2

The Constitutive Equations of Masonry-Like Materials

The purpose of this section is to characterize the mechanical behavior of so-called masonry-like or no-tension materials. In particular, we deal with a class of nonlinear elastic materials that are incapable of withstanding tensile stresses and behave like a linear elastic material when subjected to compressive stresses.

To this end, we begin in section 2.1 by assuming that the stress must be negative semidefinite and that the strain is the sum of two parts: the former depends linearly on the stress, the latter is orthogonal to the stress and positive semidefinite. The material is moreover allowed to extend freely in directions of zero stress. Later on this constitutive equation will be generalized in order to take the material's bounded compressive strength into account.

2.1 Masonry-Like Materials with Zero Tensile Strength and Infinite Compressive Strength

We shall denote by $\mathbf{E} = (\nabla\mathbf{u} + \nabla\mathbf{u}^T)/2 \in \mathrm{Sym}$ the infinitesimal strain tensor, with $\mathbf{u}$ the displacement field, and $\mathbf{T} \in \mathrm{Sym}$ the Cauchy stress tensor.

Definition 2.1. *A masonry-like material is an elastic material whose stress function* $\widehat{\mathbf{T}} : Sym \rightarrow Sym$,

$$\mathbf{T} = \widehat{\mathbf{T}}(\mathbf{E}), \quad \mathbf{E} \in Sym \tag{2.1}$$

satisfies the following properties

$$\begin{cases} \mathbf{T} \in \ Sym^-, \\ \mathbf{E} = \mathbf{E}^e + \mathbf{E}^f, \\ \mathbf{E}^f \in \ Sym^+, \\ \mathbf{T} = \mathbb{C}[\mathbf{E}^e], \\ \mathbf{E}^f \cdot \mathbf{T} = 0. \end{cases} \tag{2.2}$$

$\mathbf{E}^e$ and $\mathbf{E}^f$ are respectively the elastic and fracture[1] parts of the strain. As in the linear theory of elasticity, we call $\mathbb{C}$ the elasticity tensor, which is assumed to have both major and minor symmetry (cf. (1.81), (1.83)), and to be positive definite, in the sense that

$$\mathbf{A} \cdot \mathbb{C}[\mathbf{A}] > 0 \quad \text{for each } \mathbf{A} \in \text{ Sym}, \;\; \mathbf{A} \neq \mathbf{0}. \tag{2.3}$$

With the aim of proving the existence and uniqueness of the solution to the constitutive equation (2.2), it is convenient to characterize it further.

For every $\mathbf{E} \in$ Sym, let us consider the problem of finding $\mathbf{T} \in$ Sym$^-$ such that the variational inequality

$$(\mathbf{T} - \mathbf{T}^*) \cdot (\mathbf{E} - \mathbb{C}^{-1}[\mathbf{T}]) \geq 0 \quad \text{for each } \mathbf{T}^* \in \text{ Sym}^- \tag{2.4}$$

holds.

Proposition 2.2. *For every* $\mathbf{E} \in$ *Sym, the triplet* $(\mathbf{T}, \mathbf{E}^e, \mathbf{E}^f)$ *of elements of Sym is a solution to the constitutive equation (2.2) if and only if* $(\mathbf{T}, \mathbf{E}^e, \mathbf{E}^f)$ *satisfies* $(2.2)_2$, $(2.2)_4$, *and* $\mathbf{T}$ *belongs to Sym*$^-$ *and is a solution to the variational inequality (2.4).*

Proof. For $\mathbf{E}^f = \mathbf{E} - \mathbb{C}^{-1}[\mathbf{T}]$, let us prove that the inequality

$$(\mathbf{T} - \mathbf{T}^*) \cdot \mathbf{E}^f \geq 0 \quad \text{for each } \mathbf{T}^* \in \text{ Sym}^- \tag{2.5}$$

is equivalent to the conditions

$$\mathbf{E}^f \in \text{ Sym}^+, \qquad \mathbf{E}^f \cdot \mathbf{T} = 0. \tag{2.6}$$

If conditions (2.6) hold, then, in view of proposition 1.9(3),

$$(\mathbf{T} - \mathbf{T}^*) \cdot \mathbf{E}^f = -\mathbf{T}^* \cdot \mathbf{E}^f \geq 0. \tag{2.7}$$

Vice versa, if (2.5) is satisfied, for $\mathbf{T}^* = \mathbf{0}$, we have $\mathbf{T} \cdot \mathbf{E}^f \geq 0$. Moreover, for $\mathbf{T}^* = 2\mathbf{T}$ we have $\mathbf{T} \cdot \mathbf{E}^f \leq 0$, and $(2.6)_2$ is proved. Thus, (2.5) reduces to the inequality

$$\mathbf{T}^* \cdot \mathbf{E}^f \leq 0 \quad \text{for each } \mathbf{T}^* \in \text{ Sym}^-, \tag{2.8}$$

from which, with the help of proposition 1.9(2), $(2.6)_1$ follows. ■

Proposition 2.3. *For every* $\mathbf{E} \in$ *Sym, there exits a unique* $\mathbf{T} \in$ *Sym*$^-$ *satisfying the variational inequality (2.4).*

Proof. To prove the proposition, first of all let us show that (2.4) characterizes $\mathbf{T}$ as the $\mathbb{C}^{-1}$-orthogonal projection of $\mathbb{C}[\mathbf{E}]$ onto the closed convex set Sym$^-$. To this end, consider the inner product on Sym defined by

[1] In [68] and [70] $\mathbf{E}^f$ was called inelastic part of the strain tensor $\mathbf{E}$ and was denoted by $\mathbf{E}^a$.

$$(\mathbf{A}, \mathbf{B})_{\mathbb{C}^{-1}} = \mathbb{C}^{-1}[\mathbf{A}] \cdot \mathbf{B}. \tag{2.9}$$

In virtue of the minimum norm theorem 1.1, given $\mathbb{C}[\mathbf{E}] \in \mathrm{Sym}$, there is a unique $\mathbf{T} \in \mathrm{Sym}^-$ such that

$$(\mathbf{T} - \mathbf{T}^*, \mathbb{C}[\mathbf{E}] - \mathbf{T})_{\mathbb{C}^{-1}} \geq 0 \quad \text{for each } \mathbf{T}^* \in \ \mathrm{Sym}^-. \tag{2.10}$$

From the definition of the inner product $(\mathbf{A}, \mathbf{B})_{\mathbb{C}^{-1}}$, we obtain $(\mathbf{T}-\mathbf{T}^*, \mathbb{C}[\mathbf{E}]-\mathbf{T})_{\mathbb{C}^{-1}} = (\mathbf{T} - \mathbf{T}^*) \cdot (\mathbf{E} - \mathbb{C}^{-1}[\mathbf{T}])$, from which the thesis follows directly. ■

From the two foregoing propositions, it follows that the solution to constitutive equation (2.2) both exists and is unique.

Proposition 2.4. *For each* $\mathbf{E} \in$ *Sym, there exists a unique triplet* $(\mathbf{T}, \mathbf{E}^e, \mathbf{E}^f)$ *of tensors of Sym that satisfies (2.2).*

The constitutive equation (2.2) and, equivalently (2.4), are the formalization of a natural and intuitive characterization of masonry-like materials. Tensor $\mathbb{C}[\mathbf{E}]$, which for a linear elastic material coincides with the stress tensor associated to strain $\mathbf{E}$, is not generally negative semidefinite. Consequently, it is not an admissible stress for a masonry-like material. We therefore consider the "closest" negative semidefinite tensor to $\mathbb{C}[\mathbf{E}]$, in a suitable norm, and assume it to be the stress associated to $\mathbf{E}$. More precisely, from propositions 2.3 and 2.2 and the minimum norm theorem 1.1, it follows that, given $\mathbf{E} \in \mathrm{Sym}$, the stress $\mathbf{T} = \widehat{\mathbf{T}}(\mathbf{E})$ associated to $\mathbf{E}$ is the tensor of Sym^- having the minimum distance from $\mathbb{C}[\mathbf{E}]$, with respect to the norm induced by the inner product $(\ ,\ )_{\mathbb{C}^{-1}}$ defined in (2.9). That is to say, $\mathbf{T}$ is the projection $P_{\mathrm{Sym}^-}(\mathbb{C}[\mathbf{E}])$ of $\mathbb{C}[\mathbf{E}]$ onto Sym^- with respect to the inner product $(\ ,\ )_{\mathbb{C}^{-1}}$. This characterizes $\mathbf{T}$ as the unique minimum point of the functional $J(\mathbf{T}^*) = \mathbb{C}^{-1}[\mathbb{C}[\mathbf{E}]-\mathbf{T}^*] \cdot (\mathbb{C}[\mathbf{E}]-\mathbf{T}^*)$ defined on Sym^-, whose Euler's inequality is right (2.4).

It should be noted that, in view of inequality (2.5), fracture strain $\mathbf{E}^f$ is an element of the normal cone to Sym^- at $\mathbf{T}$,

$$\mathcal{N}(\mathbf{T}) = \{\mathbf{A} \in \mathrm{Sym} \mid (\mathbf{T} - \mathbf{T}^*) \cdot \mathbf{A} \geq 0 \quad \text{for each } \mathbf{T}^* \in \ \mathrm{Sym}^-\}. \tag{2.11}$$

Since a masonry-like material is characterized by the unilateral constraint on the stress which forces the stress $\mathbf{T}$ to be negative semidefinite, the fracture strain can be interpreted as the reactive part of the strain, satisfying the normality assumption (2.5), as prescribed by the theory of constrained materials [38]. In particular, when $\mathbf{T}$ belongs to the interior of Sym^-, then the normal cone reduces to the null element, $\mathcal{N}(\mathbf{T}) = \{\mathbf{0}\}$, and accordingly we have $\mathbf{E}^f = \mathbf{0}$.

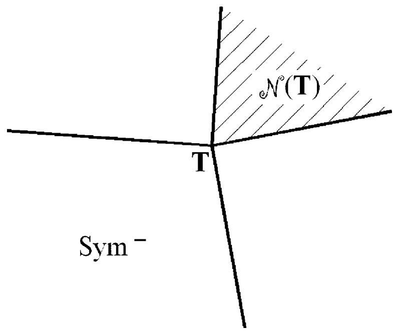

Fig. 2.1. The normal cone to Sym$^-$ at $\mathbf{T}$.

Proposition 2.5. *Tensors* $\mathbf{T}$ *and* $\mathbf{E}^f$ *satisfying (2.2) are coaxial.*

Proof. Since $\mathbf{T} \in \text{Sym}^-$, $\mathbf{E}^f \in \text{Sym}^+$ and $\mathbf{E}^f \cdot \mathbf{T} = 0$, the result follows from propositions 1.8 and 1.9. ■

In particular, in view of proposition 1.9, we have

$$\mathbf{E}^f \mathbf{T} = \mathbf{T}\mathbf{E}^f = \mathbf{0}. \tag{2.12}$$

In addition, note that, since $\mathbf{T} \in \text{Sym}^-$, if $\mathbf{Tn} \cdot \mathbf{n} = 0$, then in view of proposition 1.9, $\mathbf{Tn} = \mathbf{0}$, that is, $\mathbf{n}$ is an eigenvector of $\mathbf{T}$ corresponding to the null eigenvalue.

The following proposition sums up some properties of the nonlinear stress function $\widehat{\mathbf{T}}$ defined by the constitutive equation (2.2), or equivalently, by the variational inequality (2.4) [38].

Proposition 2.6. *Stress function* $\widehat{\mathbf{T}}$ *exhibits the following properties:*

(i) $\widehat{\mathbf{T}}$ *is not injective and therefore not invertible; indeed,*

$$\mathbf{E} \in \mathit{Sym}^+ \quad \textit{if and only if} \quad \widehat{\mathbf{T}}(\mathbf{E}) = \mathbf{0}. \tag{2.13}$$

Moreover

$$\mathbb{C}[\mathbf{E}] \in \mathit{Sym}^- \quad \textit{if and only if} \quad \widehat{\mathbf{T}}(\mathbf{E}) = \mathbb{C}[\mathbf{E}]. \tag{2.14}$$

(ii) $\widehat{\mathbf{T}}$ *is positively homogeneous of degree one,*

$$\widehat{\mathbf{T}}(\beta\mathbf{E}) = \beta\widehat{\mathbf{T}}(\mathbf{E}), \quad \textit{for each } \beta \geq 0,\ \mathbf{E} \in \mathit{Sym}. \tag{2.15}$$

(iii) $\widehat{\mathbf{T}}$ *is monotone,*

$$(\widehat{\mathbf{T}}(\mathbf{E}_1) - \widehat{\mathbf{T}}(\mathbf{E}_2)) \cdot (\mathbf{E}_1 - \mathbf{E}_2) \geq \kappa\ ||\widehat{\mathbf{T}}(\mathbf{E}_1) - \widehat{\mathbf{T}}(\mathbf{E}_2)||^2, \tag{2.16}$$

for each $\mathbf{E}_1, \mathbf{E}_2 \in \mathit{Sym}$*, with*

$$\kappa = \inf_{\mathbf{A}\in \, Sym, \, ||\mathbf{A}||=1} \mathbf{A}\cdot\mathbb{C}^{-1}[\mathbf{A}], \tag{2.17}$$

(iv) $\widehat{\mathbf{T}}$ *is Lipschitz continuous,*

$$||\widehat{\mathbf{T}}(\mathbf{E}_1) - \widehat{\mathbf{T}}(\mathbf{E}_2)|| \leq \kappa^{-1}||\mathbf{E}_1 - \mathbf{E}_2||, \tag{2.18}$$

for each $\mathbf{E}_1, \mathbf{E}_2 \in Sym$*.*

Proof. (i): (2.13) and (2.14) are immediate consequences of (2.2).

(ii): From the variational inequality (2.4), by taking into account that Sym^- is a cone, and putting $\mathbf{T}^{**} = \beta\mathbf{T}^*$, it follows that

$$(\beta\widehat{\mathbf{T}}(\mathbf{E}) - \mathbf{T}^{**})\cdot(\beta\mathbf{E} - \mathbb{C}^{-1}[\beta\widehat{\mathbf{T}}(\mathbf{E})]) \geq 0 \quad \text{for each } \mathbf{T}^{**} \in \ \mathrm{Sym}^-. \tag{2.19}$$

On the other hand, from the definition of $\widehat{\mathbf{T}}$, inequality

$$(\widehat{\mathbf{T}}(\beta\mathbf{E}) - \mathbf{T}^{*})\cdot(\beta\mathbf{E} - \mathbb{C}^{-1}[\widehat{\mathbf{T}}(\beta\mathbf{E})]) \geq 0 \quad \text{for each } \mathbf{T}^{**} \in \ \mathrm{Sym}^- \tag{2.20}$$

follows. By comparing (2.19) and (2.20), and bearing in mind the uniqueness of the solution to (2.4), we get (2.15).

(iii): In order to prove (2.16), let us put $\mathbf{T}_1 = \widehat{\mathbf{T}}(\mathbf{E}_1)$, $\mathbf{T}_2 = \widehat{\mathbf{T}}(\mathbf{E}_2)$. From (2.4) we obtain the following inequalities

$$(\mathbf{T}_1 - \mathbf{T}_2)\cdot(\mathbf{E}_1 - \mathbb{C}^{-1}[\mathbf{T}_1]) \geq 0 \tag{2.21}$$

$$(\mathbf{T}_2 - \mathbf{T}_1)\cdot(\mathbf{E}_2 - \mathbb{C}^{-1}[\mathbf{T}_2]) \geq 0 \tag{2.22}$$

which summed, yield

$$(\mathbf{T}_1 - \mathbf{T}_2)\cdot(\mathbf{E}_1 - \mathbf{E}_2) \geq (\mathbf{T}_1 - \mathbf{T}_2)\cdot\mathbb{C}^{-1}[\mathbf{T}_1 - \mathbf{T}_2]. \tag{2.23}$$

Then, from (2.23) and (2.17), we get (2.16).

(iv): From the property (2.16), by using the Schwarz inequality (1.1), we obtain

$$\begin{aligned} \kappa \, ||\widehat{\mathbf{T}}(\mathbf{E}_1) - \widehat{\mathbf{T}}(\mathbf{E}_2)||^2 &\leq (\widehat{\mathbf{T}}(\mathbf{E}_1) - \widehat{\mathbf{T}}(\mathbf{E}_2))\cdot(\mathbf{E}_1 - \mathbf{E}_2) \\ &\leq ||\widehat{\mathbf{T}}(\mathbf{E}_1) - \widehat{\mathbf{T}}(\mathbf{E}_2)|| \, ||\mathbf{E}_1 - \mathbf{E}_2||, \end{aligned} \tag{2.24}$$

whence, we directly arrive at (2.18). ■

Now, we intend to verify that the material defined by the constitutive equation (2.2) is hyperelastic, that is, there exists a differentiable function ψ, called strain energy density, defined on Sym with values in $\mathbb{R}$ such that its derivative with respect to $\mathbf{E}$ coincides with $\widehat{\mathbf{T}}(\mathbf{E})$. In fact, we shall prove that the function

$$\psi(\mathbf{E}) = \frac{1}{2}\widehat{\mathbf{T}}(\mathbf{E}) \cdot \mathbf{E}, \quad \text{for each } \mathbf{E} \in \ \mathrm{Sym}, \tag{2.25}$$

is a strain energy density. Let us start by proving the following preliminary result [38].

Proposition 2.7. *Let ψ be the function defined in (2.25). The following inequalities hold for any $\mathbf{E}_1, \mathbf{E}_2 \in Sym$*

$$\frac{1}{2}\kappa||\widehat{\mathbf{T}}(\mathbf{E}_1) - \widehat{\mathbf{T}}(\mathbf{E}_2)||^2 \le \psi(\mathbf{E}_1) - \psi(\mathbf{E}_2) - \widehat{\mathbf{T}}(\mathbf{E}_2) \cdot (\mathbf{E}_1 - \mathbf{E}_2)$$
$$\le \frac{1}{2}\kappa^{-1}||\mathbf{E}_1 - \mathbf{E}_2||^2, \tag{2.26}$$

where κ is given in (2.17).

Proof. For $\mathbf{T}_1 = \widehat{\mathbf{T}}(\mathbf{E}_1)$, $\mathbf{T}_2 = \widehat{\mathbf{T}}(\mathbf{E}_2)$, from (2.25) the following identities hold

$$\psi(\mathbf{E}_1) - \psi(\mathbf{E}_2) = \mathbf{T}_2 \cdot (\mathbf{E}_1 - \mathbf{E}_2) - \mathbf{T}_2 \cdot \mathbf{E}_1$$
$$+ \frac{1}{2}\mathbf{T}_1 \cdot \mathbf{E}_1 + \frac{1}{2}\mathbf{T}_2 \cdot \mathbf{E}_2 = \mathbf{T}_2 \cdot (\mathbf{E}_1 - \mathbf{E}_2) - \mathbf{T}_2 \cdot (\mathbb{C}^{-1}[\mathbf{T}_1] + \mathbf{E}_1^f)$$
$$+ \frac{1}{2}\mathbf{T}_1 \cdot \mathbb{C}^{-1}[\mathbf{T}_1] + \frac{1}{2}\mathbf{T}_2 \cdot \mathbb{C}^{-1}[\mathbf{T}_2]$$
$$= \mathbf{T}_2 \cdot (\mathbf{E}_1 - \mathbf{E}_2) + \frac{1}{2}(\mathbf{T}_1 - \mathbf{T}_2) \cdot \mathbb{C}^{-1}[\mathbf{T}_1 - \mathbf{T}_2] - \mathbf{T}_2 \cdot \mathbf{E}_1^f. \tag{2.27}$$

Since $\mathbf{T}_2 \cdot \mathbf{E}_1^f \le 0$, then

$$\psi(\mathbf{E}_1) - \psi(\mathbf{E}_2) - \mathbf{T}_2 \cdot (\mathbf{E}_1 - \mathbf{E}_2) \ge \frac{1}{2}(\mathbf{T}_1 - \mathbf{T}_2) \cdot \mathbb{C}^{-1}[\mathbf{T}_1 - \mathbf{T}_2]. \tag{2.28}$$

The former inequality in (2.26) comes from the definition of κ given in (2.17). In order to prove the latter inequality, we only have to interchange $\mathbf{E}_1$ and $\mathbf{E}_2$ in the former inequality, rewrite it in the form

$$\psi(\mathbf{E}_1) - \psi(\mathbf{E}_2) - \mathbf{T}_2 \cdot (\mathbf{E}_1 - \mathbf{E}_2)$$
$$\le (\mathbf{T}_1 - \mathbf{T}_2) \cdot (\mathbf{E}_1 - \mathbf{E}_2) - \frac{1}{2}\kappa||\mathbf{T}_1 - \mathbf{T}_2||^2 \tag{2.29}$$

and use the inequality

$$(\mathbf{T}_1 - \mathbf{T}_2) \cdot (\mathbf{E}_1 - \mathbf{E}_2) \le \frac{1}{2}\kappa||\mathbf{T}_1 - \mathbf{T}_2||^2$$
$$+ \frac{1}{2}\kappa^{-1}||\mathbf{E}_1 - \mathbf{E}_2||^2. \tag{2.30}$$

The algebraic relation (2.30) is proved by the following chain of inequalities

$$\frac{1}{2}\kappa||\mathbf{T}_1 - \mathbf{T}_2||^2 + \frac{1}{2}\kappa^{-1}||\mathbf{E}_1 - \mathbf{E}_2||^2$$
$$= \frac{1}{2}\kappa^{-1}(\kappa||\mathbf{T}_1 - \mathbf{T}_2|| - ||\mathbf{E}_1 - \mathbf{E}_2||)^2 + ||\mathbf{T}_1 - \mathbf{T}_2|| \, ||\mathbf{E}_1 - \mathbf{E}_2||$$
$$\ge ||\mathbf{T}_1 - \mathbf{T}_2|| \, ||\mathbf{E}_1 - \mathbf{E}_2|| \ge (\mathbf{T}_1 - \mathbf{T}_2) \cdot (\mathbf{E}_1 - \mathbf{E}_2). \blacksquare \tag{2.31}$$

We are now in a position to prove the desired result.

Proposition 2.8. *Function ψ defined in (2.25) is continuously differentiable, convex and*

$$D_E\psi(\mathbf{E}) = \widehat{\mathbf{T}}(\mathbf{E}). \tag{2.32}$$

Proof. From (2.26) with $\mathbf{E}_1 = \mathbf{E}$ and $\mathbf{E}_2 = \mathbf{E} + \mathbf{H}$, we deduce that

$$\psi(\mathbf{E}+\mathbf{H}) - \psi(\mathbf{E}) - \widehat{\mathbf{T}}(\mathbf{E}) \cdot \mathbf{H} = o(\mathbf{H}), \quad \text{as } \mathbf{H} \to \mathbf{0}, \tag{2.33}$$

thus proving (2.32). From the first inequality in (2.26), by taking (2.32) into account, we obtain

$$\psi(\mathbf{E}_1) - \psi(\mathbf{E}_2) - D_E\psi(\mathbf{E}_2) \cdot (\mathbf{E}_1 - \mathbf{E}_2) \geq 0, \tag{2.34}$$

for each $\mathbf{E}_1, \mathbf{E}_2 \in \mathrm{Sym}$. By virtue of proposition 1.13, this allows concluding that function ψ is convex. ■

We point out that from proposition 1.14, it follows that $\widehat{\mathbf{T}}$ is monotone, as already proved in proposition 2.6.

2.1.1 Isotropic Materials

This subsection deals with materials for which the elasticity tensor $\mathbb{C}$ is isotropic (cf. (1.85)). As we shall see, for this class of materials the solution to the constitutive equation (2.2) can be calculated explicitly.

We recall that stress function $\widehat{\mathbf{T}}$ is said to be isotropic if, for every $\mathbf{E} \in \mathrm{Sym}$ and $\mathbf{Q} \in \mathrm{Orth}$, it holds that

$$\widehat{\mathbf{T}}(\mathbf{Q}\mathbf{E}\mathbf{Q}^T) = \mathbf{Q}\widehat{\mathbf{T}}(\mathbf{E})\mathbf{Q}^T. \tag{2.35}$$

Moreover, if $\widehat{\mathbf{T}}$ is isotropic then the following representation

$$\mathbf{T} = \widehat{\mathbf{T}}(\mathbf{E}) = \beta_0\mathbf{I} + \beta_1\mathbf{E} + \beta_2\mathbf{E}^2, \tag{2.36}$$

holds, where coefficients β_i are functions of the principal invariants of $\mathbf{E}$ [52].

Proposition 2.9. *If the elasticity tensor $\mathbb{C}$ in (2.2) is isotropic, then stress function $\widehat{\mathbf{T}}$ is isotropic.*

Proof. Taking into account that Sym^- is invariant under Orth, i.e.,

$$\text{if } \mathbf{A} \in \mathrm{Sym}^-, \text{ then } \mathbf{Q}\mathbf{A}\mathbf{Q}^T \in \mathrm{Sym}^-, \text{ for each } \mathbf{Q} \in \mathrm{Orth}, \tag{2.37}$$

and that the trace is isotropic, i.e.,

$$tr(\mathbf{Q}\mathbf{A}\mathbf{Q}^T) = tr\mathbf{A}, \quad \text{for each } \mathbf{Q} \in \mathrm{Orth}, \quad \mathbf{A} \in \mathrm{Sym},$$

from (2.4), in view of the isotropy of $\mathbb{C}$, we obtain

$$(\mathbf{Q}\widehat{\mathbf{T}}(\mathbf{E})\mathbf{Q}^T - \mathbf{T}^{**})\cdot(\mathbf{QEQ}^T - \mathbb{C}^{-1}[\mathbf{Q}\widehat{\mathbf{T}}(\mathbf{E})\mathbf{Q}^T]) \geq 0 \tag{2.38}$$

for each $\mathbf{T}^{**} \in \mathrm{Sym}^-$. On the other hand $\widehat{\mathbf{T}}(\mathbf{QEQ}^T)$ is defined by the inequality

$$(\widehat{\mathbf{T}}(\mathbf{QEQ}^T) - \mathbf{T}^{*})\cdot(\mathbf{QEQ}^T - \mathbb{C}^{-1}[\widehat{\mathbf{T}}(\mathbf{QEQ}^T)]) \geq 0 \tag{2.39}$$

for each $\mathbf{T}^{*} \in \mathrm{Sym}^-$. From the uniqueness of the solution to (2.4), by comparing (2.38) and (2.39), (2.35) follows. ■

We say that a masonry-like material with constitutive equation (2.2) is isotropic, if its elasticity tensor $\mathbb{C}$ is isotropic. In this case, as for linear elastic materials, we call the two real numbers λ and μ defined in (1.86), the Lamé moduli. In view of condition (2.3), the inequalities

$$\mu > 0, \quad 2\mu + 3\lambda > 0 \tag{2.40}$$

hold and we have

$$\mathbb{C}^{-1}[\mathbf{A}] = \frac{1}{2\mu}\mathbf{A} - \frac{\lambda}{2\mu(2\mu+3\lambda)}\,(tr\mathbf{A})\mathbf{I}. \tag{2.41}$$

Proposition 2.10. *If* $\mathbb{C}$ *is isotropic, then* $\mathbf{E}$, $\mathbf{T}$, $\mathbf{E}^e$ *and* $\mathbf{E}^f$ *are coaxial.*

Proof. $\mathbf{T}$ and $\mathbf{E}^f$ are coaxial in view of proposition 2.5. Since $\mathbb{C}$ is isotropic, from (2.41) it follows that $\mathbb{C}^{-1}$ is isotropic as well. Moreover, in view of the corollary 1.19, for $\mathbf{A} \in \mathrm{Sym}$, $\mathbf{A}\mathbb{C}^{-1}[\mathbf{A}] = \mathbb{C}^{-1}[\mathbf{A}]\mathbf{A}$. From both the isotropy of $\mathbb{C}^{-1}$ and the coaxiality of $\mathbf{T}$ and $\mathbf{E}^f$, it follows that

$$\mathbf{ET} = \mathbf{E}^f\mathbf{T} + \mathbf{E}^e\mathbf{T} = \mathbf{TE}^f + \mathbb{C}^{-1}[\mathbf{T}]\mathbf{T} = \mathbf{TE}^f + \mathbf{T}\mathbb{C}^{-1}[\mathbf{T}] = \mathbf{TE}; \tag{2.42}$$

$$\begin{aligned}\mathbf{EE}^f &= \mathbf{E}^e\mathbf{E}^f + (\mathbf{E}^f)^2 = \mathbb{C}^{-1}[\mathbf{T}]\mathbf{E}^f + (\mathbf{E}^f)^2 \\ &= \mathbf{E}^f\mathbb{C}^{-1}[\mathbf{T}] + (\mathbf{E}^f)^2 = \mathbf{E}^f\mathbf{E}.\end{aligned} \tag{2.43}$$

Analogously, it is a simple matter to prove that

$$\mathbf{E}^e\mathbf{E} = \mathbf{EE}^e, \quad \mathbf{E}^e\mathbf{E}^f = \mathbf{E}^f\mathbf{E}^e \tag{2.44}$$

and then, the thesis is a consequence of proposition 1.8. ■

Because $\mathbb{C}$ is isotropic, in view of the preceding proposition, system (2.2) can be rewritten as a linear complementarity problem, thereby allowing its solution to be calculated explicitly.

Let $\{\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3\}$ be orthonormal eigenvectors of $\mathbf{E}$, which, in view of proposition 2.10, are eigenvectors of $\mathbf{T}$ and $\mathbf{E}^f$ as well. Moreover, let (e_1, e_2, e_3), (e_1^f, e_2^f, e_3^f), and (t_1, t_2, t_3) be the eigenvalues of $\mathbf{E}, \mathbf{E}^f$ and $\mathbf{T}$, respectively. Then, by virtue of (1.86), system (2.2) is equivalent to

$$\begin{cases} t_1 = \mu[2(e_1 - e_1^f) + \alpha(e_1 + e_2 + e_3 - e_1^f - e_2^f - e_3^f)] \\ t_2 = \mu[2(e_2 - e_2^f) + \alpha(e_1 + e_2 + e_3 - e_1^f - e_2^f - e_3^f)] \\ t_3 = \mu[2(e_3 - e_3^f) + \alpha(e_1 + e_2 + e_3 - e_1^f - e_2^f - e_3^f)] \\ t_1 \le 0,\ t_2 \le 0,\ t_3 \le 0 \\ e_1^f \ge 0,\ e_2^f \ge 0,\ e_3^f \ge 0 \\ t_1 e_1^f = t_2 e_2^f = t_3 e_3^f = 0 \end{cases} \tag{2.45}$$

with $\alpha = \lambda/\mu$ (in the following we assume $\lambda \ge 0$, so that we have $\alpha \ge 0$). Putting $\mathbf{e} = (e_1, e_2, e_3)$, $\mathbf{e}^f = (e_1^f, e_2^f, e_3^f)$ and $\mathbf{t} = (t_1, t_2, t_3)$, (2.45) is a linear complementarity problem of the type (1.29), where

$$M = D, \quad \mathbf{q} = -D\mathbf{e}, \quad \mathbf{w} = -\mathbf{t}, \quad \mathbf{z} = \mathbf{e}^f, \tag{2.46}$$

with

$$D = \mu \begin{bmatrix} 2+\alpha & \alpha & \alpha \\ \alpha & 2+\alpha & \alpha \\ \alpha & \alpha & 2+\alpha \end{bmatrix}. \tag{2.47}$$

Since by virtue of (2.40), D is positive definite, proposition 1.6 guarantees that given $\mathbf{e}$, there exist unique $\mathbf{e}^f$ and $\mathbf{t}$ satisfying (2.45). This result provides an alternative proof of the existence and uniqueness of the solution to the constitutive equation (2.2), for isotropic $\mathbb{C}$.

Now let us proceed to explicitly calculating the eigenvalues (e_1^f, e_2^f, e_3^f), and (t_1, t_2, t_3) as functions of (e_1, e_2, e_3). To this aim, we suppose that the eigenvalues of $\mathbf{E}$ are ordered in such a way that

$$e_1 \le e_2 \le e_3. \tag{2.48}$$

In view of (2.48), the solution of (2.45) takes one of the following forms

$$\mathbf{t} < \mathbf{0}, \quad \mathbf{e}^f = \mathbf{0}, \tag{2.49}$$

$$\mathbf{t} = \mathbf{0}, \quad \mathbf{e}^f > \mathbf{0}, \tag{2.50}$$

$$t_1 < 0, \quad t_2 = t_3 = 0, \qquad e_1^f = 0, \quad e_2^f > 0, \quad e_3^f > 0, \tag{2.51}$$

$$t_1 < 0, \quad t_2 < 0, \quad t_3 = 0, \qquad e_1^f = e_2^f = 0, \quad , \quad e_3^f > 0, \tag{2.52}$$

depending on the values of e_1, e_2 and e_3. In conformity with (2.49)-(2.52), it is natural to define the following subsets of Sym,

$$\mathcal{R}_1 = \{\mathbf{E} \in \mathrm{Sym} \mid 2e_3 + \alpha\ tr\mathbf{E} < 0\}, \tag{2.53}$$

$$\mathcal{R}_2 = \{\mathbf{E} \in \mathrm{Sym} \mid e_1 > 0\}, \tag{2.54}$$

$$\mathcal{R}_3 = \{\mathbf{E} \in \mathrm{Sym} \mid e_1 < 0,\ \alpha e_1 + 2(1+\alpha)e_2 > 0\}, \tag{2.55}$$

$$\mathcal{R}_4 = \{\mathbf{E} \in \mathrm{Sym} \mid \alpha e_1 + 2(1+\alpha)e_2 < 0,\ 2e_3 + \alpha\ tr\mathbf{E} > 0\}, \tag{2.56}$$

where $tr\mathbf{E} = e_1 + e_2 + e_3$. Moreover, we define the following interfaces between regions $\mathcal{R}_1, \mathcal{R}_4$; $\mathcal{R}_2, \mathcal{R}_3$ and $\mathcal{R}_3, \mathcal{R}_4$, respectively, as

$$\mathcal{I}^{\mathcal{R}}_{1,4} = \{\mathbf{E} \in \text{Sym} \mid g_{14}(\mathbf{E}) = 2e_3 + \alpha\, tr\mathbf{E} = 0\}, \tag{2.57}$$

$$\mathcal{I}^{\mathcal{R}}_{2,3} = \{\mathbf{E} \in \text{Sym} \mid g_{23}(\mathbf{E}) = e_1 = 0\}, \tag{2.58}$$

$$\mathcal{I}^{\mathcal{R}}_{3,4} = \{\mathbf{E} \in \text{Sym} \mid g_{34}(\mathbf{E}) = \alpha e_1 + 2(1+\alpha)e_2 = 0\}. \tag{2.59}$$

Regions $\mathcal{R}_i$ characterize the different types of behavior exhibited by the material. In $\mathcal{R}_1$ the material is compressed in all directions and behaves like a linear elastic material. On the contrary, in $\mathcal{R}_2$ it is subjected to a positive semidefinite strain, and the stress is zero. Regions $\mathcal{R}_3$ and $\mathcal{R}_4$ present mixed behavior; in particular, they respectively contain two directions and one direction along which the stress is zero and the material can fracture orthogonally to these directions. For later use, we observe that from (2.55) and (2.56), it clearly follows $e_1 \neq e_2$ and $e_2 \neq e_3$, respectively in $\mathcal{R}_3$ and $\mathcal{R}_4$. By solving system (2.45) we obtain the eigenvalues of $\mathbf{E}^f$ and $\mathbf{T}$,

$$\text{if } \mathbf{E} \in \mathcal{R}_1 \cup \mathcal{I}^{\mathcal{R}}_{1,4} \text{ then } \begin{aligned} e_1^f &= 0,\\ e_2^f &= 0,\\ e_3^f &= 0,\\ t_1 &= \mu[(2+\alpha)e_1 + \alpha(e_2+e_3)],\\ t_2 &= \mu[(2+\alpha)e_2 + \alpha(e_1+e_3)],\\ t_3 &= \mu[(2+\alpha)e_3 + \alpha(e_1+e_2)]; \end{aligned} \tag{2.60}$$

$$\text{if } \mathbf{E} \in \mathcal{R}_2 \cup \mathcal{I}^{\mathcal{R}}_{2,3} \text{ then } \begin{aligned} e_1^f &= e_1,\\ e_2^f &= e_2,\\ e_3^f &= e_3,\\ t_1 &= 0,\\ t_2 &= 0,\\ t_3 &= 0; \end{aligned} \tag{2.61}$$

$$\text{if } \mathbf{E} \in \mathcal{R}_3 \cup \mathcal{I}^{\mathcal{R}}_{3,4} \text{ then } \begin{aligned} e_1^f &= 0,\\ e_2^f &= e_2 + \tfrac{\alpha}{2(1+\alpha)}e_1,\\ e_3^f &= e_3 + \tfrac{\alpha}{2(1+\alpha)}e_1,\\ t_1 &= E\ e_1,\\ t_2 &= 0,\\ t_3 &= 0; \end{aligned} \tag{2.62}$$

$$\text{if } \mathbf{E} \in \mathcal{R}_4 \text{ then } \begin{aligned} e_1^f &= 0,\\ e_2^f &= 0,\\ e_3^f &= e_3 + \tfrac{\alpha}{2+\alpha}(e_1+e_2),\\ t_1 &= \tfrac{2\mu}{2+\alpha}[2(1+\alpha)e_1 + \alpha e_2],\\ t_2 &= \tfrac{2\mu}{2+\alpha}[\alpha e_1 + 2(1+\alpha)e_2],\\ t_3 &= 0, \end{aligned} \tag{2.63}$$

where $E = \mu(2\mu + 3\lambda)/(\mu + \lambda)$ is Young's modulus.

Therefore, given a symmetric tensor $\mathbf{E} = \sum_{i=1}^{3} e_i \mathbf{q}_i \otimes \mathbf{q}_i$ and having determined the region to which $\mathbf{E}$ belongs, the solution to the constitutive equation (2.2) is given by

$$\mathbf{E}^f = \sum_{i=1}^{3} e_i^f \mathbf{q}_i \otimes \mathbf{q}_i, \tag{2.64}$$

$$\mathbf{T} = \widehat{\mathbf{T}}(\mathbf{E}) = \sum_{i=1}^{3} t_i \mathbf{q}_i \otimes \mathbf{q}_i, \tag{2.65}$$

where e_i^f, t_i are the functions of (e_1, e_2, e_3) given in (2.60)-(2.63). It should be noted that in view of (1.36), relation (2.65) is consistent with the isotropy of $\widehat{\mathbf{T}}$ expressed by (2.35).

Proposition 2.11. *The coefficients β_0, β_1 and β_2 in the representation formula (2.36) are given by*

$$\begin{aligned} \textit{if } \mathbf{E} \in \mathcal{R}_1 \cup \mathcal{I}^{\mathcal{R}}_{1,4} \textit{ then } \beta_0 &= \lambda\, tr\mathbf{E}, \\ \beta_1 &= 2\mu, \\ \beta_2 &= 0, \end{aligned} \tag{2.66}$$

$$\begin{aligned} \textit{if } \mathbf{E} \in \mathcal{R}_2 \cup \mathcal{I}^{\mathcal{R}}_{2,3} \textit{ then } \beta_0 &= 0, \\ \beta_1 &= 0, \\ \beta_2 &= 0, \end{aligned} \tag{2.67}$$

$$\begin{aligned} \textit{if } \mathbf{E} \in \mathcal{R}_3 \cup \mathcal{I}^{\mathcal{R}}_{3,4} \textit{ then } \beta_0 &= E\ \tfrac{e_1 e_2 e_3}{(e_3-e_1)(e_2-e_1)}, \\ \beta_1 &= -E\ \tfrac{e_1(e_2+e_3)}{(e_3-e_1)(e_2-e_1)}, \\ \beta_2 &= E\ \tfrac{e_1}{(e_3-e_1)(e_2-e_1)}, \end{aligned} \tag{2.68}$$

$$\begin{aligned} \textit{if } \mathbf{E} \in \mathcal{R}_4 \textit{ then } \beta_0 &= \tfrac{2\mu}{2+\alpha}\ \tfrac{-(2+3\alpha)e_1e_2e_3+\alpha e_3[e_1(e_3-e_1)+e_2(e_3-e_2)]}{(e_3-e_2)(e_3-e_1)}, \\ \beta_1 &= \tfrac{2\mu}{2+\alpha}\ \tfrac{\alpha(e_1^2+e_2^2+e_3^2)+2e_3^2+(2+3\alpha)e_1e_2}{(e_3-e_2)(e_3-e_1)}, \\ \beta_2 &= -\tfrac{2\mu}{2+\alpha}\ \tfrac{2e_3+\alpha(e_1+e_2+e_3)}{(e_3-e_2)(e_3-e_1)}. \end{aligned} \tag{2.69}$$

Proof. By substituting (2.65) into (2.36), we obtain the linear system

$$\begin{cases} \beta_0 + e_1\beta_1 + e_1^2\beta_2 = t_1, \\ \beta_0 + e_2\beta_1 + e_2^2\beta_2 = t_2, \\ \beta_0 + e_3\beta_1 + e_3^2\beta_2 = t_3. \end{cases} \tag{2.70}$$

This can be solved in the different regions $\mathcal{R}_i$ by replacing t_1, t_2, t_3 with their values given in (2.60)-(2.63) and explicitly calculating the coefficients (2.66)-(2.69). ■

Note that, in view of (2.48), e_1, e_2, e_3 and then $\beta_0, \beta_1, \beta_2$ can be expressed as functions of the principal invariants of $\mathbf{E}$.

Explicit knowledge of the solution to the constitutive equation allows us to calculate the strain energy density defined in (2.25). In particular, it holds that

if $\mathbf{E} \in \mathcal{R}_1 \cup \mathcal{I}^{\mathcal{R}}_{1,4}$ then

$$\psi(\mathbf{E}) = \mu||\mathbf{E}||^2 + \frac{\lambda}{2}(tr\mathbf{E})^2, \tag{2.71}$$

if $\mathbf{E} \in \mathcal{R}_2 \cup \mathcal{I}^{\mathcal{R}}_{2,3}$ then

$$\psi(\mathbf{E}) = 0, \tag{2.72}$$

if $\mathbf{E} \in \mathcal{R}_3 \cup \mathcal{I}^{\mathcal{R}}_{3,4}$ then

$$\psi(\mathbf{E}) = \frac{1}{2}Ee_1^2, \tag{2.73}$$

if $\mathbf{E} \in \mathcal{R}_4$ then

$$\psi(\mathbf{E}) = \frac{2\mu}{2+\alpha}[(1+\alpha)(e_1^2 + e_2^2) + \alpha e_1 e_2]. \tag{2.74}$$

2.1.2 The Two-Dimensional Case

Firstly, let us consider the case in which the infinitesimal strain tensor $\mathbf{E}$ has a null eigenvalue. We assume that $\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3$ is an orthonormal basis of $\mathcal{V}$ constituted by eigenvectors of $\mathbf{E}$ with $\mathbf{E}\mathbf{q}_3 = \mathbf{0}$.

Proposition 2.12. *Let $\mathbb{C}$ be isotropic; for $\mathbf{E} = \sum_{i=1}^{3} e_i \mathbf{q}_i \otimes \mathbf{q}_i$, let $(\mathbf{T}, \mathbf{E}^e, \mathbf{E}^f)$ be the solution to system (2.2), with $\mathbf{E}^f = \sum_{i=1}^{3} e_i^f \mathbf{q}_i \otimes \mathbf{q}_i$. Then, $e_3 = \mathbf{q}_3 \cdot \mathbf{E}\mathbf{q}_3 = 0$ implies $e_3^f = \mathbf{q}_3 \cdot \mathbf{E}^f \mathbf{q}_3 = 0$.*

Proof. The *ab absurdo* proof of the proposition is as follows. Let us assume that $e_3^f > 0$. From the orthogonality of $\mathbf{E}^f$ and $\mathbf{T}$, we obtain the condition $t_3 = 0$ which, together with $(2.45)_3$ yields

$$e_3^f = \frac{\alpha}{2+\alpha}(e_1 + e_2 - e_1^f - e_2^f). \tag{2.75}$$

If $\alpha = 0$, we immediately obtain $e_3^f = 0$. On the other hand, if $\alpha > 0$, then the quantity e_3^f in (2.75) is positive, if and only if

$$e_1 + e_2 - e_1^f - e_2^f > 0. \tag{2.76}$$

Now, by substituting (2.75) into the expressions of t_1 and t_2 in $(2.45)_1$ and $(2.45)_2$, respectively, and summing, we obtain

$$0 \geq t_1 + t_2 = 2\mu\frac{2+3\alpha}{2+\alpha}(e_1 + e_2 - e_1^f - e_2^f) \tag{2.77}$$

which is incompatible with (2.76). ■

In particular, under the hypotheses of the preceding proposition, we have $t_3 = \alpha(t_1 + t_2)/(2(1+\alpha))$.

Let us indicate with the same symbols $\mathbf{E}$, $\mathbf{E}^f$ and $\mathbf{T}$, the restrictions of $\mathbf{E}$, $\mathbf{E}^f$ and $\mathbf{T}$ to the two-dimensional subspace of $\mathcal{V}$ orthogonal to $\mathbf{q}_3$. Calculation of e_1^f, e_2^f, t_1 and t_2 satisfying (2.45) requires definition of the following subsets of Sym, shown in Figure 2.2,

$$\mathcal{S}_1 = \{\mathbf{E} \in \mathrm{Sym} \mid \alpha e_1 + (2+\alpha)e_2 < 0\}, \tag{2.78}$$

$$\mathcal{S}_2 = \{\mathbf{E} \in \mathrm{Sym} \mid e_1 > 0\}, \tag{2.79}$$

$$\mathcal{S}_3 = \{\mathbf{E} \in \mathrm{Sym} \mid e_1 < 0,\ \alpha e_1 + (2+\alpha)e_2 > 0\}, \tag{2.80}$$

with $e_1 \leq e_2$. Moreover, we define the following interfaces between regions $\mathcal{S}_1, \mathcal{S}_3$ and $\mathcal{S}_2, \mathcal{S}_3$, respectively as

$$\mathcal{I}^{\mathcal{S}}_{1,3} = \{\mathbf{E} \in \mathrm{Sym} \mid \alpha e_1 + (2+\alpha)e_2 = 0\}, \tag{2.81}$$

$$\mathcal{I}^{\mathcal{S}}_{2,3} = \{\mathbf{E} \in \mathrm{Sym} \mid e_1 = 0\}, \tag{2.82}$$

Note that in $\mathcal{S}_3$, e_1 and e_2 are distinct and different from zero.
Under these circumstances, from system (2.45) we deduce

$$\text{if } \mathbf{E} \in \mathcal{S}_1 \cup \mathcal{I}^{\mathcal{S}}_{1,3} \text{ then } \begin{aligned} e_1^f &= 0, \\ e_2^f &= 0, \\ t_1 &= \mu[(2+\alpha)e_1 + \alpha e_2], \\ t_2 &= \mu[(2+\alpha)e_2 + \alpha e_1]; \end{aligned} \tag{2.83}$$

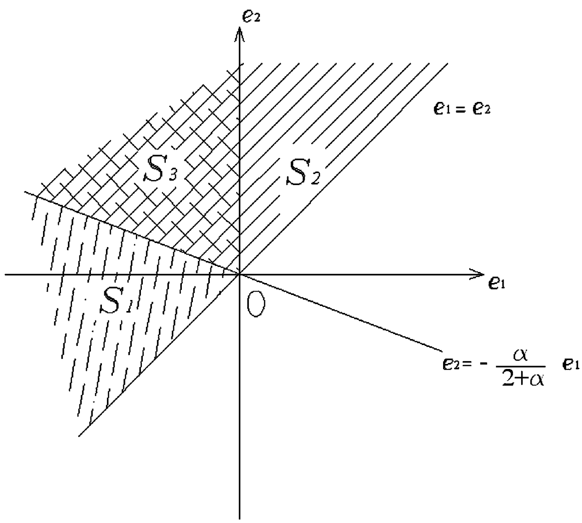

Fig. 2.2. Subdivision of the half-plane $e_1 \leq e_2$ into the regions $\mathcal{S}_1, \mathcal{S}_2$ and $\mathcal{S}_3$

$$\begin{aligned} \text{if } \mathbf{E} \in \mathcal{S}_2 \cup \mathcal{I}^{\mathcal{S}}_{2,3} \text{ then } e^f_1 &= e_1, \\ e^f_2 &= e_2, \\ t_1 &= 0, \\ t_2 &= 0; \end{aligned} \tag{2.84}$$

$$\begin{aligned} \text{if } \mathbf{E} \in \mathcal{S}_3 \text{ then } e^f_1 &= 0, \\ e^f_2 &= e_2 + \tfrac{\alpha}{2+\alpha} e_1, \\ t_1 &= 4\mu \tfrac{1+\alpha}{2+\alpha} e_1, \\ t_2 &= 0. \end{aligned} \tag{2.85}$$

Since $\widehat{\mathbf{T}}$ is an isotropic function of $\mathbf{E}$, we have

$$\mathbf{T} = \widehat{\mathbf{T}}(\mathbf{E}) = \beta_0 \mathbf{I} + \beta_1 \mathbf{E}; \tag{2.86}$$

and β_0 and β_1 can be obtained from (2.70), with $\beta_2 = 0$,

$$\begin{aligned} \text{if } \mathbf{E} \in \mathcal{S}_1 \cup \mathcal{I}^{\mathcal{S}}_{1,3} \text{ then } \beta_0 &= \lambda(e_1 + e_2), \\ \beta_1 &= 2\mu, \end{aligned} \tag{2.87}$$

$$\begin{aligned} \text{if } \mathbf{E} \in \mathcal{S}_2 \cup \mathcal{I}^{\mathcal{S}}_{2,3} \text{ then } \beta_0 &= 0, \\ \beta_1 &= 0, \end{aligned} \tag{2.88}$$

$$\begin{aligned} \text{if } \mathbf{E} \in \mathcal{S}_3 \text{ then } \beta_0 &= 4\mu \tfrac{1+\alpha}{2+\alpha} \tfrac{e_1 e_2}{e_2 - e_1}, \\ \beta_1 &= -4\mu \tfrac{1+\alpha}{2+\alpha} \tfrac{e_1}{e_2 - e_1}. \end{aligned} \tag{2.89}$$

Lastly, from (2.83)-(2.85) we deduce the following expressions for the strain energy density.

If $\mathbf{E} \in \mathcal{S}_1 \cup \mathcal{I}^{\mathcal{S}}_{1,3}$ then

$$\psi(\mathbf{E}) = \mu ||\mathbf{E}||^2 + \frac{\lambda}{2} (tr\mathbf{E})^2, \tag{2.90}$$

if $\mathbf{E} \in \mathcal{S}_2 \cup \mathcal{I}^{\mathcal{S}}_{2,3}$ then

$$\psi(\mathbf{E}) = 0, \tag{2.91}$$

if $\mathbf{E} \in \mathcal{S}_3$ then

$$\psi(\mathbf{E}) = 2\mu \frac{1+\alpha}{2+\alpha} e_1^2. \tag{2.92}$$

Now, let us consider the case in which $\mathbf{T}$ has a null eigenvalue. For $\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3$, orthonormal eigenvectors of $\mathbf{T}$, with $\mathbf{T}\mathbf{q}_3 = \mathbf{0}$, from $(2.45)_3$ we get

$$e_3 - e^f_3 = \frac{\alpha}{2+\alpha} (e^f_1 + e^f_2 - e_1 - e_2). \tag{2.93}$$

Moreover, as e^f_3 $(2.45)_6$, it is assumed to be equal to zero. Putting

$$\mathcal{T}_1 = \{\mathbf{E} \in \mathrm{Sym} \mid \alpha e_1 + 2(1+\alpha) e_2 < 0\}, \tag{2.94}$$

$$\mathcal{T}_2 = \{\mathbf{E} \in \mathrm{Sym} \mid e_1 > 0\}, \tag{2.95}$$

$$\mathcal{T}_3 = \{\mathbf{E} \in \mathrm{Sym} \mid e_1 < 0,\ \alpha e_1 + 2(1+\alpha)e_2 > 0\}, \tag{2.96}$$

$$\mathcal{I}^{\mathcal{T}}_{1,3} = \{\mathbf{E} \in \mathrm{Sym} \mid \alpha e_1 + 2(1+\alpha)e_2 = 0\}, \tag{2.97}$$

$$\mathcal{I}^{\mathcal{T}}_{2,3} = \{\mathbf{E} \in \mathrm{Sym} \mid e_1 = 0\}, \tag{2.98}$$

we have

$$\begin{aligned} \text{if } \mathbf{E} \in \mathcal{T}_1 \cup \mathcal{I}^{\mathcal{T}}_{1,3} \text{ then } e^f_1 &= 0,\\ e^f_2 &= 0,\\ t_1 &= \tfrac{2\mu}{2+\alpha}[2(1+\alpha)e_1 + \alpha e_2],\\ t_2 &= \tfrac{2\mu}{2+\alpha}[2(1+\alpha)e_2 + \alpha e_1]; \end{aligned} \tag{2.99}$$

$$\begin{aligned} \text{if } \mathbf{E} \in \mathcal{T}_2 \cup \mathcal{I}^{\mathcal{T}}_{2,3} \text{ then } e^f_1 &= e_1,\\ e^f_2 &= e_2,\\ t_1 &= 0,\\ t_2 &= 0; \end{aligned} \tag{2.100}$$

$$\begin{aligned} \text{if } \mathbf{E} \in \mathcal{T}_3 \text{ then } e^f_1 &= 0,\\ e^f_2 &= e_2 + \tfrac{\alpha}{2(1+\alpha)} e_1,\\ t_1 &= E e_1,\\ t_2 &= 0. \end{aligned} \tag{2.101}$$

In this case the coefficients of the representation (2.86) are

$$\begin{aligned} \text{if } \mathbf{E} \in \mathcal{T}_1 \cup \mathcal{I}^{\mathcal{T}}_{1,3} \text{ then } \beta_0 &= \tfrac{2\lambda}{2\mu+\lambda}(e_1 + e_2),\\ \beta_1 &= 2\mu, \end{aligned} \tag{2.102}$$

$$\begin{aligned} \text{if } \mathbf{E} \in \mathcal{T}_2 \cup \mathcal{I}^{\mathcal{T}}_{2,3} \text{ then } \beta_0 &= 0,\\ \beta_1 &= 0, \end{aligned} \tag{2.103}$$

$$\begin{aligned} \text{if } \mathbf{E} \in \mathcal{T}_3 \text{ then } \beta_0 &= E\tfrac{e_1 e_2}{e_2 - e_1},\\ \beta_1 &= -E\tfrac{e_1}{e_2 - e_1}, \end{aligned} \tag{2.104}$$

and the strain energy density has the following expressions,
if $\mathbf{E} \in \mathcal{T}_1 \cup \mathcal{I}^{\mathcal{T}}_{1,3}$ then

$$\psi(\mathbf{E}) = \mu ||\mathbf{E}||^2 + \frac{\lambda}{2+\alpha}(tr\mathbf{E})^2, \tag{2.105}$$

if $\mathbf{E} \in \mathcal{T}_2 \cup \mathcal{I}^{\mathcal{T}}_{2,3}$ then

$$\psi(\mathbf{E}) = 0, \tag{2.106}$$

if $\mathbf{E} \in \mathcal{T}_3$ then

$$\psi(\mathbf{E}) = \frac{1}{2} E e_1^2. \tag{2.107}$$

2.2 Masonry-Like Materials with Small Tensile Strength and Bounded Compressive Strength

In many applications masonry materials can be considered infinitely resistant to compression because the stresses in the structure are definitely less than the maximum compressive strength [56]. However, in some cases it is important to have a constitutive model that takes into account the limited compressive strength of the material, since re-distribution of the stresses in the structure, which occurs when the compressive strength is reached in large parts of it, may be significant and may sometimes considerably modify its behavior.

The reasons for introducing low tensile strength stem from different considerations. Indeed, it is well known that, although some masonry components may have non-negligible tensile strength, the frequent presence of fractures, which often occur during a building's construction, suggest viewing masonry materials as not withstanding tension at all. On the other hand, in some circumstances, providing for low tensile strength can facilitate the numerical analysis without significantly modifying the results.

In this section the constitutive equation presented in the preceding section is generalized by allowing the material to possess some, albeit low, tensile strength and setting a limit to its compressive strength.

We shall limit ourselves to the particular case in which the dependence of the stress on the elastic strain is isotropic. Let $\mathbb{C}$ be an isotropic tensor, $\mathbb{C}[\mathbf{A}] = 2\mu\mathbf{A}+\lambda tr(\mathbf{A})\mathbf{I}$, $\mathbf{A}\in$Sym, where λ and μ satisfy the inequalities (2.40).

Definition 2.13. *Let σ^t and σ^c be two positive material constants representing the maximum resistance to tension and compression, respectively. A masonry-like material with bounded compressive strength (and bounded tensile strength) is an elastic material whose stress function $\widehat{\mathbf{T}}_{BC} : Sym \to Sym$,*

$$\mathbf{T} = \widehat{\mathbf{T}}_{BC}(\mathbf{E}), \quad \mathbf{E} \in Sym \tag{2.108}$$

satisfies the following conditions

$$\begin{cases} \mathbf{T} - \sigma^t\mathbf{I} \in \ \mathrm{Sym}^-, \\ \mathbf{T} + \sigma^c\mathbf{I} \in \ \mathrm{Sym}^+, \\ \mathbf{E} = \mathbf{E}^e + \mathbf{E}^f + \mathbf{E}^c, \\ \mathbf{T} = \mathbb{C}[\mathbf{E}^e], \\ \mathbf{E}^f \in \ \mathrm{Sym}^+, \\ \mathbf{E}^c \in \ \mathrm{Sym}^-, \\ \mathbf{E}^f \cdot \mathbf{E}^c = 0, \\ (\mathbf{T} - \sigma^t\mathbf{I})\cdot\mathbf{E}^f = (\mathbf{T} + \sigma^c\mathbf{I})\cdot\mathbf{E}^c = 0. \end{cases} \tag{2.109}$$

Once again we shall call $\mathbb{C}$ the elasticity tensor and λ and μ the Lamé moduli. $\mathbf{E}^e, \mathbf{E}^f$ and $\mathbf{E}^c$ are respectively the elastic, fracture and crushing parts of the strain. When the crushing stress σ^c goes to infinity, and tensile stress σ^t goes to zero, relations (2.109) reduce to the constitutive equation of the masonry-like material described in Section 2.1. A material whose behavior is described by (2.109) will be also called a BCS material, for the sake of brevity.

Proposition 2.14. *Tensors* $\mathbf{E}$, $\mathbf{E}^f$, $\mathbf{E}^c$, $\mathbf{E}^e$, $\mathbf{T}$, $\mathbf{T}-\sigma^t\mathbf{I}$ *and* $\mathbf{T}+\sigma^c\mathbf{I}$ *are coaxial.*

Proof. Because $\mathbb{C}$ is isotropic, the proof is analogous to that of proposition 2.10. ■

Due to the preceding proposition, the constitutive equation (2.109) can be rewritten with respect to the basis $\{\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3\}$ of the eigenvectors of $\mathbf{E}$. Let (e_1, e_2, e_3), (e_1^f, e_2^f, e_3^f), (e_1^c, e_2^c, e_3^c) and (t_1, t_2, t_3) be the eigenvalues of $\mathbf{E}$, $\mathbf{E}^f$, $\mathbf{E}^c$ and $\mathbf{T}$, respectively. It is an easy matter to show that the constitutive equation (2.109) is equivalent to the system

$$\begin{cases} t_1 = \mu\{2(e_1 - e_1^f - e_1^c) + \alpha[tr(\mathbf{E}) - tr(\mathbf{E}^f) - tr(\mathbf{E}^c)]\} \\ t_2 = \mu\{2(e_2 - e_2^f - e_2^c) + \alpha[tr(\mathbf{E}) - tr(\mathbf{E}^f) - tr(\mathbf{E}^c)]\} \\ t_3 = \mu\{2(e_3 - e_3^f - e_3^c) + \alpha[tr(\mathbf{E}) - tr(\mathbf{E}^f) - tr(\mathbf{E}^c)]\} \\ (t_1 - \sigma^t)e_1^f = 0 \\ (t_2 - \sigma^t)e_2^f = 0 \\ (t_3 - \sigma^t)e_3^f = 0 \\ (t_1 + \sigma^c)e_1^c = 0 \\ (t_2 + \sigma^c)e_2^c = 0 \\ (t_3 + \sigma^c)e_3^c = 0 \\ e_1^f \geq 0,\ e_2^f \geq 0,\ e_3^f \geq 0 \\ e_1^c \leq 0,\ e_2^c \leq 0,\ e_3^c \leq 0 \\ t_1 - \sigma^t \leq 0,\ t_2 - \sigma^t \leq 0,\ t_3 - \sigma^t \leq 0 \\ t_1 + \sigma^c \geq 0,\ t_2 + \sigma^c \geq 0,\ t_3 + \sigma^c \geq 0 \\ e_1^f e_1^c = e_2^f e_2^c = e_3^f e_3^c = 0 \end{cases} \tag{2.110}$$

where $tr(\mathbf{E}) = e_1 + e_2 + e_3$, $tr(\mathbf{E}^f) = e_1^f + e_2^f + e_3^f$, $tr(\mathbf{E}^c) = e_1^c + e_2^c + e_3^c$ and $\alpha = \lambda/\mu$.

Given the elastic moduli λ and μ and the material constants σ^t and σ^c, the principal components (e_1^f, e_2^f, e_3^f), (e_1^c, e_2^c, e_3^c) and (t_1, t_2, t_3) satisfying (2.110) can be calculated as functions of the eigenvalues e_1, e_2, e_3 of $\mathbf{E}$, which are presumed to be ordered so that $e_1 \leq e_2 \leq e_3$. The solution to (2.110) can be calculated explicitly by using a procedure similar to that used in subsection 2.1.1. To this end, we define the following subsets of Sym,

$$\mathcal{C}_1 = \{\mathbf{E} \in \text{ Sym} \mid 2e_3 + \alpha\ tr(\mathbf{E}) - \omega^t < 0, \quad 2e_1 + \alpha\ tr(\mathbf{E}) + \omega^c > 0\}, \tag{2.111}$$

$$\begin{gathered}\mathcal{C}_2 = \{\mathbf{E} \in \text{Sym} \mid 2e_1 + \alpha\ tr(\mathbf{E}) + \omega^c < 0, \\ 2\alpha e_2 + 4(1+\alpha)e_3 - \alpha\omega^c - (2+\alpha)\omega^t < 0, \quad 2(1+\alpha)e_2 + \alpha e_3 + \omega^c > 0\},\end{gathered} \tag{2.112}$$

$$\begin{gathered}\mathcal{C}_3 = \{\mathbf{E} \in \text{Sym} \mid 2(1+\alpha)e_2 + \alpha e_3 + \omega^c < 0, \\ (2+3\alpha)e_3 - \alpha\omega^c - (1+\alpha)\omega^t < 0, \quad (2+3\alpha)e_3 + \omega^c > 0\},\end{gathered} \tag{2.113}$$

$$\mathcal{C}_4 = \{\mathbf{E} \in \text{Sym} \mid (2+3\alpha)e_3 + \omega^c < 0\}, \tag{2.114}$$

$$\begin{gathered}\mathcal{C}_5 = \{\mathbf{E} \in \text{Sym} \mid 2e_3 + \alpha\ tr(\mathbf{E}) - \omega^t > 0, \\ 2(1+\alpha)e_2 + \alpha e_1 - \omega^t < 0, \quad 4(1+\alpha)e_1 + 2\alpha e_2 + \alpha\omega^t + (2+\alpha)\omega^c > 0\},\end{gathered} \tag{2.115}$$

$$\begin{gathered}\mathcal{C}_6 = \{\mathbf{E} \in \text{Sym} \mid 2(1+\alpha)e_2 + \alpha e_1 - \omega^t > 0, \\ (2+3\alpha)e_1 - \omega^t < 0, \quad (2+3\alpha)e_1 + \alpha\omega^t + (1+\alpha)\omega^c > 0\},\end{gathered} \tag{2.116}$$

$$\mathcal{C}_7 = \{\mathbf{E} \in \text{Sym} \mid (2+3\alpha)e_1 - \omega^t > 0\}, \tag{2.117}$$

$$\begin{gathered}\mathcal{C}_8 = \{\mathbf{E} \in \text{Sym} \mid 2(2+3\alpha)e_2 - \alpha\omega^c - (2+\alpha)\omega^t > 0, \\ (2+3\alpha)e_1 + \alpha\omega^t + (1+\alpha)\omega^c < 0\},\end{gathered} \tag{2.118}$$

$$\begin{gathered}\mathcal{C}_9 = \{\mathbf{E} \in \text{Sym} \mid 2(2+3\alpha)e_2 + \alpha\omega^t + (2+\alpha)\omega^c < 0, \\ (2+3\alpha)e_3 - \alpha\omega^c - (1+\alpha)\omega^t > 0\},\end{gathered} \tag{2.119}$$

$$\begin{gathered}\mathcal{C}_{10} = \{\mathbf{E} \in \text{Sym} \mid 4(1+\alpha)e_1 + 2\alpha e_2 + \alpha\omega^t + (2+\alpha)\omega^c < 0, \\ 4(1+\alpha)e_3 + 2\alpha e_2 - \alpha\omega^c - (2+\alpha)\omega^t > 0, \\ 2(2+3\alpha)e_2 - \alpha\omega^c - (2+\alpha)\omega^t < 0, \\ 2(2+3\alpha)e_2 + \alpha\omega^t + (2+\alpha)\omega^c > 0\},\end{gathered} \tag{2.120}$$

where $\omega^t = \sigma^t/\mu$ and $\omega^c = \sigma^c/\mu$. In addition, just as for the case of masonry-like materials with zero tensile strength and infinite compressive strength, we define the following interfaces

$$\mathcal{I}^{\mathcal{C}}_{1,2} = \{\mathbf{E} \in \mathrm{Sym} \mid 2e_1 + \alpha\ tr(\mathbf{E}) + \omega^c = 0\}, \tag{2.121}$$

$$\mathcal{I}^{\mathcal{C}}_{1,5} = \{\mathbf{E} \in \mathrm{Sym} \mid 2e_3 + \alpha\ tr(\mathbf{E}) - \omega^t = 0\}, \tag{2.122}$$

$$\mathcal{I}^{\mathcal{C}}_{2,3} = \{\mathbf{E} \in \mathrm{Sym} \mid 2(1+\alpha)e_2 + \alpha e_3 + \omega^c = 0\}, \tag{2.123}$$

$$\mathcal{I}^{\mathcal{C}}_{3,4} = \{\mathbf{E} \in \mathrm{Sym} \mid (2+3\alpha)e_3 + \omega^c = 0\}, \tag{2.124}$$

$$\mathcal{I}^{\mathcal{C}}_{5,6} = \{\mathbf{E} \in \mathrm{Sym} \mid \alpha e_1 + 2(1+\alpha)e_2 - \omega^t = 0\}, \tag{2.125}$$

$$\mathcal{I}^{\mathcal{C}}_{6,7} = \{\mathbf{E} \in \mathrm{Sym} \mid (2+3\alpha)e_1 - \omega^t = 0\}, \tag{2.126}$$

$$\mathcal{I}^{\mathcal{C}}_{6,8} = \{\mathbf{E} \in \mathrm{Sym} \mid (2+3\alpha)e_1 + \alpha\omega^t + (1+\alpha)\omega^c = 0\}, \tag{2.127}$$

$$\mathcal{I}^{\mathcal{C}}_{8,10} = \{\mathbf{E} \in \mathrm{Sym} \mid 2(2+3\alpha)e_2 - \alpha\omega^c - (2+\alpha)\omega^t = 0\}, \tag{2.128}$$

$$\mathcal{I}^{\mathcal{C}}_{9,10} = \{\mathbf{E} \in \mathrm{Sym} \mid 2(2+3\alpha)e_2 + \alpha\omega^t + (2+\alpha)\omega^c = 0\}, \tag{2.129}$$

It is easy to prove that in regions $\mathcal{C}_2, \mathcal{C}_6$ and $\mathcal{C}_8$ we have $e_1 < e_2 \le e_3$ and that in $\mathcal{C}_3, \mathcal{C}_5$ and $\mathcal{C}_9$, $e_1 \le e_2 < e_3$. Lastly, the eigenvalues of $\mathbf{E}$ are distinct in $\mathcal{C}_{10}$. By solving system (2.110), we obtain the principal components of $\mathbf{E}^f, \mathbf{E}^c$ and $\mathbf{T}$.

$$\text{If } \mathbf{E} \in \mathcal{C}_1 \cup \mathcal{I}^{\mathcal{C}}_{1,2} \cup \mathcal{I}^{\mathcal{C}}_{1,3} \text{ then } \begin{aligned} e^f_1 &= 0, \\ e^f_2 &= 0, \\ e^f_3 &= 0, \\ e^c_1 &= 0, \\ e^c_2 &= 0, \\ e^c_3 &= 0, \\ t_1 &= \mu[(2+\alpha)e_1 + \alpha(e_2+e_3)], \\ t_2 &= \mu[(2+\alpha)e_2 + \alpha(e_1+e_3)], \\ t_3 &= \mu[(2+\alpha)e_3 + \alpha(e_1+e_2)]; \end{aligned} \tag{2.130}$$

$$\text{if } \mathbf{E} \in \mathcal{C}_2 \cup \mathcal{I}^{\mathcal{C}}_{2,3} \text{ then } \begin{aligned} e^f_1 &= 0, \\ e^f_2 &= 0, \\ e^f_3 &= 0, \\ e^c_1 &= e_1 + \tfrac{\alpha}{2+\alpha}(e_2+e_3) + \tfrac{1}{2+\alpha}\omega^c, \\ e^c_2 &= 0, \\ e^c_3 &= 0, \\ t_1 &= -\sigma^c, \\ t_2 &= \mu\{2e_2 + \tfrac{\alpha}{2+\alpha}[2(e_2+e_3) - \omega^c]\}, \\ t_3 &= \mu\{2e_3 + \tfrac{\alpha}{2+\alpha}[2(e_2+e_3) - \omega^c]\}; \end{aligned} \tag{2.131}$$

$$\begin{aligned} \text{if } \mathbf{E} \in \mathcal{C}_3 \cup \mathcal{I}^{\mathcal{C}}_{3,4} \text{ then } e_1^f &= 0, \\ e_2^f &= 0, \\ e_3^f &= 0, \\ e_1^c &= e_1 + \tfrac{\alpha}{2(1+\alpha)} e_3 + \tfrac{1}{2(1+\alpha)} \omega^c, \\ e_2^c &= e_2 + \tfrac{\alpha}{2(1+\alpha)} e_3 + \tfrac{1}{2(1+\alpha)} \omega^c, \\ e_3^c &= 0, \\ t_1 &= -\sigma^c, \\ t_2 &= -\sigma^c, \\ t_3 &= \tfrac{\mu}{1+\alpha}[(2+3\alpha)e_3 - \alpha\omega^c]; \end{aligned} \tag{2.132}$$

$$\begin{aligned} \text{if } \mathbf{E} \in \mathcal{C}_4 \text{ then } e_1^f &= 0, \\ e_2^f &= 0, \\ e_3^f &= 0, \\ e_1^c &= e_1 + \tfrac{1}{2+3\alpha} \omega^c, \\ e_2^c &= e_2 + \tfrac{1}{2+3\alpha} \omega^c, \\ e_3^c &= e_3 + \tfrac{1}{2+3\alpha} \omega^c, \\ t_1 &= -\sigma^c, \\ t_2 &= -\sigma^c, \\ t_3 &= -\sigma^c; \end{aligned} \tag{2.133}$$

$$\begin{aligned} \text{if } \mathbf{E} \in \mathcal{C}_5 \cup \mathcal{I}^{\mathcal{C}}_{5,6} \text{ then } e_1^f &= 0, \\ e_2^f &= 0, \\ e_3^f &= e_3 + \tfrac{\alpha}{2+\alpha}(e_1 + e_2) - \tfrac{1}{2+\alpha} \omega^t, \\ e_1^c &= 0, \\ e_2^c &= 0, \\ e_3^c &= 0, \\ t_1 &= \tfrac{\mu}{2+\alpha}[4(1+\alpha)e_1 + 2\alpha e_2 + \alpha\omega^t], \\ t_2 &= \tfrac{\mu}{2+\alpha}[4(1+\alpha)e_2 + 2\alpha e_1 + \alpha\omega^t], \\ t_3 &= \sigma^t; \end{aligned} \tag{2.134}$$

$$\begin{aligned} \text{if } \mathbf{E} \in \mathcal{C}_6 \cup \mathcal{I}^{\mathcal{C}}_{6,7} \cup \mathcal{I}^{\mathcal{C}}_{6,8} \text{ then } e_1^f &= 0, \\ e_2^f &= e_2 + \tfrac{\alpha}{2(1+\alpha)} e_1 - \tfrac{\omega^t}{2(1+\alpha)}, \\ e_3^f &= e_3 + \tfrac{\alpha}{2(1+\alpha)} e_1 - \tfrac{\omega^t}{2(1+\alpha)}, \\ e_1^c &= 0, \\ e_2^c &= 0, \\ e_3^c &= 0, \\ t_1 &= \tfrac{\mu}{1+\alpha}[(2+3\alpha)e_1 + \alpha\omega^t], \\ t_2 &= \sigma^t, \\ t_3 &= \sigma^t; \end{aligned} \tag{2.135}$$

$$
\begin{aligned}
\text{if } \mathbf{E} \in \mathcal{C}_7 \text{ then } e_1^f &= e_1 - \tfrac{1}{2+3\alpha}\omega^t, \\
e_2^f &= e_2 - \tfrac{1}{2+3\alpha}\omega^t, \\
e_3^f &= e_3 - \tfrac{1}{2+3\alpha}\omega^t, \\
e_1^c &= 0, \\
e_2^c &= 0, \\
e_3^c &= 0, \\
t_1 &= \sigma^t, \\
t_2 &= \sigma^t, \\
t_3 &= \sigma^t;
\end{aligned} \tag{2.136}
$$

$$
\begin{aligned}
\text{if } \mathbf{E} \in \mathcal{C}_8 \cup \mathcal{I}_{8,10}^{\mathcal{C}} \text{ then } e_1^f &= 0, \\
e_2^f &= e_2 - \tfrac{\alpha\omega^c}{2(2+3\alpha)} - \tfrac{2+\alpha}{2(2+3\alpha)}\omega^t, \\
e_3^f &= e_3 - \tfrac{\alpha\omega^c}{2(2+3\alpha)} - \tfrac{2+\alpha}{2(2+3\alpha)}\omega^t, \\
e_1^c &= e_1 + \tfrac{(1+\alpha)\omega^c}{2+3\alpha} + \tfrac{\alpha}{2+3\alpha}\omega^t, \\
e_2^c &= 0, \\
e_3^c &= 0, \\
t_1 &= -\sigma^c, \\
t_2 &= \sigma^t, \\
t_3 &= \sigma^t;
\end{aligned} \tag{2.137}
$$

$$
\begin{aligned}
\text{if } \mathbf{E} \in \mathcal{C}_9 \cup \mathcal{I}_{9,10}^{\mathcal{C}} \text{ then } e_1^f &= 0, \\
e_2^f &= 0, \\
e_3^f &= e_3 - \tfrac{\alpha}{2+3\alpha}\omega^c - \tfrac{1+\alpha}{2+3\alpha}\omega^t, \\
e_1^c &= e_1 + \tfrac{2+\alpha}{2(2+3\alpha)}\omega^c + \tfrac{\alpha}{2(2+3\alpha)}\omega^t, \\
e_2^c &= e_2 + \tfrac{2+\alpha}{2(2+3\alpha)}\omega^c + \tfrac{\alpha}{2(2+3\alpha)}\omega^t, \\
e_3^c &= 0, \\
t_1 &= -\sigma^c, \\
t_2 &= -\sigma^c, \\
t_3 &= \sigma^t;
\end{aligned} \tag{2.138}
$$

$$
\begin{aligned}
\text{if } \mathbf{E} \in \mathcal{C}_{10} \text{ then } e_1^f &= 0, \\
e_2^f &= 0, \\
e_3^f &= e_3 + \tfrac{\alpha}{2(1+\alpha)}e_2 - \tfrac{2+\alpha}{4(1+\alpha)}\omega^t - \tfrac{\alpha}{4(1+\alpha)}\omega^c, \\
e_1^c &= e_1 + \tfrac{\alpha}{2(1+\alpha)}e_2 + \tfrac{2+\alpha}{4(1+\alpha)}\omega^c + \tfrac{\alpha}{4(1+\alpha)}\omega^t, \\
e_2^c &= 0, \\
e_3^c &= 0, \\
t_1 &= -\sigma^c, \\
t_2 &= \tfrac{\mu}{2(1+\alpha)}[2(2+3\alpha)e_2 + \alpha(\omega^t - \omega^c)], \\
t_3 &= \sigma^t.
\end{aligned} \tag{2.139}
$$

We are now in a position to prove the following theorem.

Theorem 2.15. *Given the Lamé moduli of the material and the constants σ^t and σ^c, for each $\mathbf{E} \in$ Sym the constitutive equation (2.109) has a unique solution* $(\mathbf{T}, \mathbf{E}^e, \mathbf{E}^f, \mathbf{E}^c)$.

Proof. As far as the existence of the solution is concerned, formulae (2.130)-(2.139) provide an explicit solution as $\mathbf{E}$ varies in Sym. In order to prove its uniqueness, let $(\mathbf{T}_1, \mathbf{E}_1^e, \mathbf{E}_1^f, \mathbf{E}_1^c)$ and $(\mathbf{T}_2, \mathbf{E}_2^e, \mathbf{E}_2^f, \mathbf{E}_2^c)$ be two different solutions. For the elastic parts, we have $\mathbf{E}_1^e = \mathbf{E} - \mathbf{E}_1^f - \mathbf{E}_1^c$ and $\mathbf{E}_2^e = \mathbf{E} - \mathbf{E}_2^f - \mathbf{E}_2^c$ and thus

$$\mathbb{C}^{-1}[\mathbf{T}_1 - \mathbf{T}_2] = \mathbf{E}_1^e - \mathbf{E}_2^e = \mathbf{E}_2^c - \mathbf{E}_1^c + \mathbf{E}_2^f - \mathbf{E}_1^f. \tag{2.140}$$

By virtue of $(2.109)_1$, $(2.109)_2$ and $(2.109)_8$, the positive definiteness of $\mathbb{C}$ and proposition 1.9, we can write

$$\begin{aligned} 0 \le (\mathbf{T}_1 - \mathbf{T}_2)\cdot\mathbb{C}^{-1}[\mathbf{T}_1 - \mathbf{T}_2] \\ = (\mathbf{T}_1 - \mathbf{T}_2)\cdot(\mathbf{E}_2^c - \mathbf{E}_1^c) + (\mathbf{T}_1 - \mathbf{T}_2)\cdot(\mathbf{E}_2^f - \mathbf{E}_1^f) \\ = (\mathbf{T}_1 - \sigma^t\mathbf{I})\cdot\mathbf{E}_2^f + (\mathbf{T}_2 - \sigma^t\mathbf{I})\cdot\mathbf{E}_1^f \\ +(\mathbf{T}_1 + \sigma^c\mathbf{I})\cdot\mathbf{E}_2^c + (\mathbf{T}_2 + \sigma^c\mathbf{I})\cdot\mathbf{E}_1^c \le 0. \end{aligned} \tag{2.141}$$

Consequently, $\mathbf{T}_1 = \mathbf{T}_2$, $\mathbf{E}_1^e = \mathbf{E}_2^e$ and $\mathbf{E}_2^f + \mathbf{E}_2^c = \mathbf{E}_1^f + \mathbf{E}_1^c$. On the other hand, by using $(2.109)_7$ and proposition 1.9, we obtain $0 \le (\mathbf{E}_2^c - \mathbf{E}_1^c)\cdot(\mathbf{E}_1^f - \mathbf{E}_2^f) = \mathbf{E}_2^c\cdot\mathbf{E}_1^f + \mathbf{E}_1^c\cdot\mathbf{E}_2^f \le 0$, and finally, $\mathbf{E}_2^c - \mathbf{E}_1^c = \mathbf{E}_1^f - \mathbf{E}_2^f = \mathbf{0}$. ■

Therefore, given a symmetric tensor $\mathbf{E} = \sum_{i=1}^{3} e_i \mathbf{q}_i \otimes \mathbf{q}_i$ and having determined the region to which $\mathbf{E}$ belongs, the solution to the constitutive equation (2.109) is

$$\mathbf{T} = \widehat{\mathbf{T}}_{BC}(\mathbf{E}) = \sum_{i=1}^{3} t_i \mathbf{q}_i \otimes \mathbf{q}_i, \tag{2.142}$$

$$\mathbf{E}^f = \sum_{i=1}^{3} e_i^f \mathbf{q}_i \otimes \mathbf{q}_i, \quad \mathbf{E}^c = \sum_{i=1}^{3} e_i^c \mathbf{q}_i \otimes \mathbf{q}_i, \tag{2.143}$$

where e_i^f, e_i^c, t_i are functions of e_i, given in (2.130)-(2.139). In particular, $\widehat{\mathbf{T}}_{BC} = \widehat{\mathbf{T}}$ if $\sigma^t = 0$ and $\sigma^c = \infty$.

As in the case of a masonry-like material with zero tensile strength and infinite compressive strength, the solution to the constitutive equation (2.109) can be characterized as the solution of a suitable variational inequality. To this end, put

$$\mathrm{Sym}^{(\sigma^t,\sigma^c)} = \{\mathbf{A} \in \mathrm{Sym} \mid \mathbf{A} - \sigma^t\mathbf{I} \in \mathrm{Sym}^-, \quad \mathbf{A} + \sigma^c\mathbf{I} \in \mathrm{Sym}^+\}; \tag{2.144}$$

$\mathrm{Sym}^{(\sigma^t,\sigma^c)}$ is a closed, convex subset of Sym, which is not a cone.

For $\mathbf{E} \in \mathrm{Sym}$, let us consider the problem of finding $\mathbf{T} \in \mathrm{Sym}^{(\sigma^t,\sigma^c)}$ such that the variational inequality

$$(\mathbf{T} - \mathbf{T}^*)\cdot(\mathbf{E} - \mathbb{C}^{-1}[\mathbf{T}]) \geq 0 \quad \text{for each } \mathbf{T}^* \in \ \mathrm{Sym}^{(\sigma^t,\sigma^c)} \tag{2.145}$$

holds.

Proposition 2.16. *For every* $\mathbf{E} \in$ *Sym, there exists a unique* $\mathbf{T} \in$ *Sym*$^{(\sigma^t,\sigma^c)}$ *which satisfies the variational inequality (2.145).*

Proof. Inequality (2.145) characterizes $\mathbf{T}$ as the $\mathbb{C}^{-1}$-orthogonal projection of $\mathbb{C}[\mathbf{E}]$ onto the closed convex set $\mathrm{Sym}^{(\sigma^t,\sigma^c)}$, whose existence and uniqueness is guaranteed by the minimum norm theorem 1.1. ■

Proposition 2.17. *For* $\mathbf{E} \in$ *Sym,* $(\mathbf{T}, \mathbf{E}^e, \mathbf{E}^f, \mathbf{E}^c)$ *is the solution to the constitutive equation (2.109) if and only if* $\mathbf{T} \in$ *Sym*$^{(\sigma^t,\sigma^c)}$ *and satisfies (2.145).*

Proof. Let $(\mathbf{T}, \mathbf{E}^e, \mathbf{E}^f, \mathbf{E}^c)$ be the solution to (2.109); conditions $(2.109)_1$ and $(2.109)_2$ imply that $\mathbf{T} \in \mathrm{Sym}^{(\sigma^t,\sigma^c)}$. For $\mathbf{T}^* \in \mathrm{Sym}^{(\sigma^t,\sigma^c)}$, from $(2.109)_8$, $(2.109)_4$ and proposition 1.9, it follows that

$$\begin{aligned}(\mathbf{T} - \mathbf{T}^*)\cdot\mathbf{E}^f &= [(\mathbf{T} - \sigma^t\mathbf{I}) - (\mathbf{T}^* - \sigma^t\mathbf{I})]\cdot\mathbf{E}^f \\ &= -(\mathbf{T}^* - \sigma^t\mathbf{I})\cdot\mathbf{E}^f \geq 0.\end{aligned} \tag{2.146}$$

Analogously, from $(2.109)_8$, $(2.109)_3$ and proposition 1.9, we have

$$\begin{aligned}(\mathbf{T} - \mathbf{T}^*)\cdot\mathbf{E}^c &= [(\mathbf{T} + \sigma^c\mathbf{I}) - (\mathbf{T}^* + \sigma^c\mathbf{I})]\cdot\mathbf{E}^c \\ &= -(\mathbf{T}^* + \sigma^c\mathbf{I})\cdot\mathbf{E}^c \geq 0.\end{aligned} \tag{2.147}$$

Thus, by adding (2.146) and (2.147), bearing in mind that $\mathbf{E}^f + \mathbf{E}^c = \mathbf{E} - \mathbb{C}^{-1}[\mathbf{T}]$, (2.145) follows.

Conversely, from propositions 2.15 and 2.16, it follows that if $\mathbf{T} \in \mathrm{Sym}^{(\sigma^t,\sigma^c)}$ is the solution to (2.145), then $\mathbf{T}$ satisfies (2.109). ■

For a BCS material with constitutive equation (2.108), we define the strain energy density as

$$\psi_{BC}(\mathbf{E}) = \frac{1}{2}\mathbb{C}[\mathbf{E}^e] \cdot \mathbf{E}^e + \mathbf{T} \cdot (\mathbf{E}^f + \mathbf{E}^c), \quad \text{for each } \mathbf{E} \in \ \mathrm{Sym}. \tag{2.148}$$

Proposition 2.18. *Stress function* $\widehat{\mathbf{T}}_{BC}$ *is monotone,*

$$(\widehat{\mathbf{T}}_{BC}(\mathbf{E}_1) - \widehat{\mathbf{T}}_{BC}(\mathbf{E}_2))\cdot(\mathbf{E}_1 - \mathbf{E}_2) \geq \kappa||\widehat{\mathbf{T}}_{BC}(\mathbf{E}_1) - \widehat{\mathbf{T}}_{BC}(\mathbf{E}_2)||^2, \tag{2.149}$$

for each $\mathbf{E}_1, \mathbf{E}_2 \in$ *Sym, and Lipschitz continuous,*

$$||\widehat{\mathbf{T}}_{BC}(\mathbf{E}_1) - \widehat{\mathbf{T}}_{BC}(\mathbf{E}_2)|| \leq \kappa^{-1}||\mathbf{E}_1 - \mathbf{E}_2||, \tag{2.150}$$

for each $\mathbf{E}_1, \mathbf{E}_2 \in$*Sym. Moreover, function* ψ_{BC} *in (2.148) is continuously differentiable, convex and*

$$D_E\psi_{BC}(\mathbf{E}) = \widehat{\mathbf{T}}_{BC}(\mathbf{E}). \tag{2.151}$$

Proof. The proof of (2.149), (2.150) and (2.151) is similar to the proof of analogous results for $\widehat{\mathbf{T}}$ in Section 2.1 and is based on the fact that $\widehat{\mathbf{T}}_{BC}(\mathbf{E})$ is the solution of the variational inequality (2.145). ■

The two-dimensional case will be dealt with in the Appendix.

2.3 Masonry-Like Materials Under Non-Isothermal Conditions

There are many problems in which the presence of thermal dilatation must be taken into account. Consider, for example, the influence of thermal variations on the stress field in masonry bridges [49], or the thermo-mechanical behavior of the refractory materials used in the iron and steel industry [60], and, lastly, geological problems connected with the presence of volcanic calderas, such as that at Pozzuoli [29]. In the last two cases, the thermal variation during the thermo-mechanical process under examination can be so high that the dependence of the material constants on temperature cannot be ignored.

In this section we recall the constitutive equation for isotropic masonry-like materials with zero tensile strength and infinite compressive strength under non-isothermal conditions developed in [71].

Let $\vartheta \in [\vartheta_1, \vartheta_2]$, with $\vartheta_1 > 0$, be the absolute temperature and $\vartheta_0 \in [\vartheta_1, \vartheta_2]$ the reference temperature. In view of the target applications, no limitations are placed on the range of temperature variation. We assume that the thermal dilatation due to the temperature variation $\vartheta - \vartheta_0$ is the spherical tensor $\beta(\vartheta)\mathbf{I}$, where $\beta(\vartheta)$ is a material function of the temperature called thermal expansion, with $\beta(\vartheta_0) = 0$.

Moreover, we assume that there exist $\delta \in [0, 1)$ such that

$$||\mathbf{E} - \beta(\vartheta)\mathbf{I}|| \leq \delta, \quad \text{for each } \vartheta \in [\vartheta_1, \vartheta_2]. \tag{2.152}$$

Condition (2.152) is equivalent to requiring that the norm of the deviatoric part of $\mathbf{E}$ and the scalar function $\beta(\vartheta) - tr(\mathbf{E})/3$ be $O(\delta)$[2].

Let $E(\vartheta)$ and $\nu(\vartheta)$ be temperature-dependent functions such that

$$E(\vartheta) > 0, \qquad 0 \leq \nu(\vartheta) < 1/2, \qquad \text{for each } \vartheta \in [\vartheta_1, \vartheta_2], \tag{2.153}$$

and let us set

$$\gamma(\vartheta) = \frac{\nu(\vartheta)}{1 - 2\nu(\vartheta)}. \tag{2.154}$$

[2] Given a mapping B from a neighbourhood of 0 in $\mathbb{R}$ into a vector space with norm $||\ ||$, we write $B(\delta) = O(\delta)$ if there exist $k > 0$ and $k' > 0$ such that $||B(\delta)|| < k\delta$ whenever $|\delta| < k'$.

By generalizing the constitutive equation described in section 2.1, we present a nonlinear elastic constitutive equation which associate a negative semidefinite stress $\mathbf{T}$ to each strain $\mathbf{E} - \beta(\vartheta)\mathbf{I}$.

Let $\mathbb{C}(\vartheta)$ be the positive definite fourth-order tensor

$$\mathbb{C}(\vartheta) = \frac{E(\vartheta)}{1+\nu(\vartheta)}(\mathbb{I} + \gamma(\vartheta)\mathbf{I} \otimes \mathbf{I}). \tag{2.155}$$

Definition 2.19. *A masonry-like material under non-isothermal conditions is an elastic material whose stress function* $\widehat{\mathbf{T}}_{NI} : Sym \times [\vartheta_1, \vartheta_2] \to Sym$,

$$\mathbf{T} = \widehat{\mathbf{T}}_{NI}(\mathbf{E}, \vartheta), \quad (\mathbf{E}, \vartheta) \in Sym \times [\vartheta_1, \vartheta_2] \tag{2.156}$$

satisfies the system

$$\begin{cases} \mathbf{T} \in \ Sym^-, \\ \mathbf{E} - \beta(\vartheta)\mathbf{I} = \mathbf{E}^e + \mathbf{E}^f, \\ \mathbf{E}^f \cdot \mathbf{T} = 0, \\ \mathbf{T} = \mathbb{C}(\vartheta)[\mathbf{E}^e], \\ \mathbf{E}^f \in \ Sym^+. \end{cases} \tag{2.157}$$

As in the isothermal case, $\mathbf{T}$ is the projection of $\mathbb{C}(\vartheta)[\mathbf{E} - \beta(\vartheta)\mathbf{I}]$ onto Sym$^-$ with respect to the inner product $(\mathbf{A}, \mathbf{B}) = \mathbb{C}(\vartheta)^{-1}[\mathbf{A}] \cdot \mathbf{B}$ in Sym, and $\mathbf{E}^f = \mathbf{E} - \beta(\vartheta)\mathbf{I} - \mathbb{C}(\vartheta)^{-1}[\mathbf{T}]$ belongs to the normal cone $\mathcal{N}(\mathbf{T})$ to Sym$^-$ at $\mathbf{T}$ defined in (2.11). Once again denoting $\{\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3\}$ as an orthonormal basis constituted by eigenvectors of $\mathbf{E}$, and with (e_1, e_2, e_3) the eigenvalues of $\mathbf{E}$, we can show that for each $(\mathbf{E}, \vartheta) \in$ Sym $\times [\vartheta_1, \vartheta_2]$, the unique solution $(\mathbf{T}, \mathbf{E}^f)$ to (2.157) is given by

$$\mathbf{T} = \mathbb{C}(\vartheta)[\mathbf{E} - \beta(\vartheta)\mathbf{I}], \tag{2.158}$$

$$\mathbf{E}^f = \mathbf{0}, \tag{2.159}$$

for $(\mathbf{E}, \vartheta) \in \mathcal{W}_1 \cup \mathcal{I}^{\mathcal{W}}_{1,4}$;

$$\mathbf{T} = \mathbf{0}, \tag{2.160}$$

$$\mathbf{E}^f = \mathbf{E} - \beta(\vartheta)\mathbf{I}, \tag{2.161}$$

for $(\mathbf{E}, \vartheta) \in \mathcal{W}_2 \cup \mathcal{I}^{\mathcal{W}}_{2,3}$;

$$\mathbf{T} = E(\vartheta)(e_1 - \beta(\vartheta))\mathbf{q}_1 \otimes \mathbf{q}_1, \tag{2.162}$$

$$\begin{aligned} \mathbf{E}^f &= [e_2 - \beta(\vartheta) + \nu(\vartheta)(e_1 - \beta(\vartheta))]\mathbf{q}_2 \otimes \mathbf{q}_2 \\ &+ [e_3 - \beta(\vartheta) + \nu(\vartheta)(e_1 - \beta(\vartheta))]\mathbf{q}_3 \otimes \mathbf{q}_3, \end{aligned} \tag{2.163}$$

for $(\mathbf{E}, \vartheta) \in \mathcal{W}_3 \cup \mathcal{I}^{\mathcal{W}}_{3,4}$;

$$\begin{aligned} \mathbf{T} = \frac{E(\vartheta)}{1-\nu^2(\vartheta)} \{ &[e_1 - \beta(\vartheta) + \nu(\vartheta)(e_2 - \beta(\vartheta))]\mathbf{q}_1 \otimes \mathbf{q}_1 \\ &+ [e_2 - \beta(\vartheta) + \nu(\vartheta)(e_1 - \beta(\vartheta))]\mathbf{q}_2 \otimes \mathbf{q}_2 \}, \end{aligned} \tag{2.164}$$

$$\mathbf{E}^f = \frac{1}{1-\nu(\vartheta)}[e_3 - \beta(\vartheta) \\ +\nu(\vartheta)(e_1 + e_2 - e_3 - \beta(\vartheta))]\mathbf{q}_3 \otimes \mathbf{q}_3, \tag{2.165}$$

for $(\mathbf{E}, \vartheta) \in \mathcal{W}_4$; where

$$\mathcal{W}_1 = \{(\mathbf{E}, \vartheta) \mid e_3 - \beta(\vartheta) + \gamma(\vartheta)(tr(\mathbf{E}) - 3\beta(\vartheta)) < 0\}, \tag{2.166}$$

$$\mathcal{W}_2 = \{(\mathbf{E}, \vartheta) \mid e_1 - \beta(\vartheta) > 0\}, \tag{2.167}$$

$$\mathcal{W}_3 = \{(\mathbf{E}, \vartheta) \mid e_1 - \beta(\vartheta) < 0, \\ \nu(\vartheta)(e_1 - \beta(\vartheta)) + e_2 - \beta(\vartheta) > 0\}, \tag{2.168}$$

$$\mathcal{W}_4 = \{(\mathbf{E}, \vartheta) \mid \nu(\vartheta)(e_1 - \beta(\vartheta)) + e_2 - \beta(\vartheta) < 0, \\ e_3 - \beta(\vartheta) + \gamma(\vartheta)(tr(\mathbf{E}) - 3\beta(\vartheta)) > 0\}, \tag{2.169}$$

$$\mathcal{I}^{\mathcal{W}}_{1,4} = \{(\mathbf{E}, \vartheta) \mid e_3 - \beta(\vartheta) + \gamma(\vartheta)(tr(\mathbf{E}) - 3\beta(\vartheta)) = 0\}, \tag{2.170}$$

$$\mathcal{I}^{\mathcal{W}}_{2,3} = \{(\mathbf{E}, \vartheta) \mid e_1 - \beta(\vartheta) = 0\}, \tag{2.171}$$

$$\mathcal{I}^{\mathcal{W}}_{3,4} = \{(\mathbf{E}, \vartheta) \mid \nu(\vartheta)(e_1 - \beta(\vartheta)) + e_2 - \beta(\vartheta) = 0\}. \tag{2.172}$$

It is an easy matter to verify that in the absence of temperature variations, the material characterized by the constitutive equation (2.157) conforms to the isothermal masonry-like material dealt with in section 2.1. For a masonry-like material under non-isothermal conditions we define the strain energy density

$$\psi_{NI}(\mathbf{E}, \vartheta) = \frac{1}{2}\widehat{\mathbf{T}}_{NI}(\mathbf{E}, \vartheta) \cdot (\mathbf{E} - \beta(\vartheta)\mathbf{I}). \tag{2.173}$$

Function ψ_{NI} is continuously differentiable, convex and

$$D_E\psi_{NI}(\mathbf{E}, \vartheta) = \widehat{\mathbf{T}}_{NI}(\mathbf{E}, \vartheta). \tag{2.174}$$

The reader is referred to [71] for a more detailed description of the behavior of masonry-like materials under non-isothermal conditions. Starting from the explicit expression of the stress as function of strain and temperature, the paper deduces the free energy, internal energy, entropy and enthalpy, and defines the specific heat at constant strain. By assuming the classical Fourier hypothesis for heat flux, the material presented therein is characterized completely by five functions of the temperature: Young's modulus, Poisson's ratio, thermal expansion, conductivity and specific heat. Indeed, when these material functions are known, the thermodynamic potentials (and consequently the thermo-mechanical behavior) of the material can be determined. Just as in the linear case, the basic equations of the thermoelastic theory for no-tension materials are: the strain-displacement relation, the equilibrium equation, the constitutive equations for stress and heat flux, and the equilibrium energy equation. The system obtained is coupled because the temperature coefficient and the coefficient of the derivative of temperature with respect to time in the energy equation depend on strain and strain rate. However, without any hypothesis on the range of variation of temperature, if we assume small values for the displacement gradient, the thermal expansion and its derivative with respect to temperature, strain rate and temperature rate are small, then the thermoelastic equilibrium equations are uncoupled and can be integrated separately.

It is worth noting that if the further temperature condition

$$\vartheta = \vartheta_0 + O(\delta) \tag{2.175}$$

holds, with ϑ_0 the reference temperature, we then have

$$\beta(\vartheta) = \beta(\vartheta_0) + \beta'(\vartheta_0)(\vartheta - \vartheta_0) + o(\delta) \tag{2.176}$$

and thus, by taking into account that $\beta(\vartheta_0) = 0$, we obtain that in a neighborhood of $\vartheta = \vartheta_0$

$$\beta(\vartheta) = \beta'(\vartheta_0)(\vartheta - \vartheta_0), \tag{2.177}$$

within an error of order $o(\delta)$. The quantity

$$\alpha = \beta'(\vartheta_0) \tag{2.178}$$

is the linear coefficient of thermal expansion. In this case, functions E and ν must be presumed to be temperature-independent and coincident with their value at ϑ_0 [22].

2.4 The Derivative of the Stress Function

In this section we calculate the derivative of functions $\widehat{\mathbf{T}}$, $\widehat{\mathbf{T}}_{BC}$ and $\widehat{\mathbf{T}}_{NI}$ with respect to $\mathbf{E}$, under the assumption that $\mathbb{C}$ is isotropic. These derivatives will be used in the next chapter, dealing with numerical solution of the equilibrium problem via the finite elements method. In particular, knowing the explicit expressions of $D_E\widehat{\mathbf{T}}, D_E\widehat{\mathbf{T}}_{BC}$ and $D_E\widehat{\mathbf{T}}_{BNI}$ allows us to calculate for each strain tensor $\mathbf{E}$ the tangent stiffness matrix used in the numerical procedure described in chapter 4.

Proposition 2.20. *Stress function $\widehat{\mathbf{T}}$ given in (2.65) is differentiable in each region $\mathcal{R}_i$, $i = 1, ..., 4$.*

Proof. With the help of the results given in subsection 1.7 we are in a position to calculate $D_E\widehat{\mathbf{T}}$ in the four regions $\mathcal{R}_i$. Let $e_1 \le e_2 \le e_3$ and $\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3$ be the eigenvalues and the eigenvectors of $\mathbf{E}$, respectively. We consider the orthonormal basis of Sym,

$$\mathbf{O}_{11} = \mathbf{q}_1 \otimes \mathbf{q}_1, \quad \mathbf{O}_{22} = \mathbf{q}_2 \otimes \mathbf{q}_2, \quad \mathbf{O}_{33} = \mathbf{q}_3 \otimes \mathbf{q}_3, \tag{2.179}$$

$$\mathbf{O}_{12} = \frac{1}{\sqrt{2}}(\mathbf{q}_1 \otimes \mathbf{q}_2 + \mathbf{q}_2 \otimes \mathbf{q}_1), \tag{2.180}$$

$$\mathbf{O}_{13} = \frac{1}{\sqrt{2}}(\mathbf{q}_1 \otimes \mathbf{q}_3 + \mathbf{q}_3 \otimes \mathbf{q}_1), \tag{2.181}$$

$$\mathbf{O}_{23} = \frac{1}{\sqrt{2}}(\mathbf{q}_2 \otimes \mathbf{q}_3 + \mathbf{q}_3 \otimes \mathbf{q}_2), \tag{2.182}$$

and (2.65) then becomes

$$\mathbf{T} = \widehat{\mathbf{T}}(\mathbf{E}) = \sum_{i=1}^{3} t_i \mathbf{O}_{ii}. \tag{2.183}$$

From (2.60) and (2.61), it follows that the expression of $D_E\widehat{\mathbf{T}}$ for $\mathbf{E}$ belonging to $\mathcal{R}_1$ and $\mathcal{R}_2$ can be calculated easily. Calculation of $D_E\widehat{\mathbf{T}}$ when $\mathbf{E}$ belongs to the two other regions is slightly more complex and requires differentiating expression (2.183) with respect to $\mathbf{E}$ by using proposition 1.20. As a single example, we shall calculate $D_E\widehat{\mathbf{T}}$ when $\mathbf{E} \in \mathcal{R}_3$, where $e_1 < e_2 \leq e_3$. Let us begin by supposing that $e_1 < e_2 < e_3$; from equations (2.183), (2.62) and (1.92)-(1.97), using the relation

$$D_E\widehat{\mathbf{T}} = \sum_{i=1}^{3}(D_E t_i \mathbf{O}_{ii} + t_i D_E \mathbf{O}_{ii}), \tag{2.184}$$

we obtain

$$\begin{aligned} D_E\widehat{\mathbf{T}} = E(&\mathbf{O}_{11} \otimes \mathbf{O}_{11} - \frac{e_1}{e_2 - e_1}\mathbf{O}_{12} \otimes \mathbf{O}_{12} \\ &- \frac{e_1}{e_3 - e_1}\mathbf{O}_{13} \otimes \mathbf{O}_{13}), \quad \mathbf{E} \in \mathcal{R}_3, \end{aligned} \tag{2.185}$$

which is well-defined also when $e_2 = e_3$. ■

Let us then summarize the expressions of $D_E\widehat{\mathbf{T}}$ in the four regions $\mathcal{R}_i$,

$$D_E\widehat{\mathbf{T}} = \mathbb{T}_1 = 2\mu\mathbb{I} + \lambda\mathbf{I} \otimes \mathbf{I}, \quad \mathbf{E} \in \mathcal{R}_1, \tag{2.186}$$

$$D_E\widehat{\mathbf{T}} = \mathbb{T}_2 = \mathbb{O}, \quad \mathbf{E} \in \mathcal{R}_2, \tag{2.187}$$

$$\begin{aligned} D_E\widehat{\mathbf{T}} = \mathbb{T}_3 = E(&\mathbf{O}_{11} \otimes \mathbf{O}_{11} - \frac{e_1}{e_2 - e_1}\mathbf{O}_{12} \otimes \mathbf{O}_{12} \\ &- \frac{e_1}{e_3 - e_1}\mathbf{O}_{13} \otimes \mathbf{O}_{13}), \quad \mathbf{E} \in \mathcal{R}_3, \end{aligned} \tag{2.188}$$

$$D_E\widehat{\mathbf{T}} = \mathbb{T}_4 = 2\mu\mathbf{O}_{12} \otimes \mathbf{O}_{12} - \frac{2\mu}{2+\alpha}\frac{2(1+\alpha)e_1 + \alpha e_2}{e_3 - e_1}\mathbf{O}_{13} \otimes \mathbf{O}_{13}$$

$$-\frac{2\mu}{2+\alpha}\frac{2(1+\alpha)e_2+\alpha e_1}{e_3-e_2}\mathbf{O}_{23}\otimes\mathbf{O}_{23}$$
$$+\frac{\mu(2+3\alpha)}{2+\alpha}(\mathbf{O}_{11}+\mathbf{O}_{22})\otimes(\mathbf{O}_{11}+\mathbf{O}_{22})$$
$$+\mu(\mathbf{O}_{11}-\mathbf{O}_{22})\otimes(\mathbf{O}_{11}-\mathbf{O}_{22}), \quad \mathbf{E}\in\mathcal{R}_4, \tag{2.189}$$

where $\mathbb{I}$ and $\mathbb{O}$ are the fourth-order identity tensor and the fourth-order null tensor, respectively. It is worthwhile noting that the expressions given in (2.187)-(2.189) are the spectral representations of $D_E\widehat{\mathbf{T}}$ in the three regions $\mathcal{R}_2, \mathcal{R}_2$ and $\mathcal{R}_4$. Moreover, it can be easily verified that the eigenvalues of $D_E\widehat{\mathbf{T}}$ are non-negative, a result which conforms with convexity of the strain energy density defined in (2.25), already proved in Section 2.1.

On the interfaces $\mathcal{I}^{\mathcal{R}}_{1,4}, \mathcal{I}^{\mathcal{R}}_{2,3}$ and $\mathcal{I}^{\mathcal{R}}_{3,4}$ respectively defined in (2.57), (2.58) and (2.59), tensor $D_E\widehat{\mathbf{T}}(\mathbf{E})$ does not exist and, in view of the Lipschitz continuity of $\widehat{\mathbf{T}}$, it is possible to define the subgradient of $\widehat{\mathbf{T}}$ constituted by the convex combination $\mathbb{S}(\mathbf{E})$ of tensors $\mathbb{T}_i$ [33], [26]

$$\mathbb{S}(\mathbf{E}) = \{\xi\mathbb{T}_3 + (1-\xi)\mathbb{T}_4 \mid \xi\in[0,1]\}, \quad \mathbf{E}\in\mathcal{I}^{\mathcal{R}}_{3,4}, \tag{2.190}$$

$$\mathbb{S}(\mathbf{E}) = \{\xi\mathbb{T}_2 + (1-\xi)\mathbb{T}_3 \mid \xi\in[0,1]\}, \quad \mathbf{E}\in\mathcal{I}^{\mathcal{R}}_{2,3}, \tag{2.191}$$

$$\mathbb{S}(\mathbf{E}) = \{\xi\mathbb{T}_1 + (1-\xi)\mathbb{T}_4 \mid \xi\in[0,1]\}, \quad \mathbf{E}\in\mathcal{I}^{\mathcal{R}}_{14}. \tag{2.192}$$

Proposition 2.21. *For each $\mathbf{E}\in Sym$ in which $\widehat{\mathbf{T}}$ is differentiable, $D_E\widehat{\mathbf{T}}$ has the following properties*

$$D_E\widehat{\mathbf{T}}(\mathbf{E})[\mathbf{E}] = \widehat{\mathbf{T}}(\mathbf{E}), \tag{2.193}$$

$$D_E\widehat{\mathbf{T}}(\mathbf{E})[\mathbf{E}^e] = \widehat{\mathbf{T}}(\mathbf{E}), \quad D_E\widehat{\mathbf{T}}(\mathbf{E})[\mathbf{E}^f] = \mathbf{0}, \tag{2.194}$$

$$D_E\widehat{\mathbf{T}}(\beta\mathbf{E}) = D_E\widehat{\mathbf{T}}(\mathbf{E}), \quad \textit{for each } \beta\geq 0, \tag{2.195}$$

$$D_E\widehat{\mathbf{T}}(\mathbf{E}) = D_E\widehat{\mathbf{T}}(\mathbf{E})^T. \tag{2.196}$$

Proof. From

$$\widehat{\mathbf{T}}(\mathbf{E}+\mathbf{H}) = \widehat{\mathbf{T}}(\mathbf{E}) + D_E\widehat{\mathbf{T}}(\mathbf{E})[\mathbf{H}] + o(\mathbf{H}) \quad \text{as } \mathbf{H}\to\mathbf{0}, \tag{2.197}$$

putting $\mathbf{H} = h\mathbf{E}$, $h\geq 0$, and taking (2.15) into account, we get

$$(1+h)\widehat{\mathbf{T}}(\mathbf{E}) = \widehat{\mathbf{T}}(\mathbf{E}) + hD_E\widehat{\mathbf{T}}(\mathbf{E})[\mathbf{E}] + o(h) \quad \text{as } h\to 0, \tag{2.198}$$

whence (2.193) follows directly. The proof of (2.195) is similar and is also based on (2.15). Relations (2.194) and the major symmetry in (2.196) follow directly from the (2.186)-(2.189). ■

In addition, the jump $[D_E\widehat{\mathbf{T}}(\mathbf{E})]$ of $D_E\widehat{\mathbf{T}}(\mathbf{E})$ across the interfaces $\mathcal{I}^{\mathcal{R}}_{3,4}$, $\mathcal{I}^{\mathcal{R}}_{2,3}$ and $\mathcal{I}^{\mathcal{R}}_{1,4}$ satisfies the conditions given in [33], which express the absence of tangential discontinuity of the derivative of the stress with respect to the strain, as stated in the following proposition.

Proposition 2.22. *Tensor $D_E\widehat{\mathbf{T}}(\mathbf{E})$ satisfies the jump conditions*

$$[D_E\widehat{\mathbf{T}}(\mathbf{E})] = -\frac{\mu}{(1+\alpha)(2+\alpha)}\ \nabla g_{34}(\mathbf{E}) \otimes \nabla g_{34}(\mathbf{E}), \qquad \mathbf{E} \in \mathcal{I}^{\mathcal{R}}_{3,4}, \quad (2.199)$$

$$[D_E\widehat{\mathbf{T}}(\mathbf{E})] = E\ \nabla g_{23}(\mathbf{E}) \otimes \nabla g_{23}(\mathbf{E}), \qquad \mathbf{E} \in \mathcal{I}^{\mathcal{R}}_{2,3}, \quad (2.200)$$

$$[D_E\widehat{\mathbf{T}}(\mathbf{E})] = \frac{\mu}{2+\alpha}\ \nabla g_{14}(\mathbf{E}) \otimes \nabla g_{14}(\mathbf{E}), \qquad \mathbf{E} \in \mathcal{I}^{\mathcal{R}}_{1,4} \quad (2.201)$$

where g_{34}, g_{23} and g_{14} are given in (2.59), (2.58) and (2.57), respectively. In particular, it holds that

$$[D_E\widehat{\mathbf{T}}(\mathbf{E})][\mathbf{E}] = \mathbf{0}, \quad (2.202)$$

for $\mathbf{E} \in \mathcal{I}^{\mathcal{R}}_{3,4}$ or $\mathbf{E} \in \mathcal{I}^{\mathcal{R}}_{2,3}$, or $\mathbf{E} \in \mathcal{I}^{\mathcal{R}}_{1,4}$.

Proof. The proof is based on simple calculations involving relations (2.186)-(2.189) and the derivative with respect to $\mathbf{E}$ of functions g_{34}, g_{23} and g_{14}. ■

Proposition 2.23. *Function $\widehat{\mathbf{T}}_{BC}$ given in (2.142) is differentiable in each region $\mathcal{C}_i$, $i = 1, ..., 10$. Function $\widehat{\mathbf{T}}_{NI}$ given in (2.156) is differentiable in each region $\mathcal{W}_i$, $i = 1, ..., 4$.*

Proof. The derivatives of stress functions $\widehat{\mathbf{T}}_{BC}$ and $\widehat{\mathbf{T}}_{NI}$ can be calculated by following a procedure analogous to that sketched out for proposition 2.20. ■

In the case of material having constitutive equation (2.109), the expressions of $D_E\widehat{\mathbf{T}}_{BC}$ in the ten regions $\mathcal{C}_i$, are

$$D_E\widehat{\mathbf{T}}_{BC} = 2\mu\mathbb{I} + \lambda\mathbf{I} \otimes \mathbf{I}, \quad \mathbf{E} \in \mathcal{C}_1, \quad (2.203)$$

$$\begin{aligned} D_E\widehat{\mathbf{T}}_{BC} = {} & \frac{2\mu}{2+\alpha}\frac{\omega^c + 2(1+\alpha)e_2 + \alpha e_3}{e_2 - e_1}\mathbf{O}_{12} \otimes \mathbf{O}_{12} \\ & + \frac{2\mu}{2+\alpha}\frac{\omega^c + 2(1+\alpha)e_3 + \alpha e_2}{e_3 - e_1}\mathbf{O}_{13} \otimes \mathbf{O}_{13} + 2\mu\mathbf{O}_{23} \otimes \mathbf{O}_{23} \\ & + \frac{\mu(2+3\alpha)}{2+\alpha}(\mathbf{O}_{22} + \mathbf{O}_{33}) \otimes (\mathbf{O}_{22} + \mathbf{O}_{33}) \\ & + \mu(\mathbf{O}_{22} - \mathbf{O}_{33}) \otimes (\mathbf{O}_{22} - \mathbf{O}_{33}), \quad \mathbf{E} \in \mathcal{C}_2, \end{aligned} \quad (2.204)$$

$$\begin{aligned} D_E\widehat{\mathbf{T}}_{BC} = {} & E\mathbf{O}_{33} \otimes \mathbf{O}_{33} + \frac{\mu}{1+\alpha}\frac{\omega^c + (2+3\alpha)e_3}{e_3 - e_1}\mathbf{O}_{13} \otimes \mathbf{O}_{13} \\ & + \frac{\mu}{1+\alpha}\frac{\omega^c + (2+3\alpha)e_3}{e_3 - e_2}\mathbf{O}_{23} \otimes \mathbf{O}_{23}, \quad \mathbf{E} \in \mathcal{C}_3, \end{aligned} \quad (2.205)$$

$$D_E\widehat{\mathbf{T}}_{BC} = \mathbb{O}, \quad \mathbf{E} \in \mathcal{C}_4, \quad (2.206)$$

$$D_E\widehat{\mathbf{T}}_{BC} = 2\mu\mathbf{O}_{12}\otimes\mathbf{O}_{12} + \frac{2\mu}{2+\alpha}\frac{\omega^t - 2(1+\alpha)e_1 - \alpha e_2}{e_3 - e_1}\mathbf{O}_{13}\otimes\mathbf{O}_{13}$$

$$+\frac{2\mu}{2+\alpha}\frac{\omega^t - 2(1+\alpha)e_2 - \alpha e_1}{e_3 - e_2}\mathbf{O}_{23}\otimes\mathbf{O}_{23}$$

$$+\frac{\mu(2+3\alpha)}{2+\alpha}(\mathbf{O}_{11}+\mathbf{O}_{22})\otimes(\mathbf{O}_{11}+\mathbf{O}_{22})$$

$$+\mu(\mathbf{O}_{11}-\mathbf{O}_{22})\otimes(\mathbf{O}_{11}-\mathbf{O}_{22}), \quad \mathbf{E}\in\mathcal{C}_5, \tag{2.207}$$

$$D_E\widehat{\mathbf{T}}_{BC} = E\mathbf{O}_{11}\otimes\mathbf{O}_{11} + \frac{\mu}{1+\alpha}\frac{\omega^t - (2+3\alpha)e_1}{e_2 - e_1}\mathbf{O}_{12}\otimes\mathbf{O}_{12}$$

$$+\frac{\mu}{1+\alpha}\frac{\omega^t - (2+3\alpha)e_1}{e_3 - e_1}\mathbf{O}_{13}\otimes\mathbf{O}_{13}, \quad \mathbf{E}\in\mathcal{C}_6, \tag{2.208}$$

$$D_E\widehat{\mathbf{T}}_{BC} = \mathbb{O}, \quad \mathbf{E}\in\mathcal{C}_7, \tag{2.209}$$

$$D_E\widehat{\mathbf{T}}_{BC} = \frac{\sigma^t+\sigma^c}{e_2 - e_1}\mathbf{O}_{12}\otimes\mathbf{O}_{12} + \frac{\sigma^t+\sigma^c}{e_3 - e_1}\mathbf{O}_{13}\otimes\mathbf{O}_{13}, \quad \mathbf{E}\in\mathcal{C}_8, \tag{2.210}$$

$$D_E\widehat{\mathbf{T}}_{BC} = \frac{\sigma^t+\sigma^c}{e_3 - e_1}\mathbf{O}_{13}\otimes\mathbf{O}_{13} + \frac{\sigma^t+\sigma^c}{e_3 - e_2}\mathbf{O}_{23}\otimes\mathbf{O}_{23}, \quad \mathbf{E}\in\mathcal{C}_9, \tag{2.211}$$

$$D_E\widehat{\mathbf{T}}_{BC} = E\mathbf{O}_{22}\otimes\mathbf{O}_{22} + \frac{\mu}{2(1+\alpha)}\frac{\alpha\omega^t + (2+\alpha)\omega^c + 2(2+3\alpha)e_2}{e_2 - e_1}\mathbf{O}_{12}\otimes\mathbf{O}_{12}$$

$$+\frac{\sigma^t+\sigma^c}{e_3 - e_1}\mathbf{O}_{13}\otimes\mathbf{O}_{13}$$

$$+\frac{\mu}{2(1+\alpha)}\frac{\alpha\omega^c + (2+\alpha)\omega^t - 2(2+3\alpha)e_2}{e_3 - e_2}\mathbf{O}_{23}\otimes\mathbf{O}_{23}, \quad \mathbf{E}\in\mathcal{C}_{10}. \tag{2.212}$$

As for the derivative of $\widehat{\mathbf{T}}_{NI}(\mathbf{E},\vartheta)$ with respect to $\mathbf{E}$, we have

$$D_E\widehat{\mathbf{T}}_{NI} = \frac{E(\vartheta)}{1+\nu(\vartheta)}\{\mathbb{I} + \frac{\nu(\vartheta)}{1-2\nu(\vartheta)}\mathbf{I}\otimes\mathbf{I}\}, \quad (\mathbf{E},\vartheta)\in\mathcal{W}_1, \tag{2.213}$$

$$D_E\widehat{\mathbf{T}}_{NI} = \mathbb{O}, \quad (\mathbf{E},\vartheta)\in\mathcal{W}_2, \tag{2.214}$$

$$D_E\widehat{\mathbf{T}}_{NI} = E(\vartheta)(\mathbf{O}_{11}\otimes\mathbf{O}_{11} - \frac{e_1 - \beta(\vartheta)}{e_2 - e_1}\mathbf{O}_{12}\otimes\mathbf{O}_{12}$$

$$-\frac{e_1 - \beta(\vartheta)}{e_3 - e_1}\mathbf{O}_{13}\otimes\mathbf{O}_{13}), \quad (\mathbf{E},\vartheta)\in\mathcal{W}_3, \tag{2.215}$$

$$D_E\widehat{\mathbf{T}}_{NI} = \frac{E(\vartheta)}{2(1+\nu(\vartheta))}((\mathbf{O}_{11}-\mathbf{O}_{22})\otimes(\mathbf{O}_{11}-\mathbf{O}_{22})$$

$$+\frac{1+\nu(\vartheta)}{1-\nu(\vartheta)}(\mathbf{O}_{11}+\mathbf{O}_{22})\otimes(\mathbf{O}_{11}+\mathbf{O}_{22}) + 2\mathbf{O}_{12}\otimes\mathbf{O}_{12}$$

$$-2\frac{e_1 + \nu(\vartheta)e_2 - \beta(\vartheta)(1+\nu(\vartheta))}{(1-\nu(\vartheta))(e_3 - e_1)}\mathbf{O}_{13} \otimes \mathbf{O}_{13}$$

$$-2\frac{e_2 + \nu(\vartheta)e_1 - \beta(\vartheta)(1+\nu(\vartheta))}{(1-\nu(\vartheta))(e_3 - e_2)}\mathbf{O}_{23} \otimes \mathbf{O}_{23}), \quad (\mathbf{E}, \vartheta) \in \mathcal{W}_4. \tag{2.216}$$

The derivatives of $\widehat{\mathbf{T}}$, $\widehat{\mathbf{T}}_{BC}$ and $\widehat{\mathbf{T}}_{NI}$ for the plane cases are summarized in the Appendix.

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