2

On the Positive Solution to a Linear System with Nonnegative Coefficients

This chapter deals with a positive solution $\mathbf{p}$ to the following system of linear equations with nonnegative coefficients:

$$\mathbf{p} = \mathbf{u} + \mathbf{X}\mathbf{p}\,. \tag{2.1}$$

Here and hereafter, $\mathbf{u} \in \mathbb{R}^K_{++}$ is a given *positive* vector, $\mathbf{X} \in \mathbb{R}^{K\times K}_{+}$ is a given nonnegative matrix (not necessarily irreducible), and $\mathbf{p} \in \mathbb{R}^K_{++}$ is a sought vector, provided that it exists.

2.1 Basic Concepts and Definitions

Before starting with the analysis, we need to address the fundamental problem of the existence of a positive solution $\mathbf{p}$ to (2.1). This problem is addressed in App. A.4.4. In particular, by Theorem A.51, we know that a necessary and sufficient condition for $\mathbf{p} \geq 0, \mathbf{p} \neq \mathbf{0}$, to exist is that $\rho(\mathbf{X}) < 1$ where $\rho(\mathbf{X})$ is the spectral radius of $\mathbf{X}$. Moreover, as $\mathbf{u}$ is positive, there is a unique solution $\mathbf{p}$, which is strictly positive and given by

$$\mathbf{p} = (\mathbf{I} - \mathbf{X})^{-1}\mathbf{u}\,.$$

Theorem A.39 asserts that $\lambda_p := \rho(\mathbf{X})$ is an eigenvalue of $\mathbf{X}$, that is to say $\lambda_p \in \sigma(\mathbf{X})$ where $\sigma(\mathbf{X})$ is used to denote the spectrum of $\mathbf{X}$ (Definition A.10).

Remark 2.1. Note that except for the nonnegativity, there are no additional constraints on $\mathbf{X}$. In particular, $\mathbf{X}$ does not need to be irreducible. However, it is worth pointing out that if $\mathbf{X}$ is irreducible and its Perron root $\lambda_p = \rho(\mathbf{X}) > 0$ satisfies $\lambda_p < 1$, then $\mathbf{u} \neq \mathbf{0}$ does not need to be positive for (2.1) to have a unique positive solution $\mathbf{p}$. This is one part of the assertion of Theorem A.52.

Analogous to the previous chapter, we allow the entries of $\mathbf{X}$ to continuously depend on some parameter vector $\boldsymbol{\omega} \in \Omega$ where the parameter set Ω is

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defined by (1.53) and is an open convex subset of $\mathbb{R}^K$. The only difference is that here the matrix is not required to be irreducible for all parameter vectors. In fact, $\mathbf{X}(\boldsymbol{\omega})$ can even be identically the zero matrix, in which case, however, the problems addressed in this chapter are trivial. To be precise, let

$$\mathbf{X}(\boldsymbol{\omega}) := (x_{k,l}(\boldsymbol{\omega}))_{1 \le k,l \le K}$$

be a matrix-valued function whose entries $x_{k,l} : \Omega \to \mathbb{R}_+$ are *continuous* functions defined on Ω. Considering Definition 1.35, this is formally written as $\mathbf{X} \in \mathrm{N}_K(\Omega)$, in which case $\mathbf{X}$ is said to be nonnegative on Ω. To conform with the applications in wireless networks, we let each entry of the vector $\mathbf{u}$ in (2.1) be a continuous positive function of the parameter vector $\boldsymbol{\omega}$ as well. We indicate this by writing $\mathbf{u} \in \mathbb{R}^K_{++}(\Omega)$.

Now it follows from (2.1) and Theorem A.51 that, for any fixed $\boldsymbol{\omega} \in \Omega$, there exists a unique positive vector $\mathbf{p}(\boldsymbol{\omega})$ satisfying[1]

$$\mathbf{p}(\boldsymbol{\omega}) = \mathbf{X}(\boldsymbol{\omega})\mathbf{p}(\boldsymbol{\omega}) + \mathbf{u}(\boldsymbol{\omega}) \tag{2.2}$$

if and only if

$$\lambda_p(\boldsymbol{\omega}) := \rho(\mathbf{X}(\boldsymbol{\omega})) < 1\,. \tag{2.3}$$

Moreover, for any $\boldsymbol{\omega} \in \Omega$ with $\lambda_p(\boldsymbol{\omega}) < 1$,

$$\mathbf{p}(\boldsymbol{\omega}) = \big(\mathbf{I} - \mathbf{X}(\boldsymbol{\omega})\big)^{-1}\mathbf{u}(\boldsymbol{\omega})\,. \tag{2.4}$$

Let F be the set of those parameter vectors $\boldsymbol{\omega} \in \Omega$ for which a positive solution $\mathbf{p}(\boldsymbol{\omega})$ to (2.2) exists. Formally, we have

$$\mathrm{F} := \{\boldsymbol{\omega} \in \Omega : \lambda_p(\boldsymbol{\omega}) < 1\}\,. \tag{2.5}$$

Note that each entry of the vector $\mathbf{p}(\boldsymbol{\omega})$ is a *continuous* map from F into the set of positive reals $\mathbb{R}_{++}$. This is because if $\boldsymbol{\omega} \in \mathrm{F}$, then the Neumann series $\sum_{l=0}^{\infty}(\mathbf{X}(\boldsymbol{\omega}))^l$ converges (Theorem A.16) and $(\mathbf{I} - \mathbf{X}(\boldsymbol{\omega}))^{-1} = \sum_{l=0}^{\infty}(\mathbf{X}(\boldsymbol{\omega}))^l$. Therefore, since a composition of continuous maps is continuous, it follows from

$$p_k(\boldsymbol{\omega}) = \mathbf{e}_k^T\big(\mathbf{I} - \mathbf{X}(\boldsymbol{\omega})\big)^{-1}\mathbf{u}(\boldsymbol{\omega}),\ \boldsymbol{\omega} \in \mathrm{F}, 1 \le k \le K \tag{2.6}$$

that $p_k : \mathrm{F} \to \mathbb{R}_{++}$ is continuous. In particular, this implies that the l^1-norm of $\mathbf{p}(\boldsymbol{\omega})$ given by

$$\|\mathbf{p}(\boldsymbol{\omega})\|_1 = \sum_{k \in \mathcal{K}} p_k(\boldsymbol{\omega}) = \mathbf{1}^T(\mathbf{I} - \mathbf{X}(\boldsymbol{\omega}))^{-1}\mathbf{u}(\boldsymbol{\omega}),\ \boldsymbol{\omega} \in \mathrm{F} \tag{2.7}$$

is a continuous function on F as well.

[1] However, as $x_{k,l} : \Omega \to \mathbb{R}_+$ are not one-to-one maps, there may exist $\hat{\boldsymbol{\omega}}, \check{\boldsymbol{\omega}} \in \Omega, \hat{\boldsymbol{\omega}} \neq \check{\boldsymbol{\omega}}$, such that $\mathbf{p}(\hat{\boldsymbol{\omega}}) = \mathbf{p}(\check{\boldsymbol{\omega}})$.

In this chapter, we analyze both $p_k(\boldsymbol{\omega})$ and $\|\mathbf{p}(\boldsymbol{\omega})\|_1$ as functions of the parameter vector $\boldsymbol{\omega} \in \mathrm{F}$. In doing so, most of our interest is devoted to matrix-valued functions $\mathbf{X}(\boldsymbol{\omega})$ of the form $\mathbf{X}(\boldsymbol{\omega}) = \boldsymbol{\Gamma}(\boldsymbol{\omega})\mathbf{V}$ with $\mathbf{V} \in \mathrm{N}_K$ and

$$\boldsymbol{\Gamma}(\boldsymbol{\omega}) = \mathrm{diag}\big(\gamma_1(\omega_1), \ldots, \gamma_K(\omega_K)\big)\,.$$

Here and hereafter, $\gamma_k : \mathrm{Q}_k \to \mathbb{R}_{++}$ is a continuous strictly monotonic (bijective) function and $\mathrm{Q}_k \subseteq \mathbb{R}$ is some open interval (see also Sect. 1.3.1). Formally, this is denoted by $\mathbf{X} \in \mathrm{N}_{K,\boldsymbol{\Gamma}}(\Omega)$ where

$$\mathrm{N}_{K,\boldsymbol{\Gamma}}(\Omega) := \{\boldsymbol{\Gamma}(\boldsymbol{\omega})\mathbf{V}, \boldsymbol{\omega} \in \Omega : \mathbf{V} \in \mathrm{N}_K\} \subset \mathrm{N}_K(\Omega) \tag{2.8}$$

is the set of all nonnegative matrix-valued functions $\mathbf{X}(\boldsymbol{\omega})$ of the form $\mathbf{X}(\boldsymbol{\omega}) = \boldsymbol{\Gamma}(\boldsymbol{\omega})\mathbf{V}$ for some given $\gamma_k : \mathrm{Q}_k \to \mathbb{R}_{++}, k = 1, \ldots, K$. In this special case, it will also be assumed that $\mathbf{u}(\boldsymbol{\omega}) = (\gamma_1(\omega_1), \ldots, \gamma_K(\omega_K))$. Exceptions are only Sects. 2.3.1 and 2.3.2, where $\mathbf{X}(\boldsymbol{\omega})$ and $\mathbf{u}(\boldsymbol{\omega})$ are not confined to this special form.

2.2 Feasibility Sets

The set F defined by (2.5) contains all parameter vectors such that a positive solution to our system of linear equations exists. For this reason, if there are no additional constraints on $\mathbf{p}$, F is referred to as the feasibility set. Notice that the definition is analogous to Definition 1.41, except that now the spectral radius must be strictly smaller than 1. Therefore, the parameter vectors satisfying $\lambda_p(\boldsymbol{\omega}) = 1$ are not members of F.[2]

In wireless networks, however, some additional constraints on $\mathbf{p}$ are imposed, which gives rise to the definition of some subset of F as the feasibility set. Constraints on the l^1-norm of $\mathbf{p}(\boldsymbol{\omega})$ are common to applications in wireless communications networks. More precisely, we say that $\mathbf{p}(\boldsymbol{\omega})$ is constrained in the l^1-norm if

$$\|\mathbf{p}(\boldsymbol{\omega})\|_1 \le P_{\mathrm{t}}, \quad \boldsymbol{\omega} \in \Omega$$

must hold for some given constant $P_{\mathrm{t}} > 0$, referred to as a sum (or total) constraint. Consequently, in this case, the parameter vector $\boldsymbol{\omega} \in \Omega$ is feasible if and only if $\boldsymbol{\omega} \in \mathrm{F}(P_{\mathrm{t}})$ where

$$\mathrm{F}(\alpha) = \{\boldsymbol{\omega} \in \mathrm{F} : \|\mathbf{p}(\boldsymbol{\omega})\|_1 \le \alpha, \alpha > 0\} \subseteq \mathrm{F}\,. \tag{2.9}$$

Notice that due to continuity of $\|\mathbf{p}(\boldsymbol{\omega})\|_1$, $\mathrm{F}(\alpha)$ is monotonic in $\alpha > 0$ with respect to set inclusion in the following sense: For any $0 < \alpha \le \beta$, there holds $\mathrm{F}(\alpha) \subseteq \mathrm{F}(\beta)$. Therefore, since $\mathrm{F}(\alpha) \subseteq \mathrm{F}$ for all $\alpha > 0$, we have

[2] In the previous chapter, F is the set of all the parameter vectors for which the *homogenous* system of linear equations $(\mathbf{I} - \mathbf{X}(\boldsymbol{\omega}))\mathbf{p}(\boldsymbol{\omega}) = 0$, with $\mathbf{X}(\boldsymbol{\omega})$ being irreducible for all $\boldsymbol{\omega} \in \Omega$, has a positive solution $\mathbf{p}(\boldsymbol{\omega})$.

$$\mathrm{F} = \bigcup_{\alpha>0} \mathrm{F}(\alpha) \tag{2.10}$$

where the union is taken with respect to all $\alpha > 0$.

Another common situation encountered in wireless networks is that of constraining each element of $\mathbf{p}(\boldsymbol{\omega})$ individually. Therefore, if there are positive constants $P_1, \ldots, P_K$ such that $p_k(\boldsymbol{\omega}) \leq P_k$ must hold for each $1 \leq k \leq K$, we say that $\mathbf{p}(\boldsymbol{\omega})$ is subject to *individual constraints*. Clearly, in this case, the set of all feasible parameter vectors is given by

$$\mathrm{F}(P_1, \ldots, P_K) := \bigcap_{\alpha \in \{P_1, \ldots, P_K\}} \mathrm{F}_k(\alpha) \tag{2.11}$$

where

$$\mathrm{F}_k(\alpha) := \{\boldsymbol{\omega} \in \mathrm{F} : p_k(\boldsymbol{\omega}) \leq \alpha\}\,. \tag{2.12}$$

These two types of constraints are often combined by imposing both individual and sum constraints on $\mathbf{p}(\boldsymbol{\omega})$. Therefore, in this case, the feasibility set becomes

$$\mathrm{F}(P_{\mathrm{t}}; P_1, \ldots, P_K) := \mathrm{F}(P_{\mathrm{t}}) \cap \mathrm{F}(P_1, \ldots, P_K)\,. \tag{2.13}$$

Note that $\mathrm{F}(P_{\mathrm{t}}; P_1, \ldots, P_K) = \mathrm{F}(P_{\mathrm{t}})$ if $P_{\mathrm{t}} \leq P_k$ for each $1 \leq k \leq K$, and $\mathrm{F}(P_{\mathrm{t}}; P_1, \ldots, P_K) = \mathrm{F}(P_1, \ldots, P_K)$ if $\sum_k P_k \leq P_{\mathrm{t}}$. Thus, both $\mathrm{F}(P_{\mathrm{t}})$ and $\mathrm{F}(P_1, \ldots, P_K)$ can be viewed as special cases of $\mathrm{F}(P_{\mathrm{t}}; P_1, \ldots, P_K)$.

Remark 2.2. In what follows, we exclude the trivial case where the feasibility set is an empty set.

The next observation immediately follows from the connectedness of Ω, continuity of $\lambda_p(\boldsymbol{\omega}), \boldsymbol{\omega} \in \Omega$, and $\mathbf{p}(\boldsymbol{\omega}), \boldsymbol{\omega} \in \mathrm{F}$, as well as [39, Theorem 4.22] .

Observation 2.3. $\mathrm{F}(P_{\mathrm{t}}; P_1, \ldots, P_K)$ *is a connected set (see Definition B.1).*

It is important to emphasize that the geometry of the feasibility sets depends on the choice of $\mathbf{X}(\boldsymbol{\omega})$ and $\mathbf{u}(\boldsymbol{\omega}), \boldsymbol{\omega} \in \Omega$. In particular, the feasibility set is not convex in general. To illustrate the definitions, let us consider an elementary example.

Example 2.4. Let $\mathbf{X}(\boldsymbol{\omega}) = 0$ for all $\boldsymbol{\omega} \in \Omega$ and $\mathbf{u}(\boldsymbol{\omega}) = (\gamma(\omega_1), \ldots, \gamma(\omega_K))$ where $\gamma : \mathrm{Q} \to \mathbb{R}_{++}$ is any continuous bijective function. We see that (2.4) reduces to $\mathbf{p}(\boldsymbol{\omega}) = (\gamma(\omega_1), \ldots, \gamma(\omega_K))$, and hence one obtains

$$\mathrm{F} = \Omega = \mathrm{Q}^K$$
$$\mathrm{F}(P_{\mathrm{t}}) = \{\boldsymbol{\omega} \in \mathrm{F} : \textstyle\sum_k \gamma(\omega_k) \leq P_{\mathrm{t}}\}$$
$$\mathrm{F}(P_1, \ldots, P_K) = \{\boldsymbol{\omega} \in \mathrm{F} : \gamma(\omega_k) \leq P_k, 1 \leq k \leq K\}\,.$$

Clearly, F and $\mathrm{F}(P_1, \ldots, P_K)$ are both convex sets, regardless of the choice of $\gamma(x)$. In contrast, $\mathrm{F}(P_{\mathrm{t}})$ is not convex in general. A sufficient condition for $\mathrm{F}(P_{\mathrm{t}})$ (and also $\mathrm{F}(P_{\mathrm{t}}; P_1, \ldots, P_K)$) to be a convex set is that $\gamma(x)$ is convex.

An important example of a convex function is $\gamma(x) = e^x - 1, x > 0$. Assuming $\mathbf{X}(\boldsymbol{\omega}) = 0$ for all $\boldsymbol{\omega} \in \Omega = \mathbb{R}^2_{++}$ and $\mathbf{u}(\boldsymbol{\omega}) = (e^{\omega_1} - 1, e^{\omega_2} - 1)$, Fig. 2.1 depicts the feasibility set $\mathrm{F}(P_\mathrm{t}; P_1, P_2) \subset \mathbb{R}^2_{++}$ defined by (2.13) for some P_1, P_2 and P_t.

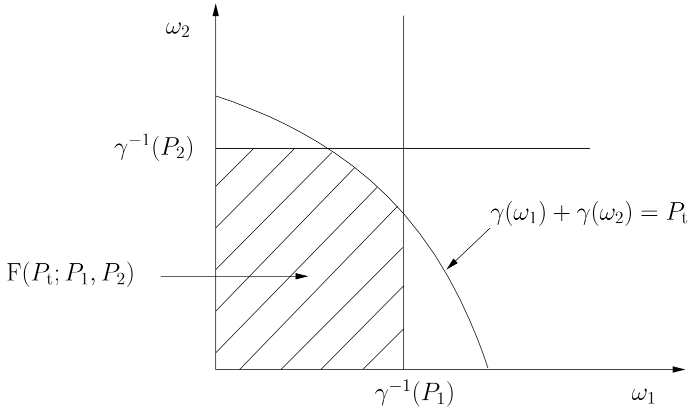

Fig. 2.1: Illustration of Example 2.4: The feasibility set $\mathrm{F}(P_\mathrm{t}; P_1, P_2)$ with $\mathbf{X}(\boldsymbol{\omega}) \equiv 0$, $\gamma(x) = e^x - 1, x > 0$, and $\mathbf{u}(\boldsymbol{\omega}) = (e^{\omega_1} - 1, e^{\omega_2} - 1)$. The constraints P_1, P_2 and P_t are chosen to satisfy $0 < P_1, P_2 < P_\mathrm{t}$ and $P_\mathrm{t} < P_1 + P_2$.

Unfortunately, as the example below shows, convexity of $\gamma(x)$ is not sufficient for $\mathrm{F}(P_\mathrm{t})$ to be a convex set if $\mathbf{X}(\boldsymbol{\omega}) = \boldsymbol{\Gamma}(\boldsymbol{\omega})\mathbf{V} \neq \mathbf{0}$.

Example 2.5. Suppose that $\mathbf{X}(\boldsymbol{\omega}) = \left(\begin{smallmatrix} 0 & \gamma(\omega_1)\varrho \\ \gamma(\omega_2)\varrho & 0 \end{smallmatrix}\right)$ for some $\varrho \geq 0$. Furthermore, assume that $\mathbf{u}(\boldsymbol{\omega}) = (\gamma(\omega_1), \gamma(\omega_2))$ and $\gamma(x) = e^x - 1, x > 0$. Thus,

$$\begin{aligned} \Omega &= \mathbb{R}^K_{++} \\ \mathrm{F} &= \{\boldsymbol{\omega} \in \Omega : \lambda_p(\boldsymbol{\omega}) = \varrho\sqrt{(e^{\omega_1} - 1)(e^{\omega_2} - 1)} < 1\}\,. \end{aligned}$$

Now we claim that F is not a convex set if $\varrho > 0$. To see this, we write $\lambda_p(\boldsymbol{\omega}) = 1$ with $\varrho > 0$ as a function of $\omega_1 > 0$ to obtain $\omega_2 = f(\omega_1) = \log \frac{1+\varrho^2 e^{\omega_1} - \varrho^2}{\varrho^2(e^{\omega_1} - 1)}, \varrho > 0$. The function $f(x), x > 0$, is twice differentiable and its second derivative is strictly positive for all $x > 0$. Consequently, instead of the feasibility set F, its complement in $\mathbb{R}^K_{++}$ ($\mathrm{F}^c = \mathbb{R}^K_{++} \setminus \mathrm{F}$) is convex.

Now let us consider $\mathrm{F}(P_\mathrm{t})$ with $\varrho \geq 0$. Applying (2.7) to our special case yields

$$\|\mathbf{p}(\boldsymbol{\omega})\|_1 = \frac{e^{\omega_1} + e^{\omega_2} - 2 + 2\varrho(e^{\omega_1} - 1)(e^{\omega_2} - 1)}{1 - \varrho^2(e^{\omega_1} - 1)(e^{\omega_2} - 1)}, \ \boldsymbol{\omega} \in \mathrm{F}\,.$$

Hence, writing $\|\mathbf{p}(\boldsymbol{\omega})\|_1 = P_\mathrm{t}$ as a function of $\omega_1 \in [0, \log(1 + P_\mathrm{t})]$, one obtains

$$\omega_2 = g(\omega_1) = \log \frac{(1-\varrho)\,(2+P_{\mathrm{t}}+P_{\mathrm{t}}\,\varrho) + e^{\omega_1}\,(\varrho\,(2+P_{\mathrm{t}}\,\varrho)-1)}{1+\varrho\big(e^{\omega_1}-1\big)\big(2+P_{\mathrm{t}}\varrho\big)}$$

where the argument under the logarithm is positive. Now if $\varrho = 0$, $g(x)$ is concave on $x \in [0, \log(1+P_{\mathrm{t}})]$ since then $g''(x) = -\frac{e^x(2+P_{\mathrm{t}})}{(2+P_{\mathrm{t}}-e^x)^2}$ is strictly negative on $[0, \log(1+P_{\mathrm{t}})]$. This implies that $\mathrm{F}(P_{\mathrm{t}})$ is a convex set, which is in total agreement with the preceding example. On the other hand, if $\varrho = 1$, the second derivative of $g(x), x \in [0, \log(1+P_{\mathrm{t}})]$, is

$$g''(x) = \frac{e^x\,(1+P_{\mathrm{t}})\,(2+P_{\mathrm{t}})}{\left(1+P_{\mathrm{t}}-e^x\,(2+P_{\mathrm{t}})\right)^2}, \quad ,x \in [0, \log(1+P_{\mathrm{t}})]$$

which is positive. Thus, if $\varrho = 1$, $\mathrm{F}(P_{\mathrm{t}})$ is not convex but its complement $\mathrm{F}^c(P_{\mathrm{t}}) = \mathbb{R}_+^K \setminus \mathrm{F}(P_{\mathrm{t}})$ is a convex set. An examination of the second derivative of $g(x), x \in [0, \log(1+P_{\mathrm{t}})]$, shows that

$$g''(x) \begin{cases} <0 & \varrho < h(P_{\mathrm{t}}) \\ >0 & \varrho > h(P_{\mathrm{t}}) \\ =0 & \varrho = h(P_{\mathrm{t}}) \end{cases} \qquad h(x) = \frac{\sqrt{1+x}-1}{x}.$$

Since $h(x) \to 0$ as $x \to \infty$, we have $g''(x) > 0$ for any $\varrho > 0$, which complies with the above discussion that $f(x)$ is convex for any $\varrho > 0$. On the other hand, if $x \to 0$, then $h(x) \to 1/2$. So, at small values of P_{t}, convexity of $\mathrm{F}(P_{\mathrm{t}})$ changes to convexity of $\mathrm{F}^c(P_{\mathrm{t}})$ around the value $\varrho \approx 1/2$.

The example above demonstrates that the feasibility set may be a non-convex set even if each entry of $\mathbf{X}(\boldsymbol{\omega})$ is convex on Ω. As a consequence, a stronger property than convexity is necessary to guarantee convexity of F. In the following section, we show that if $\mathbf{X}(\boldsymbol{\omega})$ is log-convex on Ω (see Definition 1.37), then $p_k(\boldsymbol{\omega})$ is a log-convex function of $\boldsymbol{\omega} \in \mathrm{F}$ for each $1 \le k \le K$.

2.3 Convexity Results

In this section, we show that if $\mathbf{X} \in \mathrm{N}_K(\Omega)$ and $\mathbf{u} \in \mathbb{R}_{++}^K(\Omega)$ are both log-convex on Ω, then $p_k : \mathrm{F} \to \mathbb{R}_{++}$ given by (2.6) is log-convex for each $1 \le k \le K$. This in turn implies that the feasibility set $\mathrm{F}(P_{\mathrm{t}}; P_1, \ldots, P_K)$ is a convex set, regardless of the choice of $P_1, \ldots, P_K > 0$ and $P_{\mathrm{t}} > 0$. Following that, we consider the problem of strict convexity.

Recall that according to Definition 1.37, the notation $\mathbf{X} \in \mathrm{LC}_K(\Omega)$ means that $\mathbf{X} \in \mathrm{N}_K(\Omega)$ is log-convex on Ω. Furthermore, note that by this definition, the identically zero function is a log-convex function (see also the remark in Sect. 1.3). In an analogous manner, we say that $\mathbf{u} \in \mathbb{R}_{++}^K(\Omega)$ is log-convex on Ω if each entry of the vector $\mathbf{u}(\boldsymbol{\omega})$ is a continuous log-convex function defined on Ω. Let us indicate this by writing $\mathbf{u} \in \mathrm{lc}(\Omega) \subset \mathbb{R}_{++}^K(\Omega)$.

2.3.1 Log-Convexity of the Positive Solution

Let $\boldsymbol{\omega}(\mu)$ with $\mu \in [0,1]$ be a convex combination of two arbitrary vectors $\hat{\boldsymbol{\omega}}, \check{\boldsymbol{\omega}} \in \Omega$:

$$\boldsymbol{\omega}(\mu) = (1-\mu)\hat{\boldsymbol{\omega}} + \mu\check{\boldsymbol{\omega}}, \quad \mu \in [0,1].$$

Unless otherwise stated, assume that $\hat{\boldsymbol{\omega}}, \check{\boldsymbol{\omega}} \in \mathrm{F} \subseteq \Omega$, which implies that both $p_k(\hat{\boldsymbol{\omega}}) > 0$ and $p_k(\check{\boldsymbol{\omega}}) > 0$ exists.

Theorem 2.6. *Let* $\mathbf{X} \in \mathrm{LC}_K(\Omega) \subset \mathrm{N}_K(\Omega)$ *and* $\mathbf{u} \in \mathrm{lc}(\Omega) \subset \mathbb{R}^K_{++}(\Omega)$ *be arbitrary. Then,* $p_k(\boldsymbol{\omega})$ *is log-convex on* F *for each* $1 \le k \le K$*, i.e., we have*

$$p_k(\boldsymbol{\omega}(\mu)) \le p_k(\hat{\boldsymbol{\omega}})^{1-\mu} p_k(\check{\boldsymbol{\omega}})^{\mu}, \quad 1 \le k \le K \tag{2.14}$$

for all $\mu \in (0,1)$ *and* $\hat{\boldsymbol{\omega}}, \check{\boldsymbol{\omega}} \in \mathrm{F}$*.*

Proof. Let $\hat{\boldsymbol{\omega}}, \check{\boldsymbol{\omega}} \in \mathrm{F}$ be arbitrary. Then, by Theorem 1.39 as well as by Sect. 1.7, we know that $\lambda_p(\boldsymbol{\omega}(\mu)) < 1$ for all $\mu \in (0,1)$. Thus, for every $\mu \in (0,1)$, there exists a unique positive $p_k(\boldsymbol{\omega}(\mu))$ given by (see (2.6))

$$p_k(\boldsymbol{\omega}(\mu)) = \mathbf{e}_k^T \big[\mathbf{I} - \mathbf{X}(\boldsymbol{\omega}(\mu))\big]^{-1} \mathbf{u}(\boldsymbol{\omega}(\mu)), \quad 1 \le k \le K.$$

Now let $\mu \in (0,1)$ be arbitrary but fixed. Since $\boldsymbol{\omega}(\mu) \in \mathrm{F}$, we can expand $(\mathbf{I} - \mathbf{X}(\boldsymbol{\omega}(\mu)))^{-1}$ into a Neumann series (see Theorem A.16) to obtain

$$\big[\mathbf{I} - \mathbf{X}(\boldsymbol{\omega}(\mu))\big]^{-1} = \sum_{l=0}^{\infty} \big(\mathbf{X}(\boldsymbol{\omega}(\mu))\big)^l.$$

From this it follows that

$$\begin{aligned} p_k(\boldsymbol{\omega}(\mu)) &= \mathbf{e}_k^T \sum_{l=0}^{\infty} \big(\mathbf{X}(\boldsymbol{\omega}(\mu))\big)^l \mathbf{u}(\boldsymbol{\omega}(\mu)) = \sum_{l=0}^{\infty} \mathbf{e}_k^T \big(\mathbf{X}(\boldsymbol{\omega}(\mu))\big)^l \mathbf{u}(\boldsymbol{\omega}(\mu)) \\ &= \sum_{l=0}^{\infty} g_l\big(\boldsymbol{\omega}(\mu)\big). \end{aligned}$$

By assumption, all the entries of $\mathbf{X}(\boldsymbol{\omega})$ and $\mathbf{u}(\boldsymbol{\omega})$ are log-convex on F. Hence, (2.14) immediately follows from the above equation when one considers the following properties of log-convex functions:

(i) If two positive functions f and g are log-convex, then $f+g$ and $f \cdot g$ are log-convex.
(ii) For any convergent sequence f_n of log-convex functions, the limit $f = \lim_{n\to\infty} f_n$ is log-convex provided that the limit is strictly positive.

Due to (i), $g_l : \mathrm{F} \to \mathbb{R}_{++}$ is log-convex for each $l \ge 0$ and $\sum_{l=0}^{M} g_l(\boldsymbol{\omega})$ is log-convex for any $M > 0$. Furthermore, since $\sum_{l=0}^{M} g_l(\boldsymbol{\omega})$ is monotonically increasing in M and g_l is positive, it must converge to a positive limit as $M \to +\infty$. Hence, by (ii), $p_k(\boldsymbol{\omega})$ is log-convex on F and (2.14) must hold.

Remark 2.7. Recall that the spectral radius of $\mathbf{X} \in \mathrm{N}_K(\Omega)$ can be expressed as follows (Theorem A.15)

$$\lambda_p(\boldsymbol{\omega}) = \lim_{m\to+\infty} \|\mathbf{X}(\boldsymbol{\omega})^m\|^{1/m} .$$

Thus, considering the two properties (i) and (ii) of log-convex functions in the proof of Theorem 2.6 and the fact that if f is log-convex, so also is f^α for every positive α, shows that if the entries of $\mathbf{X}(\boldsymbol{\omega})$ are log-convex functions on Ω, then $\lambda_p(\boldsymbol{\omega})$ is log-convex on Ω. This leads to an alternative proof of log-convexity of the spectral radius (see for instance [32]).

A trivial but important consequence of the theorem is the following.

Corollary 2.8. *If* $\mathbf{X} \in \mathrm{LC}_K(\Omega) \subset \mathrm{N}_K(\Omega)$ *and* $\mathbf{u} \in \mathrm{lc}(\Omega) \subset \mathbb{R}^K_{++}$*, then* $\|\mathbf{p}(\boldsymbol{\omega})\|_1 = \sum_{k\in\mathcal{K}} p_k(\boldsymbol{\omega})$ *is log-convex on* F*, that is to say,*

$$\|\mathbf{p}(\boldsymbol{\omega}(\mu))\|_1 \le \|\mathbf{p}(\hat{\boldsymbol{\omega}})\|_1^{1-\mu}\|\mathbf{p}(\check{\boldsymbol{\omega}})\|_1^{\mu} \tag{2.15}$$

for all $\mu \in (0,1)$ *and* $\hat{\boldsymbol{\omega}}, \check{\boldsymbol{\omega}} \in \mathrm{F}$.

Proof. As log-convex functions are closed under addition, it is clear that the log-convexity property carries over to the l^1-norm of $\mathbf{p}(\boldsymbol{\omega})$.

More generally, we can say that if $\mathbf{X} \in \mathrm{LC}_K(\Omega)$ and $\mathbf{u} \in \mathrm{lc}(\Omega)$, then

$$F(p_1(\boldsymbol{\omega}(\mu)),\dots,p_K(\boldsymbol{\omega}(\mu))) \le F(p_1(\hat{\boldsymbol{\omega}}),\dots,p_K(\hat{\boldsymbol{\omega}}))^{1-\mu}F(p_1(\check{\boldsymbol{\omega}}),\dots,p_K(\check{\boldsymbol{\omega}}))^{\mu}$$

for all $\mu \in (0,1)$ and $\hat{\boldsymbol{\omega}}, \check{\boldsymbol{\omega}} \in \mathrm{F}$ where $F : \mathbb{R}^K_{++} \to \mathbb{R}_{++}$ is any function that preserves log-convexity. Standard examples of such functions are

1. weighted sum: $F(x_1,\dots,x_K) = \sum_{k\in\mathcal{K}} w_k x_k$,
2. weighted pointwise multiplication: $F(x_1,\dots,x_K) = \prod_{k=1}^K w_k x_k$, and
3. pointwise maximum and supremum: $F(x_1,\dots,x_K) = \max_{1\le k\le K} x_k$.

The weighted sum operation and the pointwise multiplication operation preserve log-convexity as log-convex functions are closed under both addition and multiplication. The claim about the pointwise maximum operation follows since

$$\begin{aligned}\max\{p_k(\boldsymbol{\omega}(\mu)) :& 1 \le k \le K\}\\ &\le \max\{p_k(\hat{\boldsymbol{\omega}})^{1-\mu}p_k(\check{\boldsymbol{\omega}})^{\mu} : 1 \le k \le K\}\\ &\le \max\{p_k(\hat{\boldsymbol{\omega}})^{1-\mu} : 1 \le k \le K\}\max\{p_k(\check{\boldsymbol{\omega}})^{\mu} : 1 \le k \le K\}\\ &= \max\{p_k(\hat{\boldsymbol{\omega}}) : 1 \le k \le K\}^{1-\mu}\max\{p_k(\check{\boldsymbol{\omega}}) : 1 \le k \le K\}^{\mu}\end{aligned}$$

for all $\mu \in (0,1)$ and $\hat{\boldsymbol{\omega}}, \check{\boldsymbol{\omega}} \in \mathrm{F}$.

2.3.2 Convexity of the Feasibility Set

Since the geometric mean is bounded above by the arithmetic mean (B.18), we have

$$p_k(\hat{\boldsymbol{\omega}})^{1-\mu} p_k(\check{\boldsymbol{\omega}})^{\mu} \leq (1-\mu) p_k(\hat{\boldsymbol{\omega}}) + \mu p_k(\check{\boldsymbol{\omega}}) \leq \max\{p_k(\hat{\boldsymbol{\omega}}), p_k(\check{\boldsymbol{\omega}})\}$$

for all $\hat{\boldsymbol{\omega}}, \check{\boldsymbol{\omega}} \in \mathrm{F}$ and $\mu \in (0,1)$. Thus, if $p_k(\boldsymbol{\omega})$ is log-convex on F, then the above inequality implies that $\mathrm{F}_k(P_k)$ defined by (2.12) is a convex set. By Theorem 2.6, we know that if $\mathbf{X}(\boldsymbol{\omega})$ and $\mathbf{u}(\boldsymbol{\omega})$ are both log-convex on Ω, then $p_k : \mathrm{F} \to \mathbb{R}_{++}$ is log-convex for each $1 \leq k \leq K$. Consequently, since the intersection of convex sets is convex, it follows from (2.11) that $\mathrm{F}(P_1, \ldots, P_K)$ is a convex set if $\mathbf{X} \in \mathrm{LC}_K(\Omega)$ and $\mathbf{u} \in \mathrm{lc}(\Omega)$. By Corollary 2.8 and (2.13), we see that this also true for $\mathrm{F}(P_{\mathrm{t}})$ and $\mathrm{F}(P_{\mathrm{t}}; P_1, \ldots, P_K)$. We summarize these observations in a corollary.

Corollary 2.9. *Suppose that* $\mathbf{X} \in \mathrm{LC}_K(\Omega) \subset \mathrm{N}_K(\Omega)$ *and* $\mathbf{u} \in \mathrm{lc}(\Omega) \subset \mathbb{R}^K_{++}(\Omega)$. *Then,* $\mathrm{F}(P_1, \ldots, P_K), \mathrm{F}(P_{\mathrm{t}})$ *and* $\mathrm{F}(P_{\mathrm{t}}; P_1, \ldots, P_K)$ *are convex sets, regardless of the choice of* $P_{\mathrm{t}}, P_1, \ldots, P_K > 0$.

To illustrate the results, let us consider a simple example.

Example 2.10. Let $\mathbf{X}(\boldsymbol{\omega})$ and $\mathbf{u}(\boldsymbol{\omega})$ be defined as in Example 2.5 except that now $\gamma(x) = e^x, x \in \mathbb{R}$. Clearly, the exponential function is log-convex on $\mathbb{R}$. Thus, by Theorem 1.39 (note that the matrix $\mathbf{X}(\boldsymbol{\omega})$ is irreducible for all $\boldsymbol{\omega} \in \mathbb{R}^2$), the Perron root is log-convex and, by Corollary 1.42, F is a convex set. In contrast to the previous example, all pairs satisfying $\lambda_p(\boldsymbol{\omega}) = \varrho\sqrt{e^{\omega_1} e^{\omega_2}} = 1$ lie on a line given by $\omega_2 = -\omega_1 - 2\log\varrho$, which, of course, is both convex and concave.

The nonnegative solution (2.4) yields

$$\mathbf{p}(\boldsymbol{\omega}) = \begin{pmatrix} \frac{e^{\omega_1} + \varrho e^{\omega_1+\omega_2}}{1-\varrho^2 e^{\omega_1+\omega_2}} \\ \frac{e^{\omega_2} + \varrho e^{\omega_1+\omega_2}}{1-\varrho^2 e^{\omega_1+\omega_2}} \end{pmatrix}, \quad \varrho^2 e^{\omega_1+\omega_2} < 1\,.$$

By Theorem 2.6, both entries are log-convex on $\mathbb{R}^2$. All pairs (ω_1, ω_2) satisfying $p_1(\boldsymbol{\omega}) = P_1$ and $p_2(\boldsymbol{\omega}) = P_2$ are $\omega_2 = f(\omega_1) = \log[(P_1 - e^{\omega_1})/(\varrho(1+\varrho P_1))] - \omega_1, \omega_1 < \log P_1$, and $\omega_2 = g(\omega_1) = \log[P_2/(1 + e^{\omega_1}\varrho + e^{\omega_1}\varrho^2 P_2)], \omega_2 < \log P_2$, respectively. It may be verified that $f(x)$ is concave on $(-\infty, \log P_1)$ and $g(x)$ is concave on $\mathbb{R}$ implying that $\mathrm{F}_1(P_1), \mathrm{F}_2(P_2)$ and $\mathrm{F}(P_1, P_2)$ are all convex sets. Similarly, $\|\mathbf{p}(\boldsymbol{\omega})\|_1 = P_{\mathrm{t}}$ can be rewritten to give $\omega_2 = h(\omega_1) = \log[(P_{\mathrm{t}} - e^{\omega_1})/(1+2e^{\omega_1}\varrho + e^{\omega_1}\varrho^2 P_2)], \omega_1 < \log P_{\mathrm{t}}$. Again, $h(x)$ can be seen to be concave on $(-\infty, \log P_{\mathrm{t}})$, from which convexity of $\mathrm{F}(P_{\mathrm{t}})$ follows.

In the preceding example, instead of $\gamma(x) = \gamma_1(x) = \gamma_2(x) = e^x, x \in \mathbb{R}$, we could consider any log-convex functions $\gamma_1 : \mathrm{Q}_1 \to \mathbb{R}_{++}$ and $\gamma_2 : \mathrm{Q}_2 \to \mathbb{R}_{++}$. In such a case, the unique positive solution $\mathbf{p}(\boldsymbol{\omega})$ exists if and only if $\boldsymbol{\omega} \in \mathrm{F} = \{\boldsymbol{\omega} \in \Omega : \lambda_p(\boldsymbol{\omega}) = \varrho\sqrt{\gamma_1(\omega_1)\gamma_2(\omega_2)} < 1\}$ and is given by

$$\mathbf{p}(\boldsymbol{\omega}) = \begin{pmatrix} \frac{\gamma_1(\omega_1)+\varrho\gamma_1(\omega_1)\gamma_2(\omega_2)}{1-\varrho^2\gamma_1(\omega_1)\gamma_2(\omega_2)} \\ \frac{\gamma_2(\omega_2)+\varrho\gamma_1(\omega_1)\gamma_2(\omega_2)}{1-\varrho^2\gamma_1(\omega_1)\gamma_2(\omega_2)} \end{pmatrix}, \quad \boldsymbol{\omega} \in \mathrm{F}\,. \tag{2.16}$$

It may be verified that if γ_1 and γ_2 are both log-convex, then each entry of $\mathbf{p}(\boldsymbol{\omega})$ is log-convex on F. This in turn implies that the feasibility set $\mathrm{F}(P_{\mathrm{t}}; P_1, \ldots, P_K)$ is a convex set, regardless of the choice of $P_{\mathrm{t}} > 0$ and $P_1, \ldots, P_K > 0$.

Finally, it is worth pointing out that the results presented in this chapter straightforwardly extends to the case when $p_k(\boldsymbol{\omega})$ is either subject to $\|\mathbf{p}(\boldsymbol{\omega})\|_1 \leq P_{\mathrm{t}}(\boldsymbol{\omega})$ or $p_k(\boldsymbol{\omega}) \leq P_k(\boldsymbol{\omega}), 1 \leq k \leq K$, for all $\boldsymbol{\omega} \in \Omega$ where $P_{\mathrm{t}} : \Omega \to \mathbb{R}_{++}$ and $P_k : \Omega \to \mathbb{R}_{++}$ are given *concave* functions. So if $p_k(\boldsymbol{\omega})$ is convex for each $1 \leq k \leq K$, then $\{\boldsymbol{\omega} \in \Omega : \|\mathbf{p}(\boldsymbol{\omega})\|_1 \leq P_{\mathrm{t}}(\boldsymbol{\omega})\}$ and $\{\boldsymbol{\omega} \in \Omega : p_k(\boldsymbol{\omega}) \leq P_k(\boldsymbol{\omega})\}, 1 \leq k \leq K$, are convex sets.

2.3.3 Strict Log-Convexity

When $\mathbf{X}(\boldsymbol{\omega})$ and $\mathbf{u}(\boldsymbol{\omega})$ are log-convex on Ω, Theorem 2.6 asserts that $p_k(\boldsymbol{\omega})$ is a log-convex function of $\boldsymbol{\omega} \in \mathrm{F}$. In this section, we strengthen this result by proving conditions on strict log-convexity. In the second part of the book, we will exploit these results to prove some interesting properties of the addressed power control problem.

For the analysis in this section, it is assumed that $\mathbf{X} \in \mathrm{N}_K(\Omega)$ and $\mathbf{u} \in \mathbb{R}^K_{++}(\Omega)$ are restricted to be of the following form:

$$\begin{aligned} \mathbf{u}(\boldsymbol{\omega}) &= \boldsymbol{\Gamma}(\boldsymbol{\omega})\mathbf{z} \\ \mathbf{X}(\boldsymbol{\omega}) &= \boldsymbol{\Gamma}(\boldsymbol{\omega})\mathbf{V} \quad \text{with} \quad \mathrm{trace}(\mathbf{V}) = 0\,. \end{aligned} \tag{2.17}$$

Here and hereafter, $\mathbf{z} = (z_1, \ldots, z_K)$ is any fixed *positive* vector, $\mathbf{V} \in \mathrm{N}_K$ and $\gamma_k : \mathrm{Q}_k \to \mathbb{R}_{++}, k = 1 \ldots K$ are continuous and strictly monotonic (bijective) functions. Formally, we have $\mathbf{X} \in \mathrm{N}^0_{K,\boldsymbol{\Gamma}}(\Omega)$ which is the subset of $\mathrm{N}_{K,\boldsymbol{\Gamma}}(\Omega)$ defined by (2.8) such that $\mathrm{trace}(\mathbf{V}) = 0$.

Lemma 2.11. *Let* $\mathbf{X} \in \mathrm{N}^0_{K,\boldsymbol{\Gamma}}(\Omega)$ *and* $u(\boldsymbol{\omega}) = \boldsymbol{\Gamma}(\boldsymbol{\omega})\mathbf{z}, \boldsymbol{\omega} \in \Omega$, *be arbitrary. Then,* $\mathbf{p} : \mathrm{F} \to \mathbb{R}^K_{++}$ *defined by (2.4) is a bijection.*

Proof. By Theorem A.51, $\mathbf{z} > 0$ and (2.5), $\mathbf{p}(\boldsymbol{\omega}) > 0$ exists and is unique if and only if $\boldsymbol{\omega} \in \mathrm{F}$. Thus, $\mathbf{p}(\boldsymbol{\omega})$ is a function from F into $\mathbb{R}^K_{++}$. It is a bijection as $\gamma_k : \mathrm{Q}_k \to \mathbb{R}_{++}$ is bijective, in which case there is a function $\boldsymbol{\phi} : \mathbb{R}^K_{++} \to \mathrm{F}$ given by

$$\boldsymbol{\phi}(\mathbf{p}) = \Big(\gamma_1^{-1}\big(p_1/(\mathbf{V}\mathbf{p}+\mathbf{z})_1\big), \ldots, \gamma_K^{-1}\big(p_K/(\mathbf{V}\mathbf{p}+\mathbf{z})_K\big)\Big)$$

such that $\mathbf{p}(\boldsymbol{\phi}(\mathbf{p})) = \mathbf{p}, \mathbf{p} > 0$, and $\boldsymbol{\phi}(\mathbf{p}(\boldsymbol{\omega})) = \boldsymbol{\omega}, \boldsymbol{\omega} \in \mathrm{F}$. So, the lemma follows from Theorem B.7.

It is important to emphasize that the positivity of the vector $\mathbf{z}$ is crucial for the results to hold. In contrast, the assumption $\operatorname{trace}(\mathbf{V}) = 0$ is merely motivated by practical applications and could be easily dropped. Note that due to this assumption, $(\mathbf{Vs})_k$ for any $\mathbf{s} \in \mathbb{R}^K$ is independent of s_k for each $1 \le k \le K$.

In what follows, we extensively exploit the following special form of Hölder's inequality (Theorem A.4): For any $\mu \in (0,1)$ and $\mathbf{u}, \mathbf{v} \in \mathbb{R}^K_+$, there holds

$$\langle \mathbf{u}, \mathbf{v} \rangle \le \|\mathbf{u}\|_p \|\mathbf{v}\|_q, \quad p = \frac{1}{1-\mu} \text{ and } q = \frac{1}{\mu}, \tag{2.18}$$

with equality if and only if there exists a constant $c > 0$ such that

$$v_k = c\, u_k^{p-1} = c\, u_k^{\frac{\mu}{1-\mu}}, \quad 1 \le k \le K.$$

Finally, recall that $\gamma_k : \mathrm{Q}_k \to \mathbb{R}_{++}, 1 \le k \le K$, is said to be strictly log-convex if $\gamma_k(x(\mu)) < \gamma_k(\hat{x})^{1-\mu}\gamma_k(\check{x})^{\mu}$ for all $\mu \in (0,1)$ and $\hat{x}, \check{x} \in \mathrm{Q}_k$ with $\hat{x} \ne \check{x}$ and $x(\mu) = (1-\mu)\hat{x} + \mu\check{x}$. Similarly, we say that $p_k : \mathrm{F} \to \mathbb{R}_{++}$ given by (2.6) is strictly log-convex for some $1 \le k \le K$ if $p_k(\boldsymbol{\omega}(\mu)) < p_k(\hat{\boldsymbol{\omega}})^{1-\mu} p_k(\check{\boldsymbol{\omega}})^{\mu}$ for all $\mu \in (0,1)$ and $\hat{\boldsymbol{\omega}}, \check{\boldsymbol{\omega}} \in \mathrm{F}$ with $\hat{\boldsymbol{\omega}} \ne \check{\boldsymbol{\omega}}$. The following result is a straightforward extension of Theorem 2.6 to the case of strictly log-convex functions $\gamma_1, \ldots, \gamma_K$.

Theorem 2.12. *Let $\mathbf{V} \ge 0$ be arbitrary, and let $\gamma_k : \mathrm{Q} \to \mathbb{R}_{++}$ be strictly log-convex for each $1 \le k \le K$. Then, for all $\hat{\boldsymbol{\omega}}, \check{\boldsymbol{\omega}} \in \mathrm{F}$ with $\hat{\boldsymbol{\omega}} \ne \check{\boldsymbol{\omega}}$, there exists an index $1 \le k_0 \le K$ such that $p_{k_0}(\boldsymbol{\omega}(\mu)) < p_{k_0}(\hat{\boldsymbol{\omega}})^{1-\mu} p_{k_0}(\check{\boldsymbol{\omega}})^{\mu}$ for all $\mu \in (0,1)$.*

Proof. Let $\hat{\boldsymbol{\omega}}, \check{\boldsymbol{\omega}} \in \mathrm{F}$ be arbitrary, and let k_0 be an index such that $\hat{\omega}_{k_0} \ne \check{\omega}_{k_0}$. By Theorem 2.6, we know that $\boldsymbol{\omega}(\mu) = (1-\mu)\hat{\boldsymbol{\omega}} + \mu\check{\boldsymbol{\omega}} \in \mathrm{F}$ for all $\mu \in (0,1)$. Therefore, for any $\mu \in (0,1)$, it follows from (2.2) that

$$p_{k_0}(\boldsymbol{\omega}(\mu)) = \gamma_{k_0}(\omega_{k_0}(\mu))\big(\mathbf{Vp}(\boldsymbol{\omega}(\mu)) + \mathbf{z}\big)_{k_0}.$$

So, by strict log-convexity of γ_{k_0} and positivity of the vector $\mathbf{z}$, we have

$$p_{k_0}(\boldsymbol{\omega}(\mu)) < \gamma_{k_0}(\hat{\omega}_{k_0})^{1-\mu}\gamma_{k_0}(\check{\omega}_{k_0})^{\mu}\big(\mathbf{Vp}(\boldsymbol{\omega}(\mu)) + \mathbf{z}\big)_{k_0}.$$

Considering Theorem 2.6 and Hölder's inequality (2.18) yields

$$\begin{aligned} p_{k_0}(\boldsymbol{\omega}(\mu)) &< \gamma_{k_0}(\hat{\omega}_{k_0})^{1-\mu}\gamma_{k_0}(\check{\omega}_{k_0})^{\mu}\Big[\sum_{l\in\mathcal{K}}\big(v_{k_0,l}p_l(\hat{\boldsymbol{\omega}})\big)^{1-\mu}\big(v_{k_0,l}p_l(\check{\boldsymbol{\omega}})\big)^{\mu} + z_{k_0}\Big] \\ &\le \gamma_{k_0}(\hat{\omega}_{k_0})^{1-\mu}\gamma_{k_0}(\check{\omega}_{k_0})^{\mu}\Big[\big(\mathbf{Vp}(\hat{\boldsymbol{\omega}})\big)_{k_0}^{1-\mu}\big(\mathbf{Vp}(\check{\boldsymbol{\omega}})\big)_{k_0}^{\mu} + z_{k_0}^{1-\mu}z_{k_0}^{\mu}\Big] \\ &= \langle \hat{\mathbf{u}}, \check{\mathbf{u}} \rangle \end{aligned}$$

where[3]

[3] For any vector $\mathbf{u} \in \mathbb{R}^K$ and any constant $c \in \mathbb{R}$, $(\mathbf{u})_k^c = [(\mathbf{u})_k]^c = u_k^c, 1 \le k \le K$.

$$\hat{\mathbf{u}} = \begin{pmatrix} (\mathbf{\Gamma}(\hat{\boldsymbol{\omega}})\mathbf{V}\mathbf{p}(\hat{\boldsymbol{\omega}}))_{k_0}^{1-\mu} \\ (\mathbf{\Gamma}(\hat{\boldsymbol{\omega}})\mathbf{z})_{k_0}^{1-\mu} \end{pmatrix} \qquad \check{\mathbf{u}} = \begin{pmatrix} (\mathbf{\Gamma}(\check{\boldsymbol{\omega}})\mathbf{V}\mathbf{p}(\check{\boldsymbol{\omega}}))_{k_0}^{\mu} \\ (\mathbf{\Gamma}(\check{\boldsymbol{\omega}})\mathbf{z})_{k_0}^{\mu} \end{pmatrix} .$$

By repeated application of (2.18), we obtain

$$\begin{aligned} p_{k_0}(\boldsymbol{\omega}(\mu)) &< \|\hat{\mathbf{u}}\|_{\frac{1}{1-\mu}} \|\check{\mathbf{u}}\|_{\frac{1}{\mu}} \\ &= \Big(\mathbf{\Gamma}(\hat{\boldsymbol{\omega}})\mathbf{V}\mathbf{p}(\hat{\boldsymbol{\omega}}) + \mathbf{\Gamma}(\hat{\boldsymbol{\omega}})\mathbf{z}\Big)_{k_0}^{1-\mu} \Big(\mathbf{\Gamma}(\check{\boldsymbol{\omega}})\mathbf{V}\mathbf{p}(\check{\boldsymbol{\omega}}) + \mathbf{\Gamma}(\check{\boldsymbol{\omega}})\mathbf{z}\Big)_{k_0}^{\mu} \\ &= p_{k_0}(\hat{\boldsymbol{\omega}})^{1-\mu} p_{k_0}(\check{\boldsymbol{\omega}})^{\mu} . \end{aligned}$$

This completes the proof.

Remarkably, there are no additional restrictions on $\mathbf{V} \geq 0$. As shown below, we obtain a similar property if we drop the requirement on strict log-convexity of $\gamma_k, 1 \leq k \leq K$, and instead put some constraints on $\mathbf{V}$.

Theorem 2.13. *Let $\gamma_k : \mathrm{Q}_k \to \mathbb{R}_{++}$ be log-convex for each $1 \leq k \leq K$. Suppose that $\mathbf{V} \in \mathbb{R}_+^{K \times K}$ is chosen such that for each $1 \leq l \leq K$, there exists $k \neq l$ with $v_{k,l} > 0$. Then, for any fixed $\hat{\boldsymbol{\omega}}, \check{\boldsymbol{\omega}} \in \mathrm{F}$ with $\hat{\boldsymbol{\omega}} \neq \check{\boldsymbol{\omega}}$, there exists $k_0, 1 \leq k_0 \leq K$, so that $p_{k_0}(\boldsymbol{\omega}(\mu)) < p_{k_0}(\hat{\boldsymbol{\omega}})^{1-\mu} p_{k_0}(\check{\boldsymbol{\omega}})^{\mu}$ for all $\mu \in (0, 1)$.*

Proof. Let $\hat{\boldsymbol{\omega}}, \check{\boldsymbol{\omega}} \in \mathrm{F}$ with $\hat{\boldsymbol{\omega}} \neq \check{\boldsymbol{\omega}}$ be arbitrary. Since $\mathbf{p}(\boldsymbol{\omega})$ is a bijection (Lemma 2.11), we have $\mathbf{p}(\hat{\boldsymbol{\omega}}) \neq \mathbf{p}(\check{\boldsymbol{\omega}})$. Choose $l_0, 1 \leq l_0 \leq K$, such that

$$p_{l_0}(\hat{\boldsymbol{\omega}}) \neq p_{l_0}(\check{\boldsymbol{\omega}}) \tag{2.19}$$

and let $k_0 \neq l_0$ be any index with $v_{k_0,l_0} > 0$. Note that by assumption, there exists such an index. Using

$$\hat{\mathbf{u}} = \begin{pmatrix} \sum_{l \in \mathcal{K}_{l_0}} \big(\gamma_{k_0}(\hat{\omega}_{k_0}) v_{k_0,l} p_l(\hat{\boldsymbol{\omega}})\big)^{1-\mu} \\ (\gamma_{k_0}(\hat{\omega}_{k_0}) z_{k_0})^{1-\mu} \\ \big(\gamma_{k_0}(\hat{\omega}_{k_0}) v_{k_0,l_0} p_{l_0}(\hat{\boldsymbol{\omega}})\big)^{1-\mu} \end{pmatrix} \check{\mathbf{u}} = \begin{pmatrix} \sum_{l \in \mathcal{K}_{l_0}} \big(\gamma_{k_0}(\check{\omega}_{k_0}) v_{k_0,l} p_l(\check{\boldsymbol{\omega}})\big)^{\mu} \\ (\gamma_{k_0}(\check{\omega}_{k_0}) z_{k_0})^{\mu} \\ \big(\gamma_{k_0}(\check{\omega}_{k_0}) v_{k_0,l_0} p_{l_0}(\check{\boldsymbol{\omega}})\big)^{\mu} \end{pmatrix}$$

and considering log-convexity of $\gamma_k, 1 \leq k \leq K$, one obtains

$$\begin{aligned} p_{k_0}(\boldsymbol{\omega}(\mu)) = \Big(\mathbf{\Gamma}(\boldsymbol{\omega}(\mu))\mathbf{V}\mathbf{p}(\boldsymbol{\omega}(\mu)) + \mathbf{\Gamma}(\boldsymbol{\omega}(\mu))\mathbf{z}\Big)_{k_0} &\overset{(a)}{\leq} \langle \hat{\mathbf{u}}, \check{\mathbf{u}} \rangle \\ \overset{(b)}{\leq} \|\hat{\mathbf{u}}\|_{\frac{1}{1-\mu}} \|\check{\mathbf{u}}\|_{\frac{1}{\mu}} = p_{k_0}(\hat{\boldsymbol{\omega}})^{1-\mu} p_{k_0}(\check{\boldsymbol{\omega}})^{\mu} & \end{aligned}$$

for any $\mu \in (0, 1)$, where (a) follows from Theorem 2.6 and (b) from (2.18). Therefore, since $v_{k_0,l_0} > 0$ and $z_{k_0} > 0$, we can have equality in (b) only if $p_{l_0}(\hat{\boldsymbol{\omega}}) = p_{l_0}(\check{\boldsymbol{\omega}})$ which contradicts (2.19), and hence completes the proof.

It is important to emphasize that Theorems 2.12 and 2.13 do not imply the existence of an index k such that $p_k(\boldsymbol{\omega})$ is strictly log-convex on F. In fact, the theorems only assert that for any fixed $\hat{\boldsymbol{\omega}}, \check{\boldsymbol{\omega}} \in \mathrm{F}, \hat{\boldsymbol{\omega}} \neq \check{\boldsymbol{\omega}}$, there is an index k such that $p_k(\boldsymbol{\omega}(\mu)) < p_k(\hat{\boldsymbol{\omega}})^{1-\mu} p_k(\check{\boldsymbol{\omega}})^{\mu}$ for all $\mu \in (0, 1)$. However, this is sufficient to deduce the following corollary.

Corollary 2.14. *Suppose that at least one of the following holds.*

(i) For each $1 \le k \le K$*,* $\gamma_k : \mathrm{Q}_k \to \mathbb{R}_{++}$ *is strictly log-convex.*
(ii) Each column of the matrix $\mathbf{V}$ *has at least one positive entry.*

Then $\|\mathbf{p}(\boldsymbol{\omega})\|_1$ *is strictly log-convex on* F*.*

Proof. Let $\hat{\boldsymbol{\omega}}, \check{\boldsymbol{\omega}} \in \mathrm{F}$ with $\hat{\boldsymbol{\omega}} \neq \check{\boldsymbol{\omega}}$ be arbitrary. For any fixed $\mu \in (0,1)$, we have

$$\begin{aligned}\|\mathbf{p}(\boldsymbol{\omega}(\mu))\|_1 &= \sum_{k\in\mathcal{K}} p_k(\boldsymbol{\omega}(\mu)) \overset{(a)}{<} \sum_{k\in\mathcal{K}} (p_k(\hat{\boldsymbol{\omega}}))^{1-\mu}(p_k(\check{\boldsymbol{\omega}}))^{\mu} \\ &\overset{(b)}{\le} \Big(\sum_{k\in\mathcal{K}} p_k(\hat{\boldsymbol{\omega}})\Big)^{1-\mu}\Big(\sum_{k\in\mathcal{K}} p_k(\check{\boldsymbol{\omega}})\Big)^{\mu} = \|\mathbf{p}(\hat{\boldsymbol{\omega}})\|_1^{1-\mu}\|\mathbf{p}(\check{\boldsymbol{\omega}})\|_1^{\mu}\end{aligned}$$

where (a) is either due to Theorem 2.12 or due to Theorem 2.13 depending on whether (i) or (ii) holds, and (b) follows from (2.18).

Obviously, condition (ii) of the corollary, which is equivalent to the condition of Theorem 2.13, is weaker than irreducibility. For instance, the following two reducible matrices satisfy the condition of Theorem 2.13:

$$\mathbf{V} = \begin{pmatrix} 0&1&0&0\\ 1&0&0&0\\ 0&0&0&1\\ 0&0&1&0 \end{pmatrix} \qquad \text{and} \qquad \mathbf{V} = \begin{pmatrix} 0&1&1\\ 1&0&0\\ 0&0&0 \end{pmatrix} .$$

With these particular choices of $\mathbf{V}$ and with $\gamma_1(x) = \gamma_2(x) = \gamma_3(x) = e^x, x \in \mathbb{R}$, we have (respectively)

$$\begin{aligned} p_1(\boldsymbol{\omega}) &= e^{\omega_1}p_2(\boldsymbol{\omega}) + e^{\omega_1}z_1\\ p_2(\boldsymbol{\omega}) &= e^{\omega_2}p_1(\boldsymbol{\omega}) + e^{\omega_2}z_2\\ p_3(\boldsymbol{\omega}) &= e^{\omega_3}p_4(\boldsymbol{\omega}) + e^{\omega_3}z_3\\ p_4(\boldsymbol{\omega}) &= e^{\omega_4}p_3(\boldsymbol{\omega}) + e^{\omega_4}z_4 \end{aligned} \qquad \begin{aligned} p_1(\boldsymbol{\omega}) &= e^{\omega_1}p_2(\boldsymbol{\omega}) + e^{\omega_1}p_3(\boldsymbol{\omega}) + e^{\omega_1}z_1\\ p_2(\boldsymbol{\omega}) &= e^{\omega_2}p_1(\boldsymbol{\omega}) + e^{\omega_2}z_2\\ p_3(\boldsymbol{\omega}) &= e^{\omega_3}z_3 . \end{aligned}$$

In the first case, we see that $p_1(\boldsymbol{\omega})$ and $p_2(\boldsymbol{\omega})$ are strictly log-convex with respect to (ω_1, ω_2) but they are independent of (ω_3, ω_4). For $p_3(\boldsymbol{\omega})$ and $p_4(\boldsymbol{\omega})$, the situation is reversed so that $\|\mathbf{p}(\boldsymbol{\omega})\|_1$ remains strictly log-convex on F. In the second example, we can write $p_1(\boldsymbol{\omega})$ as

$$p_1(\boldsymbol{\omega}) = e^{\omega_1}\frac{e^{\omega_2}z_2 + e^{\omega_3}z_3 + z_1}{1 - e^{\omega_1}e^{\omega_2}}$$

which is strictly log-convex on F. In contrast, the transpose matrix $\mathbf{V} = \left(\begin{smallmatrix} 0&1&0\\ 1&0&0\\ 1&0&0 \end{smallmatrix}\right)$ does not satisfy the conditions of Theorem 2.13 since $v_{k,3} = 0$ for each $1 \le k \le K$. In this case, the nonnegative solution $\mathbf{p}(\boldsymbol{\omega}), \boldsymbol{\omega} \in \mathbb{R}^3$, is given by

$$
\begin{aligned}
p_1(\boldsymbol{\omega}) &= e^{\omega_1} p_2(\boldsymbol{\omega}) + e^{\omega_1} z_1 \\
p_2(\boldsymbol{\omega}) &= e^{\omega_2} p_1(\boldsymbol{\omega}) + e^{\omega_2} z_2 \\
p_3(\boldsymbol{\omega}) &= e^{\omega_3} p_1(\boldsymbol{\omega}) + e^{\omega_3} z_3 \ .
\end{aligned}
$$

We see that whereas $p_1(\boldsymbol{\omega})$ and $p_2(\boldsymbol{\omega})$ are independent of ω_3, $p_3(\boldsymbol{\omega})$ is a log-convex function of ω_3, though not strictly log-convex. Therefore, there is no index k such that $p_k(\boldsymbol{\omega})$ is strictly log-convex along the third coordinate of $\boldsymbol{\omega}$ (with ω_1 and ω_2 being fixed).

Finally we show that if $\mathbf{V}$ is irreducible, $p_k(\boldsymbol{\omega})$ is strictly log-convex on F for each $1 \le k \le K$, regardless of whether γ_k is strictly log-convex or only log-convex.

Theorem 2.15. *Let $\gamma_k : \mathrm{Q}_k \to \mathbb{R}_{++}, 1 \le k \le K$, be log-convex, and let $\mathbf{V} \in \mathrm{X}_K$. Then, $p_k(\boldsymbol{\omega})$ is strictly log-convex on F for each $1 \le k \le K$.*

Proof. Let $\hat{\boldsymbol{\omega}}, \check{\boldsymbol{\omega}} \in \mathrm{F}$ with $\hat{\boldsymbol{\omega}} \neq \check{\boldsymbol{\omega}}$ be arbitrary. Suppose that the theorem is false. Then, there exists k_0 and $\mu_0 \in (0,1)$ such that

$$p_{k_0}(\boldsymbol{\omega}(\mu_0)) = p_{k_0}(\hat{\boldsymbol{\omega}})^{1-\mu_0} p_{k_0}(\check{\boldsymbol{\omega}})^{\mu_0} \ .$$

So, by log-convexity of γ_k, Theorem 2.6 and Hölder's inequality,

$$
\begin{aligned}
& p_{k_0}(\boldsymbol{\omega}(\mu_0)) \\
&= \gamma_{k_0}(\omega_{k_0}(\mu_0)) \Big(\sum_{l \in \mathcal{K}} v_{k_0,l} p_l(\boldsymbol{\omega}(\mu_0)) + z_{k_0} \Big) \\
&\overset{(a)}{\le} \gamma_{k_0}(\omega_{k_0}(\mu_0)) \Big(\sum_{l \in \mathcal{K}} v_{k_0,l} p_l(\hat{\boldsymbol{\omega}})^{1-\mu_0} p_l(\check{\boldsymbol{\omega}})^{\mu_0} + z_{k_0} \Big) \\
&\le \gamma_{k_0}(\hat{\omega}_{k_0})^{1-\mu_0} \gamma_{k_0}(\check{\omega}_{k_0})^{\mu_0} \Big(\sum_{l \in \mathcal{K}} (v_{k_0,l} p_l(\hat{\boldsymbol{\omega}}))^{1-\mu_0} (v_{k_0,l} p_l(\check{\boldsymbol{\omega}}))^{\mu_0} + z_{k_0} \Big) \\
&\le \gamma_{k_0}(\hat{\omega}_{k_0})^{1-\mu_0} \gamma_{k_0}(\check{\omega}_{k_0})^{\mu_0} \\
&\qquad \cdot \Big[\Big(\sum_{l \in \mathcal{K}} v_{k_0,l} p_l(\hat{\boldsymbol{\omega}}) \Big)^{1-\mu_0} \Big(\sum_{l \in \mathcal{K}} v_{k_0,l} p_l(\check{\boldsymbol{\omega}}) \Big)^{\mu_0} + z_{k_0}^{1-\mu_0} z_{k_0}^{\mu_0} \Big] \\
&\overset{(b)}{\le} \Big(\gamma_{k_0}(\hat{\omega}_{k_0}) \big(\sum_{l \in \mathcal{K}} v_{k_0,l} p_l(\hat{\boldsymbol{\omega}}) + z_{k_0} \big) \Big)^{1-\mu_0} \Big(\gamma_{k_0}(\check{\omega}_{k_0}) \big(\sum_{l \in \mathcal{K}} v_{k_0,l} p_l(\check{\boldsymbol{\omega}}) + z_{k_0} \big) \Big)^{\mu_0} \\
&= p_{k_0}(\hat{\boldsymbol{\omega}})^{1-\mu_0} p_{k_0}(\check{\boldsymbol{\omega}})^{\mu_0} \ .
\end{aligned}
$$

So, in each step, we have equality. Now let $\mathcal{N}_1 \subset \{1, \dots, K\}$ be a set of those indices l for which $v_{k_0,l} > 0$. As $\mathbf{V}$ is irreducible, we have $\mathcal{N}_1 \neq \emptyset$. Hence, since $\mathbf{z}$ is positive, it follows from (2.18) that there can be equality in (b) only if $\forall_{l \in \mathcal{N}_1} \, p_l(\hat{\boldsymbol{\omega}}) = p_l(\check{\boldsymbol{\omega}})$. Now suppose that $\mathcal{N}_2 \subset \{1, \dots, K\}$ with $\mathcal{N}_2 \neq \mathcal{N}_1$ is a set of all indices l such that there exists $k_1 \in \mathcal{N}_1, k_1 \neq k_0$, with $v_{k_1,l} > 0$. Again, due to irreducibility of $\mathbf{V}$, it holds $\mathcal{N}_2 \neq \emptyset$. Moreover, since there is equality in (a) if only if $p_{k_1}(\boldsymbol{\omega}(\mu_0)) = p_{k_1}(\hat{\boldsymbol{\omega}})^{1-\mu_0} p_{k_1}(\check{\boldsymbol{\omega}})^{\mu_0}$ for each $k_1 \in \mathcal{N}_1$, we can reason along the same lines as above to show that $\forall_{l \in \mathcal{N}_2}\, p_l(\hat{\boldsymbol{\omega}}) = p_l(\check{\boldsymbol{\omega}})$. Now since $\mathbf{V}$ is irreducible, we can proceed in this way until there are no indices left to obtain

$$\forall_{1 \le k \le K}\ p_k(\hat{\boldsymbol{\omega}}) = p_k(\check{\boldsymbol{\omega}})\,.$$

Clearly, since $\mathbf{z}$ is positive and $\mathbf{p}(\boldsymbol{\omega})$ is a bijection, this implies that $\hat{\boldsymbol{\omega}} = \check{\boldsymbol{\omega}}$, which contradicts $\hat{\boldsymbol{\omega}} \neq \check{\boldsymbol{\omega}}$ and therefore completes the proof.

Figure 2.2 depicts $\|\mathbf{p}(\boldsymbol{\omega}(\mu))\|_1$ as a function of $\mu \in [0,1]$ for three different log-convex functions $\gamma(x) = \gamma_1(x) = \ldots = \gamma_K(x), x > 0$, and a randomly chosen irreducible matrix $\mathbf{V}$. Since $\gamma(x) = e^x/(1-e^x)$ is strictly log-convex on $\mathrm{Q} = (-\infty, 0)$ and $\gamma(x) = 1/x$ is strictly log-convex on $(0, +\infty)$, it follows from Theorem 2.12 that $\|\mathbf{p}(\boldsymbol{\omega})\|_1$ is strictly log-convex on Q^K. In contrast, $\gamma(x) = e^x$ is not strictly log-convex on $\mathbb{R}$. Nevertheless, since $\mathbf{V}$ is irreducible, Theorem 2.15 asserts that the l^1-norm is strictly log-convex.

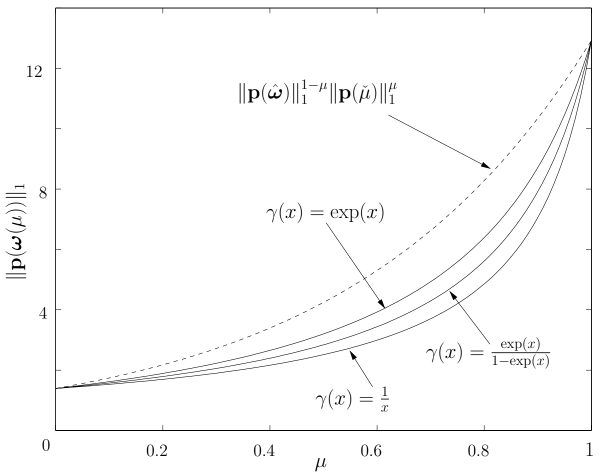

Fig. 2.2: The l^1-norm $\|\mathbf{p}(\boldsymbol{\omega}(\mu))\|_1$ as a function of $\mu \in [0,1]$ for some fixed $\hat{\boldsymbol{\omega}}, \check{\boldsymbol{\omega}} \in \mathrm{Q}^K$ chosen such that $\|\mathbf{p}(\hat{\boldsymbol{\omega}})\|_1$ and $\|\mathbf{p}(\check{\boldsymbol{\omega}})\|_1$ are independent of the choice of γ.

2.3.4 Strict Convexity of the Feasibility Sets

The results in the preceding section may be used to deduce strict convexity of the feasibility set in the following sense (see also Definition 1.44).

Definition 2.16. $\mathrm{F}(P_{\mathrm{t}})$ *(respectively,* $\mathrm{F}(P_1, \ldots, P_K)$*) is said to be strictly convex (or s-convex) if* $\boldsymbol{\omega}(\mu) = (1-\mu)\hat{\boldsymbol{\omega}} + \mu\check{\boldsymbol{\omega}}$ *is interior to* $\mathrm{F}(P_{\mathrm{t}})$ *(respectively,* $\mathrm{F}(P_1, \ldots, P_K)$*) for all* $\mu \in (0,1)$ *and* $\hat{\boldsymbol{\omega}}, \check{\boldsymbol{\omega}} \in \partial\mathrm{F}(P_{\mathrm{t}})$ *(respectively,* $\hat{\boldsymbol{\omega}}, \check{\boldsymbol{\omega}} \in \partial\mathrm{F}(P_1, \ldots, P_K)$*),* $\hat{\boldsymbol{\omega}} \neq \check{\boldsymbol{\omega}}$*, where*

$$\begin{aligned}\partial\mathrm{F}(P_{\mathrm{t}}) &= \{\boldsymbol{\omega}\in\mathrm{F} : \|\mathbf{p}(\boldsymbol{\omega})\|_1 = P_{\mathrm{t}}\}\\ \partial\mathrm{F}(P_1,\ldots,P_K) &= \{\boldsymbol{\omega}\in\mathrm{F} : \exists_{1\le k\le K}\; p_k(\boldsymbol{\omega}) = P_k\}\,.\end{aligned}\tag{2.20}$$

Under the setup of Corollary 2.14, $\mathrm{F}(P_{\mathrm{t}})$ is a strictly convex set for all $P_{\mathrm{t}} > 0$ since then $\|\mathbf{p}(\boldsymbol{\omega})\|_1$ is strictly log-convex. These conditions however are not necessary for $\mathrm{F}(P_{\mathrm{t}})$ to be a strictly convex set (see Example 2.4). As far as $\mathrm{F}(P_1,\ldots,P_K)$ is concerned, the set is strictly convex when $p_k(\boldsymbol{\omega})$ is strictly log-convex for each $1\le k\le K$. Therefore, we have the following corollary.

Corollary 2.17. *Under the setup of Theorem 2.15, $F(P_1,\ldots,P_K)$ is a strictly convex set for any $P_1,\ldots,P_K > 0$.*

Of course, if $\mathrm{F}(P_1,\ldots,P_K)$ is strictly convex, so also is $F(P_{\mathrm{t}};P_1,\ldots,P_K)$.

2.4 The Linear Case

In this section, we further focus on the special case (2.17) except that now

$$\gamma(x) = \gamma_1(x) = \cdots = \gamma_K(x) = x, \quad x > 0\,.$$

Hence, we have $\Omega = \mathrm{Q}^K = \mathbb{R}^K_{++}$.

The linear case has already been considered in Sect. 1.5 where it is shown that F^c is not a convex set in general. More precisely, Theorem 1.60 asserts that there exist $\mathbf{V}\in\mathrm{X}_K$ and $K > 1$ such that neither F nor its complement $\mathrm{F}^c = \mathbb{R}^K_{++}\backslash\mathrm{F}$ is a convex set. In this section, we will use this result to show that $\mathrm{F}^c(P_{\mathrm{t}}) = \mathbb{R}^K_{++}\setminus\mathrm{F}(P_{\mathrm{t}})$ is *in general* not convex either. However, note that this does not exclude the possibility of convexity of $\mathrm{F}^c(P_{\mathrm{t}})$ for some special choices of P_{t}, K and $\mathbf{V}$. For instance, consider $K = 2, \mathbf{z} = (1,1)$ and $\mathbf{V} = \left(\begin{smallmatrix}0 & \varrho\\ \varrho & 0\end{smallmatrix}\right)$ for any fixed $\varrho > 0$. Then, we see that the set of pairs $(\omega_1,\omega_2)\in\partial\mathrm{F}(P_{\mathrm{t}})$ (see Definition 2.16) must satisfy $\omega_2 = f(\omega_1) = (P_{\mathrm{t}}-\omega_1)/(1+2\varrho\omega_1+\varrho^2\omega_1 P_{\mathrm{t}})$. Now it may be verified that

$$f'(x) = \frac{-(1+\varrho P_{\mathrm{t}})^2}{(1+\varrho(2+\varrho P_{\mathrm{t}})x)^2}, \quad x > 0\,.$$

Thus, as the numerator is independent of x and the denominator is monotonically increasing in $x > 0$, we must have $f''(x)\ge 0$ for every $x > 0$. From this, it follows that $f(x)$ is not concave but convex on $\mathbb{R}_{++}$. As a consequence of this, $\mathrm{F}^c(P_{\mathrm{t}}) = \mathbb{R}^2_{++}\setminus\mathrm{F}(P_{\mathrm{t}})$ is a convex set if $K = 2$ and $\gamma_1(x) = \gamma_2(x) = x, x > 0$.

As in Sect. 1.5, this simple example might suggest that $\mathrm{F}^c(P_{\mathrm{t}})$ is a convex set in general, which in turn would allow us to draw some interesting conclusions with respect to optimal link scheduling in wireless networks. Unfortunately, simple reasoning shows that such a general statement is not possible.

Theorem 2.18. *There exist at least one $P_{\mathrm{t}} > 0$ and an irreducible matrix $\mathbf{V}\ge 0$ for some $K > 1$ such that $\mathrm{F}^c(P_{\mathrm{t}})$ is not convex.*

Proof. The proof is by contradiction. So, assume that $\mathrm{F}^c(P_\mathrm{t})$ is convex for all $P_\mathrm{t} > 0, K > 1$ and all $\mathbf{V} \in \mathrm{X}_K$. Therefore, as the intersection of convex sets is convex, it follows from (see (2.10))

$$\mathrm{F}^c = \bigcap_{P_\mathrm{t}>0} \mathrm{F}^c(P_\mathrm{t})$$

that F^c is a convex set for all $K > 1$ and all $\mathbf{V} \in \mathrm{X}_K$. However, this contradicts Theorem 1.60, and therefore prove the assertion.

Notice that the theorem only deals with the feasibility set when $\mathbf{p}(\boldsymbol{\omega})$ is constrained in the l^1-norm. When each element of $\mathbf{p}(\boldsymbol{\omega})$ is constrained individually, the complement of the feasibility set defined by (2.11) is not convex even if $K = 2$. Indeed, proceeding essentially as before shows that $p_1(\boldsymbol{\omega}) = P_1$ and $p_2(\boldsymbol{\omega}) = P_2$ are both convex if they are written explicitly as functions of ω_1. However, even though $\mathrm{F}_1^c(P_1)$ and $\mathrm{F}_2^c(P_2)$ are both convex sets, the set

$$\mathrm{F}^c(P_1, P_2) = (\mathrm{F}_1(P_1) \cap \mathrm{F}_2(P_2))^c = \mathrm{F}_1^c(P_1) \cup \mathrm{F}_2^c(P_2)$$

does not need to be convex as the union of convex sets is not convex in general. This is illustrated in Fig. 2.3. Obviously, the same reasoning applies to hybrid

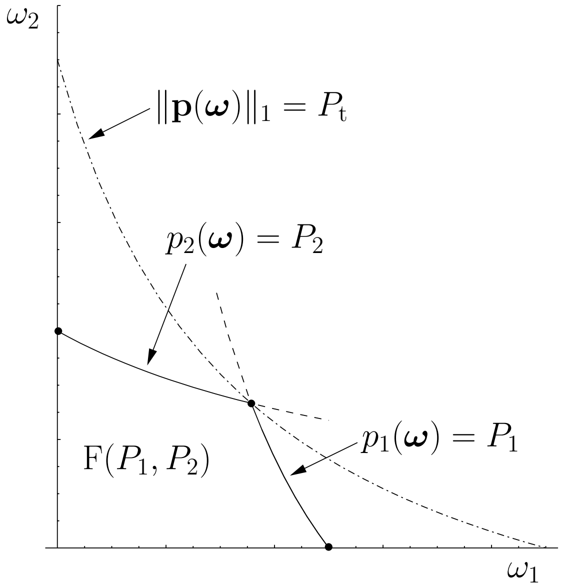

Fig. 2.3: $\mathrm{F}(P_1, P_2)$ is equal to the intersection of $\mathrm{F}_1(P_1)$ and $\mathrm{F}_2(P_2)$. Thus, $\mathrm{F}^c(P_1, P_2)$ is equal to the union of $\mathrm{F}_1^c(P_1)$ and $\mathrm{F}_2^c(P_2)$, each of which is a convex set if $\gamma(x) = x, x > 0$. However, the union of these sets is not convex in general.

constraints, in which case neither the feasibility set $\mathrm{F}(P_{\mathrm{t}}; P_1, \ldots, P_K)$ given by (2.13) nor its complement is a convex set in general. This immediately follows from (2.13) and the above discussion.

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