2

Scalarization Approaches

For determining solutions of the multiobjective optimization problem (MOP)

$$
\begin{aligned}
\text{(MOP)} \qquad & \min f(x) \\
& \text{subject to the constraints} \\
& g(x) \in C, \\
& h(x) = 0_q, \\
& x \in S
\end{aligned}
$$

with the constraint set $\Omega = \{x \in S \mid g(x) \in C,\ h(x) = 0_q\}$ a wide-spread approach is the transformation of this problem to a scalar-valued parameter dependent optimization problem. This is done for instance in the weighted sum method ([245]). There the scalar problems

$$\min_{x \in \Omega} \sum_{i=1}^{m} w_i f_i(x)$$

with weights $w \in K^* \setminus \{0_m\}$ and K^* the dual cone to the cone K, i. e. $K^* = \{y^* \in \mathbb{R}^m \mid (y^*)^\top y \geq 0 \text{ for all } y \in K\}$, are solved. Another scalarization especially for calculating EP-minimal points is based on the minimization of only one of the m objectives while all the other objectives are transformed into constraints by introducing upper bounds. This scalarization is called ε-constraint method ([98, 159]) and is given by

$$
\begin{aligned}
& \min f_k(x) \\
& \text{subject to the constraints} \\
& f_i(x) \leq \varepsilon_i, \quad i \in \{1, \ldots, m\} \setminus \{k\}, \\
& x \in \Omega.
\end{aligned}
\tag{2.1}
$$

Here the parameters are the upper bounds $\varepsilon_i, i \in \{1, \ldots, m\} \setminus \{k\}$ for a $k \in \{1, \ldots, m\}$. Surveys about different scalarization approaches can be found in [60, 112, 124, 138, 165, 189]. Other solution approaches use e. g. stochastic methods as it is done by Schäffler, Schultz and Weinzierl ([194]) or evolutionary algorithms (surveys for these types of methods can be found in [112, p.19] and in [41, 42, 31, 228, 246]). In this book only procedures based on a scalarization of the multiobjective optimization problem are considered.

By solving the scalar problems for a variety of parameters for instance for different weights, several solutions of the multiobjective optimization problem are generated. In the last decades the main focus was on finding one minimal solution e. g. by interactive methods ([166, 165]) whereas objective numerical calculations alternate with subjective decisions done by the decision maker. Based on much better computer performances it is now possible to represent the whole efficient set. Having the whole solution set available the decision maker gets a useful insight in the problem structure. For engineering tasks, it is especially interesting to have all design alternatives available ([119]). The aim is to generate an approximation of the whole efficient set as it is the aim for instance in [40, 47, 48, 81, 106, 110, 111, 164, 196] and many more.

The information provided by this approximation depends mainly on the quality of the approximation. Many approximation points cause a high numerical effort, however approximations with only few points neglect large areas of the efficient set. Thus, the aim of this book is to generate an approximation with a high quality.

A wide variety of scalarizations exist based on which one can determine single approximation points. However not all methods are appropriate for non-convexity or arbitrary partial orderings. For instance the weighted sum method has the disadvantage that it is in general only possible for convex problems to determine all efficient points by an appropriate parameter choice (see [39, 138]). The ε-constraint method as given in (2.1) is only suited for the calculation of EP-minimal points. Yet problems arising in applications are often non-convex. Further it is also of interest to consider more general partial orderings than the natural ordering. Thus we concentrate on a scalarization by Pascoletti and Serafini, 1984 ([181]) which we present and discuss in the following sections. An advantage of this scalarization is that many other scalarization approaches as the mentioned weighted sum method or the ε-constraint method are included in this more general formulation. The relationship to other scalarization problems are examined in the last section of this chapter.

2.1 Pascoletti-Serafini Scalarization

Pascoletti and Serafini propose the following scalar optimization problem with parameters $a \in \mathbb{R}^m$ and $r \in \mathbb{R}^m$ for determining minimal solutions of (MOP) w. r. t. the cone K:

$$(\mathrm{SP(a,r)}) \qquad \begin{array}{c} \min t \\ \text{subject to the constraints} \\ a + t\,r - f(x) \in K, \\ g(x) \in C, \\ h(x) = 0_q, \\ t \in \mathbb{R},\ x \in S. \end{array}$$

This problem has the parameter dependent constraint set

$$\Sigma(a,r) := \{(t,x) \in \mathbb{R}^{n+1} \mid a + t\,r - f(x) \in K,\ x \in \Omega\}.$$

We assume that the cone K is a nonempty closed pointed convex cone. The formulation of this scalar optimization problem corresponds to the definition of K-minimality. A point $\bar{x} \in \Omega$ with $\bar{y} = f(\bar{x})$ is K-minimal if

$$(\bar{y} - K) \cap f(\Omega) = \{\bar{y}\},$$

(see Fig. 2.1 for $m = 2$ and $K = \mathbb{R}^2_+$). If we rewrite the problem (SP(a,r)) as follows

$$\begin{array}{c} \min t \\ \text{subject to the constraints} \\ f(x) \in a + tr - K, \\ x \in \Omega, \\ t \in \mathbb{R}, \end{array}$$

we see that for solving this problem the ordering cone $-K$ is moved in direction $-r$ on the line $a + t\,r$ starting in the point a till the set $(a + t\,r - K) \cap f(\Omega)$ is reduced to the empty set. The smallest value $\bar{t}$ for which $(a + \bar{t}\,r - K) \cap f(\Omega) \neq \emptyset$ is the minimal value of (SP(a,r)) (see Fig. 2.2 with $m = 2$ and $K = \mathbb{R}^2_+$).

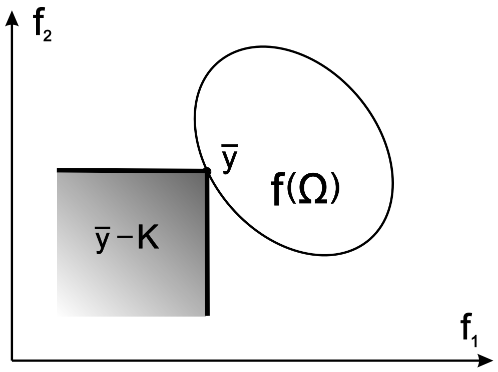

Fig. 2.1. K-minimality.

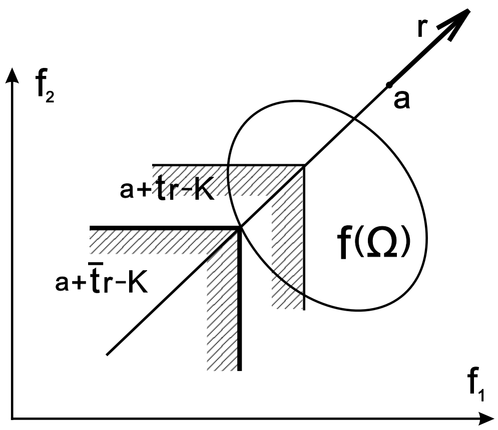

Fig. 2.2. Moving the ordering cone in the Pascoletti-Serafini problem.

The scalar problem (SP(a, r)) features all important properties a scalarization approach for determining minimal solutions of (MOP) should have. If $(\bar{t}, \bar{x})$ is a minimal solution of (SP(a, r)) then the point $\bar{x}$ is an at least weakly K-minimal solution of the multiobjective optimization problem (MOP) and by a variation of the parameters $(a, r) \in \mathbb{R}^m \times \mathbb{R}^m$ all K-minimal points of (MOP) can be found as solutions of (SP(a, r)). We will discuss these important properties among others in the following section.

Problem (SP(a, r)) is also discussed by Helbig in [104]. He interprets the point a as a reference point and the parameter r as a direction. For $r \in \operatorname{int}(\mathbb{R}^m_+)$ this corresponds to the interpretation of r as a weighting of the objective functions with the weights $w_i := \frac{1}{r_i}$, $i = 1, \ldots, m$

(compare with the weighted Chebyshev norm). Then for a minimal solution $\bar{x}$ of the scalar problem the point $f(\bar{x}) = \bar{y}$ is the (weakly) K-minimal point (see Theorem 2.1,c)) which is closest to the reference point. The Pascoletti-Serafini problem is also related to a scalarization introduced by Gerstewitz in [91] as well as to the problem discussed in [92, 237] by Tammer, Weidner and Winkler. Further, in [74] Engau and Wiecek examine the Pascoletti-Serafini scalarization concerning ε-efficiency.

Pascoletti and Serafini allow for the parameter r only $r \in L(K)$ with $L(K)$ the smallest linear subspace in $\mathbb{R}^m$ including K. Sterna-Karwat discusses in [213] and [214] also this problem. Helbig ([104, 106]) assumes $r \in \operatorname{rint}(K)$, i. e. he assumes r to be an element of the relative interior of the closed pointed convex cone K.

In [106] Helbig varies not only the parameters a and r, but he also varies the cone K. For $s \in K^*$ he defines parameter dependent cones $K(s)$ with $K \subset K(s)$. For these cones he solves the scalar problems (SP(a,r)). The solutions are then weakly $K(s)$-minimal, yet w. r. t. the cone K they are even minimal. However in this book we concentrate on a variation of the parameters a and r. We will see that by an appropriate controlling of the parameters high-quality approximations in the sense of nearly equidistant approximations of the efficient set can be generated.

2.2 Properties of the Pascoletti-Serafini Scalarization

We examine the Pascoletti-Serafini problem in this section more detailed and we start with the main properties of this scalarization (see also [181]). We assume again that K is a nonempty closed pointed ordering cone in $\mathbb{R}^m$.

Theorem 2.1. *We consider the scalar optimization problem (SP(a,r)) to the multiobjective optimization problem (MOP). Let int$(K) \neq \emptyset$.*

a) Let $\bar{x}$ be a weakly K-minimal solution of the multiobjective optimization problem (MOP), then $(0,\bar{x})$ is a minimal solution of (SP(a,r)) for the parameter $a := f(\bar{x})$ and for arbitrary $r \in \operatorname{int}(K)$.

b) Let $\bar{x}$ be a K-minimal solution of the multiobjective optimization problem (MOP), then $(0,\bar{x})$ is a minimal solution of (SP(a,r)) for the parameter $a := f(\bar{x})$ and for arbitrary $r \in K \setminus \{0_m\}$.

c) *Let* $(\bar{t}, \bar{x})$ *be a minimal solution of the scalar problem (SP(a, r)), then* $\bar{x}$ *is a weakly* K*-minimal solution of the multiobjective optimization problem (MOP) and* $a + \bar{t}\, r - f(\bar{x}) \in \partial K$ *with* ∂K *the boundary of the cone* K*.*

d) *Let* $\bar{x}$ *be a locally weakly* K*-minimal solution of the multiobjective optimization problem (MOP), then* $(0, \bar{x})$ *is a local minimal solution of (SP(a, r)) for the parameter* $a := f(\bar{x})$ *and for arbitrary* $r \in \operatorname{int}(K)$*.*

e) *Let* $\bar{x}$ *be a locally* K*-minimal solution of the multiobjective optimization problem (MOP), then* $(0, \bar{x})$ *is a local minimal solution of (SP(a, r)) for the parameter* $a := f(\bar{x})$ *and for arbitrary* $r \in K \setminus \{0_m\}$*.*

f) *Let* $(\bar{t}, \bar{x})$ *be a local minimal solution of (SP(a, r)), then* $\bar{x}$ *is a locally weakly* K*-minimal solution of the multiobjective optimization problem (MOP) and* $a + \bar{t}\, r - f(\bar{x}) \in \partial K$*.*

Proof. a) Set $a = f(\bar{x})$ and choose $r \in \operatorname{int}(K)$ arbitrarily. Then the point $(0, \bar{x})$ is feasible for $(\mathrm{SP}(a, r))$. It is also minimal, because otherwise there exists a feasible point (t', x') with $t' < 0$ and a $k' \in K$ with

$$a + t'\, r - f(x') = k'.$$

Hence we have $f(\bar{x}) = f(x') + k' - t'\, r$. It is $k' - t'\, r \in \operatorname{int}(K)$ and thus it follows $f(\bar{x}) \in f(x') + \operatorname{int}(K)$ in contradiction to $\bar{x}$ weakly K-minimal.

b) Set $a = f(\bar{x})$ and choose $r \in K \setminus \{0_m\}$ arbitrarily. Then the point $(0, \bar{x})$ is feasible for $(\mathrm{SP}(a, r))$. It is also a minimal solution because otherwise there exists a scalar $t' < 0$ and a point $x' \in \Omega$, with (t', x') feasible for $(\mathrm{SP}(a, r))$, and a $k' \in K$ with $a + t'\, r - f(x') = k'$. This leads to

$$f(\bar{x}) = f(x') + k' - t'\, r \in f(x') + K.$$

Because of the K-minimality of $\bar{x}$ we conclude $f(\bar{x}) = f(x')$ and thus $k' = t'\, r$. Due to the pointedness of the ordering cone K, $k' \in K$ and $t'\, r \in -K$ it follows $t'\, r = k' = 0_m$ in contradiction to $t' < 0$ and $r \neq 0_m$.

c) Assume $\bar{x}$ is not weakly K-minimal. Then there is a point $x' \in \Omega$ and a $k' \in \operatorname{int}(K)$ with $f(\bar{x}) = f(x') + k'$. As $(\bar{t}, \bar{x})$ is a minimal solution and hence feasible for $(\mathrm{SP}(a, r))$ there is a $\bar{k} \in K$ with $a + \bar{t}\, r - f(\bar{x}) = \bar{k}$. Because of $\bar{k} + k' \in \operatorname{int}(K)$ there is a $\varepsilon > 0$ with $\bar{k} + k' - \varepsilon\, r \in \operatorname{int}(K)$. Then we conclude from $a + \bar{t}\, r - f(x') = \bar{k} + k'$

$$a + (\bar{t} - \varepsilon)\, r - f(x') \in \operatorname{int}(K).$$

Then the point $(\bar{t} - \varepsilon, x')$ is feasible for (SP(a, r)), too, with $\bar{t} - \varepsilon < \bar{t}$ in contradiction to $(\bar{t}, \bar{x})$ minimal. Using the same arguments we can show $\bar{k} \in \partial K$.

d) We assume $(0, \bar{x})$ is not a local minimal solution of (SP(a, r)). Then in any neighborhood $U = U_t \times U_x \subset \mathbb{R}^{n+1}$ of $(0, \bar{x})$ there exists a feasible point (t', x') with $t' < 0$ and a $k' \in K$ with $a + t'\, r - f(x') = k'$. With $a = f(\bar{x})$ we get

$$f(\bar{x}) = f(x') + k' - t'\, r.$$

Since $k' - t'\, r \in \operatorname{int}(K)$ we have $f(\bar{x}) \in f(x') + \operatorname{int}(K)$ and because the neighborhood U_x is arbitrarily chosen $\bar{x}$ cannot be locally weakly K-minimal.

e) With the same arguments as in the preceding proof we conclude again that if there exists a feasible point (t', x') with $t' < 0$ and x' in a neighborhood of $\bar{x}$ this leads to $f(\bar{x}) = f(x') + k' - t'\, r$ with $r \in K \setminus \{0_m\}$. Hence we have $f(\bar{x}) \in f(x') + K \setminus \{0_m\}$ in contradiction to $\bar{x}$ locally K-minimal.

f) Let $U = U_t \times U_x \subset \mathbb{R}^{n+1}$ be a neighborhood such that $(\bar{t}, \bar{x})$ is a local minimal solution of (SP(a, r)). Then there exists a $\bar{k} \in K$ with

$$a + \bar{t}\, r - f(\bar{x}) = \bar{k}. \tag{2.2}$$

We assume $\bar{x}$ is not a locally weakly K-minimal point of the multiobjective optimization problem (MOP). Then there exists no neighborhood $\bar{U}_x$ of $\bar{x}$ such that

$$f(\Omega \cap \bar{U}_x) \cap (f(\bar{x}) - \operatorname{int}(K)) = \emptyset.$$

Hence for U_x there exists a point $x' \in \Omega \cap U_x$ with $f(x') \in f(\bar{x}) - \operatorname{int}(K)$ and thus there is a $k' \in \operatorname{int}(K)$ with $f(x') = f(\bar{x}) - k'$. Together with (2.2) we get

$$f(x') = a + \bar{t}\, r - \bar{k} - k'.$$

Because of $\bar{k} + k' \in \operatorname{int}(K)$ there exists a $\varepsilon > 0$ with $\bar{t} - \varepsilon \in U_t$ and $\bar{k} + k' - \varepsilon\, r \in \operatorname{int}(K)$. We conclude

$$f(x') = a + (\bar{t} - \varepsilon)\, r - (\bar{k} + k' - \varepsilon\, r)$$

and thus $(\bar{t} - \varepsilon, x') \in U$ is feasible for (SP(a, r)) with $\bar{t} - \varepsilon < \bar{t}$ in contradiction to $(\bar{t}, \bar{x})$ a local minimal solution. $\square$

Remark 2.2. Note, that for the statement of Theorem 2.1,b) we need the pointedness of the ordering cone K. This is for instance not the case for the statement c).

Note also that it is not a consequence of Theorem 2.1,c) that we get always a weakly K-minimal point $\bar{x}$ by solving (SP(a,r)) for arbitrary parameters. It is possible that the problem (SP(a,r)) has no minimal solution at all as in the example shown in Fig. 2.3 for the case $m = 2$ and $K = \mathbb{R}^2_+$. There the minimal value of (SP(a,r)) is not bounded from below.

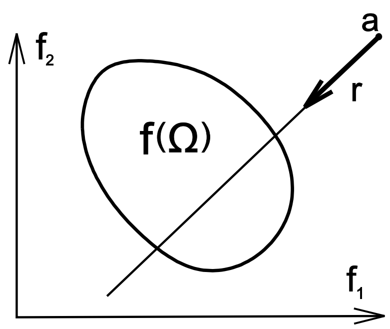

Fig. 2.3. For $K = \mathbb{R}^2_+$ there exists no minimal solution of problem (SP(a,r)).

For $\mathrm{int}(K) = \emptyset$ we cannot apply the preceding theorem. However we can still consider the case of finding minimal points w. r. t. the relative algebraic interior (or intrinsic core, see also p.11) $\mathrm{icr}(K)$. For a closed convex cone $K \subset \mathbb{R}^m$ it is $\mathrm{icr}(K) \neq \emptyset$ (compare [181, p.503]).

Theorem 2.3. *We consider the scalar optimization problem (SP(a,r)) to the multiobjective optimization problem (MOP) with $a \in \mathbb{R}^m$, $r \in L(K)$. Let $(\bar{t},\bar{x})$ be a minimal solution, then $\bar{x}$ is minimal w. r. t. $icr(K) \cup \{0_m\}$.*

For the proof of this theorem we refer to [181]. If for a choice of parameters $a \in \mathbb{R}^m$ and $r \in \mathrm{int}(K)$ the optimization problem (SP(a,r))

has no minimal solution, we can conclude under some additional assumptions that the related multiobjective optimization problem has no K-minimal solution at all. This is stated in the following theorem proven by Helbig in [104, Theorem 2.2].

Theorem 2.4. *Let $K \subset \mathbb{R}^m$ be a closed pointed convex cone with $int(K) \neq \emptyset$ and let the set $f(\Omega) + K$ be closed and convex. Assume $\mathcal{E}(f(\Omega), K) \neq \emptyset$. Then*

$$\{(a,r) \in \mathbb{R}^m \times int(K) \mid \Sigma(a,r) \neq \emptyset\} = \mathbb{R}^m \times int(K),$$

i. e. for any choice of parameters $(a,r) \in \mathbb{R}^m \times int(K)$ the scalar optimization problem (SP(a,r)) has feasible points.

Besides for all parameters $(a,r) \in \mathbb{R}^m \times int(K)$ there exists a minimal solution of (SP(a,r)).

As a direct consequence it follows:

Corollary 2.5. *Let $K \subset \mathbb{R}^m$ be a closed pointed convex cone with $int(K) \neq \emptyset$ and let the set $f(\Omega) + K$ be closed and convex. If there is a parameter $(a,r) \in \mathbb{R}^m \times int(K)$ such that (SP(a,r)) has no minimal solution then $\mathcal{E}(f(\Omega), K) = \emptyset$.*

Hence, if we solve the scalar problem (SP(a,r)) related to the multiobjective optimization problem (MOP) fulfilling the assumptions of Corollary 2.5 for an arbitrary choice of parameters $(a,r) \in \mathbb{R}^m \times \mathrm{int}(K)$, then we either get a weakly K-minimal solution or we get the information that there are no efficient points of the problem (MOP). This property is not satisfied by all scalarization problems as e. g. not by the ε-constraint method as we will see later in Sect. 2.5.1.

For the special case of the natural ordering we have also the following similar theorem by Bernau ([15, Lemma 1.3]) not assuming the set $f(\Omega) + K$ to be closed and convex.

Theorem 2.6. *Let $\mathcal{M}_w(f(\Omega), \mathbb{R}^m_+) \neq \emptyset$. Then the objective function of the optimization problem (SP(a,r)) is bounded from below for arbitrary parameters $a \in \mathbb{R}^m$ and $r \in int(\mathbb{R}^m_+)$.*

Proof. Let $\bar{x} \in \mathcal{M}_w(f(\Omega), \mathbb{R}^m_+)$. We set

$$\bar{t} := \min_{1 \leq i \leq m} \frac{f_i(\bar{x}) - a_i}{r_i}. \tag{2.3}$$

Then $\bar{t} \leq \frac{1}{r_i}(f_i(\bar{x}) - a_i)$ for $i = 1, \ldots, m$. Next we assume there is a feasible point (t,x) of (SP(a,r)) with $t < \bar{t}$. Then $x \in \Omega$ and together with $r_i > 0,\ i = 1, \ldots, m$, and (2.3) it follows

$$f_i(x) \leq a_i + t\, r_i < a_i + \bar{t}\, r_i \leq f_i(\bar{x}) \qquad \text{for } i = 1, \ldots, m,$$

in contradiction to $\bar{x}$ weakly EP-minimal. Hence $\bar{t}$ is a lower bound for the objective function of the problem (SP(a, r)). □

We conclude:

Corollary 2.7. *If the objective value of the optimization problem (SP(a, r)) for $a \in \mathbb{R}^m$, $r \in$ int$(\mathbb{R}^m_+)$ is not bounded from below then $\mathcal{M}(f(\Omega), \mathbb{R}^m_+) = \emptyset$, i. e. there exists no EP-minimal point of the related multiobjective optimization problem.*

An important property of the discussed scalarization approach is the possibility to generate all (weakly) K-minimal points of the multiobjective optimization problem (compare Theorem 2.1,a)). If $(\bar{t}, \bar{x})$ is a minimal solution of the scalar problem by Pascoletti-Serafini with $\bar{x}$ being weakly K-minimal but not K-minimal we have the following property for the points dominating the point $f(\bar{x})$ ([181, Theorem 3.3]).

Theorem 2.8. *If the point $(\bar{t}, \bar{x})$ is a minimal solution of (SP(a, r)) with $\bar{k} := a + \bar{t}\, r - f(\bar{x})$ and if there is a point $y = f(x) \in f(\Omega)$ dominating the point $f(\bar{x})$ w. r. t. the cone K, then the point $(\bar{t}, x)$ is also a minimal solution of (SP(a, r)) and there exists a $k \in \partial K$, $k \neq 0_m$, with $a + \bar{t}\, r - f(x) = \bar{k} + k$.*

From that we can immediately conclude:

Corollary 2.9. *If the point $(\bar{t}, \bar{x})$ is an image-unique minimal solution of the scalar problem (SP(a, r)) w. r. t. f, i. e. there is no other minimal solution (t, x) with $f(x) = f(\bar{x})$, then $\bar{x}$ is a K-minimal solution of the multiobjective optimization problem (MOP).*

Pascoletti and Serafini ([181, Theorem 3.7]) derive a criterion for checking whether a point is K-minimal or not.

Corollary 2.10. *A point $\bar{x}$ is a K-minimal solution of the multiobjective optimization problem (MOP) if*

i) there is some $\bar{t} \in \mathbb{R}$ so that $(\bar{t}, \bar{x})$ is a minimal solution of (SP(a, r)) for some parameters $a \in \mathbb{R}$ and $r \in \operatorname{int}(K)$ and
ii) for $k := a + \bar{t}\, r - f(\bar{x})$ it is

$$((a + \bar{t}\, r) - \partial K) \cap (f(\bar{x}) - \partial K) \cap f(\Omega) = \{f(\bar{x})\}.$$

Hence if $(\bar{t}, \bar{x})$ is a minimal solution of $(\mathrm{SP}(a,r))$ with $r \in \operatorname{int}(K)$, then $\bar{x}$ is a weakly K-minimal solution and for checking if $\bar{x}$ is also K-minimal it is sufficient to test the points $((a + \bar{t}\,r) - \partial K) \cap (f(\bar{x}) - \partial K)$ of the set $f(\Omega)$.

2.3 Parameter Set Restriction for the Pascoletti-Serafini Scalarization

Our general aim is an approximation of the whole efficient set of the multiobjective optimization problem (MOP) by solving the problem $(\mathrm{SP}(a,r))$ for several parameters. In Theorem 2.1,b) we have seen that we can find all K-minimal points for a constant parameter $r \in K \setminus \{0_m\}$ by varying the parameter $a \in \mathbb{R}^m$ only. In this section we show that we do not have to consider all parameters $a \in \mathbb{R}^m$. We can restrict the set from which we have to choose the parameter a such that we can still find all K-minimal points of the multiobjective optimization problem. We start by showing that it is sufficient to vary the parameter a on a hyperplane $H = \{y \in \mathbb{R}^m \mid b^\top y = \beta\}$ with $b \in \mathbb{R}^m \setminus \{0_m\},\ \beta \in \mathbb{R}$.

Theorem 2.11. *Let $\bar{x}$ be K-minimal for (MOP) and define a hyperplane*

$$H = \{y \in \mathbb{R}^m \mid b^\top y = \beta\}$$

with $b \in \mathbb{R}^m \setminus \{0_m\}$ and $\beta \in \mathbb{R}$. Let $r \in K$ with $b^\top r \neq 0$ be arbitrarily given. Then there is a parameter $a \in H$ and some $\bar{t} \in \mathbb{R}$ so that $(\bar{t}, \bar{x})$ is a minimal solution of (SP(a,r)). This holds for instance for

$$\bar{t} = \frac{b^\top f(\bar{x}) - \beta}{b^\top r}$$

and

$$a = f(\bar{x}) - \bar{t}\,r.$$

Proof. For

$$\bar{t} = \frac{b^\top f(\bar{x}) - \beta}{b^\top r} \quad \text{and} \quad a = f(\bar{x}) - \bar{t}\,r$$

we have $a \in H$ and the point $(\bar{t}, \bar{x})$ is feasible for $(\mathrm{SP}(a,r))$. We assume that $(\bar{t}, \bar{x})$ is not a minimal solution of $(\mathrm{SP}(a,r))$. Then there is a $t' \in \mathbb{R}$, $t' < \bar{t}$, and points $x' \in \Omega$ and $k' \in K$ with

$$a + t'\,r - f(x') = k'.$$

With the definition of a it follows

$$f(\bar{x}) - \bar{t}\, r + t'\, r - f(x') = k'.$$

Hence

$$f(\bar{x}) = f(x') + k' + \underbrace{(\bar{t} - t')}_{>0}\ \underbrace{r}_{\in K}$$

and because of the convexity of the cone K

$$f(\bar{x}) \in f(x') + K. \tag{2.4}$$

As the cone K is pointed and because $r \neq 0_m$ it is

$$k' + \underbrace{(\bar{t} - t')}_{>0}\ r \neq 0_m$$

and thus $f(\bar{x}) \neq f(x')$. With (2.4) we conclude $f(\bar{x}) \in f(x') + K \setminus \{0_m\}$ for $x' \in \Omega$ in contradiction to $\bar{x}$ K-minimal. □

In the following we give a stricter restriction of the set from which we have to choose the parameter a such that we are still able to find all K-minimal points. We first consider the bicriteria case before we come to the more general case of an arbitrary multiobjective optimization problem.

2.3.1 Bicriteria Case

In this section we only consider biobjective problems, i. e. let $m = 2$ except when otherwise stated. In the preceding theorem we have seen that it is sufficient to choose the parameter $r \in K \setminus \{0_m\}$ constant and to vary the parameter a only in a hyperplane

$$H = \{y \in \mathbb{R}^2 \mid b_1 y_1 + b_2 y_2 = \beta\}$$

(here a line) with $b = (b_1, b_2) \in \mathbb{R}^2$, $b^\top r \neq 0$ and $\beta \in \mathbb{R}$. For example we can choose $b = r$ and $\beta = 0$, then $b^\top r = r^\top r = r_1^2 + r_2^2 \neq 0$ for $r \neq 0_2$.

In the bicriteria case we have the property that any closed pointed ordering cone in $\mathbb{R}^2$ is polyhedral (see Lemma 1.20). By using this property we can show that it is sufficient to consider only a subset H^a of the hyperplane H. In the following we assume $r \in K \setminus \{0_2\}$.

We first consider the case that the ordering cone K has a nonempty interior and is thus given by

$$K = \left\{ y \in \mathbb{R}^2 \;\middle|\; l^{1\top} y \geq 0,\ l^{2\top} y \geq 0 \right\} \tag{2.5}$$

with $l^1,\ l^2 \in \mathbb{R}^2 \setminus \{0_2\},\ l^1, l^2$ linearly independent. Then the interior of the cone is $\operatorname{int}(K) = \{y \in \mathbb{R}^2 \mid l^{1\top} y > 0,\ l^{2\top} y > 0\}$. Assuming the set $f(\Omega)$ to be compact there exists a minimal solution $\bar{x}^1$ of the scalar-valued problem

$$\min_{x \in \Omega} l^{1\top} f(x) \tag{2.6}$$

and a minimal solution $\bar{x}^2$ of the scalar-valued problem

$$\min_{x \in \Omega} l^{2\top} f(x). \tag{2.7}$$

These minimal solutions are also weakly K-minimal solutions of the multiobjective optimization problem (MOP) with the vector-valued objective function $f \colon \mathbb{R}^n \to \mathbb{R}^2$ as it is shown in the following lemma. This result can be generalized to the case with more than two objectives, too, for a finitely generated ordering cone $K \subset \mathbb{R}^m$. Besides, as the minimal solutions of (2.6) and (2.7) are weakly K-minimal, there are also parameters a and r such that these points are minimal solutions of the scalar problem $(\mathrm{SP}(a, r))$.

Lemma 2.12. *We consider the multiobjective optimization problem (MOP) for $m \in \mathbb{N}$, $m \geq 2$. Let $K \subset \mathbb{R}^m$ be a finitely generated cone with nonempty interior given by*

$$K = \{y \in \mathbb{R}^m \mid l^{i\top} y \geq 0,\ i = 1, \ldots, s\}$$

($s \in \mathbb{N}$). Let $\bar{x}^j$ be a minimal solution of

$$\min_{x \in \Omega} l^{j\top} f(x) \tag{2.8}$$

for a $j \in \{1, \ldots, s\}$. Then $\bar{x}^j$ is weakly K-minimal.

If we consider now the scalarization problem $(SP(a, r))$ with parameters $r \in K$, with $l^{j\top} r > 0$ (e. g. satisfied for $r \in int(K)$), and $a := \bar{a}^j$ given by

$$\bar{a}^j := f(\bar{x}^j) - \bar{t}^j\, r \quad \text{with} \quad \bar{t}^j := \frac{b^\top f(\bar{x}^j) - \beta}{b^\top r}$$

for $b \in \mathbb{R}^m$, $b^\top r \neq 0$, $\beta \in \mathbb{R}$, then $(\bar{t}^j, \bar{x}^j)$ is a minimal solution of $(SP(\bar{a}^j, r))$.

Proof. We first show $\bar{x}^j \in \mathcal{M}_w(f(\Omega), K)$. For that we assume that the point $\bar{x}^j$ is not weakly K-minimal. Then there is a point $x \in \Omega$ with

$$f(\bar{x}^j) \in f(x) + \operatorname{int}(K).$$

Then it follows $l^{j\top}(f(\bar{x}^j) - f(x)) > 0$ and hence $l^{j\top} f(\bar{x}^j) > l^{j\top} f(x)$ in contradiction to $\bar{x}^j$ a minimal solution of (2.8).

Next we show that $(\bar{t}^j, \bar{x}^j)$ is a minimal solution of $(\mathrm{SP}(\bar{a}^j, r))$. Because of

$$\bar{a}^j + \bar{t}^j\, r - f(\bar{x}^j) = 0_m$$

the point $(\bar{t}^j, \bar{x}^j)$ is a feasible point. We now assume that this point is not a minimal solution. Then there exists a feasible point (t', x') with $t' < \bar{t}^j$. Because of the feasibility of (t', x') for $(\mathrm{SP}(\bar{a}^j, r))$ it holds

$$\bar{a}^j + t'\, r - f(x') \in K.$$

Together with the definition of $\bar{a}^j$ we conclude

$$f(\bar{x}^j) - \bar{t}^j\, r + t'\, r - f(x') \in K.$$

Then

$$l^{j\top} \left(f(\bar{x}^j) + (t' - \bar{t}^j)\, r - f(x') \right) \geq 0$$

and thus

$$\begin{aligned} l^{j\top} f(\bar{x}^j) &\geq l^{j\top} f(x') + \underbrace{(\bar{t}^j - t')}_{>0}\, \underbrace{l^{j\top} r}_{>0} \\ &> l^{j\top} f(x') \end{aligned}$$

in contradiction to $\bar{x}^j$ a minimal solution of (2.8). □

The second result of this lemma is no longer true for arbitrary $r \in \partial K = K \setminus \operatorname{int}(K)$ with $l^{j\top} r = 0$ as it is demonstrated in the following example.

Example 2.13. We consider the bicriteria optimization problem

$$\min \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

subject to the constraints

$$1 \leq x_1 \leq 3,$$
$$1 \leq x_2 \leq 3,$$
$$x \in \mathbb{R}^2$$

w. r. t. the ordering cone $K = \mathbb{R}^2_+ = \{y \in \mathbb{R}^2 \mid (1,0)\,y \geq 0,\ (0,1)\,y \geq 0\}$.

Then for $l^1 = (1,0)^\top$ the point $\bar{x}^1 = (1,2)$ is a minimal solution of (2.8) for $j = 1$ and also a weakly EP-minimal solution of the bicriteria problem. For the hyperplane (here a line) $H := \{y \in \mathbb{R}^2 \mid (0,1)\,y = 0\}$, i. e. $b = (0,1)^\top$, $\beta = 0$, and the parameter $r := (0,1)^\top$ we get according to Lemma 2.12 $\bar{t} = 2$ and $\bar{a}^1 = (1,0)^\top$. However the point $(\bar{t}^1, \bar{x}^1) = (2,1,2)$ is not a minimal solution of $(\mathrm{SP}(\bar{a}^1, r))$. The point $(1,1,1)$ is the unique minimal solution of $(\mathrm{SP}(\bar{a}^1, r))$. For $r = (0,1)^\top$ there is no parameter $a \in H$ at all such that there exists a t with $(t, \bar{x}^1)$ a minimal solution of $(\mathrm{SP}(a, r))$.

Remark 2.14. If we extend the assumptions in Lemma 2.12 by the assumption that the minimal solution of (2.8) is unique then we can drop the condition $l^{j\top} r > 0$ and the result is also valid for r with $l^{j\top} r \geq 0$ what is already fulfilled for $r \in K$.

We now concentrate again on the bicriteria case for which we get further results.

Lemma 2.15. *We consider the multiobjective optimization problem (MOP) for* $m = 2$ *with the ordering cone* $K \subset \mathbb{R}^2$ *given by*

$$K = \{y \in \mathbb{R}^2 \mid l^{i\top} y \geq 0,\ i = 1,2\}.$$

Let $\bar{x}^1$ *be a minimal solution of (2.6) and let* $\bar{x}^2$ *be a minimal solution of (2.7). Then for all* $x \in \mathcal{M}(f(\Omega), K)$ *it is*

$$l^{1\top} f(\bar{x}^1) \leq l^{1\top} f(x) \leq l^{1\top} f(\bar{x}^2)$$

and

$$l^{2\top} f(\bar{x}^2) \leq l^{2\top} f(x) \leq l^{2\top} f(\bar{x}^1).$$

Proof. As $\bar{x}^1$ and $\bar{x}^2$ are minimal solutions of (2.6) and (2.7) we have of course for all $x \in \mathcal{M}(f(\Omega), K) \subset \Omega$

$$l^{1\top} f(x) \geq l^{1\top} f(\bar{x}^1) \text{ and } l^{2\top} f(x) \geq l^{2\top} f(\bar{x}^2).$$

Let us now suppose that there is a point $x \in \mathcal{M}(f(\Omega), K)$ with $l^{2\top} f(x) > l^{2\top} f(\bar{x}^1)$, i. e. with $l^{2\top}(f(x) - f(\bar{x}^1)) > 0$. Together with $l^{1\top}(f(x) - f(\bar{x}^1)) \geq 0$ we get $f(x) - f(\bar{x}^1) \in K \setminus \{0_2\}$ in contradiction to x K-minimal. Thus we have shown that $l^{2\top} f(x) \leq l^{2\top} f(\bar{x}^1)$ has to be true. The same for $l^{1\top} f(x) \leq l^{1\top} f(\bar{x}^2)$. □

We conclude:

Lemma 2.16. *Let the assumptions of Lemma 2.15 hold. If the efficient set* $\mathcal{E}(f(\Omega), K)$ *consists of more than one point, what is generally the case, then*

$$l^{1\top} f(\bar{x}^1) < l^{1\top} f(\bar{x}^2)$$

and

$$l^{2\top} f(\bar{x}^2) < l^{2\top} f(\bar{x}^1).$$

Proof. As $\bar{x}^1$ is a minimal solution of (2.6) we already have $l^{1\top} f(\bar{x}^1) \leq l^{1\top} f(\bar{x}^2)$. We assume now $l^{1\top} f(\bar{x}^1) = l^{1\top} f(\bar{x}^2)$. Applying Lemma 2.15 we get $l^{1\top} f(x) = l^{1\top} f(\bar{x}^2)$ for all $x \in \mathcal{M}(f(\Omega), K)$ and thus $l^{1\top}(f(x) - f(\bar{x}^2)) = 0$. Further, according to Lemma 2.15 it is $l^{2\top} f(x) \geq l^{2\top} f(\bar{x}^2)$ and hence $l^{2\top}(f(x) - f(\bar{x}^2)) \geq 0$ for all $x \in \mathcal{M}(f(\Omega), K)$.

Summarizing this results in $f(x) - f(\bar{x}^2) \in K$. As x is K-minimal we conclude $f(x) = f(\bar{x}^2)$ for all $x \in \mathcal{M}(f(\Omega), K)$ and thus $\mathcal{E}(f(\Omega), K) = \{f(\bar{x}^2)\}$.

Analogously $l^{2\top} f(\bar{x}^1) = l^{2\top} f(\bar{x}^2)$ implies $\mathcal{E}(f(\Omega), K) = \{f(\bar{x}^1)\}$.
□

We project the points $f(\bar{x}^1)$ and $f(\bar{x}^2)$ in direction r onto the line H (compare Fig. 2.4 for $l^1 = (1, 0)$ and $l^2 = (0, 1)$, i. e. $K = \mathbb{R}^2_+$). The projection points $\bar{a}^1 \in H = \{y \in \mathbb{R}^2 \mid b^\top y = \beta\}$ and $\bar{a}^2 \in H$ are given by

$$\bar{a}^i := f(\bar{x}^i) - \bar{t}^i\, r \quad \text{with} \quad \bar{t}^i := \frac{b^\top f(\bar{x}^i) - \beta}{b^\top r}, \qquad i = 1, 2. \tag{2.9}$$

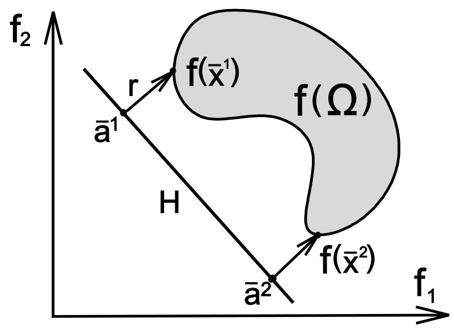

Fig. 2.4. Projection of the points $f(\bar{x}^1)$ and $f(\bar{x}^2)$ in direction r onto H.

We show that it is sufficient to consider parameters $a \in H^a$ with the set H^a given by

$$H^a = \{y \in H \mid y = \lambda \bar{a}^1 + (1-\lambda)\bar{a}^2,\ \lambda \in [0,1]\}, \tag{2.10}$$

i. e. it is sufficient to consider parameters on the line H between the points $\bar{a}^1$ and $\bar{a}^2$.

Theorem 2.17. *We consider the multiobjective optimization problem (MOP) with $m = 2$ and K as in (2.5). Further let $\bar{a}^1$ and $\bar{a}^2$ be given as in (2.9) with $\bar{x}^1$ and $\bar{x}^2$ minimal solutions of (2.6) and (2.7) respectively. Then we have for the set H^a as defined in (2.10) $H^a \subset H$ and for any K-minimal solution $\bar{x}$ of (MOP) there exists a parameter $a \in H^a$ and some $\bar{t} \in \mathbb{R}$ so that $(\bar{t}, \bar{x})$ is a minimal solution of (SP(a, r)).*

Proof. Because of $\bar{a}^1, \bar{a}^2 \in H$ it is $H^a \subset H$. According to Theorem 2.11 we already have that for any $\bar{x} \in \mathcal{M}(f(\Omega), K)$ there exists a parameter $a \in H$ and a $\bar{t} \in \mathbb{R}$ given by

$$\bar{t} = \frac{b^\top f(\bar{x}) - \beta}{b^\top r} \quad \text{and} \quad a = f(\bar{x}) - \bar{t}\, r$$

so that $(\bar{t}, \bar{x})$ is a minimal solution of (SP(a, r)). Hence it is sufficient to show that the parameter a lies on the line segment between the points $\bar{a}^1$ and $\bar{a}^2$, i. e. that $a = \lambda \bar{a}^1 + (1-\lambda)\bar{a}^2$ for a $\lambda \in [0,1]$. Using the definitions of a, $\bar{a}^1$ and $\bar{a}^2$ the equation $a = \lambda \bar{a}^1 + (1-\lambda)\bar{a}^2$ is equivalent to

$$f(\bar{x}) - \bar{t}\, r = \lambda\,(f(\bar{x}^1) - \bar{t}^1\, r) + (1-\lambda)\,(f(\bar{x}^2) - \bar{t}^2\, r). \tag{2.11}$$

If the efficient set of the multiobjective optimization problem consists of one point only and thus of $f(\bar{x}^1)$ or $f(\bar{x}^2)$ only, then (2.11) is satisfied for $\lambda = 1$ or $\lambda = 0$ respectively. Otherwise we have according to the Lemma 2.16

$$l^{1\top} f(\bar{x}^1) < l^{1\top} f(\bar{x}^2) \tag{2.12}$$

and

$$l^{2\top} f(\bar{x}^2) < l^{2\top} f(\bar{x}^1). \tag{2.13}$$

We reformulate the equation (2.11) as

$$f(\bar{x}) = \lambda\, f(\bar{x}^1) + (1-\lambda)\, f(\bar{x}^2) + (\bar{t} - \lambda\, \bar{t}^1 - (1-\lambda)\, \bar{t}^2)\, r. \tag{2.14}$$

Then we can do a case differentiation for

$$\bar{t} - \lambda\,\bar{t}^1 - (1-\lambda)\,\bar{t}^2 = \frac{1}{b^\top r}(b^\top(f(\bar{x}) - \lambda\, f(\bar{x}^1) - (1-\lambda)\, f(\bar{x}^2)) \geq 0$$

and $\bar{t} - \lambda\,\bar{t}^1 - (1-\lambda)\,\bar{t}^2 < 0$ respectively.

For the case $\bar{t} - \lambda\,\bar{t}^1 - (1-\lambda)\,\bar{t}^2 \geq 0$ we start by assuming that (2.14) is only satisfied for $\lambda < 0$. By applying the linear map l^1 to (2.14) we get because of $r \in K$ and together with (2.12)

$$\begin{aligned} l^{1\top} f(\bar{x}) &= \lambda\, l^{1\top} f(\bar{x}^1) + (1-\lambda)\, l^{1\top} f(\bar{x}^2) + \underbrace{(\bar{t} - \lambda\,\bar{t}^1 - (1-\lambda)\,\bar{t}^2)}_{\geq 0}\underbrace{l^{1\top} r}_{\geq 0} \\ &\geq \underbrace{\lambda}_{<0}\underbrace{l^{1\top} f(\bar{x}^1)}_{<l^{1\top} f(\bar{x}^2)} + (1-\lambda) l^{1\top} f(\bar{x}^2) \\ &> \lambda\, l^{1\top} f(\bar{x}^2) + (1-\lambda)\, l^{1\top} f(\bar{x}^2) \\ &= l^{1\top} f(\bar{x}^2) \end{aligned}$$

in contradiction to Lemma 2.15.

Now we suppose (2.14) is only satisfied for $\lambda > 1$. By applying the linear map l^2 to (2.14) and together with (2.13) we conclude

$$\begin{aligned} l^{2\top} f(\bar{x}) &\geq \lambda\, l^{2\top} f(\bar{x}^1) + \underbrace{(1-\lambda)}_{<0}\, \underbrace{l^{2\top} f(\bar{x}^2)}_{<l^{2\top} f(\bar{x}^1)} \\ &> l^{2\top} f(\bar{x}^1) \end{aligned}$$

in contradiction to Lemma 2.15. Thus we have shown that for the case $\bar{t} - \lambda\,\bar{t}^1 - (1-\lambda)\,\bar{t}^2 \geq 0$ it is $\lambda \in [0,1]$.

For the case $\bar{t} - \lambda\,\bar{t}^1 - (1-\lambda)\,\bar{t}^2 < 0$ one can show analogously that $\lambda \in [0,1]$. □

Remark 2.18. An even more strict restriction of the parameter set H^a is possible by minimizing in (2.6) and (2.7) over the set $\mathcal{M}(f(\Omega), K)$ instead of over Ω. Then we still have that for any $\bar{x} \in \mathcal{M}(f(\Omega), K)$ there exists a parameter $a \in H^a$ and some scalar $\bar{t} \in \mathbb{R}$ so that $(\bar{t}, \bar{x})$ is a minimal solution of $(\mathrm{SP}(a,r))$. However the set $\mathcal{M}(f(\Omega), K)$ is generally not known and thus we cannot optimize over the set of minimal solutions of the multiobjective optimization problem.

Next we come to the case that the ordering cone K has an empty interior, i. e. is given by $K = \{\lambda k \mid \lambda \geq 0\}$ for a $k \in \mathbb{R}^2 \setminus \{0_2\}$. Then the scalar optimization problem $(\mathrm{SP}(a,r))$ with $r \in K \setminus \{0_2\}$, i. e. $r = \lambda^r k$ for a $\lambda^r > 0$, can be formulated as

$$\begin{aligned}
&\min t\\
&\text{subject to the constraints}\\
&a + (t\,\lambda^r - \lambda)\,k = f(x),\\
&t \in \mathbb{R},\ x \in \Omega,\ \lambda \geq 0
\end{aligned} \tag{2.15}$$

by introducing an additional variable $\lambda \in \mathbb{R}$. If a point $(\bar{t}, \bar{x}, \bar{\lambda})$ is a minimal solution of (2.15) we always have $\bar{\lambda} = 0$: suppose $(\bar{t}, \bar{x}, \bar{\lambda})$ is a minimal solution of (2.15) with $\bar{\lambda} > 0$. Then the point $(\bar{t} - \frac{\bar{\lambda}}{\lambda^r}, \bar{x}, 0)$ is also feasible for (2.15) with $\bar{t} - \frac{\bar{\lambda}}{\lambda^r} < \bar{t}$ in contradiction to $(\bar{t}, \bar{x}, \bar{\lambda})$ a minimal solution. Thus we can consider the problem

$$\begin{aligned}
&\min t\\
&\text{subject to the constraints}\\
&a + t\,r = f(x),\\
&t \in \mathbb{R},\ x \in \Omega,
\end{aligned} \tag{2.16}$$

instead of (2.15).

To determine the set H^a as a subset of the set H it is sufficient to project the set $f(\Omega)$ in direction r onto the hyperplane H (see Fig. 2.5). If we have $l \in \mathbb{R}^2 \setminus \{0_2\}$ with $l^\top r = 0$ then we can determine the set H^a as described in Theorem 2.17 by solving the problems (2.6) and (2.7) with $l^1 := l$ and $l^2 := -l$.

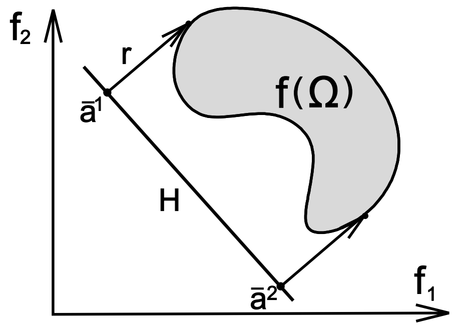

Fig. 2.5. Projection of the set $f(\Omega)$ in direction r on the hyperplane H.

2.3.2 General Case

Compared to the case with only two objective functions it is more difficult to restrict the parameter set in the case of three and more criteria. For example not any closed pointed convex cone in $\mathbb{R}^m$, $m \geq 3$, is polyhedral unlike it is in the case for $m = 2$ (Lemma 1.20). The cone of Example 1.21 which represents the Löwner partial ordering is non-polyhedral and thus not finitely generated. However even for polyhedral cones the results from Sect. 2.3.1 cannot be generalized to $\mathbb{R}^3$. A finitely generated cone $K \subset \mathbb{R}^3$ given by

$$K = \{y \in \mathbb{R}^3 \mid l^{i\top} y \geq 0,\ i = 1, \ldots, s\}$$

with $l^i \in \mathbb{R}^3 \setminus \{0_3\},\ i = 1, \ldots, s,\ s \in \mathbb{N}$, does not need to be generated by $s = m = 3$ vectors l^i only. Instead it is possible that $s > m$, as it is shown in Example 1.19.

Even if the ordering cone K is finitely generated by three vectors, i. e. $s = 3$ as it is the case for the ordering cone $K = \mathbb{R}^3_+$ inducing the natural ordering, we cannot generalize the results gained in the preceding section for determining the set H^a. This is illustrated with the following example.

Example 2.19. We consider the objective function $f\colon \mathbb{R}^3 \to \mathbb{R}^3$ with $f(x) := x$ for all $x \in \mathbb{R}^3$ and the constraint set $\Omega \subset \mathbb{R}^3$ defined by

$$\Omega := \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^2 + x_2^2 + x_3^2 \leq 1\}$$

which equals the unit ball in $\mathbb{R}^3$. We assume that the ordering is induced by the cone $K := \mathbb{R}^3_+$ which is finitely generated by

$$l^1 := (1, 0, 0)^\top,\ l^2 := (0, 1, 0)^\top,\ \text{and}\ l^3 := (0, 0, 1)^\top.$$

Thus $K = \{y \in \mathbb{R}^3 \mid l^{i\top} y \geq 0,\ i = 1, 2, 3\}$. The tricriteria optimization problem

$$\min_{x \in \Omega} f(x)$$

has the solution set

$$\mathcal{M}(f(\Omega), \mathbb{R}^3_+) = \{x \in \mathbb{R}^3 \mid x_1^2 + x_2^2 + x_3^2 = 1,\ x_i \leq 0,\ i = 1, 2, 3\}.$$

By solving the three scalar optimization problems

$$\min_{x \in \Omega} l^{i\top} f(x)$$

($i = 1, 2, 3$) corresponding to the problems (2.6) and (2.7) we get the three minimal solutions

$$\bar{x}^1 = (-1, 0, 0)^\top,\ \bar{x}^2 = (0, -1, 0)^\top,\ \text{and } \bar{x}^3 = (0, 0, -1)^\top.$$

Further define a hyperplane by

$$H := \{y \in \mathbb{R}^3 \mid (-1, -1, -1) \cdot y = 1\}$$

with $b = (-1, -1, -1)^\top, \beta = 1$. Then it is $f(\bar{x}^i) = \bar{x}^i \in H$ for $i = 1, 2, 3$.

For $r := (1, 1, 1)^\top$ the points $\bar{a}^i \in H$, $i = 1, 2, 3$, gained analogously to the points in (2.9), are

$$\bar{a}^1 = (-1, 0, 0)^\top,\ \bar{a}^2 = (0, -1, 0)^\top,\ \text{and } \bar{a}^3 = (0, 0, -1)^\top.$$

By defining the set H^a as the set of all convex combinations of the points $\bar{a}^i$, $i = 1, 2, 3$, as it is done in the bicriteria case, we get

$$H^a := \left\{ y \in \mathbb{R}^3 \;\middle|\; y = \sum_{i=1}^{3} \lambda_i \bar{a}^i,\ \lambda_i \geq 0,\ i = 1, 2, 3,\ \sum_{i=1}^{3} \lambda_i = 1 \right\}.$$

Then it is no longer true that to any K-minimal point $\bar{x}$ of the multiobjective optimization problem there is a parameter $\bar{a} \in H^a$ and a $\bar{t} \in \mathbb{R}$ so that $(\bar{t}, \bar{x})$ is a minimal solution of $(\mathrm{SP}(\bar{a}, r))$. For example the point $\bar{x} = (-1/\sqrt{2}, -1/\sqrt{2}, 0)^\top$ is EP-minimal but there is no parameter $\bar{a} \in H^a$ such that we get the point $\bar{x}$ by solving $(\mathrm{SP}(\bar{a}, r))$. For $\bar{a} = -1/(3\sqrt{2}) \cdot (1 + \sqrt{2}, 1 + \sqrt{2}, \sqrt{2} - 2)^\top$ and $\bar{t} = (1 - \sqrt{2})/3$ the point $(\bar{t}, \bar{x})$ is a minimal solution of $(\mathrm{SP}(\bar{a}, r))$, but it is $\bar{a} \notin H^a$.

Das and Dennis are confronted with the same problem during their examinations of the normal boundary intersection method in [40, pp.635f].

Due to these difficulties we determine a weaker restriction of the set H for the parameter a by projecting the image set $f(\Omega)$ in direction r onto the set H. Thus we determine the set

$$\tilde{H} := \{y \in H \mid y + t\, r = f(x),\ t \in \mathbb{R},\ x \in \Omega\} \subset H \tag{2.17}$$

(see Fig. 2.6 for $m = 3$).

We would again (see Remark 2.18) get a better result in the sense of a stronger restriction of the set H by projecting the efficient points of the set $f(\Omega)$, i. e. the set $\mathcal{E}(f(\Omega), K)$, only onto the hyperplane H, i. e.

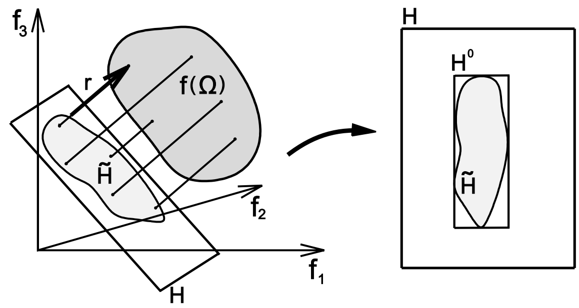

Fig. 2.6. Determination of the sets $\tilde{H}$ and H^0.

by determining $\{y \in H \mid y + t\,r = f(x),\ t \in \mathbb{R},\ x \in \mathcal{M}(f(\Omega), K)\} \subset H$. However the set of K-minimal points is generally not known. It is the aim of our method to approximate this set.

The set $\tilde{H} \subset H$ has in general an irregular boundary and is therefore not suitable for a systematic procedure. Hence we embed the set $\tilde{H}$ in a $(m-1)$-dimensional cuboid $H^0 \subset \mathbb{R}^m$ which is chosen as minimal as possible. For calculating the set H^0 we first determine $m-1$ vectors $v^1, \ldots, v^{m-1}$, which span the hyperplane H with $\tilde{H} \subset H$ and which are orthogonal and normalized by one, i. e.

$$v^{i\top} v^j = \begin{cases} 0, & \text{for } i \neq j, \quad i,j \in \{1,\ldots,m-1\}, \\ 1, & \text{for } i = j, \quad i,j \in \{1,\ldots,m-1\}. \end{cases} \tag{2.18}$$

These vectors form an orthonormal basis of the smallest subspace of $\mathbb{R}^m$ containing H. We have the condition $v^i \in H$, $i = 1, \ldots, m-1$, i. e.

$$b^\top v^i = \beta, \qquad i = 1, \ldots, m-1. \tag{2.19}$$

For example for $m = 3$ we can choose v^1 and v^2 dependent on $b = (b_1, b_2, b_3)^\top$ as follows

$$\tilde{v}^1 := \begin{cases} (\frac{b_1}{b_3}, -\frac{b_3}{b_2} - \frac{b_1^2}{b_2 b_3}, 1)^\top & \text{if } b_2 \neq 0,\ b_3 \neq 0, \\ (b_3, 0, -b_1)^\top & \text{if } b_2 = 0,\ b_3 \neq 0, \\ (0, 0, -b_1)^\top & \text{if } b_1 \neq 0,\ b_3 = 0, \\ (1, 0, 0)^\top & \text{if } b_1 = 0,\ b_3 = 0, \end{cases} \quad \text{and } v^1 := \frac{\tilde{v}^1}{\|\tilde{v}^1\|_2}$$

as well as

$$\tilde{v}^2 := \begin{cases} (b_3, 0, -b_1)^\top & \text{if } b_2 \neq 0,\ b_3 \neq 0, \\ (0, 1, 0)^\top & \text{if } b_2 = 0,\ b_3 \neq 0, \\ (-\frac{b_2}{b_1}, 1, 0)^\top & \text{if } b_1 \neq 0,\ b_3 = 0, \\ (0, 0, 1)^\top & \text{if } b_1 = 0,\ b_3 = 0, \end{cases} \quad \text{and hence } v^2 := \frac{\tilde{v}^2}{\|\tilde{v}^2\|_2}.$$

This leads to the representation

$$H = \{y \in \mathbb{R}^m \mid y = \sum_{i=1}^{m-1} s_i v^i,\ s \in \mathbb{R}^{m-1}\}. \tag{2.20}$$

of the hyperplane H. Then, for the set $\tilde{H}$ as in (2.17), we can determine the searched cuboid by solving the following $2(m-1)$ scalar-valued optimization problems

$$\begin{gathered} \min s_j \\ \text{subject to the constraints} \\ \sum_{i=1}^{m-1} s_i v^i + t\, r = f(x), \\ t \in \mathbb{R}, \\ x \in \Omega, \\ s \in \mathbb{R}^{m-1} \end{gathered} \tag{2.21}$$

for $j \in \{1, \ldots, m-1\}$ with minimal solution $(t^{\min,j}, x^{\min,j}, s^{\min,j})$ and minimal value $s_j^{\min,j}$ and

$$\begin{gathered} \min -s_j \\ \text{subject to the constraints} \\ \sum_{i=1}^{m-1} s_i v^i + t\, r = f(x), \\ t \in \mathbb{R}, \\ x \in \Omega, \\ s \in \mathbb{R}^{m-1} \end{gathered} \tag{2.22}$$

for $j \in \{1, \ldots, m-1\}$ with minimal solution $(t^{\max,j}, x^{\max,j}, s^{\max,j})$ and minimal value $-s_j^{\max,j}$. We get

$$H^0 := \left\{ y \in \mathbb{R}^m \;\middle|\; y = \sum_{i=1}^{m-1} s_i v^i,\ s_i \in [s_i^{\min,i}, s_i^{\max,i}],\ i = 1, \ldots, m-1 \right\}$$

with $\tilde{H} \subset H^0$. This is a suitable restriction of the parameter set H as the following lemma shows.

Lemma 2.20. *Let $\bar{x}$ be a K-minimal solution of the multiobjective optimization problem (MOP). Let $r \in K \setminus \{0_m\}$. Then there is a parameter $\bar{a} \in H^0$ and some $\bar{t} \in \mathbb{R}$ so that $(\bar{t}, \bar{x})$ is a minimal solution of (SP($\bar{a}, r$)).*

Proof. According to Theorem 2.11 the point $(\bar{t}, \bar{x})$ with

$$\bar{t} := \frac{b^\top f(\bar{x}) - \beta}{b^\top r}$$

is a minimal solution of (SP$(\bar{a}, r)$) for $\bar{a} := f(\bar{x}) - \bar{t}\, r \in H$. Because of $H^0 \subset H$ it suffices to show $\bar{a} \in H^0$. As $\bar{a} \in H$ there is according to the representation in (2.20) a point $\bar{s} \in \mathbb{R}^{m-1}$ with

$$\bar{a} = \sum_{i=1}^{m-1} \bar{s}_i v^i.$$

Because of $\bar{a} + \bar{t}\, r = f(\bar{x})$ the point $(\bar{t}, \bar{x}, \bar{s})$ is feasible for the optimization problems (2.21) and (2.22). Thus it is $s_i^{\min,i} \leq \bar{s}_i \leq s_i^{\max,i}$ for $i = 1, \ldots, m-1$ and it follows $\bar{a} \in H^0$. □

Hence we can also restrict the parameter set for the case of more than two objectives and arbitrary ordering cones K.

2.4 Modified Pascoletti-Serafini Scalarization

For theoretical reasons we are also interested in the following modification of the Pascoletti-Serafini problem named ($\overline{\mathrm{SP}}(a, r)$) which is given by

$$\begin{aligned} &\min t \\ &\text{subject to the constraints} \\ &a + t\, r - f(x) = 0_m, \\ &t \in \mathbb{R}, x \in \Omega. \end{aligned} \tag{2.23}$$

Here the inequality constraint $a + t\, r - f(x) \in K$ is replaced by an equality constraint. For the connection between the problem (SP(a, r)) and the problem ($\overline{\mathrm{SP}}(a, r)$) the following theorem is important.

Theorem 2.21. *Let a hyperplane* $H = \{y \in \mathbb{R}^m \mid b^\top y = \beta\}$ *with* $b \in \mathbb{R}^m \setminus \{0_m\}$ *and* $\beta \in \mathbb{R}$ *be given. Let* $(\bar{t}, \bar{x})$ *be a minimal solution of the scalar optimization problem (SP*(a, r)*) for the parameters* $a \in \mathbb{R}^m$ *and* $r \in \mathbb{R}^m$ *with* $b^\top r \neq 0$*. Hence there is a* $\bar{k} \in K$ *with*

$$a + \bar{t}\, r - f(\bar{x}) = \bar{k}.$$

Then there is a parameter $a' \in H$ *and some* $t' \in \mathbb{R}$ *so that* $(t', \bar{x})$ *is a minimal solution of (SP*(a', r)*) with*

$$a' + t'\, r - f(\bar{x}) = 0_m.$$

Proof. We set

$$t' := \frac{b^\top f(\bar{x}) - \beta}{b^\top r}$$

and

$$a' := a + (\bar{t} - t')\, r - \bar{k} = f(\bar{x}) - t'\, r.$$

Then $a' \in H$ and $a' + t'\, r - f(\bar{x}) = 0_m$. The point $(t', \bar{x})$ is feasible for (SP(a', r)) and it is also a minimal solution, because otherwise there exists a feasible point $(\hat{t}, \hat{x})$ of (SP(a', r)) with $\hat{t} < t'$, $\hat{x} \in \Omega$, and some $\hat{k} \in K$ with

$$a' + \hat{t}\, r - f(\hat{x}) = \hat{k}.$$

Together with the definition of a' we conclude

$$a + (\bar{t} - t' + \hat{t})\, r - f(\hat{x}) = \hat{k} + \bar{k} \in K.$$

Hence $(\bar{t} - t' + \hat{t}, \hat{x})$ is feasible for (SP(a, r)) with $\bar{t} - t' + \hat{t} < \bar{t}$ in contradiction to the minimality of $(\bar{t}, \bar{x})$ for (SP(a, r)). $\square$

Remark 2.22. Thus for a minimal solution $(\bar{t}, \bar{x})$ of the scalar optimization problem (SP(a, r)) with

$$a + \bar{t}\, r - f(\bar{x}) = \bar{k}, \qquad \bar{k} \neq 0_m,$$

there is a parameter $a' \in H$ and some $t' \in \mathbb{R}$ so that $(t', \bar{x})$ is a minimal solution of (SP(a', r)) with

$$a' + t'\, r - f(\bar{x}) = 0_m$$

(see Fig. 2.7) and hence $(t', \bar{x})$ is also a minimal solution of ($\overline{\text{SP}}(a', r)$).

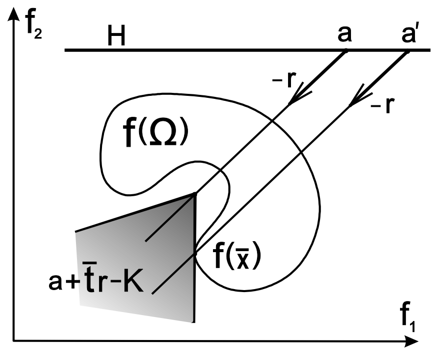

Fig. 2.7. Visualization of the Remark 2.22.

For the scalarization $(\overline{\mathrm{SP}}(a,r))$ the property c) of Theorem 2.1 is no longer valid, which means that minimal solutions of $(\overline{\mathrm{SP}}(a,r))$ are not necessarily weakly K-minimal points of the multiobjective optimization problem (MOP). However due to Theorem 2.21 we can still find all K-minimal points only by varying the parameter a on a hyperplane:

Theorem 2.23. *Let a hyperplane $H = \{y \in \mathbb{R}^m \mid b^\top y = \beta\}$ with $b \in \mathbb{R}^m$, $\beta \in \mathbb{R}$ be given an let $\bar{x} \in \mathcal{M}(f(\Omega), K)$ and $r \in K \setminus \{0_m\}$ with $b^\top r \neq 0$. Then there is a parameter $a \in H$ and some $\bar{t} \in \mathbb{R}$ so that $(\bar{t}, \bar{x})$ is a minimal solution of $(\overline{\mathrm{SP}}(a,r))$.*

Proof. According to Theorem 2.1,b) there is a parameter a' and some $t' \in \mathbb{R}$ so that $(t', \bar{x})$ is a minimal solution of $(\mathrm{SP}(a',r))$. According to Theorem 2.21 there is then a point $a \in H$ and some $\bar{t} \in \mathbb{R}$ so that $(\bar{t}, \bar{x})$ is a minimal solution of $(\overline{\mathrm{SP}}(a,r))$. □

Analogously to the Pascoletti-Serafini method we can do a parameter set restriction for the modified Pascoletti-Serafini problem, too. We demonstrate this for the bicriteria case and we show, that it is again sufficient to consider parameters a from a line segment of the hyperplane H to be able to detect all K-minimal points of the original problem.

Lemma 2.24. *Let $m = 2$ and let the ordering cone K be given as in (2.5). Let $\bar{a}^1$ and $\bar{a}^2$ and the set H^a be given as in Theorem 2.17.*

Then for any K-minimal solution $\bar{x}$ of the multiobjective optimization problem (MOP) there exists a parameter $a \in H^a$ and some $\bar{t} \in \mathbb{R}$ so that $(\bar{t}, \bar{x})$ is a minimal solution of $(\overline{\mathrm{SP}}(a,r))$.

Proof. According to Theorem 2.17 there exists a parameter $a \in H^a$ and some $\bar{t} \in \mathbb{R}$ so that $(\bar{t}, \bar{x})$ is a minimal solution of $(\mathrm{SP}(a, r))$. As shown in the proof of Theorem 2.17 we can choose a and $\bar{t}$ so that $a + \bar{t}\, r = f(\bar{x})$ and hence, $(\bar{t}, \bar{x})$ is a minimal solution of $(\overline{\mathrm{SP}}(a, r))$, too. $\square$

For the sensitivity studies in the following chapter the Lagrange function and the Lagrange multipliers will be of interest. For that we first recapitulate these notions and then we formulate an extension of Theorem 2.21 taking the Lagrange multipliers into account. We start with a general scalar-valued optimization problem

$$\begin{aligned} &\min F(x) \\ &\text{subject to the constraints} \\ &G(x) \in C, \\ &H(x) = 0_q, \\ &x \in S \end{aligned}$$

with a closed convex cone $C \subset \mathbb{R}^p$, an open subset $\hat{S} \subset \mathbb{R}^n$, a closed convex set $S \subset \hat{S}$ and continuously differentiable functions $F\colon \hat{S} \to \mathbb{R}$, $G\colon \hat{S} \to \mathbb{R}^p$, and $H\colon \hat{S} \to \mathbb{R}^q$ $(n, p, q \in \mathbb{N}_0)$. Then the related Lagrange function is given by $\mathcal{L}\colon \mathbb{R}^n \times C^* \times \mathbb{R}^q \to \mathbb{R}$ (with $C^* = \{y \in \mathbb{R}^p \mid y^\top x \geq 0 \text{ for all } x \in C\}$ the dual cone to C),

$$\mathcal{L}(x, \mu, \xi) := F(x) - \mu^\top G(x) - \xi^\top H(x).$$

If the point x is feasible and if there exists $(\mu, \xi) \in C^* \times \mathbb{R}^q$ with

$$\nabla_x \mathcal{L}(x, \mu, \xi)^\top (s - x) \geq 0 \qquad \forall s \in S$$

and

$$\mu^\top G(x) = 0,$$

then μ and ξ are called Lagrange multipliers to the point x.

We need the following assumptions:

Assumption 2.25. Let K be a closed pointed convex cone in $\mathbb{R}^m$ and C a closed convex cone in $\mathbb{R}^p$. Let $\hat{S}$ be a nonempty open subset of $\mathbb{R}^n$ and assume $S \subset \hat{S}$ to be closed and convex. Let the functions $f\colon \hat{S} \to \mathbb{R}^m$, $g\colon \hat{S} \to \mathbb{R}^p$, and $h\colon \hat{S} \to \mathbb{R}^q$ be continuously differentiable on $\hat{S}$.

We now formulate an extended version of Theorem 2.21:

Lemma 2.26. *We consider the scalar optimization problem (SP(a,r)) under the Assumption 2.25. Let $(\bar{t},\bar{x})$ be a minimal solution and assume there exist Lagrange multipliers $(\mu,\nu,\xi)\in K^*\times C^*\times\mathbb{R}^q$ to the point $(\bar{t},\bar{x})$. According to Theorem 2.21 there exists a parameter $a'\in H$ and some $t'\in\mathbb{R}$ so that $(t',\bar{x})$ is a minimal solution of (SP(a',r)) and $a'+t'\,r=f(\bar{x})$.*

Then (μ,ν,ξ) are Lagrange multipliers to the point $(t',\bar{x})$ for the problem (SP(a',r)), too.

Proof. The Lagrange function $\mathcal{L}$ to the scalar optimization problem (SP(a,r)) related to the multiobjective optimization problem (MOP) is given by

$$\mathcal{L}(t,x,\mu,\nu,\xi,a,r)=t-\mu^\top(a+t\,r-f(x))-\nu^\top g(x)-\xi^\top h(x).$$

If (μ,ν,ξ) are Lagrange multipliers to the point $(\bar{t},\bar{x})$ then it follows

$$\begin{aligned}&\nabla_{(t,x)}\mathcal{L}(\bar{t},\bar{x},\mu,\nu,\xi,a,r)^\top\begin{pmatrix}t-\bar{t}\\x-\bar{x}\end{pmatrix}=\\&\left[\begin{pmatrix}1\\0\end{pmatrix}-\sum_{i=1}^m\mu_i\begin{pmatrix}r_i\\-\nabla_x f_i(\bar{x})\end{pmatrix}-\sum_{j=1}^p\nu_j\begin{pmatrix}0\\\nabla_x g_j(\bar{x})\end{pmatrix}\right.\\&\left.-\sum_{k=1}^q\xi_k\begin{pmatrix}0\\\nabla_x h_k(\bar{x})\end{pmatrix}\right]^\top\begin{pmatrix}t-\bar{t}\\x-\bar{x}\end{pmatrix}\geq 0\qquad\forall t\in\mathbb{R},\ x\in S.\end{aligned}$$

Hence $1-\mu^\top r=0$ and $\left(\mu^\top\nabla_x f(\bar{x})-\nu^\top\nabla_x g(\bar{x})-\xi^\top\nabla_x h(\bar{x})\right)(x-\bar{x})\geq 0$ for all $x\in S$. Further we have $\mu^\top(a+\bar{t}\,r-f(\bar{x}))=0$ and $\nu^\top g(\bar{x})=0$. For the minimal solution $(t',\bar{x})$ of the problem (SP(a',r)) it is

$$a'+t'\,r-f(\bar{x})=0_m,$$

and thus $\mu^\top(a'+t'\,r-f(\bar{x}))=0$. Because of

$$\nabla_{(t,x)}\mathcal{L}(t',\bar{x},\mu,\nu,\xi,a',r)=\nabla_{(t,x)}\mathcal{L}(\bar{t},\bar{x},\mu,\nu,\xi,a,r)$$

we also have

$$\nabla_{(t,x)}\mathcal{L}(t',\bar{x},\mu,\nu,\xi,a',r)^\top\begin{pmatrix}t-t'\\x-\bar{x}\end{pmatrix}\geq 0\qquad\forall t\in\mathbb{R},\ x\in S.$$

Thus (μ,ν,ξ) are also Lagrange multipliers to the point $(t',\bar{x})$ for the problem (SP(a',r)). □

In the chapter about sensitivity considerations the Lagrange multipliers play an important role.

2.5 Relations Between Scalarizations

We have seen in the preceding sections that the Pascoletti-Serafini problem features many interesting and important properties. It is a very general formulation allowing two parameters to vary arbitrarily. Due to this, many other well-known and wide spread scalarization approaches can be seen as a special case and can be subsumed under this general problem. The connections will be studied in this section. The relations are important for applying the results about an adaptive parameter control gained in the following chapters for the general scalarization to the special problems, too.

2.5.1 ε-Constraint Problem

We start with a common method called ε-constraint method ([54, 98, 60, 159, 165]). It is a very wide spread method especially in engineering design for finding EP-minimal points, because the method is very intuitive and the parameters are easy to interpret as upper bounds. In [147, 148, 186] this method is used for solving multiobjective optimization problems via evolutionary algorithms.

For an arbitrary $k \in \{1, \dots, m\}$ and parameters $\varepsilon_i \in \mathbb{R}$, $i \in \{1, \dots, m\} \setminus \{k\}$, the scalarized problem called $(P_k(\varepsilon))$ reads as follows (compare (2.1)):

$$\begin{aligned} &\min f_k(x) \\ &\text{subject to the constraints} \\ &f_i(x) \le \varepsilon_i, \quad i \in \{1, \dots, m\} \setminus \{k\}, \\ &x \in \Omega. \end{aligned} \tag{2.24}$$

It is easy to see that this is just a special case of the Pascoletti-Serafini scalarization for the ordering cone $K = \mathbb{R}^m_+$. We even get a connection w. r. t. the Lagrange multipliers:

Theorem 2.27. *Let Assumption 2.25 hold and let $K = \mathbb{R}^m_+$, $C = \mathbb{R}^p_+$, and $\hat{S} = S = \mathbb{R}^n$. A point $\bar{x}$ is a minimal solution of $(P_k(\varepsilon))$ with Lagrange multipliers $\bar{\mu}_i \in \mathbb{R}_+$ for $i \in \{1, \dots, m\} \setminus \{k\}$, $\bar{\nu} \in \mathbb{R}^p_+$, and $\bar{\xi} \in \mathbb{R}^q$, if and only if $(f_k(\bar{x}), \bar{x})$ is a minimal solution of (SP(a, r)) with Lagrange multipliers $(\bar{\mu}, \bar{\nu}, \bar{\xi})$ with $\bar{\mu}_k = 1$, and*

$$a_i = \varepsilon_i, \quad \forall i \in \{1, \dots, m\} \setminus \{k\}, \quad a_k = 0 \text{ and } r = e_k \tag{2.25}$$

with e_k the kth unit vector in $\mathbb{R}^m$.

Proof. By introducing the additional variable $t \in \mathbb{R}$ the scalar optimization problem $(P_k(\varepsilon))$ can be formulated as

$$\begin{array}{c} \min t \\ \text{subject to the constraints} \end{array}$$
$$\begin{array}{lcll} \varepsilon_i - f_i(x) & \geq & 0, & i \in \{1,\ldots,m\} \setminus \{k\}, \\ t - f_k(x) & \geq & 0, & \\ g_j(x) & \geq & 0, & j = 1,\ldots,p, \\ h_l(x) & = & 0, & l = 1,\ldots,q, \\ t \in \mathbb{R},\ x \in \mathbb{R}^n. & & & \end{array} \tag{2.26}$$

If $\bar{x}$ is a minimal solution of $(P_k(\varepsilon))$ then $(\bar{t}, \bar{x}) := (f_k(\bar{x}), \bar{x})$ is a minimal solution of the problem (2.26). However problem (2.26) is equivalent to the Pascoletti-Serafini problem $(\mathrm{SP}(a,r))$ with a and r as in (2.25).

Because $\bar{\mu}_i$, $i \in \{1,\ldots,m\} \setminus \{k\}$, $\bar{\nu}_j$, $j = 1,\ldots,p$, $\bar{\xi}_l$, $l = 1,\ldots,q$, are Lagrange multipliers to $\bar{x}$ for $(P_k(\varepsilon))$, we have

$$\begin{aligned} \bar{\mu}_i(\varepsilon_i - f_i(\bar{x})) = 0 \qquad & \text{for all } i \in \{1,\ldots,m\} \setminus \{k\}, \\ \bar{\nu}_j(g_j(\bar{x})) = 0 \qquad & \text{for all } j \in \{1,\ldots,p\}, \end{aligned}$$

and

$$\nabla f_k(\bar{x}) + \sum_{\substack{i=1 \\ i \neq k}}^{m} \bar{\mu}_i \nabla f_i(\bar{x}) - \sum_{j=1}^{p} \bar{\nu}_j \nabla g_j(\bar{x}) - \sum_{l=1}^{q} \bar{\xi}_l \nabla h_l(\bar{x}) = 0_n. \tag{2.27}$$

The derivative of the Lagrange function $\mathcal{L}(t, x, \mu, \nu, \xi, a, r)$ to $(\mathrm{SP}(a,r))$ with a and r as in (2.25) in the point $(f_k(\bar{x}), \bar{x})$ reads as follows:

$$\begin{aligned} \nabla_{(t,x)} \mathcal{L}(f_k(\bar{x}), \bar{x}, \mu, \nu, \xi, a, r) &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \mu_k \begin{pmatrix} 1 \\ -\nabla f_k(\bar{x}) \end{pmatrix} \\ - \sum_{\substack{i=1 \\ i \neq k}}^{m} \mu_i \begin{pmatrix} 0 \\ -\nabla f_i(\bar{x}) \end{pmatrix} &- \sum_{j=1}^{p} \nu_j \begin{pmatrix} 0 \\ \nabla g_j(\bar{x}) \end{pmatrix} - \sum_{l=1}^{q} \xi_l \begin{pmatrix} 0 \\ \nabla h_l(\bar{x}) \end{pmatrix}. \end{aligned}$$

By choosing $\bar{\mu}_k = 1$ and applying (2.27) we get

$$\nabla_{(t,x)} \mathcal{L}(f_k(\bar{x}), \bar{x}, \bar{\mu}, \bar{\nu}, \bar{\xi}, a, r) = 0_{n+1},$$

and hence $(\bar{\mu}, \bar{\nu}, \bar{\xi})$ are Lagrange multipliers to the point $(f_k(\bar{x}), \bar{x})$ for the problem $(\mathrm{SP}(a,r))$, too. The proof of the converse direction can be done analogously. □

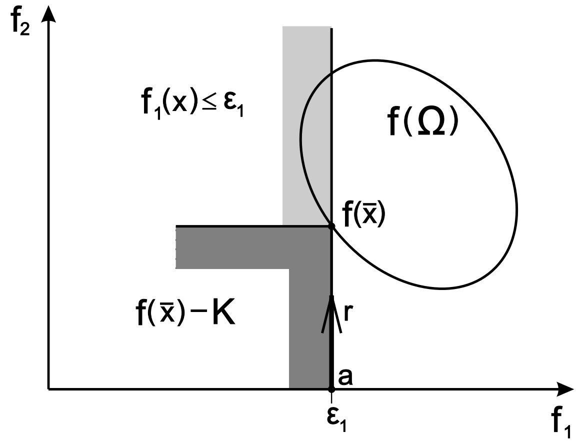

Fig. 2.8. Connection between the ε-constraint and the Pascoletti-Serafini method.

The statement of Theorem 2.27 is visualized in Fig. 2.8 on a bicriteria optimization problem with $k = 2$ in the ε-constraint method.

For the choice of the parameters as in (2.25) it follows that the constraint $a_k + t\, r_k - f_k(x) \geq 0$ is always active in $(f_k(\bar{x}), \bar{x})$, i. e. it is $a_k + t\, r_k - f_k(x) = 0$. The ε-constraint method is a restriction of the Pascoletti-Serafini problem with the parameter a chosen only from the hyperplane $H = \{y \in \mathbb{R}^m \mid y_k = 0\}$ and the parameter $r = e_k$ constant. From Theorem 2.27 together with Theorem 2.11 we conclude:

Corollary 2.28. *If $\bar{x} \in \mathcal{M}(f(\Omega), \mathbb{R}^m_+)$, then $\bar{x}$ is a minimal solution of $(P_k(\varepsilon))$ for $\varepsilon_i = f_i(\bar{x}),\ i \in \{1, \ldots, m\} \setminus \{k\}$.*

In contrast to the Pascoletti-Serafini method in general not any weakly EP-minimal solution can be found by solving $(P_k(\varepsilon))$ because we choose $r \in \partial K = \partial \mathbb{R}^m_+$ for the ε-constraint method. However weakly EP-minimal points which are not also EP-minimal are not of practical interest. As a consequence of Theorem 2.1,c) we have:

Corollary 2.29. *If $\bar{x}$ is a minimal solution of $(P_k(\varepsilon))$, then $\bar{x} \in \mathcal{M}_w(f(\Omega), \mathbb{R}^m_+)$.*

A direct proof of this result can be found for instance in [62, Prop. 4.3] or in [165, Theorem 3.2.1].

The ε-constraint method has a big drawback against the more general Pascoletti-Serafini problem. According to Corollary 2.5, if we have

$\mathcal{E}(f(\Omega), \mathbb{R}^m_+) \neq \emptyset$ and $f(\Omega)+K$ is closed and convex, then for any choice of the parameters $(a, r) \in \mathbb{R}^m \times \operatorname{int}(K)$ there exists a minimal solution of the problem (SP(a, r)). This is no longer true for the ε-constraint method as the following example demonstrates.

Example 2.30. Let $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ with $f(x) := x$ for all $x \in \mathbb{R}^2$ be given. We consider the bicriteria optimization problem

$$\begin{gathered} \min f(x) = x \\ \text{subject to the constraints} \\ \|x\|_2 \leq 1, \\ x \in \mathbb{R}^2 \end{gathered}$$

w. r. t. the natural ordering $K = \mathbb{R}^m_+$. The set $f(\Omega) + K$ is convex and the efficient set is

$$\mathcal{E}(f(\Omega), \mathbb{R}^2_+) = \{x = (x_1, x_2)^\top \in \mathbb{R}^2 \mid \|x\|_2 = 1,\ x_1 \leq 0,\ x_2 \leq 0\} \neq \emptyset.$$

The ε-constraint scalarization for $k = 2$ is given by

$$\begin{gathered} \min f_2(x) \\ \text{subject to the constraints} \\ f_1(x) \leq \varepsilon_1, \\ \|x\|_2 \leq 1, \\ x \in \mathbb{R}^2, \end{gathered}$$

but for $\varepsilon_1 < -1$ there exists no feasible point and thus no minimal solution.

Hence it can happen that the ε-constraint problem is solved for a large number of parameters without getting any solution, and with that weakly EP-minimal points, or at least the information $\mathcal{M}(f(\Omega), \mathbb{R}^m_+) = \emptyset$. This is due to the fact that this is a special case of the Pascoletti-Serafini problem with $r \in \partial K = \partial \mathbb{R}^m_+$.

We can apply the results of Theorem 2.17 for a restriction of the parameter set for the ε-constraint problem, too. For $m = 2$ and e. g. $k = 2$ we have according to Theorem 2.27

$$r = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{and} \quad a \in H = \left\{y \in \mathbb{R}^2 \mid y_2 = 0\right\} = \left\{ \begin{pmatrix} \varepsilon \\ 0 \end{pmatrix} \;\middle|\; \varepsilon \in \mathbb{R} \right\}$$

(i. e. $b = (0,1)^\top$, $\beta = 0$). The ordering cone $K = \mathbb{R}^2_+$ is finitely generated by $l^1 = (1,0)^\top$ and $l^2 = (0,1)^\top$ and hence (2.6) and (2.7) are equal to

$$\min_{x\in\Omega} f_1(x) \quad \text{and} \quad \min_{x\in\Omega} f_2(x).$$

For $\bar{x}^1$ and $\bar{x}^2$ respectively minimal solutions of these problems we get

$$H^a := \{y \in H \mid y = \lambda \bar{a}^1 + (1-\lambda)\bar{a}^2,\ \lambda \in [0,1]\}$$

with

$$\bar{a}^i := f(\bar{x}^i) - \frac{b^\top f(\bar{x}^i) - \beta}{b^\top r}\, r = \begin{pmatrix} f_1(\bar{x}^i) \\ 0 \end{pmatrix},\ i = 1,2,$$

and hence

$$\begin{aligned} H^a &= \{y = (\varepsilon, 0)^\top \mid \varepsilon = \lambda\, f_1(\bar{x}^1) + (1-\lambda)\, f_1(\bar{x}^2),\ \lambda \in [0,1]\} \\ &= \{y = (\varepsilon, 0)^\top \mid f_1(\bar{x}^1) \le \varepsilon \le f_1(\bar{x}^2)\}. \end{aligned}$$

We conclude:

Corollary 2.31. *Let $\bar{x}$ be an EP-minimal solution of the multiobjective optimization problem (MOP) with $m = 2$. Let $\bar{x}^1$ be a minimal solution of $\min_{x\in\Omega} f_1(x)$ and $\bar{x}^2$ a minimal solution of $\min_{x\in\Omega} f_2(x)$. Then there is a parameter $\varepsilon \in \{y \in \mathbb{R} \mid f_1(\bar{x}^1) \le y \le f_1(\bar{x}^2)\}$ with $\bar{x}$ a minimal solution of $(P_2(\varepsilon))$. The same result holds for $(P_1(\varepsilon))$.*

2.5.2 Normal Boundary Intersection Problem

We start with a short recapitulation of this method introduced by Das and Dennis in [38, 40]. For determining EP-minimal points the scalar optimization problems

$$\begin{gathered} \max s \\ \text{subject to the constraints} \\ \Phi\beta + s\,\bar{n} = f(x) - f^*, \\ s \in \mathbb{R},\ x \in \Omega, \end{gathered} \tag{2.28}$$

named (NBI(β)) for parameters $\beta \in \mathbb{R}^m_+$, $\sum_{i=1}^m \beta_i = 1$, are solved. Here f^* denotes the so-called ideal point defined by $f_i^* := f_i(x^i) := \min_{x\in\Omega} f_i(x),\ i = 1,\ldots,m$. The matrix $\Phi \in \mathbb{R}^{m\times m}$ consists of the columns $f(x^i) - f^*$ $(i = 1,\ldots,m)$ and the set

$$f^* + \{\Phi\beta \mid \beta \in \mathbb{R}^m_+,\ \sum_{i=1}^m \beta_i = 1\} \tag{2.29}$$

is then the set of all convex combinations of the extremal points $f(x^i),\ i = 1,\ldots,m$, the so-called CHIM (convex hull of individual minima). The vector $\bar{n}$ is defined as normal unit vector to the hyperplane extending the CHIM directing to the negative orthant.

The idea of this method is that by solving the problem (NBI(β)) for an equidistant choice of parameters β an equidistant approximation of the efficient set is generated. However already for the case $m \geq 3$ generally not all EP-minimal points can be found as a solution of (NBI(β)) (see [40, Fig. 3]) and what is more, the maximal solutions of (NBI(β)) are not necessarily weakly EP-minimal ([231, Ex. 7.9.1]).

There is a direct connection between the normal boundary intersection (NBI) method and the modified version ($\overline{\mathrm{SP}}(a,r)$) of the Pascoletti-Serafini problem.

Lemma 2.32. *A point $(\bar{s},\bar{x})$ is a maximal solution of (NBI(β)) with $\beta \in \mathbb{R}^m$, $\sum_{i=1}^m \beta_i = 1$, if and only if $(-\bar{s},\bar{x})$ is a minimal solution of ($\overline{\mathrm{SP}}(a,r)$) with $a = f^* + \Phi\beta$ and $r = -\bar{n}$.*

Proof. By setting $a = f^* + \Phi\beta$, $t = -s$ and $r = -\bar{n}$ we see immediately that solving problem (NBI(β)) is equivalent to solve

$$\begin{gathered} -\min t \\ \text{subject to the constraints} \\ a + t\,r - f(x) = 0_m, \\ t \in \mathbb{R},\ x \in \Omega, \end{gathered}$$

being again equivalent to solve ($\overline{\mathrm{SP}}(a,r)$). □

Hence, the NBI method is a restriction of the modified Pascoletti-Serafini method as the parameter a is chosen only from the CHIM and the parameter $r = -\bar{n}$ is chosen constant. In the bicriteria case ($m = 2$) the CHIM consists of the points:

$$\begin{aligned} f^* + \Phi\,\beta &= \begin{pmatrix} f_1(x^1) \\ f_2(x^2) \end{pmatrix} + \begin{pmatrix} 0 & f_1(x^2) - f_1(x^1) \\ f_2(x^1) - f_2(x^2) & 0 \end{pmatrix} \begin{pmatrix} \beta_1 \\ 1 - \beta_1 \end{pmatrix} \\ &= \beta_1 \begin{pmatrix} f_1(x^1) \\ f_2(x^1) \end{pmatrix} + (1 - \beta_1) \begin{pmatrix} f_1(x^2) \\ f_2(x^2) \end{pmatrix} \end{aligned}$$

for $\beta = (\beta_1, 1-\beta_1)^\top \in \mathbb{R}^2_+$. Then the hyperplane H including the CHIM is

$$H := \left\{ f^* + \Phi\,\beta \mid \beta \in \mathbb{R}^2,\ \beta_1 + \beta_2 = 1 \right\}$$

$$= \left\{ \beta\, f(x^1) + (1-\beta)\, f(x^2) \mid \beta \in \mathbb{R} \right\}.$$

The set H^a according to Lemma 2.24 is then

$$H^a = \left\{ \beta\, f(x^1) + (1-\beta)\, f(x^2) \mid \beta \in [0,1] \right\}$$

which equals (only here in the bicriteria case) the set CHIM proposed by Das and Dennis.

In the case of more then three objective functions it is no longer sufficient to consider convex combinations of the extremal points as we have already seen in Example 2.19. That is the reason why in general not all EP-minimal points can be found with the NBI method (as proposed by Das and Dennis) for the case $m \geq 3$. However by allowing the parameter β to vary arbitrarily, and with that the parameter a to vary arbitrarily on the hyperplane including the CHIM, all EP-minimal points of (MOP) can be found by solving $(\overline{\mathrm{SP}}(a,r))$ and $(\mathrm{NBI}(\beta))$ respectively (see Theorem 2.23).

For a discussion of the NBI method see also [140, 209]. In [208] a modification of the NBI method is proposed. There, the equality constraint in $(\mathrm{NBI}(\beta))$ is replaced by the inequality

$$\Phi\beta + s\,\bar{n} \geq_m f(x) - f^*.$$

This modified problem guarantees thus weakly efficient points. It is then a special case of the Pascoletti-Serafini scalarization $(\mathrm{SP}(a,r))$ with the parameters as in Lemma 2.32. In [208] also the connection between that modified problem and the weighted sum as well as the ε-constraint problem are discussed.

2.5.3 Modified Polak Problem

The modified Polak method ([112, 128, 182] and an application in [127]) has a similar connection to the Pascoletti-Serafini problem as the normal boundary intersection method has. We restrict the presentation of the modified Polak method here to the bicriteria case. Then, for different values of the parameter $y_1 \in \mathbb{R}$, the scalar optimization problems called $(\mathrm{MP}(y_1))$

$$\begin{aligned} &\min f_2(x) \\ &\text{subject to the constraints} \\ &f_1(x) = y_1, \\ &x \in \Omega \end{aligned} \tag{2.30}$$

are solved. Here the objectives are transformed to equality constraints, like in the normal boundary intersection problem, and not to inequality constraints, like in the general Pascoletti-Serafini method. Besides the constraint $f_1(x) = y_1$ shows a similarity to the ε-constraint method with the constraint $f_1(x) \leq \varepsilon_1$.

Lemma 2.33. *Let $m = 2$. A point $\bar{x}$ is a minimal solution of (MP(y_1)) if and only if $(f_2(\bar{x}), \bar{x})$ is a minimal solution of ($\overline{\mathrm{SP}}(a,r)$) with $a = (y_1, 0)$ and $r = (0, 1)$.*

Proof. With the parameters a and r as defined in the theorem problem ($\overline{\mathrm{SP}}(a,r)$) reads as follows:

$$\begin{aligned} &\min t \\ &\text{subject to the constraints} \\ &y_1 - f_1(x) = 0, \\ &t - f_2(x) = 0, \\ &t \in \mathbb{R},\ x \in \Omega, \end{aligned}$$

and it can immediately be seen that solving this problem is equivalent to solve problem (MP(y_1)). □

Of course a generalization to the case $m \geq 3$ can be done as well.

The modified Polak method is, like the NBI method, a restriction of the modified Pascoletti-Serafini method. However because the parameter $a = (y_1, 0)$ is allowed to vary arbitrarily in the hyperplane $H = \{y \in \mathbb{R}^2 \mid y_2 = 0\}$ in contrast to the NBI method all EP-minimal points can be found. We can apply Lemma 2.24 and get again a result on the restriction of the parameter set:

Lemma 2.34. *Let $m = 2$, $K = \mathbb{R}^2_+$ and let $f_1(\bar{x}^1) := \min_{x \in \Omega} f_1(x)$ and $f_2(\bar{x}^2) := \min_{x \in \Omega} f_2(x)$ be given. Then, for any EP-minimal solution $\bar{x}$ of the multiobjective optimization problem (MOP) there exists a parameter y_1 with $f_1(\bar{x}^1) \leq y_1 \leq f_1(\bar{x}^2)$ such that $\bar{x}$ is a minimal solution of (MP(y_1)).*

Proof. We determine the set H^a according to Lemma 2.24. It is $H = \{y \in \mathbb{R}^2 \mid (0,1)y = 0\}$ and $r = (0,1)^\top$. Then it follows

$$\bar{a}^1 = \begin{pmatrix} f_1(\bar{x}^1) \\ 0 \end{pmatrix} \quad \text{and} \quad \bar{a}^2 = \begin{pmatrix} f_1(\bar{x}^2) \\ 0 \end{pmatrix}$$

and thus $H^a = \{y = (y_1, 0) \in \mathbb{R}^2 \mid f_1(\bar{x}^1) \leq y_1 \leq f_2(\bar{x}^2)\}$. As a consequence of Lemma 2.24 there exists some $\bar{t} \in \mathbb{R}$ and a parameter $a = (y_1, 0) \in H^a$ with $(\bar{t}, \bar{x})$ a minimal solution of $(\overline{\mathrm{SP}}(a,r))$. Here, $\bar{t}$ is according to the proof of Theorem 2.17 given as $\bar{t} = f_2(\bar{x})$. Thus $(f_2(\bar{x}), \bar{x})$ is a minimal solution of $(\overline{\mathrm{SP}}(a,r))$ and with Lemma 2.33 we conclude that $\bar{x}$ is a minimal solution of $(\mathrm{MP}(y_1))$ with $f_1(\bar{x}^1) \leq y_1 \leq f_1(\bar{x}^2)$. □

This result is also used in the algorithm for the modified Polak method presented in [124, p.314].

2.5.4 Weighted Chebyshev Norm Problem

In this scalarization method ([56, 151, 158, 216], [112, p.13]) for determining EP-minimal points we have weights $w_i > 0$, $i = 1, \ldots, m$, and a reference point ([25, 167, 236]) $a \in \mathbb{R}^m$ with $a_i < \min_{x \in \Omega} f_i(x)$, $i = 1, \ldots, m$, (assuming solutions exist), i. e. $f(\Omega) \subset a + \mathrm{int}(\mathbb{R}^m_+)$, as parameters. For scalarizing the multiobjective optimization problem the weighted Chebyshev norm of the function $f(\cdot) - a$ is minimized:

$$\min_{x \in \Omega} \max_{i \in \{1,\ldots,m\}} w_i(f_i(x) - a_i). \tag{2.31}$$

This problem has the following connection to the Pascoletti-Serafini problem:

Theorem 2.35. *A point $(\bar{t}, \bar{x}) \in \mathbb{R} \times \Omega$ is a minimal solution of (SP(a, r)) with $K = \mathbb{R}^m_+$ and with parameters $a \in \mathbb{R}^m$, $a_i < \min_{x \in \Omega} f_i(x)$, $i = 1, \ldots, m$, and $r \in$ int$(\mathbb{R}^m_+)$ if and only if $\bar{x}$ is a solution of* (2.31) *with reference point a and weights $w_i = \frac{1}{r_i} > 0$, $i = 1, \ldots, m$.*

Proof. If we set $r_i = \frac{1}{w_i} > 0$ and $K = \mathbb{R}^m_+$ problem $(\mathrm{SP}(a,r))$ reads as follows:

$$\begin{gathered} \min t \\ \text{subject to the constraints} \\ a_i + t\,\frac{1}{w_i} - f_i(x) \geq 0, \quad i = 1, \ldots, m, \\ t \in \mathbb{R},\ x \in \Omega. \end{gathered}$$

This is because of $w_i > 0$, $i = 1, \ldots, m$, equivalent to

$$\begin{aligned} &\min t \\ &\text{subject to the constraints} \\ &w_i(f_i(x) - a_i) \leq t, \quad i = 1, \ldots, m, \\ &t \in \mathbb{R},\ x \in \Omega, \end{aligned}$$

but this is according to [124, p.305], or [151, p.14] just a reformulation of (2.31) with an additional variable introduced. □

Thus a variation of the weights in the norm corresponds to a variation of the direction r, and a variation of the reference point is like a variation of the parameter a in the Pascoletti-Serafini method. Similar in the goal attainment method by Gembicki and Haimes, [89], for determining EP-minimal points for the case $m = 2$ the parameter a is interpreted as a goal and the parameter r with $r_i > 0,\ i = 1, 2$, as weights of the deviation of the objective function to the goal. For generating various efficient points the parameter $r \in \operatorname{int}(\mathbb{R}^m_+)$ with $r_1 + r_2 = 1$ is varied.

As a consequence of Theorem 2.1 any solution of (2.31) is at least weakly EP-minimal. Besides, any weakly EP-minimal point can be found as a solution of (2.31). In [231, Example 7.7.1] it is shown, that $f(\Omega) \subset a + \mathbb{R}^m_+$ is necessary for the last statement.

For $a = 0_m$ problem (2.31) reduces to the weighted minimax method as discussed in [149].

2.5.5 Problem According to Gourion and Luc

This problem is lately developed by Gourion and Luc, [95], for finding EP-maximal points of a multiobjective optimization problem with $f(\Omega) \subset \mathbb{R}^m_+$. This corresponds to the multiobjective optimization problem (MOP) w. r. t. the ordering cone $K = \mathbb{R}^m_-$. The parameter dependent scalar optimization problems according to Gourion and Luc read as follows

$$\begin{aligned} &\max s \\ &\text{subject to the constraints} \\ &f(x) \geq s\,\alpha, \\ &s \in \mathbb{R},\ x \in \Omega \end{aligned} \tag{2.32}$$

introducing the new variable $s \in \mathbb{R}$ and with the parameter $\alpha \in \mathbb{R}^m_+$. We will see that the scalarization of Gourion and Luc can be seen as a special case of the Pascoletti-Serafini method with a variation of the parameter $r = -\alpha$ only and with the constant parameter $a = 0_m$.

Theorem 2.36. *A point* $(\bar{s}, \bar{x})$ *is a maximal solution of* (2.32) *with parameter* $\alpha \in \mathbb{R}^m_+$ *if and only if* $(-\bar{s}, \bar{x})$ *is a minimal solution of (SP(a, r)) with* $a = 0_m$, $r = -\alpha \in \mathbb{R}^m_-$ *and* $K = \mathbb{R}^m_-$.

Proof. By defining $r = -\alpha \in \mathbb{R}^m_-$ and $t = -s$ problem (2.32) can be written as

$$\max\ (-t)$$
$$\text{subject to the constraints}$$
$$f(x) \geq (-r) \cdot (-t),$$
$$t \in \mathbb{R},\ x \in \Omega$$

being equivalent to

$$-\ \min\ t$$
$$\text{subject to the constraints}$$
$$t\,r - f(x) \in K,$$
$$t \in \mathbb{R},\ x \in \Omega$$

with $K = \mathbb{R}^m_-$, i. e. to the Pascoletti-Serafini scalarization (SP(a, r)) with $a = 0_m$ and $K = \mathbb{R}^m_-$. $\square$

From this theorem together with Theorem 2.1,c) it follows that if $(\bar{s}, \bar{x})$ is a maximal solution of (2.32) then $\bar{x}$ is weakly $\mathbb{R}^m_-$-minimal, i. e. $\bar{x}$ is weakly EP-maximal for the related multiobjective optimization problem.

For the choice of the parameter $\alpha = -r$ Gourion and Luc present a procedure and give a convergence proof for this method. Further for special sets $f(\Omega)$ they show that the minimal value function (in the notion of the Pascoletti-Serafini problem)

$$r \mapsto \min\{t \mid t\,r - f(x) \in K,\ t \in \mathbb{R},\ x \in \Omega\}$$

is continuous on the set $\{r \in \mathbb{R}^m_+ \mid \sum_{i=1}^{m} r_i = 1\}$ ([95, Lemma 3.1]).

2.5.6 Generalized Weighted Sum Problem

Before we come to the usual weighted sum method we consider a more general formulation having not only a weighted sum as objective function but also similar constraints as the already discussed ε-constraint method (compare [231, p.136]):

$$\begin{aligned} &\min \sum_{i=1}^{m} w_i f_i(x) = w^\top f(x) \\ &\text{subject to the constraints} \\ &f_i(x) \le \varepsilon_i \quad \text{for all } i \in P, \\ &\qquad x \in \Omega \end{aligned} \tag{2.33}$$

with $P \subsetneqq \{1, \ldots, m\}$, $\varepsilon_i \in \mathbb{R}$ for all $i \in P$, and weights $w \in \mathbb{R}^m$, $\sum_{i \notin P} w_i > 0$. We start with the connection to the Pascoletti-Serafini problem and from that we conclude some properties of the problem (2.33).

Theorem 2.37. *A point $\bar{x}$ is a minimal solution of* (2.33) *for the parameters $w \in \mathbb{R}^m$, $\sum_{i \notin P} w_i > 0$, $\varepsilon_i \in \mathbb{R}$, $i \in P$, if and only if there is some $\bar{t}$ so that $(\bar{t}, \bar{x})$ is a minimal solution of (SP(a, r)) with $a_i = \varepsilon_i$ for $i \in P$, a_i arbitrary for $i \in \{1, \ldots, m\} \setminus P$, $r_i = 0$ for $i \in P$, $r_i = 1$ for $i \in \{1, \ldots, m\} \setminus P$ and cone $K_w := \{y \in \mathbb{R}^m \mid y_i \ge 0,$ for all $i \in P,\ w^\top y \ge 0\}$, i. e. of*

$$\begin{aligned} &\min t \\ &\textit{subject to the constraints} \\ &a + t\, r - f(x) \in K_w, \\ &t \in \mathbb{R},\ x \in \Omega. \end{aligned} \tag{2.34}$$

Proof. The optimization problem (2.34) is equivalent to

$$\begin{aligned} &\min t \\ &\text{subject to the constraints} \\ &w^\top (a + t\, r - f(x)) \ge 0, \\ &a_i + t\, r_i - f_i(x) \ge 0 \quad \text{for all } i \in P, \\ &t \in \mathbb{R},\ x \in \Omega. \end{aligned} \tag{2.35}$$

As $a_i = \varepsilon_i$ and $r_i = 0$ for $i \in P$ and because of $w^\top r = \sum_{i \notin P} w_i > 0$, a point $(\bar{t}, \bar{x})$ is a minimal solution of (2.35) if and only if $\bar{x}$ is a minimal solution of

$$\begin{aligned} &\min \tfrac{w^\top f(x) - w^\top a}{w^\top r} \\ &\text{subject to the constraints} \\ &f_i(x) \le \varepsilon_i \quad \text{for all } i \in P, \\ &\qquad x \in \Omega. \end{aligned} \tag{2.36}$$

Because we can ignore the constant term $-\frac{w^\top a}{w^\top r}$ in the objective function of (2.36) and because of $w^\top r > 0$ a point $\bar{x}$ is a minimal solution of (2.36) if and only if it is a minimal solution of (2.33). $\square$

The set K_w is a closed convex cone and for $|P| = m-1$ the cone K_w is even pointed ([231, Lemma 7.11.1]).

Corollary 2.38. *Let $\bar{x}$ be a minimal solution of* (2.33) *and let* $K \subset \{y \in \mathbb{R}^m \mid y_i \geq 0 \quad \text{for all } i \in P\}$, $w \in K^* \setminus \{0_m\}$ *and* $\sum_{i \notin P} w_i > 0$, *then* $\bar{x} \in M_w(f(\Omega), K)$.

Proof. Applying Theorem 2.37 and Theorem 2.1,c) it follows that $\bar{x} \in \mathcal{M}_w(f(\Omega), K_w)$ with $K_w = \{y \in \mathbb{R}^m \mid y_i \geq 0 \text{ for all } i \in P,\ w^\top y \geq 0\}$. As for all $y \in K$ we have $w^\top y \geq 0$ it follows $K \subset K_w$ and hence, with Lemma 1.7, $M_w(f(\Omega), K_w) \subset M_w(f(\Omega), K)$. $\square$

Thus, e. g. for $K = \mathbb{R}^m_+$, $w \in \mathbb{R}^m_+$ and $w_i > 0$ for all $i \notin P$, all minimal solutions of (2.33) are at least weakly EP-minimal. For $w_i > 0$, $i = 1, \ldots, m$, they are even EP-minimal (see [231, Theorem 7.11.1c)]).

Of course we can find all EP-minimal points $\bar{x} \in \mathcal{M}(f(\Omega), \mathbb{R}^m_+)$ by solving (2.33) if we set $P = \{1, \ldots, m-1\}$, $w_i = 0$ for $i \in P$, and $w_m = 1$. Then $K_w = \mathbb{R}^m_+$ and (2.33) equals the ε-constraint problem $(P_m(\varepsilon))$. By choosing $\varepsilon_i = f_i(\bar{x})$ for all $i \in P$, it is known that $\bar{x}$ is a minimal solution of the ε-constraint problem and hence of the problem (2.33) (see Corollary 2.31).

The case $P = \{1, \ldots, m\}$ is introduced and discussed in Charnes and Cooper ([29]) and later in Wendell and Lee ([233]). See also [97]. Weidner has shown ([231, p.130]) that there is no equivalent formulation between (2.33) with $P = \{1, \ldots, m\}$ and $(\mathrm{SP}(a, r))$.

2.5.7 Weighted Sum Problem

Now we come to the usual weighted sum method ([245], see also [55, 86])

$$\begin{array}{c} \min w^\top f(x) \\ \text{subject to the constraint} \\ x \in \Omega \end{array} \tag{2.37}$$

for weights $w \in K^* \setminus \{0_m\}$ which is just a special case of (2.33) for $P = \emptyset$. Because it is such an important problem formulation we adapt Theorem 2.37 for this special case (see also Fig. 2.9):

Theorem 2.39. *A point $\bar{x}$ is a minimal solution of* (2.37) *for the parameter $w \in K^* \setminus \{0_m\}$ if and only if there is some $\bar{t}$ so that $(\bar{t}, \bar{x})$ is a minimal solution of (SP(a,r)) with $a \in \mathbb{R}^m$ arbitrarily chosen, cone $K_w := \{y \in \mathbb{R}^m \mid w^\top y \geq 0\}$ and $r \in int(K_w)$.*

Proof. Problem (SP(a,r)) with cone K_w reads as follows:

$$\begin{gathered} \min t \\ \text{subject to the constraints} \\ w^\top (a + t\,r - f(x)) \geq 0, \\ t \in \mathbb{R},\ x \in \Omega. \end{gathered} \tag{2.38}$$

Because of $r \in \operatorname{int}(K_w)$ we have $w^\top r > 0$ and hence (2.38) is equivalent to

$$\begin{gathered} \min \tfrac{w^\top f(x) - w^\top a}{w^\top r} \\ \text{subject to the constraint} \\ x \in \Omega. \end{gathered}$$

With the same arguments as used in the proof to Theorem 2.37 this is equivalent to (2.37). □

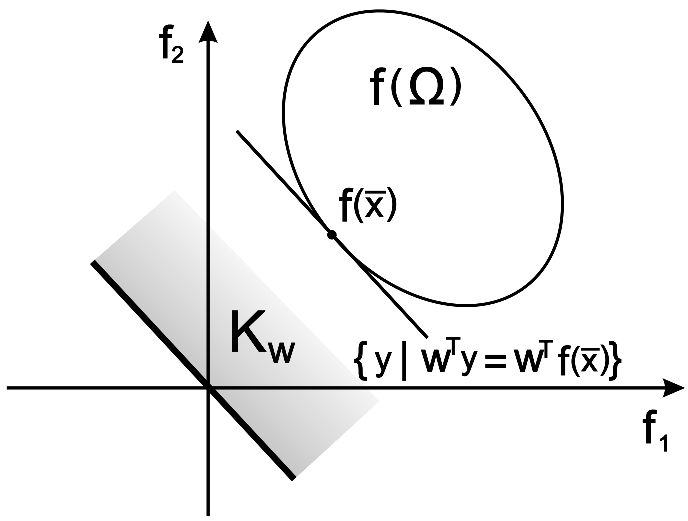

Fig. 2.9. Connection between the weighted sum and the Pascoletti-Serafini problem.

Hence a variation of the weights $w \in K^* \setminus \{0_m\}$ corresponds to a variation of the ordering cone K_w. So we get a new interpretation for the weighted sum method. The cone K_w is a closed convex polyhedral cone, but K_w is not pointed. That is the reason why the results from Theorem 2.1,b) cannot be applied to the weighted sum method and why it is in general (in the non-convex case) not possible to find all K-minimal points of (MOP) by solving the weighted sum method with appropriate weights. However it is known that in the case of a convex set $f(\Omega)$ all K-minimal points of the multiobjective optimization problem (MOP) can be found ([124, pp.299f]). In [239] the stability of the solutions of the weighted sum problem is studied.

We can conclude from Theorem 2.1,c) the following well known result:

Corollary 2.40. *Let $\bar{x}$ be a minimal solution of (2.37) with parameter $w \in K^* \setminus \{0_m\}$, then $\bar{x}$ is weakly K-minimal for (MOP).*

Proof. According to Theorem 2.39 there is some $\bar{t}$ so that $(\bar{t}, \bar{x})$ is a minimal solution of $(\mathrm{SP}(a,r))$ with cone K_w and hence $\bar{x}$ is according to Theorem 2.1,c) and Remark 2.2 a weakly K_w-minimal point. Because $w \in K^* \setminus \{0_m\}$ we have $w^\top y \geq 0$ for all $y \in K$ and hence $K \subset K_w$. Thus, according to Lemma 1.7, $\mathcal{M}_w(f(\Omega), K_w) \subset \mathcal{M}_w(f(\Omega), K)$ and hence $\bar{x}$ is a weakly K-minimal point, too. □

The weighted sum method has the same drawback against the Pascoletti-Serafini method as the ε-constraint method has, as shown in Example 2.30: not for any choice of the parameters $w \in K^* \setminus \{0_m\}$ there exists a minimal solution, even not for the case $\mathcal{M}(f(\Omega), K) \neq \emptyset$. In [24, Ex. 7.3] Brosowski gives a simple example where the weighted sum problem delivers only for one choice of weights a minimal solution and where it is not solvable for all other weights despite the set $f(\Omega) + K$ is closed and convex in contrast to Corollary 2.5:

Example 2.41. We consider the bicriteria optimization problem

$$\begin{gathered} \min f(x) = x \\ \text{subject to the constraints} \\ x_1 + x_2 \geq 1, \\ x \in \mathbb{R}^2 \end{gathered}$$

w. r. t. the natural ordering. The set of K-minimal points is given by

$$\{x = (x_1, x_2)^\top \in \mathbb{R}^2 \mid x_1 + x_2 = 1\}.$$

Here the weighted sum problem has a minimal solution only for the weights $w_1 = w_2 = 0.5$. For all other parameters there are feasible points but the scalar problem is not solvable.

We further want to mention the weighted p-power method ([149]) where the scalarization is given by

$$\min_{x \in \Omega} \sum_{i=1}^{m} w_i f_i^p(x) \tag{2.39}$$

for $p \geq 1$, $w \in \mathbb{R}^m_+ \setminus \{0_m\}$ (or $w \in K^* \setminus \{0_m\}$). For $p = 1$ (2.39) is equal to the weighted sum method. For arbitrary p the problem (2.39) can be seen as an application of the weighted sum method to the multiobjective optimization problem

$$\min_{x \in \Omega} \begin{pmatrix} f_1^p(x) \\ \vdots \\ f_m^p(x) \end{pmatrix}.$$

Another generalization of the weighted sum method is discussed in [231, pp.111f]. There k ($k \in \mathbb{N},\ k \leq m$) linearly independent weights $w^1, \ldots, w^k \in \mathbb{R}^m_+$ are allowed representing for instance the preferences of k decision makers. Besides a reference point $v \in \mathbb{R}^k$ is given. Then the problem

$$\min_{x \in \Omega} \max_{i \in \{1,\ldots,k\}} (w^i)^\top f(x) - v_i \tag{2.40}$$

is solved. The connection to the parameters of the Pascoletti-Serafini problem is given by the equations

$$\begin{aligned} (w^i)^\top a &= v_i, \qquad & i &= 1, \ldots, k, \\ (w^i)^\top r &= 1, \qquad & i &= 1. \ldots, k, \end{aligned}$$

and $K := \{y \in \mathbb{R}^m \mid (w^i)^\top y \geq 0,\ i = 1, \ldots, k\}$. The set K is a closed convex cone and K is pointed if and only if $k = m$. A minimal solution of (2.40) is not only weakly K-minimal but because of $\mathbb{R}^m_+ \subset K$ also weakly EP-minimal ([231, Theorem 7.2.1a)]). For $k = 1$ and $v = 0$ (2.40) is equivalent to (2.37). In the following section we will discuss a special case of the problem (2.40) for $k = m$.

2.5.8 Problem According to Kaliszewski

In [134] Kaliszewski discusses the following problem called (P^∞):

$$\min_{x\in\Omega}\max_{i\in\{1,\dots,m\}}\lambda_i\left((f_i(x)-y_i^*)+\rho\sum_{j=1}^m(f_j(x)-y_j^*)\right) \tag{2.41}$$

for a closed set $f(\Omega)\subset y^*+\mathrm{int}(\mathbb{R}^m_+)$, $y^*\in\mathbb{R}^m$, and $\rho>0$, $\lambda_i>0$, $i=1,\dots,m$, for determining properly efficient solutions ([134, Theorem 4.2]). The connection to the Pascoletti-Serafini problem is given by the following theorem (compare [231, pp.118f]):

Theorem 2.42. *A point $\bar{x}$ is a minimal solution of (P^∞) for $f(\Omega)\subset y^*+int(\mathbb{R}^m_+)$, $y^*\in\mathbb{R}^m$, $\rho>0$, and $\lambda_i>0$, $i=1,\dots,m$, if and only if the point $(\bar{t},\bar{x})$ with*

$$\bar{t}=\max_{i\in\{1,\dots,m\}}\lambda_i\left((f_i(\bar{x})-y_i^*)+\rho\sum_{j=1}^m(f_j(\bar{x})-y_j^*)\right) \tag{2.42}$$

is a minimal solution of (SP(a,r)) with $a=y^*$, $r\in\mathbb{R}^m$ with

$$r_i+\rho\sum_{j=1}^m r_j=\frac{1}{\lambda_i},\qquad \text{for all } i=1,\dots,m, \tag{2.43}$$

and $K=\{y\in\mathbb{R}^m\mid y_i+\rho\sum_{i=1}^m y_j\geq 0,\ i=1,\dots,m\}$.

Proof. For the parameter a and the cone K as in the theorem the problem (SP(a,r)) reads as follows

$$\min t$$
$$\text{subject to the constraints}$$
$$y_i^*+t\,r_i-f_i(x)+\rho\textstyle\sum_{j=1}^m(y_j^*+t\,r_j-f_j(x))\geq 0,\qquad i=1,\dots,m,$$
$$t\in\mathbb{R},\ x\in\Omega,$$

which is because of $r_i+\rho\sum_{j=1}^m r_j=\frac{1}{\lambda_i}>0$ equivalent to

$$\min t$$
$$\text{subject to the constraints}$$
$$t\geq\frac{f_i(x)-y_i^*+\rho\sum_{j=1}^m(f_j(x)-y_j^*)}{r_i+\rho\sum_{j=1}^m r_j},\qquad i=1,\dots,m,$$
$$t\in\mathbb{R},\ x\in\Omega.$$

Using (2.43) a point $(\bar{t}, \bar{x})$ is a minimal solution of this problem if and only if $\bar{x}$ is a solution of (2.41) with $\bar{t}$ as in (2.42). $\square$

The set K is a closed pointed convex cone and we have $r \in K$. For $m = 2$ the cone K is given by the set

$$K = \left\{ y \in \mathbb{R}^2 \;\middle|\; \begin{pmatrix} 1+\rho & \rho \\ \rho & 1+\rho \end{pmatrix} y \geq 0_2 \right\}.$$

Hence the parameter ρ controls the cone K. A variation of the parameters λ and ρ lead to a variation of the parameter r while the parameter a is chosen constant as y^*. Because of $\mathbb{R}^m_+ \subset K$ and $r \in K$ for $\lambda_i > 0$, $i = 1 \ldots, m$, we have $\mathcal{E}_w(f(\Omega), K) \subset \mathcal{E}_w(f(\Omega), \mathbb{R}^m_+)$ and thus a minimal solution of (P^∞) is an at least weakly EP-minimal point of (MOP) as a result of Lemma 1.7.

2.5.9 Further Scalarizations

We have shown that many scalarization problems can be seen as a special case of the Pascoletti-Serafini method and hence that the results for the general problem can be applied to these special cases, too. The enumeration of special cases is not complete. For example in [231] a problem called hyperbola efficiency going back to [75] is discussed. However for a connection to the Pascoletti-Serafini problem K has to be defined as a convex set which is not a cone. Also a generalization of the weighted Chebyshev norm problem is mentioned there which can be connected to the Pascoletti-Serafini problem then using a closed pointed convex cone K.

There are many other scalarization approaches, too, which cannot be subsumed under the Pascoletti-Serafini method like the hybrid method ([62, p.101]), the elastic constraint method ([62, p.102]), or Benson's method ([11]). Literature with surveys about different scalarization approaches is listed in the introduction of this chapter.

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