assigns to each vertex the number that the vertex represents when considered as a binary digit, plus one. It can be easily seen that this natural numbering produces the above algorithm and hence the first 2^{n-1} vertices in this natural order give a set of 2^{n-1} vertices with the largest possible number of edges. This proves Proposition 9. □

Lecture 5 Words with Pairwise Common Letter

In this lecture we present a problem that seems to stay apart from the topics of the other lectures. The problem deals with sets of words with pairwise common letter in different positions. It does, however, fall into the general frame of maximizing cardinalities of sets, whose members are pairwise in a certain relation like incomparable, t-intersecting, $t - \mathcal{H}_q^n$ intersecting, having distance d, independent, etc.

We start with some definitions. For an alphabet $\mathcal{X}_q = [q]$ we consider the set $\mathcal{X}_q^n$ of words of length n and also the subset W_q^n of all words without repetition of letters, that is,

$$W_q^n = \{x^n = (x_1, \dots, x_n) : \ x_t \in \mathcal{X}_q, \ x_s \neq x_t \ if \ s \neq t\}. \tag{1}$$

We say that two words x^n and y^n are in "good relation" if $x_s = y_t$ for some $s \neq t$. For this relation we write $x^n \swarrow\!\searrow y^n$. A set $G \subset \mathcal{X}_q^n$ is good if $x^n \swarrow\!\searrow y^n$ for all $x^n, y^n \in G$. We will study $\mathcal{G}_q^n$, the family of good sets in $\mathcal{X}_q^n$, and the quantity

$$g_q^n = \max\{|G| : G \in \mathcal{G}_q^n\}. \tag{2}$$

Denote by $\mathcal{F}_q^n$ the family of good sets in W_q^n and

$$f_q^n = \max\{|F| : F \in \mathcal{F}_q^n\}.$$

Also, denote the set of entries in x^n by

$$E(x^n) = \{x : \text{ for some } t,\ x_t = x\}.$$

Very little is known about the values of g_q^n and f_q^n. In the following we are going to demonstrate the asymptotical behavior of g_q^n as $q \to \infty$. The theorem we will prove states the most significant known result in this area.

§1 Asymptotical Behavior of g_q^n

Here we will show that $g_q^n \sim q^{n-2}\binom{n}{2}$ as $q \to \infty$. We will prove

Theorem 10 (Ahlswede and Cai 1991) *The following relation is valid:*

$$\lim_{q\to\infty} \frac{g_q^n}{q^{n-2}} = \binom{n}{2}. \tag{3}$$

Proof. We begin with the inequality

$$\liminf_{q\to\infty} \frac{g_q^n}{\binom{q-1}{n-2}} \geq \binom{n}{2}(n-2)! \tag{4}$$

Define

$$G_0 = \{x^n \in \mathcal{X}_q^n : |E(x^n)| = n-1 \text{ and } 1 \text{ occurs exactly twice in } x^n\}.$$

Obviously $G_0 \in \mathcal{G}_q^n,\ |G_0| = \binom{n}{2}(n-2)!.\binom{q-1}{n-2}$ $(q-1 \geq n-2)$, and (4) follows.

Next we show that

$$\limsup_{q\to\infty} \frac{g_q^n}{\binom{q-1}{n-2}} \leq \binom{n}{2}(n-2)!. \tag{5}$$

Recall that a partition of an integer n is a finite nonincreasing sequence of positive integers $\lambda_1 \geq \cdots \geq \lambda_r$ with $\sum_{i=1}^r \lambda_i = n$. Denote by $\mathcal{P}(n)$ the set of all partitions of n. We partition now $\mathcal{X}_q^n$ according to $\mathcal{P}(n)$ as follows. For $\Lambda = (\lambda_1, \dots, \lambda_r) \in \mathcal{P}(n)$, set $\mathcal{T}(\Lambda) = \{x^n \in \mathcal{X}_q^n :\ \exists z_1, \dots, z_r \in \mathcal{X}_q$ such that z_i occurs in $x^n = (x_1, \dots, x_n)$ exactly λ_i times . We subdivide $\{\mathcal{T}(\Lambda) : \Lambda \in \mathcal{P}(n)\}$ into three classes. The class 0 consists of $\mathcal{T}(\Lambda_0)$, where $\Lambda_0 = (1, \dots, 1)$. The class 1 consists of $\mathcal{T}(\Lambda)$ for $\Lambda = \Lambda_1 = (2, 1, \dots, 1)$ and the remaining sets belong to the class 2. For all $G \in \mathcal{G}_q^n$ we have by our definitions $|G \cap \mathcal{T}(\Lambda_0)| \leq f_q^n$.

It is easy to see that $f_q^n \leq (n!)^2$, the bound being independent of q. Choose any $(x_1, \dots, x_n) \in F \in \mathcal{F}_\infty^n$. For all $y^n \in F$ it holds $E(y^n) \cap \{x_1, \dots, x_n\} \neq \emptyset$ and, on the other hand, for fixed j and i, $|F \cap \{y^n : y_j = x_i\}| \leq f_\infty^{n-1}$. This implies $f_\infty^n \leq n^2 f_\infty^{n-1}$ and clearly $f_q^n \leq f_\infty^n$. Therefore,

$$|G \cap \mathcal{T}(\Lambda_0)| \leq (n!)^2. \tag{6}$$

Now consider the class 2. For all $x^n, y^n \in G$ we have $E(x^n) \cap E(y^n) \neq \emptyset$. So $\{E(x^n) : x^n \in G \cap \mathcal{T}(\Lambda)\}$ is an intersecting family of r-element sets (if $\Lambda = (\lambda_1, \ldots, \lambda_r)$). For a $\mathcal{T}(\Lambda)$ in the class 2 of partitions n into $r \leq n-2$ parts and for all $x^n \in \mathcal{T}(\Lambda)$ it holds

$$|\{y^n : E(y^n) = E(x^n)\}| \leq r^n \leq (n-2)^n. \tag{7}$$

This leads for large q to the estimate

$$\left| G \cap \left(\bigcup_{\Lambda \notin \{\Lambda_0,\, \Lambda_1\}} \mathcal{T}(\Lambda) \right) \right| \leq |\mathcal{P}(n)| \binom{q-1}{n-3} (n-2)^n. \tag{8}$$

This inequality uses also equality (3) (Lecture 1). Taking into account relations (6) and (8) ($\ln |\mathcal{P}(n)| = O(\sqrt{n})$), for verification of (5) it suffices to show that for $G \in \mathcal{G}_q^n$

$$\limsup_{q \to \infty} \frac{|G \cap \mathcal{T}(\Lambda_1)|}{\binom{q-1}{n-2}} \leq \binom{n}{2} (n-2)!. \tag{9}$$

To do this, we have to consider a partition of $\mathcal{T}(\Lambda_1) \cap G$ into a few subparts. First of all, we can assume that the intersecting system $\{E(x^n) : x^n \in G \cap \mathcal{T}(\Lambda_1)\}$ is not a 2-intersecting family, because otherwise for large q, $|G \cap \mathcal{T}(\Lambda_1)| \leq \binom{q-2}{n-3}\binom{n}{2}(n-1)! \sim q^{n-3}$, which follows from the equality

$$|\{y^n : E(y^n) = E(x^n)\}| = \binom{n}{2}(n-1)!,\ x^n \in \mathcal{T}(\Lambda_1) \tag{10}$$

and (3) (Lecture 1). From this (9) follows.

Using (6), (8), and (9) and taking into account that the set on the LHS of (10) is intersecting (which gives the factor $\binom{q-1}{n-2}$), we obtain the relation

$$\limsup_{q \to \infty} \frac{g_q^n}{\binom{q-1}{n-2}} \leq \binom{n}{2}(n-1)!. \tag{11}$$

Now suppose that $|E(a^n) \cap E(b^n)| = 1$ for some $a^n, b^n \in G \cap \mathcal{T}(\Lambda_1)$. W.l.o.g. let $E(a^n) = \{1, 2, \ldots, n-1\}$ and $E(b^n) = \{1, n, n+1, \ldots, 2n-3\}$. Denote $\mathcal{Z} = \{x^n \in G \cap \mathcal{T}(\Lambda_1) : 1 \notin E(x^n)\}$. Since $E(x^n) \cap E(a^n) \neq \emptyset$ and $E(x^n) \cap E(b^n) \neq \emptyset$ for all $x^n \in \mathcal{Z}$, we have $|\{E(x^n) : x^n \in \mathcal{Z}\}| < 2^{2(n-2)}\binom{q-2n+2}{n-3}$. Consequently, by (10),

$$|\mathcal{Z}| < 2^{2(n-1)} \binom{n}{2} (n-1)! \binom{q-2n+2}{n-3}. \tag{12}$$

Let now $C_i = \{(c_1, \ldots, c_n) \in \mathcal{T}(\Lambda_1) : c_i = 1,\ c_j \neq 1,\ j \neq i\}$ for $i = 1, \ldots, n$. Then

$$\mathcal{T}(\Lambda_1) \cap G = (G \cap G_0) \cup \mathcal{Z} \cup (C_1 \cap G) \cup \ldots \cup (C_n \cap G). \tag{13}$$

As $\{(c_1,\dots,c_{i-1},c_{i+1},\dots,c_n):\ (c_1,\dots,c_{i-1},1,c_{i+1},\dots,c_n)\in C_i\cap G\}\in \mathcal{G}_q^{n-1}$, we obtain

$$|C_i\cap G| = O(q^{n-3}),\ q\to\infty \tag{14}$$

by inequality (11).

Finally,

$$|G_0\cap G|\le |G_0| = \binom{n}{2}(n-2)!\binom{q-1}{n-2} \tag{15}$$

and (12)–(15) imply (9). This completes the proof of (5) and the theorem. ■

Lecture 6 Constant Distance Code Pairs

For an alphabet $\mathcal{X}_q=[q]$ consider the Hamming metric d_H on $\mathcal{X}_q^n:\ d_H(x^n,y^n)=|\{i:\ x_i\neq y_i\}|$.

A pair (A,B) of sets $A,B\subset\mathcal{H}_q^n$ is an (n,δ) constant distance code pair if

$$d_H(a^n,b^n)=\delta,\ \text{for all } a^n\in A,\ b^n\in B.$$

The set of all such pairs we denote by $S_q(n,\delta)$. In this lecture we give a partial solution to the problem of determining the value

$$M_q(n,\delta)=\max\{|A||B|:(A,B)\in S_q(n,\delta)\}.$$

We will find an explicit formula for $M_q(n,\delta)$ only in the cases $q=2,4,5$ and will formulate a conjecture for the values of $M_q(n,\delta)$, when $q=3$ and $q\geq 6$. The explicit formula for $M_q(n,\delta)$ will be expressed in terms of the following functions

$$F_2(n,\delta) = \max_{d_1+d_2=\delta} 4^{d_1}\binom{n-2d_1}{d_2}, \tag{1}$$

$$F_3(n,\delta) = \max_{2\ell+d=\delta} 18^{\ell}\binom{n-3\ell}{d}2^d, \tag{2}$$

$$F_q(n,\delta) = \max_{d_1+d_2=\delta} \bar{q}^{d_1}\binom{n-d_1}{d_2}(q-1)^{d_2},\ q\geq 4, \tag{3}$$

$$\bar{q} = \left\lfloor\frac{q}{2}\right\rfloor\left\lceil\frac{q}{2}\right\rceil.$$

§1 The Exact Value of $M_q(n,\delta)$

The main result we are going to prove here is contained in the following:

Theorem 11 (Ahlswede 1987) *For $q=2,4,5$ the following equality holds:*

$$M_q(n,\delta)=F_q(n,\delta). \tag{4}$$

Proof. First we show for arbitrary q the validity of the inequality

$$M_q(n,\delta) \geq F_q(n,\delta). \tag{5}$$

To do this we present explicit constructions of the sets A and B such that $(A,B) = (A,B)_{q,n,\delta}$ and $|A||B| = F_q(n,\delta)$.

First of all we define the following sets:

$$\begin{aligned}
E_1(q,m) &= \{(1,\dots,1),\dots,(q,\dots,q)\} \subset [q]^m, \\
E_2(q) &= \{\pi(1),\dots,\pi(q):\ \pi \in S_q\} \subset [q]^q, \\
E_3(q,m,d) &= \{x^m \in [q]^m:\ d_H(x^m,(1,\dots,1)) = d\}, \\
E_4 &= \{1,2,\dots,\beta\},\ \bar{E}_4 = \{\beta+1,\dots,q\},\ \beta = \left\lfloor \frac{q}{2} \right\rfloor,
\end{aligned}$$

where S_q is the set of all permutations on $[q]$.

We treat first the case $q = 2$ and consider the sets

$$\begin{aligned}
A &= (E_1(2,2))^{d_1} \times E_1(1,n-2d_1), \\
B &= (E_2(2))^{d_1} \times E_3(2,n-2d_1,d_2).
\end{aligned}$$

We have $d_H(a,b) = d_1 + d_2$, when $a \in A,\ b \in B$ and $|A| = 2^{d_1},\ |B| = 2^{d_1}\binom{n-2d_1}{d_2}$. Thus an optimal choice of d_i gives

$$|A||B| = F_2(n,\delta).$$

Next, suppose $q = 3$. This time we define the sets A and B as follows:

$$\begin{aligned}
A &= (E_1(3,3))^{\ell} \times E_1(1,n-3\ell), \\
B &= (E_2(3))^{\ell} \times E_3(3,n-3\ell,d).
\end{aligned}$$

We have $d_H(a,b) = 2\ell + d,\ a \in A,\ b \in B$, and

$$|A| = 3^{\ell},\ |B| = 6^{\ell}\binom{n-3\ell}{d}2^d.$$

An optimal choice of d,ℓ with $2\ell + d = \delta$ gives

$$|A||B| = F_3(n,\delta).$$

In the case $q \geq 4$ define

$$\begin{aligned}
A &= (E_4(q))^{d_1} \times E_1(1,n-d_1), \\
B &= (\bar{E}_4(q))^{d_1} \times E_3(d,n-d_1,d_2).
\end{aligned}$$

Again, $d_H(a,b) = d_1 + d_2,\ a \in A,\ b \in B$, and

$$|A| = \left\lfloor \frac{q}{2} \right\rfloor^{d_1},\ |B| = \left\lceil \frac{q}{2} \right\rceil^{d_1} \binom{n-d_1}{d_2} (q-1)^{d_2}.$$

An optimal choice of d_i with $d_1 + d_2 = \delta$ yields

$$|A||B| = F_q(n,\delta).$$

Now we start to prove for $q = 2,4,5$ the inequality

$$M_q(n,\delta) \le F_q(n,\delta),$$

which together with (5) gives the proof of Theorem 11. We need the following lemma, which we then use in the inductive proof of the theorem.

Lemma 13 *The following relations are valid:*

$$F_2(n,\delta) = F_2(n-2,\delta-1)\max\left(4, \frac{n(n-1)}{\delta(n-\delta)}\right), \tag{6}$$
$$n \ge 3,\ 1 \le \delta \le n-1,$$
$$F_q(n,\delta) = F_q(n-1,\delta-1)\max\left(\bar{q}, \frac{n(q-1)}{\delta}\right), \tag{7}$$
$$q \ge 4,\ n \ge 2,\ \delta \ge 1.$$

Proof. First we show that the LHS of equalities (6), (8) do not exceed their RHS. Choose d_1, d_2 such that $d_1 + d_2 = \delta$ and

$$F_2(n,\delta) = 2^{2d_1}\binom{n-2d_1}{d_2}.$$

If $d_1 = 0$, then

$$F_2(n,\delta) = \binom{n}{\delta} = \binom{n-2}{\delta-1}\frac{n(n-1)}{\delta(n-\delta)} \le F_2(n-2,\delta-1)\frac{n(n-1)}{\delta(n-\delta)},$$

and if $d_1 \ge 1$, then

$$F_2(n,\delta) = 2^2 2^{2(d_1-1)}\binom{n-2-2(d_1-1)}{d_2} \le 4F_2(n-2,\delta-1).$$

For $q \ge 4$ we have

$$F_q(n,\delta) = \bar{q}^{d_1}\binom{n-d_1}{d_2}(q-1)^{d_2}$$

and in the case $d_1 = 0$

$$F_q(n,\delta) = \binom{n}{\delta}(q-1)^{\delta} = \frac{n(q-1)}{\delta}\binom{n-1}{\delta-1}(q-1)^{\delta-1}$$
$$\leq F_q(n-1,\delta-1)\frac{n(q-1)}{\delta}.$$

If $d_1 \geq 1$, then

$$F_q(n,\delta) = \bar{q}^{d_1}\binom{n-d_1}{d_2}(q-1)^{d_2}$$
$$= \bar{q}\bar{q}^{d_1-1}\binom{(n-1)-(d-1)}{d_2}(q-1)^{d_2} \leq \bar{q}F_q(n-1,\delta-1).$$

Next we prove that the RHS of (8) does no exceed its LHS. Let d_1, d_2 satisfy $d_1 + d_2 = \delta - 1$ and

$$F_q(n-1,\delta-1) = \bar{q}^{d_1}\binom{n-1-d_1}{d_2}(q-1)^{d_2}.$$

Then $d_1 + 1 + d_2 = \delta$ and we have

$$\bar{q}F_q(n-1,\delta-1) = \bar{q}^{d_1+1}\binom{n-(d_1+1)}{d_2}(q-1)^{d_2} \leq F_q(n,\delta).$$

Furthermore, since

$$\frac{n(q-1)}{\delta}F_q(n-1,\delta-1) = \bar{q}^{d_1}\binom{n-1-d_1}{d_2}\frac{n}{\delta}(q-1)^{d_2+1},$$

it suffices to show that

$$\binom{n-1-d_1}{d_2}\frac{n}{\delta} \leq \binom{n-d_1}{d_2+1}.$$

But

$$\binom{n-1-d_1}{d_2}\frac{n}{\delta} = \binom{n-d_1}{d_2+1}\frac{d_2+1}{n-d_1}\frac{n}{\delta}.$$

Therefore, it suffices to show that

$$\frac{n}{n-d_1} \leq \frac{\delta}{d_2+1} = \frac{\delta}{\delta-d_1},$$

which is true, because for $x \geq y \geq 0, z \geq 0$ with $xyz \neq 0$ it holds $\frac{x+z}{y+z} \leq \frac{x}{y}$.

Now we prove that the RHS of (6) does not exceed its LHS. Suppose that

$$F_2(n-2,\delta-1) = 2^{2d}\binom{n-2-2d}{\delta-1-d}, \tag{8}$$

then

$$4F_2(n-2,\delta-1) = 2^{2(d+1)}\binom{n-2(d+1)}{\delta-(d+1)} \le F_2(n,\delta)$$

and, to finish the proof, we have to consider the case

$$4 < \frac{n(n-1)}{\delta(n-\delta)}. \tag{9}$$

From (8) it follows that

$$F_2(n-2,\delta-1)\frac{(n-2d)(n-2d-1)}{(\delta-d)(n-d-\delta)} = 2^{2d}\binom{n-2d}{\delta-d} < F_2(n,\delta).$$

It remains to prove that under condition (9) either

$$\frac{n(n-1)}{\delta(n-\delta)} \le \frac{(n-2d)(n-2d-1)}{(\delta-d)(n-d-\delta)}$$

or

$$\delta(n-\delta)(n^2-4nd+4d^2-n+2d) \ge (n^2-n)((n-\delta)\delta-(n-\delta)d-\delta d+d^2)$$

or

$$\frac{n(n-1)}{\delta(n-\delta)} \ge 4 - \frac{2}{n-d} \tag{10}$$

holds, which is true under condition (9). The proof of Lemma 13 is completed.

□

Next we give the following definitions. For a set $C \subset [q]^n$ and $i, j \in [q],\ J \subset [q]$, define

$$\begin{aligned}
C_i^t &= \{(c_1,\dots,c_{t-1},c_{t+1},\dots,c_n) : (c_1,\dots,c_{t-1},i,c_{t+1},\dots,c_n) \in C\},\\
C^t(J) &= \{(c_1,\dots,c_n) \in C :\ c_i \in J\} \subset \mathcal{C},\ n \ge 2,\\
C_{ij}^{st} &= \{(c_1,\dots,c_{s-1},c_{s+1},\dots,c_{t-1},c_{t+1},\dots,c_n) :\\
&\qquad (c_1,\dots,c_{s-1},i,c_{s+1},\dots,c_{t-1},j,c_{t+1},\dots,c_n) \in C\}, s \ne t,\ n \ge 3.
\end{aligned}$$

Denote also $\mathcal{J}_q = \binom{[q]}{\lfloor \frac{q}{2} \rfloor}$. We need two lemmas.

Lemma 14 *For $(A,B) \in S_2(n,\delta)$ there exist $s,t \in [n]$ such that*

$$\begin{aligned}
&(|A_{11}^{st}|+|A_{22}^{st}|)(|B_{12}^{st}|+|B_{21}^{st}|)+(|A_{12}^{st}|+|A_{21}^{st}|)(|B_{11}^{st}|+|B_{22}^{st}|) \qquad (11)\\
\ge\ &\frac{2\delta(n-\delta)}{n(n-1)}|A||B|.
\end{aligned}$$

Proof. Let

$$C_{ij}(s,t) = \{(c_1,\dots,c_n) \in C :\ c_s = i,\ c_t = j\}.$$

Then $|C_{ij}(s,t)| = |C_{ij}^{st}|$ and if $I_A(x^n)$ is the indicator function of the set A, then

$$\sum_{s\neq t}\left[(|A_{11}^{st}|+|A_{22}^{st}|)(|B_{12}^{st}|+|B_{21}^{st}|)+(|A_{12}^{st}|+|A_{21}^{st}|)(|B_{11}^{st}|+|B_{22}^{st}|)\right]$$

$$= \sum_{(x^n,y^n)\in(A,B),\ s\neq t}\Big[(I_{A_{11}(s,t)}(x^n)+I_{A_{22}(s,t)}(x^n))(I_{B_{12}(s,t)}(y^n)+I_{B_{21}(s,t)}(y^n))$$

$$+\ (I_{A_{12}(s,t)}(x^n)+I_{A_{21}(s,t)}(x^n))(I_{B_{11}(s,t)}(y^n)+I_{B_{22}(s,t)}(y^n))\Big].$$

Since $d_H(x^n,y^n)=\delta$ for $x^n\in A$ and $y^n\in B$, the contribution of (A,B) is $|A||B|\delta(n-\delta)$ and there exists at least one pair (s,t) with contribution at least $|A||B|\delta(n-\delta)/\binom{n}{2}$. The lemma is proved. □

Lemma 15 *For $(A,B)\in S_q(n,\delta)$ there exists a $t\in[n]$, such that*

$$\sum_{J\in\mathcal{J}_q}|A^t(J)||B^t(J^c)| \geq |A||B|\frac{\delta}{n(q-1)}\frac{\bar{q}}{q}\binom{q}{\lfloor\frac{q}{2}\rfloor}, \tag{12}$$

where $J^c=[q]\setminus J$.

Proof. We have

$$\begin{aligned}\sum_{t=1}^{n}\sum_{J\in\mathcal{J}_q}|A^t(J)||B^t(J^c)| &= \sum_{t=1}^{n}\sum_{J\in\mathcal{J}_q}\sum_{x^n\in A,\ y^n\in B}I_{A^t(J)}(x^n)I_{B^t(J^c)}(y^n)\\ &= \sum_{x^n\in A,\ y^n\in B}\sum_{t=1}^{n}\sum_{J\in\mathcal{J}_q}I_{A^t(J)}(x^n)I_{B^t(J^c)}(y^n)\\ &= \sum_{x^n\in A,\ y^n\in B}\delta\binom{q-2}{\lfloor\frac{q}{2}\rfloor-1} = |A||B|\delta\binom{q-2}{\lfloor\frac{q}{2}\rfloor-1}.\end{aligned}$$

Therefore, there exists a t with

$$\sum_{J\in\mathcal{J}_q}|A^t(J)||B^t(J^c)| \geq |A||B|\frac{\delta}{n}\binom{q-2}{\lfloor\frac{q}{2}\rfloor-1}$$

and (12) follows due to the identity

$$\binom{q}{\lfloor\frac{q}{2}\rfloor} = \frac{q(q-1)}{\bar{q}}\binom{q-2}{\lfloor\frac{q}{2}\rfloor-1}.$$

By symmetry also

$$\sum_{J\in\mathcal{J}_q}|A^t(J^c)||B^t(J)| \geq |A||B|\frac{\delta}{n(q-1)}\frac{\bar{q}}{q}\binom{q}{\lfloor\frac{q}{2}\rfloor}.$$

Thus there exists a t for which we have

$$\sum_{J\in\mathcal{J}_q}(|A^t(J)||B^t(J^c)|+|A^t(J^c)||B^t(J)|)\geq |A||B|\frac{2\delta}{n(q-1)}\frac{\bar{q}}{q}\binom{q}{\lfloor\frac{q}{2}\rfloor}. \tag{13}$$

□

Now we continue to prove the theorem. First we prove (4) for $q=2$. In the cases $\delta=0$ and $\delta=n$, it can be easily verified that

$$M_2(n,0)=F_2(n,0)=M_2(n,n)=F_2(n,n)=1.$$

In the other cases we proceed by induction on n and we assume that $\delta\neq 0,n$. For $n=1,2$ only the case

$$M_2(2,1)=F_2(2,1)=4$$

is relevant. An optimal configuration here is $(A,B)=(\{11,22\},\ \{21,12\})$.

Let (4) be valid for $n-2$. We show that it holds also for n. We use the sets $A^{st}_{\alpha\beta},B^{st}_{\alpha\beta}$ with property (11). For simplicity we omit the indices s,t and make the following conventions:

$$\begin{aligned}
I&=(|A_{11}|+|A_{22}|)(|B_{11}|+|B_{22}|),\\
II&=(|A_{12}|+|A_{21}|)(|B_{11}|+|B_{22}|),\\
III&=(|A_{11}|+|A_{22}|)(|B_{12}|+|B_{21}|),\\
IV&=(|A_{12}|+|A_{21}|)(|B_{12}|+|B_{21}|).
\end{aligned}$$

Lemma 14 says that

$$|A||B|\leq\frac{n(n-1)}{2\delta(n-\delta)}(II+III). \tag{14}$$

W.l.o.g. we can assume that

$$II\leq III. \tag{15}$$

First we consider the case $A_{11}\cap A_{22}\neq\emptyset$. Then

$$d_H(a^n_{11},b^n_{\beta\beta})\neq d_H a^n_{22},b^n_{\beta\beta}),\ a^n_{\alpha\alpha}\in A_{\alpha\alpha}(s,t),\ b^n_{\beta\beta}\in B_{\beta\beta}(s,t)$$

and we have $B_{11}=B_{22}=\emptyset$ and therefore $I=II=0$. If now $B_{12}\cap B_{21}\neq\emptyset$, then by the same argument $A_{12}=A_{21}=\emptyset$ and thus also $IV=0$. Therefore,

$$|A||B|=III\leq 4M_2(n-2,\delta-1)=4F_2(n-2,\delta-1)\leq F_2(n,\delta).$$

Here, the last equality follows from the induction hypothesis and the last inequality follows from (6).

On the other hand, if $B_{12}\cap B_{21}=\emptyset$, then $(A_{\alpha\alpha},B_{12}\cup B_{21})\in S_2(n-2,\delta-1),\ \alpha=1,2$ and therefore $III\leq 2M_2(n-2,\delta-1)$. Since $II=0$, we conclude that

$$II+III\leq 2M_2(n-2,\delta-1)=2F_2(n-2,\delta-1)$$

and by (14) we have

$$|A||B| \leq \frac{n(n-1)}{\delta(n-\delta)} F_2(n-2, \delta-1).$$

Then (6) implies $|A||B| \leq F_2(n,\delta)$.

Suppose now $A_{11} \cap A_{22} = \emptyset$. If $B_{12} \cap B_{21} \neq \emptyset$, then, as previously, $A_{12} = A_{21} = \emptyset$ and $II = 0,\ II + III = III \leq 2M_2(n-2, \delta-1)$, and $|A||B| \leq F_2(n,\delta)$. Finally, if $B_{12} \cap B_{21} = \emptyset$, then

$$(A_{11} \cup A_{22}, B_{12} \cup B_{21}) \in S_2(n-2, \delta-1)$$

and thus $III \leq M_2(n-2,\delta-1)$. From the assumption (15) it follows that $II + III \leq 2M_2(n-2,\delta-1)$ and the proof can be completed as in the previous case.

Now we prove (4) for $q = 4$. The case $n = 1$ is settled by inspection. We assume that $J = \{0,1\},\ J^c = \{2,3\}$ and consider the following scheme (we omit index t in the notations $A_i^t,\ B_i^t$):

For $q = 4$ inequality (13) can be written in the form

$$\begin{aligned} |A||B| \leq \ & \frac{3n}{2\delta}((|A_1| + |A_2|)(|B_3| + |B_4|) \\ & + (|A_3| + |A_4|)(|B_1| + |B_2|)) = \frac{3n}{2\delta}(II + III), \end{aligned} \tag{16}$$

where

$$\begin{aligned} I &= (|A_1| + |A_2|)(|B_1| + |B_2|), \\ II &= (|A_3| + |A_4|)(|B_1| + |B_2|), \\ III &= (|A_1| + |A_2|)(|B_3| + |B_4|), \\ IV &= (|A_3| + |A_4|)(|B_3| + |B_4|). \end{aligned}$$

Now we proceed as in the proof of the case $q = 2$ with substitutions $11 \to 1,\ 22 \to 2,\ 12 \to 3,\ 21 \to 4,\ A_{11} \to A_1,\ B_{11} \to B_1$, etc.; $F_2(n,\delta) \to F_4(n,\delta)$, (14)$\to$ (16). Repeating all arguments from the previous proof and taking into account that $\bar{q} = 4$, we are done with the case $q = 4$.

Now let $q = 5$. We need one simple preliminary result, which we state in the forthcoming Lemma 16, whose proof we leave to the reader (Exercise 12). For simplicity we again omit the index t in the notations A_i^t, B_i^t. Define the numbers r, s, p by

$$\begin{aligned} r &= |\{1 \leq i \leq q :\ |A_i||B_i| > 0\}|, \\ s &= |\{1 \leq i \leq q :\ |A_i| > 0\}| - r, \\ p &= |\{1 \leq i \leq q :\ |B_i| > 0\}| - r. \end{aligned}$$

After relabeling we have $|A_i||B_i| > 0$ for $1 \leq i \leq r$, $|A_i| > 0$ for $1 \leq i \leq r+s$, $|B_i| > 0$ for $1 \leq i \leq r$, and $r+s+1 \leq i \leq r+s+p$.

Lemma 16 *Let $n \geq 2$. If $(A,B) \in S_q(n,\delta)$ and $r+s, r+p \geq 2$, then for $1 \leq i \leq r$, $1 \leq j \leq q$, $i \neq j$ we have $A_i \cap A_j = \emptyset$ and $B_i \cap B_j = \emptyset$.*

Denote

$$X = \{1,\ldots,r\},\ Y = \{r+1,\ldots,r+s\},$$
$$Z = \{r+s+1,\ldots,r+s+p\}.$$

It is easy to see that if we replace $A_i,\ i \in Y$ by $E = \bigcup_{i\in Y} A_i$ and $B_i,\ i \in Z$ by $F = \bigcup_{i\in Z} B_i$, then we again obtain a pair in $S_q(n,\delta)$. Note also that if $s+p \neq 0$ we can enlarge Y or Z so that $r+s+p = q$.

Denote

$$e = \sum_{J\in\mathcal{J}_q} |A(J)||B(J^c)|.$$

For $J \in \mathcal{J}_q$ we define

$$U = J\cap X,\ V = J\cap Y,\ W = J\cap Z,\ E = \bigcup_{i\in Y} A_i,\ F = \bigcup_{i\in Z} B_i,$$
$$a(J) = \sum_{i\in J} |A_i|,\ b(J) = \sum_{i\in J} |B_i|,\ J \subset [q].$$

If $s+p = 0$, then

$$e = \sum_{U\subset X,\ 1\leq|U|\leq\min\{\beta k-1\}} a(U)b(X\setminus U)\binom{q-r}{\beta-|U|}. \tag{17}$$

If $r+s+p = q$, then

$$e = \sum_{U\subset X,\ V\subset Y,\ W\subset Z,\ |U|+|V|+|W|=\beta} (a(U)+|V||E|)(b(X\setminus U)+|Z\setminus W||F|).$$

Opening the brackets on the RHS of the expression for e we obtain four sums

$$\begin{aligned}
e_1 &= \sum_{U\subset X,\ V\subset Y,\ W\subset Z,\ |U|+|V|+|W|=\beta} a(U)b(X\setminus U) \\
&= \sum_{U\subset X,\ell\ ,|U|+\ell=2} \binom{s+p}{\ell} a(U)b(x\setminus U) \\
&= \sum_{U\subset X,\ 1\leq|U|\leq\min\{\beta,r-1\}} \binom{q-r}{\beta-|U|} a(U)b(X\setminus U), \\
e_2 &= \sum_{U\subset X,\ V\subset Y,\ W\subset Z,\ |U|+|V|+|W|=\beta} a(U)|Z\setminus W||F| \\
&= \sum_{U\subset X,\ |U|+|V|+|W|=\beta} \binom{s}{|V|}\binom{p}{|W|}(p-|W|)a(U)|F|
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{U\subset X,\ 1\le |U|\le \beta} \binom{q-k-1}{\beta-|U|} pa(U)|F|,\\
e_3 &= \sum_{U\subset X,\ V\subset Y,\ W\subset Z,\ |U|+|V|+|W|=\beta} b(X\setminus U)|V||E|\\
&= \sum_{U\subset X,\ |U|+|V|+|W|=\beta} \binom{s}{|V|}\binom{p}{|W|}|V|b(X\setminus U)|E|\\
&= \sum_{U\subset X, 1\le |U|\le \min\{\beta,r-1\}} \binom{q-r-1}{\beta-|U|-1} b(X\setminus U)s|E|,\\
e_4 &= \sum_{U\subset X,\ V\subset Y,\ W\subset Z,\ |U|+|V|+|W|=\beta} |V||E||Z\setminus W||F|\\
&= \sum_{U\subset X,\ |U|+|V|+|W|=\beta} \binom{s}{|V|}\binom{p}{|W|}|V||Z\setminus W||E||F|\\
&= \sum_{U\subset X,\ |U|\le \beta-1} \binom{q-r-2}{\beta-|U|-1} sp|E||F|.
\end{aligned}
$$

Note that in the case $s+p=0$ we obtain the same final relations for e_i and e. Now by (8) and Lemma 15, in the case $q=5$ the relation

$$
e \le F_5(n-1,\delta-1)12 \tag{18}
$$

is sufficient for induction to work. To prove this inequality we go through the cases defined by the value of r.
$\mathbf{r=5}$. Since $s=p=0$, we have $e_2=e_3=e_4=0$. Therefore,

$$
e = e_1 = \sum_{U\subset[5],\ 1\le|U|\le 2} \binom{5-5}{2-|U|} a(U)b(X\setminus U).
$$

As $\left(\bigcup_{i\in U} A_i, \bigcup_{i\in X\setminus U} B_i\right) = S_5(n-1,\delta-1)$ and by Lemma 16

$$
\left|\bigcup_{i\in U} A_i\right| = a(U),\ \left|\bigcup_{i\in X\setminus U} B_i\right| = b(X\setminus U),
$$

we conclude by using the induction hypothesis that

$$
e \le \binom{5}{2} M_2(n-1,\delta-1) \le 10F_5(n-2,\delta-1).
$$

$\mathbf{r=4}$. In this case either $s=1, p=0$ or $s=0, p=1$ holds. By symmetry, it suffices to consider only the first case. Then $e_2=e_4=0$ and

$$\begin{aligned} e = e_1 + e_2 = & \sum_{U \subset [4],\ 1 \le |U| \le 2} \binom{2}{2-|U|} a(U) b(X \setminus U) \\ & + \sum_{U \subset [4],\ |U| \le 2} \binom{0}{2-|U|-1} b(X \setminus U) s|E| \\ & \le \binom{4}{2} F_5(n-1,\delta-1) + \sum_{U \subset [4], |U|=1} (a(U)+|E|) b(X \setminus U). \end{aligned}$$

By Lemma 16 and the induction hypothesis the second summand is smaller than $4F_5(n-1,\delta-1)$ and therefore $e \le 10F_5(n-1,\delta-1)$.
$\mathbf{r=3}$. Here we have

$$\begin{aligned} e_1 &= \sum_{U \subset [3],\ 1 \le |U| \le 2} \binom{2}{2-|U|} a(U) b(X \setminus U) \\ &= 2(|A_1|(B_2|+|B_3|)|A_2|(|B_1|+|B_3|)+|A_3|(|B_1|+|B_2|)) \\ &+ ((|A_1|+|A_2|)|B_3|+(|A_1|+|A_3|)|B_2|+(|A_2|+|A_3|)|B_1|), \\ e_2 &= 3(|A_1|+|A_2|+|A_3|)p|F|, \\ e_3 &= 3(|B_1|+|B_2|+|B_3|)s|E|, \\ e_4 &= 3sp|E||F|. \end{aligned}$$

Now we consider a few subcases.
$\mathbf{s=2}$, $\mathbf{p=0}$. Then

$$\begin{aligned} e &= e_1+e_2 = 3(|B_1|+|B_2|)(|A_3|+|E|) \\ &+ 3(|B_1|+|B_3|)(|A_2|+|E|)+3(|B_2|+|B_3|)(|A_1|+|E|). \end{aligned}$$

Since $(B_1 \cup B_2, A_3 \cup E) \in S_5(n-1,\delta-1)$, we have $3(|B_1|+|B_2|)(|A_3|+|E|) \le F_5(n-1,\delta-1)$. The remaining terms in the expression for e are estimated in the same manner. Thus we have $e \le 9F_5(n-1,\delta-1)$.
$\mathbf{s=1}, \mathbf{p=1}$. Then

$$\begin{aligned} e &= e_1+e_2+e_3+e_4, \\ e_2 &= 3(|A_1|+|A_2|+|A_3|)|F|,\ e_3 = 3(|B_1|+|B_2|+|B_3|)|E|,\ e_4 = 3|E||F| \end{aligned}$$

and e_1 has the same expression as in the previous subcase. We can assume that $e_2 \le e_3$, because otherwise we can exchange the roles of A and B. Thus, by the previous subcase, $e_1+e_2+e_3 \le 9F_5(n-1,\delta-1)$, and since $e_4 \le 3F_5(n-1,\delta-1)$, we conclude $e \le 12F_5(n-1,\delta-1)$.
$\mathbf{s=0}, \mathbf{p=2}$. Since e_1 and e_2 are symmetric in A and B, replacement of e_3 by e_2 in the case $s=2, t=0$ gives again the bound $e \le 9F_5(n-1,\delta-1)$.

$\mathbf{r = 2}$. In this case

$$
\begin{aligned}
e_1 &= \sum_{U \subset [2], |U|=1} \binom{3}{|U|} a(U) b(X \setminus U) = 3(|A_1||B_2| + |A_2||B_1|),\\
e_2 &= \sum_{U \subset [2],\ 1 \le |U| \le 2} \binom{2}{2-|U|} a(U) p|F| = 3(|A_1| + |A_2|) p|F|,\\
e_3 &= \sum_{U \subset [2],\ |U| \le 1} \binom{2}{1-|U|} b(X \setminus U) s|E| = 3(|B_1| + |B_2|) s|E|,\\
e_4 &= \sum_{U \subset [2],\ |U| \le 1} \binom{1}{1-|U|} sp|E||F| = 3sp|E||F|.
\end{aligned}
$$

Here also we have some subcases.
$\mathbf{s = 3, p = 0}$. Then

$$
\begin{aligned}
e &= e_1 + e_2 = 3(|B_1|(|A_2| + |E|)\\
&+ 3|B_2|(|A_1| + |E|) + 6(|B_1| + |B_2|)|E|) \le 12F_5(n-1, \delta - 1).
\end{aligned}
$$

$\mathbf{s = 2, p = 1}$. Then

$$
\begin{aligned}
e &= e_1 + e_2 + e_3 + e_4 = \left(e_1 + e_2 + \frac{1}{2}e_3 + e_4\right) + \frac{1}{2}e_3\\
&= 3(|B_1| + |F|)(|A_2| + |E|) + 3(|B_2| + |F|)(|A_1| + |E|)\\
&+ 3(|B_1| + |B_2|)|E| \le 9F_5(n-1, \delta - 1).
\end{aligned}
$$

The other subcases are symmetrically the same.
$\mathbf{r = 1}$. In this case we can write

$$
\begin{aligned}
e_1 &= 0,\ e_2 = \binom{5-1}{2-1} |A_1||F|,\\
e_3 &= \binom{5-1}{2-1} |B_1| s|E|,\\
e_4 &= \binom{5-3}{2-1} sp|E||F| \binom{5-3}{2-2} sp|E||F| = 5q - 2\beta - 1sp|E||F|
\end{aligned}
$$

and

$$
e = 3(|A_1|p|F| + |B_1|s|E| + s|E|p|F|).
$$

But

$$
\begin{aligned}
\lambda &= |A_1|p|F| + |B_1|s|E| + s|E|p|F| = s|E|(|B_1| + |F|) + (|A_1| + |E|)p|F|\\
&+ (s-1)p|E||F| - s|E||F|\\
&= s|E(|B_1| + |F|) + (|A_1| + |E|)p|F| + (sp - s - p)|E||F|.
\end{aligned}
$$

As $(E,(B_1 \cup F)),\ (A_1 \cup E, F),\ (E,F) \in S_5(n-1,\delta-1)$, the induction hypothesis gives

$$\lambda \leq (s+p+(sp-s-p))F_5(n-1,\delta-1) \leq 4F_5(n-1,\delta-1)$$

and therefore $e \leq 12F_5(n-1,\delta-1)$.

$\mathbf{r=0}$. We have

$$|A||B| = sp|E||F| \leq 6|E||F| \leq 6F_5(n-1,\delta-1).$$

This completes the proof of the theorem. ∎

§2 Four-Words Property

We formulate a generalization of the property of the pair (A,B) to be a constant distance pair. We say that the pair of sets $(A,B),\ A,B \subset \mathcal{X}_q^n,\ \mathcal{X}_q = [q]$ satisfies the four-words property (4-WP) if

$$d_H(a^n,b^n) - d_H(a^n,b'^n) + d_H(a'^n,b'^n) - d_H(a'^n,b^n) \neq 1,2$$

for all $a^n, a'^n \in A,\ b^n, b'^n \in B$.

Proposition 10 *If a pair* (A,B) *satisfies the* 4-*WP, then*

$$|A||B| \leq q^{*n},\ q^* = \begin{cases} q, & q=2,3,4, \\ \bar{q} = \left\lfloor \frac{q}{2} \right\rfloor \cdot \left\lceil \frac{q}{2} \right\rceil, & q \geq 4 \end{cases}$$

and this bound is best.

Next we consider a further generalization of the 4-WP and prove Theorem 12 below, from which also follows the statement of Proposition 10.

Let $\mathcal{X}$ and $\mathcal{Y}$ be two finite sets. We consider the function

$$f:\ \mathcal{X} \times \mathcal{Y} \to \mathbb{Z}.$$

With f we associate the sum-type function $f_n:\ \mathcal{X}^n \times \mathcal{Y}^n \to \mathbb{Z}$:

$$f_n(x^n,y^n) = \sum_{i=1}^{n} f(x_i,y_i),$$

$x^n = (x_1,\dots,x_n) \in \mathcal{X}^n,\ y^n = (y_1,\dots,y_n) \in \mathcal{Y}^n$.

We say that the pair (A,B) with $A \subset \mathcal{X}^n,\ B \subset \mathcal{Y}^n$ satisfies the $\mathcal{R}-$four-word property ($\mathcal{R}-$4-WP), if

$$f_n(a^n,b^n) - f_n(a^n,b'^n) + f_n(a'^n,b'^n) - f_n(a'^n,b^n) \in \mathcal{R}, \tag{19}$$

for all $a^n, a'^n \in A$, $b^n, b'^n \in B$. Let $\mathcal{P}(f,\mathcal{R},n)$ be the set of all those pairs. We are interested in

$$M(f,\mathcal{R},n) = \max\{|A||B| : (A,B) \in \mathcal{P}(f,\mathcal{R},n)\}.$$

Let $\mathcal{P}^*(f,\mathcal{R},n)$ be the set of those pairs in $\mathcal{P}(f,\mathcal{R},n)$ on which the maximum $M(f,\mathcal{R},n)$ is achieved. The following theorem is the basis in the study of the $\mathcal{R}-4$-WP [ACZ89].

Theorem 12 (Ahlswede, Cai, and Zhang 1989) *For any* $\mathcal{R} \subset \mathbb{Z}$

$$M(f,\mathcal{R},n) \leq M^n(f,\mathcal{R},1). \tag{20}$$

Furthermore, if $0 \in \mathcal{R}$ *and* $M(f,\{0\},1) = M(f,\mathcal{R},1)$, *then equality holds in (20).*

The proof of this theorem proceeds by induction on n and is based on two simple lemmas, which we first state and prove.

For the set C of sequences of length n from some finite alphabet denote

$$\begin{aligned} C_c &= \{(c_1,\dots,c_{n-1}) : (c_1,\dots,c_{n-1},c) \in C\}, \\ J(C) &= \{c : C_c \neq \emptyset\}, \\ L(C) &= \max\left\{|D| : D \in J(C), \bigcap_{c \in D} C_c \neq \emptyset\right\}. \end{aligned}$$

Lemma 17 *For* $(A,B) \in \mathcal{P}(f,\mathcal{R},n)$ *we have* $L(A)|J(B)| \leq M(f,\mathcal{R},1)$.

Proof. It suffices to show that for every $D \subset J(A)$ with $\bigcap_{a \in D} A_a \neq \emptyset$ necessarily $(D, J(B)) \in \mathcal{P}(f,\mathcal{R},1)$.

To see this choose $a, a' \in D$, $b, b' \in J(B)$ and note that by assumptions there are a^{n-1}, b^{n-1}, b'^{n-1} such that $a^{n-1}a$, $a^{n-1}a' \in A$, and $b^{n-1}b, b'^{n-1}b' \in B$. Now obviously

$$\begin{aligned} \mathcal{R} \ni\ & f_n(a^{n-1}a, b^{n-1}b) - f_n(a^{n-1}a, b'^{n-1}b') \\ &+ f_n(a^{n-1}a', b'^{n-1}b') - f_n(a^{n-1}a', b^{n-1}b) \\ &= f(a,b) - f(a,b') + f(a',b') - f(a',b). \end{aligned}$$

□

Lemma 18 *If* $(A,B) \in \mathcal{P}(f,\mathcal{R},n)$, *then* $\left(\bigcup_{d \in J(A)} A_d, B_b\right) \in \mathcal{P}(f,\mathcal{R},n-1)$ *for all* $b \in J(B)$.

Proof. For $a^{n-1}, a'^{n-1} \in \bigcup_{d \in J(A)} A_d$ choose $a, a' \in J(A)$ such that $a^{n-1} \in A_a$, $a'^{n-1}\ A_{a'}$. Now for any $b^{n-1}, b'^{n-1} \in B_b$,

$$\begin{aligned} \mathcal{R} \ni\ & f_n(a^{n-1}a, b^{n-1}b) - f_n(a^{n-1}a, b'^{n-1}b) \\ &+ f_n(a'^{n-1}a', b'^{n-1}b) - f_n(a'^{n-1}a', b^{n-1}b) \end{aligned}$$

$$= f_{n-1}(a^{n-1}, b^{n-1}) - f_{n-1}(a^{n-1}, b'^{n-1}) + f_{n-1}(a'^{n-1}, b'^{n-1}) - f_{n-1}(a'^{n-1}, b^{n-1}).$$

□

Proof of Theorem 12. Obviously, if for $(A,B) \in \mathcal{P}^*(f,\{0\},1)$ we have $|A||B| = M(f,\mathcal{R},1)$, then $(\prod_{i=1}^n A, \prod_{i=1}^n B) \in \mathcal{P}(f,\{0\},n)$, where $\prod_{i=1}^n C$ is the set of n-tuples of elements from C. Therefore, if $0 \in \mathcal{R}$, then $M(f,\mathcal{R},n) \geq (|A||B|)^n = M^n(f,\mathcal{R},1)$.

To prove (20) we use induction. For $n = 1$ nothing needs to be proved. For $(A,B) \in \mathcal{P}(f,\mathcal{R},n)$ we have

$$\begin{aligned} |A||B| &= \sum_{a \in J(A)} |A_a| \sum_{b \in J(B)} |B_b| \\ &\leq L(A) \left| \bigcup_{a \in J(A)} A_a \right| |J(B)| \max_{b \in J(B)} |B_b| \\ &\leq M(f,\mathcal{R},1) \left| \bigcup_{a \in J(A)} A_a \right| \max_{b \in J(B)} |B_b|. \end{aligned}$$

The last inequality here follows from Lemma 17. The result $|A||B| \leq M^n(f,\mathcal{R},1)$ now follows from Lemma 18 and the induction hypothesis. ■

From this Theorem follows Proposition 10. Indeed the 4-WP means that $(A,B) \in \mathcal{P}(d_H, \mathbb{Z} - \{1,2\}, n)$. We have $\mathcal{P}(d_H, \mathbb{Z} - \{1,2\}, 1) = \mathcal{P}(d_H, \{0\}, 1)$ and therefore $M(d_H, \mathbb{Z} - \{1,2\}, n) = M^n(d_H, \{0\}, 1)$. Finally, equality $M(d_H, \{0\}, 1) = q^*$ is easily verified.

The following fact easily follows from Theorem 12 (Exercise 13). If $A, B \subset [0, q-1]^n$ and the set $[0, q-1]^n$ is equipped with a Lee metric d_L, which is defined as follows:

$$d_L(x^n, y^n) = \sum_{i=1}^n \min\{|x_i - y_i|, q - |x_i - y_i|\},$$

and for all $a^n, a'^n \in A,\ b^n, b'^n \in B$

$$d_L(a^n, b^n) - d_L(a^n, b'^n) + d_L(a'^n, b'^n) - d_L(a'^n, b^n) \neq 1, 2, \ldots q, \tag{21}$$

then

$$|A||B| \leq \left(\max\left\{ q, \left(\left\lfloor \frac{q}{4} \right\rfloor + 1 \right) \left(\left\lceil \frac{\lfloor q/2 \rfloor}{2} \right\rceil + 1 \right) \right\} \right)^n. \tag{22}$$

The next fact is also the consequence of Theorem 12 (Exercise 13). Let d_T be a Taxi metric on $[0, q-1]^n$. If $A, B \subset [0, q-1]^n$ and for $a^n, a'^n \in A,\ b^n, b'^n \in B$,

$$d_T(a^n, b^n) - d_T(a^n, b'^n) + d_T(a'^n, b'^n) - d_T(a'^n, b^n) \neq 1, 2, \ldots 2q,$$

then

$$|A||B| \leq \left(\max\left\{q, \left(\left\lfloor\frac{q}{2}\right\rfloor + 1\right)\left\lceil\frac{q}{2}\right\rceil\right\}\right)^n \tag{23}$$

and this bound is best possible.

Notes to Chapter II

As already said, Theorems 6, 7, and 8 completely solve the problem of determining the maximal sets of given diameter in the Taxi metric when all components $\mathcal{X}_j$ have even or odd lengths. In the mixed case, when some of the components have even length and some odd length, in general some partial results are known when $d < b(\mathcal{B})$ and $d \geq b(\mathcal{B}) + e(\mathcal{B}) - 1$, where $e(\mathcal{B})$ is the number of components with odd q_i's. In this case it is known that the ball of radius $d/2$ with some center in L^1 is a maximal set of diameter d. The proof here is the same as for all-even q_i's in Theorem 8, but the splitting of $\mathcal{B}$ is different. For details about such splittings we refer to [ACZ92a] and [DK90], see also [KF88]. In [BL93] a direct approach was used to the diametric problem in Taxi metric. A complete solution was presented for the problem in the space $\mathcal{B}$, where all q_i are equal. There the diametric problem on the torus also has been considered. Relation (5) (Lecture 2) was first proved in [K66a].

Theorem 3 was proved in [AK97b] and Theorem 5 was proved in [AK98]. We reproduced here their proofs. The Intersection Theorem 4 was first proved by Katona by using another method in [K64]. Relation (3) (Lecture 1) for $t \geq 15$ was first established by Frankl [F78] and subsequently by Wilson [W84] for all t. We took the proof of Proposition 9 from [H64]. Theorems 6 and 7 were proved in [ACZ92a]. Theorem 9 is taken from [AK00b].

An $\mathcal{A} \in \mathcal{I}(n,k,t)$ is called *nontrivial* if $\left|\bigcap_{A\in\mathcal{A}} A\right| < t$ and $\tilde{\mathcal{I}}(n,k,t)$ denotes all nontrivial families from $\mathcal{I}(n,k,t)$. Let

$$\tilde{M}(n,k,t) = \max_{\mathcal{A}\in\tilde{\mathcal{I}}(n,k,t)} |\mathcal{A}|, \ 1 \leq t \leq k \leq n.$$

Let also

$$\begin{aligned}\mathcal{V}_1(n,k,t) = \Bigg\{V \in \binom{[n]}{k} :\ [1,t] \subset V,\\ V \cap [1+t, k+1] \neq \emptyset\Bigg\} \cup \{[1,k+1] \setminus \{i\} :\ i \in [1,t]\}.\end{aligned}$$

In [AK96d] the following equalities are proved, which give the complete solution of the determination of the maximal cardinality of a nontrivial family. This settles the Hilton-Milner problem, whose investigation was initiated in [HM67b].

(i) If $2k-t<n\leq (t+1)(k-t+1)$, then

$$\tilde{M}(n,k,t)=M(n,k,t).$$

(ii) If $(t+1)(k-t+1)<n$ and $k\leq 2t+1$, then

$$\tilde{M}(n,k,t)=|\mathcal{F}(1)|$$

and $\mathcal{F}(1)$ is – up to permutations – the unique optimum.

(iii) If $(t+1)(k-t+1)<n$ and $k>2t+1$, then

$$\tilde{M}(n,k,t)=\max\{|\mathcal{F}(1)|,|\mathcal{V}_1|\},$$

and – up to permutations – $\mathcal{F}(1)$ or $\mathcal{V}_1$ are the only solutions.

Consider the following sets:

$$\begin{aligned}
\binom{[n]}{\geq k} &= \bigcup_{i=k}^{n}\binom{[n]}{i},\\
\binom{[n]}{\leq k} &= \bigcup_{i=0}^{k}\binom{[n]}{i},\\
\mathcal{I}(n,\geq k,t) &= \mathcal{I}(n,t)\cap 2^{\binom{[n]}{\geq k}},\\
\mathcal{I}(n,\leq k,t) &= \mathcal{I}(n,t)\cap 2^{\binom{[n]}{\leq k}},\\
\mathcal{F}(i,\geq k) &= \mathcal{G}(i)\cap\binom{[n]}{\geq k},\\
\mathcal{F}(i,\leq k) &= \mathcal{G}(i)\cap\binom{[n]}{\leq k}, i=0,\dots\left\lfloor\frac{n-t}{2}\right\rfloor.
\end{aligned}$$

The description of the following results can be found in [ABEK02]. Using Katona's Theorem 4 it is not difficult to prove that

$$\max_{\mathcal{A}\in\mathcal{I}(n,t,\geq k)}|\mathcal{A}|=\left|\mathcal{F}\left(\left\lfloor\frac{n-t}{2}\right\rfloor,\geq k\right)\right|.$$

The problem of determination of the value $\max_{\mathcal{A}\in\mathcal{I}(n,t,\leq k)}|\mathcal{A}|$ is still open. In Research Problem 3 at the end of the chapter the corresponding conjecture is formulated.

In [AAK98], Ahlswede et al. consider the problem of maximal intersecting systems for direct products. This problem was initiated by Frankl and arose in connection with a result of Sali. Let $n=n_1+\cdots+n_m$, $k=k_1+\cdots+k_m$, $[n]=[n_1]\cup[n_2]\cdots\cup[n_m]$, $\mathcal{H}=\left\{F\in\binom{[n]}{k}:|F\cap[n_i]|=k_i \text{ for } i=1,\dots,m\right\}$. For given integers t_i, $1\leq t\leq t_i\leq k_i$, $1\leq i\leq m$, we may say that $\mathcal{A}\subset\mathcal{H}$ is $(t_1,\dots,t_m)$-intersecting, if for every $A,B\in\mathcal{A}$ there exists an i, $1\leq i\leq m$, such that $|A\cap B\cap\Omega_i|\geq t_i$ holds.

Denote the set of such systems by $I(\mathcal{H}, t_1, \ldots, t_m)$. The problem is to determine $\max_{\mathcal{A} \in I(\mathcal{H}, t_1, \ldots, t_m)} |\mathcal{A}|$.

The case $t_1 = t_2 = \cdots = t_m = 1$ has been solved by Frankl. Here is the complete solution.

Theorem 13 (Ahlswede, Aydinian, and Khachatrian 1998) *Let $n_i \geq k_i \geq t_i \geq 1$ for $i = 1, \ldots, m$, then*

$$\max_{\mathcal{A} \in I(\mathcal{H}, t_1, \ldots, t_m)} |\mathcal{A}| = \max_i \frac{M(n_i, k_i, t_i)}{\binom{n_i}{k_i}} |\mathcal{H}|.$$

We emphasize that the combination of this Theorem and Theorem 3 gives an explicit value. The proof is heavily (but not only!) based on ideas and methods from [A96], in particular the method of "generated sets" (c.f. [N] Bey/Engel, "Old and New Results for the Weighted t-Intersection Problem via AK-Methods", 45-74;) takes a central role in the book [E97b].

We took Theorem 10 from [AC91]. Also the following relations were proved there:

$$\begin{aligned} f_q^{q-1} &= \frac{1}{2}|W_q^{q-1}| = \frac{1}{2}q!, \\ f_q^3 &= 12,\ q \geq 4, \\ g_q^3 &= 3q + 7,\ 3 \leq q < \infty. \end{aligned}$$

We took Theorem 11 from [A87].

For the matrix with entries $a_{ij} = d(x_i^n, y_j^n)$, where x_i^n are n-tuples with elements from some finite set $\mathcal{X}$ and $d(\cdot,\cdot)$ is a metric on $\mathcal{X}^n$, consider the area $i \cdot j$ of an $i \times j$ minors with constant entries. This concept was introduced in [Y79] for estimating communication complexity. It inspired the work reported in Lecture 6.

Proposition 10 was first proved in [AM88]. Inequality (22) was first proved in [C86]. Theorem 12 was proved in [ACZ89].

A pair $(A,B),\ A,B \subset \{0,1\}^n$ is said to be ℓ-cross-intersecting iff $\lambda(a^n, b^n) = \sum_{i=1}^n \min\{a_i, b_i\} = \ell$ for all $a^n \in A,\ b^n \in B$. If one considers a^n, b^n as subsets of $[n]$, then $\lambda(a^n, b^n)$ is their intersection. How large can $|A||B|$ be? A simple construction in [ACZ89] gives a lower bound stated in Exercise 14. Moreover, it is conjectured that the construction is best possible. In [AL06] this conjecture is proved for sufficiently large $\ell > \ell_0$.

Exercises

1. Erdös, Ko, and Rado [EKR61] proved the Theorem 3 for the case $t = 1$. Give a proof using the Kruskal/Katona Theorem ([K63],[K68],[D74]). A formulation of the result of [K63] and [K68] can be found without proof in [S59].
2. Give another proof with Katona's cycle method ([K72], see also the book [N] with the survey [K00]).

3. Prove relation (4) (Lecture 1).
4. Prove Lemma 4 and Proposition 1.
5. Determine all optimal unrestricted t-intersecting families for $t = 1$. *Hint:* Among them is always $\mathcal{A} = \{A \in 2^{[n]} : \{1\} \in A\}$, $\mathcal{A} = \bigcup_{i \geq \frac{n+1}{2}} \binom{[n]}{i}$ for n odd, and $\mathcal{A} = \bigcup_{i \geq \frac{n}{2}} \binom{[n-1]}{i}$ for n even.
6. For $t = 1$ and $q = 2$ find all optimal configurations for the Hamming distance problem. *Hint:* see considerations before Lemma 6.
7. One can see that $M(n,t) = N_2(n,t)$. Using operations T_{ji}, relation (8), and the method of the proof of Theorem 4, prove the validity of (5) (Lecture 2).
   For $t > 1$ prove that the set on which $N_2(n,t)$ is achieved is unique up to changing $0 \leftrightarrow 1$ symbols in components and permutations of components.
8. Prove equality (4) (Lecture 2) directly. Consider *mod q* componentwise summation in $\mathcal{H}_q^n$ and prove that if a^n is in intersection family $\mathcal{A}$, then $a^n + b^n \notin \mathcal{A}$ for all $b^n = (b, \ldots, b)$, $b \in \{1, \ldots, q-1\}$.
9. Prove that for $t > 1$ or $t = 1$, $q > 2$ up to permutations of the components and elements of the alphabet in the components there is only one optimal configuration in Theorem 5, unless $t > 1$, $t + 2(t-1)/(q-1) \leq n$, and $(t-1)/(q-2)$ is an integer in which case we have two optimal configurations $\mathcal{K}\left(\frac{t-1}{q-2}\right)$ and $\mathcal{K}\left(\frac{t-1}{q-2} - 1\right)$.
   In addition to the optimal configuration in Theorem 9 we have in the case $d = n-2$, $2|d$ also the optimal configuration $\mathcal{G}_{d/2-1}(n, n-d)$. Prove that up to permutations of the components and elements of the alphabet in the components these configurations are unique.
10. Prove Propositions 2, 3, 4, and 5.
11. Prove relation (18) (Lecture 2): if $E_1, E_2 \in \mathcal{M}_0(\mathcal{A})$ and $|E_1 \cap E_2| = t$, then $|E_1| + |E_2| = \ell + t$.
12. Prove Lemma 16.
13. Using Theorem 12 prove inequalities (22) and (23).
14. Give a construction of an l-cross-intersecting pair (A, B), $A, B \subset \{0,1\}^n$ with
$$|A||B| \geq \begin{cases} \binom{2l}{l} 2^{n-2l} & \text{if } n \geq 2l, \\ \binom{n}{l} & \text{if } n < 2l. \end{cases}$$
15. Actually, it was originally conjectured in [A87] that
$$\max_{\substack{A,B \subset \{0,1\}^n \\ l\text{-cross-inters.}}} |A||B| = \max_{l \leq x \leq n} 2^{n-x} \binom{x}{l}.$$
   Show that this bound equals the bound in exercise 14, which was conjectured in [ACZ89].

16. For $B \subset \{0,1\}^n,\ \mathcal{X}^t = \{0,1\}$, and $X = (x_1 < x_2 < \ldots, < x_{|X|}) \subset [n]$, we say that B has parity on X if for all $b^n \in B$, $\|X|$-tuples $b^{|X|} = (b_{x_1}, \ldots, b_{x_{|X|}})$ have number of units of the same parity. Prove that [A87]

$$\sum_{X \subset [n],\ B \text{ has parity on } X} 2^{|X|}|B| \leq (2^n+1)2^{n-1}.$$

This bound achieves equality, for instance, on the set B of all n-tuples with even number of ones.

Research Problems

1. **Conjecture** The lower bound in Exercise 14 is the maximum value for $|A||B|$. This was proved in [AL06] for large n.
2. **Conjecture** Theorem 11 holds also for values of q different from $2,4,5$.
3. **Conjecture** If $k \leq \frac{n+t}{2}$, then the following relation is valid

$$\max_{\mathcal{A} \in \mathcal{I}(n,t,\leq k)} |\mathcal{A}| = \max\left\{|\mathcal{F}(i, \leq k)| :\ i = 0, \ldots, \left\lfloor \frac{n-t}{2} \right\rfloor \right\}.$$

http://www.springer.com/978-3-540-78601-6

Lectures on Advances in Combinatorics
Ahlswede, R.; Blinovsky, V.
2008, XIV, 318 p. 3 illus., Softcover
ISBN: 978-3-540-78601-6