2

Fundamentals

Die Theorie ist das Netz, das wir auswerfen, um "die Welt" einzufangen, — sie zu rationalisieren, zu erklären und zu beherrschen. Wir arbeiten daran, die Maschen dieses Netzes immer enger zu machen.

(Karl Popper, "Logik der Forschung", 1935)

Theories are nets cast to catch what we call "the world": to rationalize, to explain and to master it. We endeavour to make the mesh ever finer and finer.

(Karl Popper, "The Logic of Scientific Discovery", 1959)

2.1 Basic Concepts

2.1.1 Mass

We consider dynamics in the sense as discussed in the Introduction. That means we shall not refer to relativistic aspects whatsoever. The only deviation from the classical mass concept consists in the effects generated by rocket systems with their time-dependent masses. Focusing our future considerations mainly to technical artefacts we usually know all relevant mass distributions and can thus define:

- Masses are always positive, also in the time-dependent case, $m > 0$.
- Masses are
  - either constant with $\dot{m} = 0$,
  - or not constant with $\dot{m} \neq 0$, where $(\dot{m} = \frac{dm}{dt})$.
- Masses can be added and divided into parts.

Another more physically oriented definition of a mass is given by Synge [257]. He states, that a mass is "a quantity of matter in a body, a measure of the reluctance of a body to change its velocity and a measure of the capacity of a body to attract another gravitationally".

Modeling masses depends on the problem under consideration. We might have rigid or elastic masses and in dynamics also interactions with fluid masses. Theoretically we always get as a matter of fact an interdependence of the selected mass model and the results we can achieve with such a model. On the other hand the experience of modeling for a huge amount of practical

cases tells us how to choose mass models. Nevertheless it makes sense keeping in mind these features. In the following we shall mainly consider systems with constant masses.

It is interesting to follow the evolution of the mass concept during the centuries [116]. It started as a matter of fact long before Newton, but Newton was the first to give with his *vis inertiae* idea a scientific basis for mass. In the following centuries the development of chemistry influenced the mass concept stating that mass represents a "quantity of matter", which requires a force to be put into motion. Euler developed a new concept defining mass as the fraction of force and acceleration, a conception, which was quickly accepted especially by French representatives of Mathematical Physics. The axiomatization of mechanics in the last two centuries made it necessary to define the mass anew within the framework of deductive and geometric forms. Completely new aspects entered the mass discussion with Einstein's relativistic mechanics.

2.1.2 Cut Principle and Forces

Before establishing a model we have to make clear what part of a system we would like to consider. It depends on the results we want to achieve, and this depends on the problem of the system under consideration. In technical artifacts a meaningful set of cuts for a machine or a car, for example, should define the kinematic or kinetic inputs into the system in the form of time-series or of spectra, and it should define the output of the system considering those positions giving typical performance characteristics. For a car's power transmission system, for example, the input might be the oscillations at the motor's crankshaft exciting the transmission system to vibrate, and the output might be the load torques at the tires coming from the environmental conditions like road quality, acceleration and car weight. Between these two cuts we have the complete power transmission, called in Figure 2.1 system, the performance of which we want to know, for example the acoustical performance. To be able to evaluate this performance we must define the limiting areas (points, lines, volumes) in a meaningful manner. In our practical example it is the load on one side (cut 2) and the input as generated by the combustion engine on the entrance side (cut 1). This makes sense from the technical point of view, indicating that positioning cuts results from experience and empirical knowledge more than from scientific arguments.

In order to apply this cut principle we have to generalize it a bit. In mechanics we are interested in the interaction of bodies of any kind with forces or torques. If we therefore separate two bodies by cutting them apart we must at the same time arrange those forces along the cut, which in the original configuration keep the two bodies together. Thus by cutting any system apart we transform internal forces to external ones acting on the cut parts with the same magnitude but opposite sign. This ingenious cut principle, first established by Euler, was characterized by Szabo [258] in a very appropriate way:

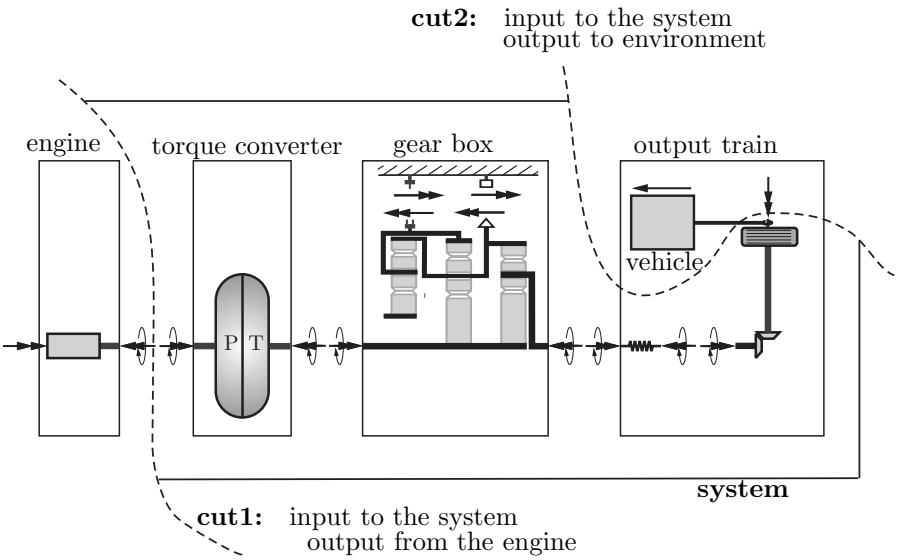

Fig. 2.1: The cut principle for a power transmission

"Euler teaches us with the imagination of an artist to look in thought into the matter, where no eye and no experiment can penetrate. With this he has laid a foundation stone for the only genuine mechanics, namely the continuum mechanics." The cut principle gives us a tool to establish for any part of a system the equations of motion, if we choose the cuts correctly and add to the applied forces and torques also the reaction forces and torques as freed by these cuts. We need in addition a sign definition, which we may choose arbitrarily, but then we must stay with it.

To illustrate the difference of internal and external forces depending on the cut positions we use a simple example [63]. Considering in Figure 2.2 the cut S1 around the three masses we see that all forces within that cut are internal forces possessing no influence on the system S1. Selecting a cut S2 we come out with two external forces F_{12} and F_{32} and with two internal forces F_{13} and F_{31}. Finally the cut S3 generates only external forces, namely F_{21} and F_{31}.

The mechanical sciences are interested in the interaction of any kind of masses with forces. Dynamics as a part of mechanics is especially interested in those forces, which generate motion. Therefore it makes sense to define as a generic concept that of active and passive forces. Active forces can be moved in their direction of action, and from there they can produce work and power. Passive forces cannot be moved with respect to their point of action. Active forces generate motion, passive forces prevent motion, they are as a matter of fact the consequence of some constraints. All other definitions of forces are subsets of this concept. Internal or external forces, applied or constraint

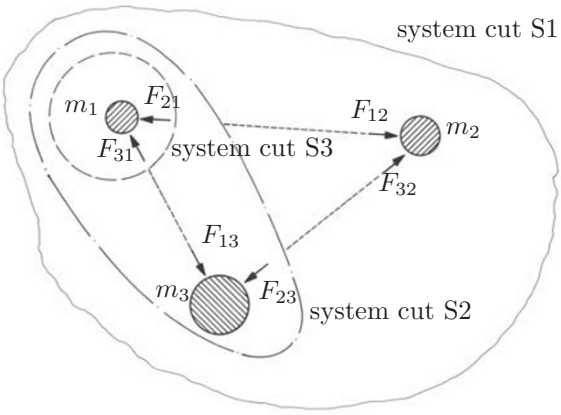

Fig. 2.2: Cut principle: internal and external forces [63]

forces, volume or surface forces, they all might be active or passive, depending on the specific system under consideration.

To give a simple example [208] we consider a block on a plane surface under the influence of an oblique external force. If the block does not move, all forces, the external applied force, the weight force and the contact forces are passive forces. If the external oblique force is big enough to move the block, then the horizontal components of the external applied force and of the contact force are active forces contributing to the motion of the block, whereas the weight force and the vertical components of the external applied force and of the contact force are passive forces adding certain loads to the block and to the ground (see figure 2.3). This simple example demonstrates already, that the property of a force becoming active or passive may depend on the dynamics of the system, which is reasonable especially in the face of unilaterally determined behavior. It sounds complicated, but we shall see later, that this concept is the only workable one with respect to a dynamical theory including all possible types of constraints, bilateral and unilateral ones.

2.1.3 Constraints and Generalized Coordinates

Constraints possess a kinematical character. They are the mechanical controllers telling systems where to go and where not to go. In mechanical engineering we do not have any machine or mechanism, which are unconstrained. Constraints realize, at least kinematically, operational requirements and, applied correctly, guarantee the function of a mechanical system. Constraints

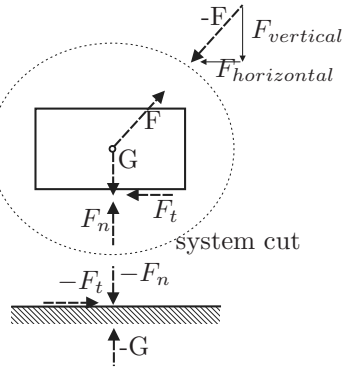

Fig. 2.3: Active and passive forces

might be bilateral or unilateral, representing in the first case an ideal connection between two adjacent bodies or between one body of the system and its environment constraining this connection to a limited number of degrees of freedom, and representing in the second case a connection, which might be open or closed, which might be sticking or sliding, depending on the dynamics of the system under consideration. Mechanical engineering includes as many bilateral as unilateral systems with a clear tendency to unilateral behavior with increasing requirements on modeling details.

Constraints depend on the coordinates of position and orientation, and they depend on velocities. Position- and orientation-dependent constraints are called holonomic, velocity-dependent constraints non-holonomic. They are rheonomic, if they depend on time, and scleronomic, if not. An important non-holonomic property says that such a constraint cannot be integrated to come out with a holonomic constraint. This leads to significant consequences.

The existence of constraints implies two difficulties. The first one concerns the independence of coordinates, which are constrained. Therefore the original coordinate set, for example in some three-dimensional workspace, does not represent the possible number of degrees of freedom. Some of the equations of motion depend on each other. The second difficulty is connected with the forces due to constraints. These constraint forces are not given a priori, they must be evaluated by the solution process. Moreover, the constraint forces do not contribute to the motion of the system, they are internal forces holding the system together, where we should keep in mind that passive forces and motion means passive forces and relative motion. Passive forces may of course move themselves withing the overall system. From the technical standpoint of view we need them as forces in bearings, guides, joints or the like. They determine system design.

Another consequence of constraints is given with the difference of their holonomic and non-holonomic properties. Constraints may be used to generate

a set of generalized coordinates representing the degrees of freedom of the system. The elimination of the dependent variables is formally possible, but not necessarily practically. In the case of non-holonomic constraints it is not possible at all to eliminate coordinates of position and orientation, but it is possible ti eliminate velocity coordinates. A well-known example is the rolling disc, which at the same time is an example with the minimal possible number of degrees of freedom for a non-holonomic system, namely three [93]. The rolling condition for rolling without sliding cannot be integrated to come out with a holonomic equality, because a change of the orientation includes also a change in the position. These properties will have significant consequences for the development of the differential principles. A more detailed discussion of constraints can be found in [180], [93], [27] or [63].

Figure 2.4 depicts some typical constraints. The pendulum on the left represents a holonomic constraint depending only on the position of its mass. As long as the mass connection remains under tension we have a bilateral constraint. The sledge example represents also a holonomic, bilateral constraint as long as the sledge does not detach from the ground. In doing so we get a unilateral constraint with contact- and detachment-phases. The wheel example includes a non-holonomic constraint, because in the general case the function $f(\dot{x}, \dot{y}, \dot{\beta}, \alpha) = 0$ cannot be integrated to give then a position-dependent constraint. This is possible only, if the wheel follows exactly a straight line by rolling without sliding. Then we can roll back the wheel coming exactly to the starting point, and only then the constraint can be integrated.

The examples in Figure 2.4 illustrate also the empirical experience, that the constraint forces F_c are in all cases perpendicular to the directions of motion, which later on will give a basis for the principle of d'Alembert-Lagrange.

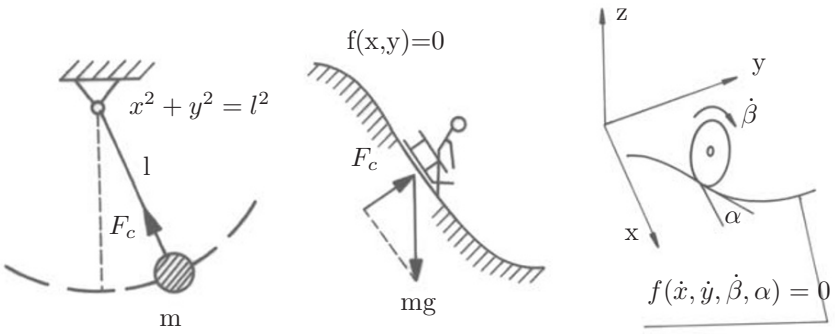

Fig. 2.4: Typical constraint examples

Constraints confine the number of degrees of freedom of a system. In general cases it is not possible to eliminate those degrees of freedom, which are

constrained. But for special cases, usually of smaller dimensions, we might succeed in reducing the coordinates to the number of degrees of freedom really existing in the system. We then call these coordinates "minimal coordinates" $\mathbf{q} \in \mathbb{R}^f$, if f are the degrees of freedom. This is the limiting case for generalized coordinates for all constraints being eliminited. We shall not distinguish that in the following. Minimal coordinates include no more constraints in the form of algebraic equations. We then come out with differential equations without any additional constraints in the form of algebraic equations. It is sometimes difficult to interprete the generalized coordinates physically, but in any case they describe the possible motion of the system under consideration. For nonholonomic problems we are usually able to eliminate the generalized velocities, because the relevant constraints are linearly dependent on the velocities, at least in all cases known so far. We shall come back to that.

2.1.4 Virtual Displacements and Velocities

The concept of virtual displacements and velocities is not only a very productive one in whole mechanics, it is for certain significant areas of mechanics an indispensable tool, some authors say "axiom", for the development of basic theories, for example for analytical dynamics. Very probable Johann Bernoulli was the first one to use the idea of virtual displacements in the year 1717 and also the word "virtual" [258].

We shall understand as virtual displacements and velocities some thought magnitudes $\delta\mathbf{r}$ or $\delta\dot{\mathbf{r}}$, which necessarily must be compatible with all constraints acting for the time t under consideration. The displacements and velocities are called virtual, firstly because they are thought magnitudes and not real ones, and secondly to distinguish them from real changes $\mathrm{d}\mathbf{r}$ and $\mathrm{d}\dot{\mathbf{r}}$, which take place during the time interval dt, where forces and constraints might change considerably. Virtual displacements and velocities are considered for a fixed time t, which always means that $\delta \mathrm{t} = 0$. As a consequence we take so-to-say a photo of the system at time t and investigate the system's behavior resulting from some virtual changes. The changes of the virtual velocities need not to be necessarily infinitesimal small, they might take on any values, but the above mentioned compatibility with the constraints is a must.

The concept of virtual displacements and velocities forms a basis for the principles of virtual work and power, which play a dominant role in all areas of mechanics.

2.2 Kinematics

Kinematics is geometry of motion and its evolution with time. It is the most important foundation of dynamics, as a matter of fact a foundation of any mechanical field. Rigid or elastic bodies must be defined in some suitable coordinate frame, the choice of which is more an art than a science strongly deciding on the complexity, or simplicity, of the mathematical model following from it. The basic movements of a rigid body are translation and rotation, each one described by three coordinates. Considering mechanical systems with many bodies requires the definition of many coordinate frames, body-fixed ones and inertial ones; where again the choice of these coordinates heavily influence the structure of the equations of motion and from there the necessary solution efforts. For most of the applications we apply orthogonal coordinate systems, but sometimes curvilinear coordinates represent the system under consideration in a more elegant way. Contact problems of rigid or elastic bodies are an example.

Dealing with systems, especially with multibody systems, includes vector spaces composed for example by the generalized coordinates. If these generalized coordinates represent the degrees of freedom of our system, then they are linearly independent and form a basis of the $\mathbb{R}$-vector-space with the dimension $\mathbb{R}^f$. The properties of these spaces are indispensable aspects for analyzing dynamical systems [155], [27], [180].

2.2.1 Coordinates

We define a coordinate system as a set of orthogonal unit vectors, for example $(\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)$, which form a basis for all vector representations to come. With respect to this base we assign an origin zero (0) or sometimes also an origin zero (O), which we assume to be fixed to some rigid or elastic body under consideration. Only this definition of an origin allows the measurement of distances or dynamic features with the help of such a coordinate system. Depending on the state of motion of the body with the coordinate system $(0, \mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)$ we call these coordinates inertial or non-inertial (body-fixed).

The coordinate system $(0, \mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)$ possesses the property "inertial", if the base vectors $(\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)$ do not change with time, which means, that such a coordinate system might only move with constant velocity with respect to our not moving postulated space. This is a question of definition. For technical dynamics it is usually sufficient to connect the earth or some building with an inertial coordinate system, for problems of space dynamics the sun might be a more suitable system.

If we connect a coordinate system with a rigid body to define a "body-fixed" coordinate system we have the choice to select for the origin any point, the center of mass or another point convenient for our evaluations, for example the center of mass and one joint in the case of robots. From this we may have several coordinate frames in one body. A basic property of a rigid body consists

in the constant distance between two material points. Rigid bodies have six degrees of freedom, three of translation and three of rotation. Therefore the three positions (x,y,z) and the three orientations (α, β, γ), given for example as Cardan-angles, define unambiguously position and orientation of a rigid body with respect to any coordinate system, inertial or body-fixed (see Figure 2.5).

From this we might define for example the position and the orientation of a body B_1 with respect to the inertial system I or with respect to the body-fixed system of B_2 by the six magnitudes (x, y, z) and (α, β, γ), where (x, y, z) are the coordinates of the mass center of the body B_1, written in the bases I or B_2, and where the Cardan-angles (α, β, γ) give the orientation between the coordinate system of B_1 and those of I or B_2, written correspondingly in the I- or B_2-bases (see Figure 2.5).

The crucial point in dealing with rotations consists in the fact, that the rotational angles or the orientation angles cannot be evaluated straightforward, but are usually given implicitly by the rotational velocities expressed by the Euler kinematical equations or any other form. Therefore we always have to regard some equations of the form (2.31) to (2.33), which usually are part of the set of differential equations of motion. Only for small rotations, where the rotation angles are allowed to be represented as a vector, the matrices of the mentioned equations degenerate to an identity matrix.

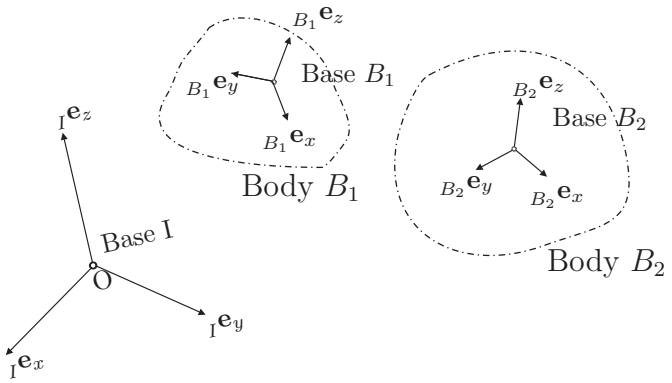

Fig. 2.5: Position and Orientation of Rigid Bodies

From this we introduce for many bodies with large rotations a vector $\mathbf{z}$ for translations and a matrix-vector equation for the rotational velocities

$$\begin{aligned} \mathbf{z} =& (x_1, y_1, z_1, x_2, y_2, z_2, \ldots, x_n, y_n, z_n), \\ \boldsymbol{\omega}_i =& \mathbf{H}_i(\mathbf{q}_i)\dot{\mathbf{q}}_i, \quad (i = 1, \cdot n), \end{aligned} \tag{2.1}$$

where we have chosen a general representation including the rotational velocities $\omega_i = (\omega_1, \omega_2, \omega_3)_i$ and some vector $\mathbf{q}_i$, which may be replaced by Euler or Cardan angles or any other set of orientation angles (see for example [146]

and [180]). For many bodies with small rotations we collect all the body coordinates in a vector $\mathbf{z}$ giving

$$\mathbf{z} = (x_1, y_1, z_1, \alpha_1, \beta_1, \gamma_1, x_2, y_2, z_2, \alpha_2, \beta_2, \gamma_2, \ldots, x_n, y_n, z_n, \alpha_n, \beta_n, \gamma_n), \quad (2.2)$$

which contains n bodies with altogether 6n coordinate elements. Considering only one body we have $\mathbf{z} = \mathbf{r} = (x_1, y_1, z_1, \alpha_1, \beta_1, \gamma_1)$. Instead of $\mathbf{z}$ we also shall use the vector $\mathbf{r}$.

As pointed out already in chapter 2.1.3 all real mechanical systems are constrained, where these constraints might be holonomic or non-holonomic, they might be scleronomic or rheonomic ([180]). In any case they constrain the motion of our system being then described by less coordinates as indicated in equation 2.2, which therefore contains some spare coordinates. Sometimes it is possible to eliminate these spare coordinates and to establish a set of coordinates, which corresponds exactly to the number of degrees of freedom, in many cases though this is not possible. Let us first consider some set of constraints, which might be of any type, but as an example we take into account a number of m holonomic and rheonomic constraint equations in the form

$$\boldsymbol{\Phi}(\mathbf{z}, t) = \mathbf{0}, \qquad \boldsymbol{\Phi} \in \mathbb{R}^m, \qquad m \leq 6n. \quad (2.3)$$

Such a set of m constraint equations reduce the free directions of motion to f = 6n - m, which we shall call in the following the number of degrees of freedom. We assign to these f degrees of freedom the coordinates $\mathbf{q} \in \mathbb{R}^f$, which are the "generalized coordinates". This elimination will not be possible for all applications under consideration leaving us with a certain rest of not fulfilled constraints. We then come out with a set of differential-algebraic equations containing f_{min} differential equations and m_{min} remaining constraints. We then still shall use the name "generalized coordinates" for the vector $\mathbf{q}$.

At this point we also should mention the special form of coordinates, which are convenient for the treatment of all kind of trajectory problems and of contact phenomena. It concerns robotics and walking on the one hand and many machine applications on the other one. To give an example: for unilateral contact problems it makes sense to represent the surface coordinates of the bodies in a parametric form (see Figure 2.6 and chapter 2.2.6):

$$\mathbf{r} = \begin{pmatrix} x(u, v) \\ y(u, v) \\ z(u, v) \end{pmatrix} \quad (2.4)$$

2.2.2 Coordinate Transformations

The basic elements of kinematics are translation and rotation. In addition, and mostly confined to special cases, we have projections and reflections (see for

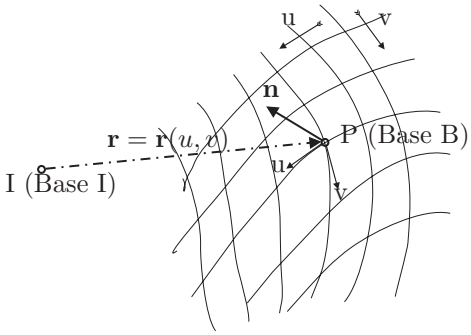

Fig. 2.6: Coordinate systems

example [3] and [155]). We shall focus on the first two movements. Considering mechanical systems requires a precise and unique definition of coordinate frames. In the following we shall use an inertial base I, and several body-fixed bases B or B_i and R or R_i (Figure 2.7). A vector $\mathbf{v}$ is a component of the vector space V, $\mathbf{v} \in \mathrm{V}$, and it can be represented in any of the mentioned coordinate systems. From the standpoint of dynamics it is convenient to describe mechanical systems in different frames, and therefore we need a transforma-

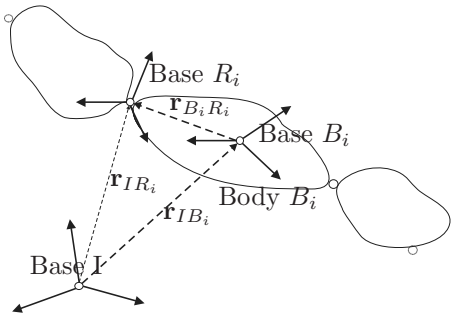

Fig. 2.7: Frame Relations

tion from one frame to any other one. From Figure 2.7 we easily can describe the vector chain in a coordinate-free form

$$\mathbf{r}_{IR_i} = \mathbf{r}_{IB_i} + \mathbf{r}_{B_iR_i}, \tag{2.5}$$

where the indices (I, R_i, B_i) stand for the origins of the bases and for the bases themselves. The property of the necessary coordinate transformation can be nicely illustrated by the transformation triangle, which relates graphically the representation in different coordinate systems (see Figure 2.8). We apply for these representations the convention:

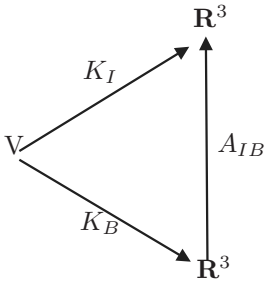

Fig. 2.8: Transformation triangle

$$
\begin{aligned}
K_I(\mathbf{v}) &= {}_I\mathbf{v} \in \mathbb{R}^3,\\
K_B(\mathbf{v}) &= {}_B\mathbf{v} \in \mathbb{R}^3,\\
K_R(\mathbf{v}) &= {}_R\mathbf{v} \in \mathbb{R}^3.
\end{aligned}
\tag{2.6}
$$

These definitions indicate that the components of the vector $\mathbf{v}$ are written in the coordinate frames I,B,R, respectively. Going from one frame to another one we must evaluate the compositions [155]

$$
\begin{aligned}
K_I &= \mathbf{A}_{IB} \circ K_B,\\
K_B &= \mathbf{A}_{BI} \circ K_I,
\end{aligned}
\tag{2.7}
$$

which according to Figure 2.8 can be performed by a linear transformation with the transformation matrices $\mathbf{A}_{IB}$ or $\mathbf{A}_{BI}$. The index "IB" has to be read from right to left in the sense of transforming the vector $\mathbf{v}$ from the B-frame to the I-frame, and for "BI" from the I-frame to the B-frame. From equation 2.7 we see immediately, that the matrix-product $\mathbf{A}_{IB}\mathbf{A}_{BI} = \mathbf{E}$ comes out with the unit-matrix, which means, that these matrices $\mathbf{A}$ are orthogonal:

$$
\mathbf{A}_{IB}\mathbf{A}_{BI} = \mathbf{E}. \tag{2.8}
$$

A typical situation in multibody system modeling consists in the necessity to

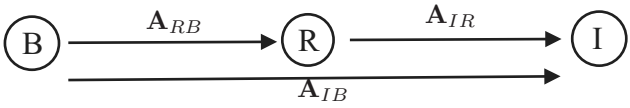

Fig. 2.9: Three successive coordinate systems

choose very many coordinate systems, sometimes several ones for only one of the body-elements. As a consequence we must consider also many successive coordinate frames performing multiple compositions as done in equations 2.7

for one frame only. Let us first consider three coordinate systems B,R,I, and let us go from B to I on the one side and from B to R to I on the other side (see 2.9). According to equation 2.7 we also get successive compositions in the form

$$K_I = \mathbf{A}_{IB} \circ K_B = \mathbf{A}_{IR}\mathbf{A}_{RB} \circ K_B \tag{2.9}$$

From equation 2.9 we get immediately that $\mathbf{A}_{IB} = \mathbf{A}_{IR}\mathbf{A}_{RB}$, which we can easily generalize by introducing a whole chain of intermediate coordinate systems, see Figure 2.10. We come out with the following chain of transforma-

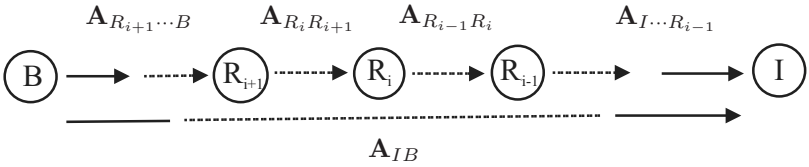

Fig. 2.10: n successive coordinate systems

tions:

$$\begin{aligned} K_I &= \mathbf{A}_{IB} \circ K_B = (\mathbf{A}_{IR_2} \cdot\cdots\cdot \mathbf{A}_{R_{i-1}R_i}\mathbf{A}_{R_iR_{i+1}} \cdot\cdots\cdot \mathbf{A}_{R_{n-1}B}) \circ K_B \\ \mathbf{A_{IB}} &= \mathbf{A}_{IR_2} \cdot\cdots\cdot \mathbf{A}_{R_{i-1}R_i}\mathbf{A}_{R_iR_{i+1}} \cdot\cdots\cdot \mathbf{A}_{R_{n-1}B} \end{aligned} \tag{2.10}$$

Numbering the chain of coordinate systems from "1" for "I" and "n" for "B" we also can write the second equation of 2.10 in the form:

$$\mathbf{A}_{1,n} = \prod_{i=1}^{n-1} \mathbf{A}_{i,i+1} \tag{2.11}$$

The basic movements of kinematics are translations and rotations. Translations can be described by vector-chains as given with equation 2.5. Rotations concern a rotation of the complete coordinate frame around its origin. Various angle-triples exist to describe such rotations ([146], [3]). Sometimes it makes sense to apply a four-dimensional representation to avoid singularities, for example in space dynamics. Such quaternions have first been introduced by Hamilton [3]. We shall limit our considerations to Euler- and Cardan-angles, which are used very often in multibody problems. They have their origin in the early works of gyro-dynamics and celestial mechanics. In both cases we must perform three elementary rotations to come from some base B, for example a body-fixed frame, to a base R, which might be an intermediate or an inertial frame.

We start with the Euler angles. In Figure 2.11 we rotate the coordinates from the orientation I to the orientation B by three successive elementary rotations ψ, φ, ϑ. In a first step we rotate the I-system around the inertially

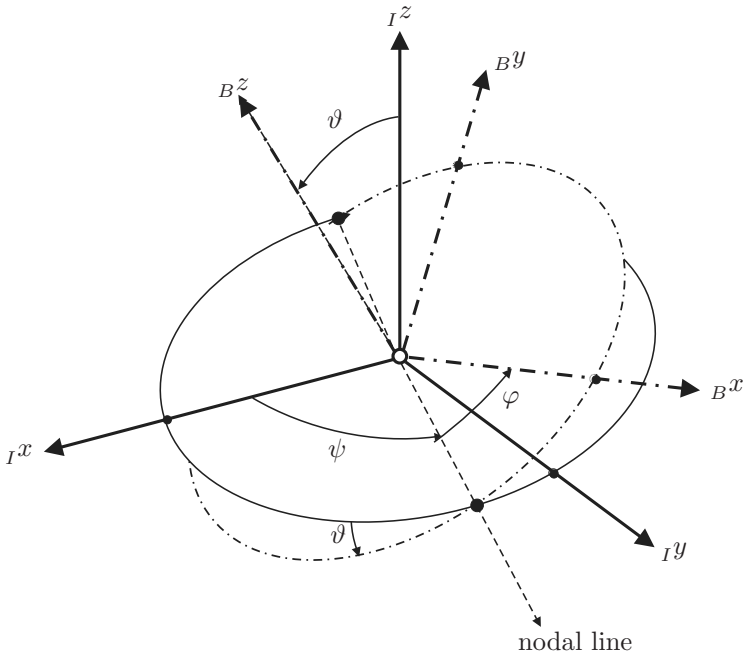

Fig. 2.11: Euler angles

fixed z-axis $_I z$ with the angle ψ. This rotation reaches the nodal line and can be represented by the linear elementary transformation

$$\mathbf{A}_{I,\psi} = \begin{pmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{2.12}$$

In a second step we tilt the system around the nodal line applying the angle ϑ. This elementary rotation follows the linear transformation

$$\mathbf{A}_{nodal,\vartheta} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\vartheta & -\sin\vartheta \\ 0 & \sin\vartheta & \cos\vartheta \end{pmatrix}, \tag{2.13}$$

Finally, in a third step we rotate around the $_B z$-axis and come out with the final orientation with the index "B", for example as a body-fixed orientation. The corresponding transformation writes

$$\mathbf{A}_{B,\varphi} = \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{2.14}$$

The complete transformation then results from the successive products of the three elementary rotations. We get

$$\mathbf{A}_{IB} = \mathbf{A}_{I,\psi}\mathbf{A}_{nodal,\vartheta}\mathbf{A}_{B,\varphi} = \begin{pmatrix} \cos\psi\cos\varphi - \sin\psi\cos\vartheta\sin\varphi & -\cos\psi\sin\varphi - \sin\psi\cos\vartheta\cos\varphi & +\sin\psi\sin\vartheta \\ \sin\psi\cos\varphi + \cos\psi\cos\vartheta\sin\varphi & -\sin\psi\sin\varphi + \cos\psi\cos\vartheta\cos\varphi & -\cos\psi\sin\vartheta \\ \sin\vartheta\sin\varphi & \sin\vartheta\cos\varphi & \cos\vartheta \end{pmatrix}. \tag{2.15}$$

Cardan-angles use a simpler sequence of rotations to go from the coordinate frame I to that of B (see Figure 2.12). We rotate firstly around the ${}_{I}x$-axis with the angle α and come to an intermediate axis by the transformation:

$$\mathbf{A}_{I,\alpha} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{pmatrix}, \tag{2.16}$$

By a second rotation around the intermediate ${}_{IB}y$-axis with the angle β we come already to the final ${}_{B}z$-axis by the transformation:

$$\mathbf{A}_{inter,\beta} = \begin{pmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{pmatrix}, \tag{2.17}$$

A last rotation around the new ${}_{B}z$-axis brings us into the final position with $({}_{B}x, {}_{B}y, {}_{B}z)$:

$$\mathbf{A}_{B,\gamma} = \begin{pmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{2.18}$$

With Cardan-angles we are able to reach any position of a body-fixed coordinate frame. Their application is especially useful for problems of machine dynamics. The complete transformation matrix is then the result of the above successive rotations. We come out with:

$$\mathbf{A}_{IB} = \mathbf{A}_{I,\alpha}\mathbf{A}_{inter,\beta}\mathbf{A}_{B,\gamma} = \begin{pmatrix} \cos\beta\cos\gamma & -\cos\beta\sin\gamma & \sin\beta \\ \cos\alpha\sin\gamma + \sin\alpha\sin\beta\cos\gamma & \cos\alpha\cos\gamma - \sin\alpha\sin\beta\sin\gamma & -\sin\alpha\cos\beta \\ \sin\alpha\sin\gamma - \cos\alpha\sin\beta\cos\gamma & \sin\alpha\cos\gamma + \cos\alpha\sin\beta\sin\gamma & +\cos\alpha\cos\beta \end{pmatrix} \tag{2.19}$$

2.2.3 Velocities and Accelerations

With the knowledge of the coordinates and coordinate transformation we have established a basis for deriving the expressions for velocities and for accelerations in the various possible coordinate systems. Let us first go back to Figure 2.7 and simplify this figure a bit for our purposes (see Figure 2.13). We go from the I-system into the B-system, or vice versa, and we consider

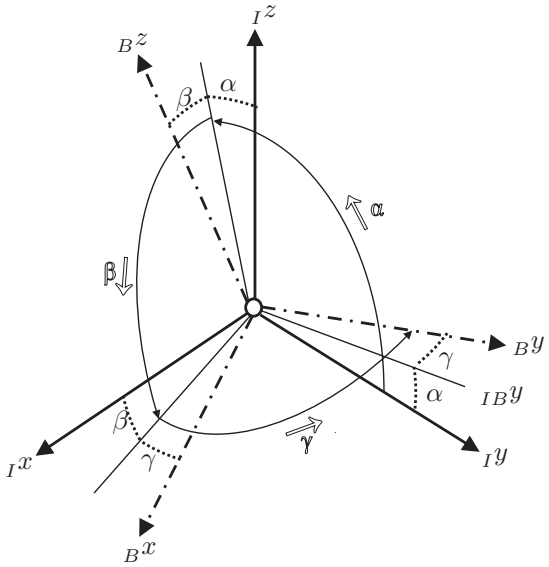

Fig. 2.12: Cardan angles

in both systems the coordinates of the point P. The translation follows from that Figure by

$$\mathbf{r}_{OP} = \mathbf{r}_{OQ} + \mathbf{r}_{QP}. \tag{2.20}$$

According to chapter 2.2.2 the rotation of coordinate system B with respect to I can be described by the matrix $\mathbf{A}_{BI}$, if we go from I to B and by $\mathbf{A}_{IB}$, if we go from B to I. Applying this matrix we can express the coordinates in one frame by those in the other frame, for example

$$\begin{aligned} {}_I\mathbf{r} &= \mathbf{A}_{IB}\ {}_B\mathbf{r} \\ {}_B\mathbf{r} &= \mathbf{A}_{BI}\ {}_I\mathbf{r}, \end{aligned} \tag{2.21}$$

which immediately confirms equation 2.8 and its properties

$$\begin{aligned} \mathbf{A}_{IB}\mathbf{A}_{BI} &= \mathbf{E} \\ \mathbf{A}_{BI} &= \mathbf{A}_{IB}^{-1} = \mathbf{A}_{IB}^{T} \end{aligned} \tag{2.22}$$

Before going to a derivation of the velocities we make a detour by considering the velocity of a point trajectory. It represents for example the case of a robot hand following a prescribed trajectory ([208]). The velocity of a point P along that trajectory is defined as

$$\mathbf{v} = \lim_{\Delta t \to 0} \frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t} = \frac{\mathrm{d}\, r}{\mathrm{d}\, t} \tag{2.23}$$

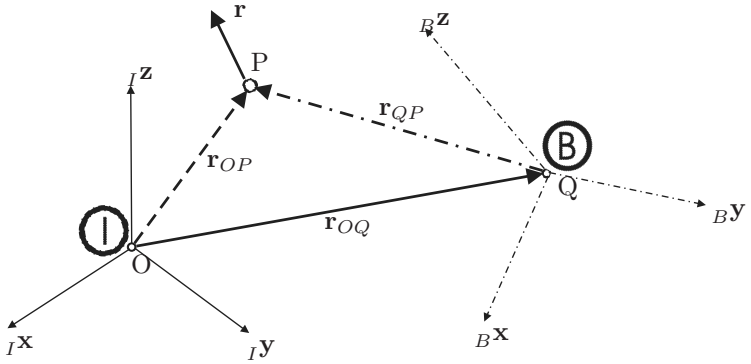

Fig. 2.13: Coordinate Relations

The velocity vector **v** has the direction of the tangent line to the point trajectory of Figure 2.14. If we consider the above defined velocity in an inertial frame as shown in Figure 2.14, we come out with

$$ {}_I\mathbf{v}_{abs} = \frac{\mathrm{d}\,{}_I\mathbf{r}}{\mathrm{d}\,t} = {}_I\dot{\mathbf{r}}, \tag{2.24} $$

which means the following: The absolute velocity of a moving point P is the derivation with respect to time of a vector $\mathbf{r}(t) = {}_I\mathbf{r}(t)$ represented in a coordinate frame I, which is assumed to be an inertial system. Note that this vector or any other mechanical object can be represented in any coordinate frame without changing its physical properties, for example the magnitude of the vector is always the same. We only look at this vector from another point of view leading to different coordinates, but the vector itself or the object itself remains the same.

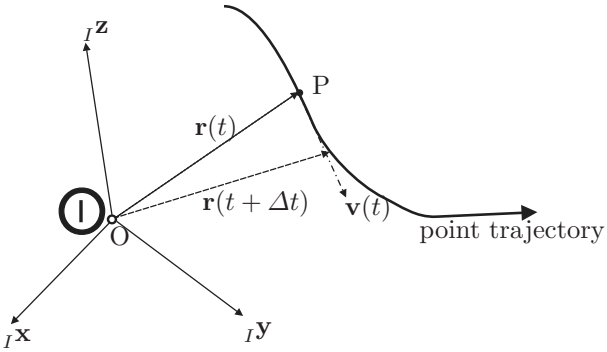

Fig. 2.14: Point Trajectory

Going for example from an inertial system to a body-fixed frame in Figure (2.7) or to a point fixed frame in form of the moving trihedral in Figure 2.14 (see [51]) we must transform the corresponding vectors from one frame to the other one applying the equation 2.21, which yields

$$ {}_I\mathbf{v}_{abs} = {}_I\dot{\mathbf{r}}, \tag{2.25} $$

$$ {}_B\mathbf{v}_{abs} = \mathbf{A}_{BI} \ {}_I\dot{\mathbf{r}}. \tag{2.26} $$

With these transformations we write the equation 2.20

$$ {}_I\mathbf{r}_{OP} = {}_I\mathbf{r}_{OQ} + {}_I\mathbf{r}_{QP} = {}_I\mathbf{r}_{OQ} + \mathbf{A}_{IB} \cdot {}_B\mathbf{r}_{QP}. \tag{2.27} $$

The main reason for doing that lies in two facts, firstly that principally all time derivations in dynamics have to be performed in an inertial frame, which requires the appropriate transformations, and secondly that in many multibody applications it is much more convenient to define some points, some mass elements for example, in body-fixed coordinates than in inertial ones. The additional transformation matrices $\mathbf{A}$ usually can be calculated in a straightforward manner. Deriving 2.27 with respect to time we come out with

$$ {}_I\dot{\mathbf{r}}_{OP} = {}_I\dot{\mathbf{r}}_{OQ} + \frac{\mathrm{d}}{\mathrm{d}\,t}(\mathbf{A}_{IB} \cdot {}_B\mathbf{r}_{QP}), $$

$$ {}_I\dot{\mathbf{r}}_{OP} = {}_I\dot{\mathbf{r}}_{OQ} + \dot{\mathbf{A}}_{IB} \cdot {}_B\mathbf{r}_{QP} + \mathbf{A}_{IB} \cdot {}_B\dot{\mathbf{r}}_{QP}. \tag{2.28} $$

Equation 2.28 is one of the most important and basic relationships of kinematics. It allows the following interpretation: the left hand term of the second line represents the absolute velocity of the point P defined in the inertial system of Figure 2.13, the first right hand term is the absolute velocity of point Q also defined in the I-system, the second right hand term is the velocity of point P resulting from the rotation of the body-fixed B-system with respect to the inertial I-system, and finally $\dot{\mathbf{r}}_{QP}$ in the third right hand term is the relative velocity of P with respect to the B-system, for example resulting from a deformation, and transformed to the I-system by $\mathbf{A}_{IB}$.

To proceed to the classical formulation we must investigate the properties of these transformation matrices. First we recall the well known fact that the matrices $\mathbf{A}$ are orthogonal (see [155] and [27]). The rows of the matrix $\mathbf{A}_{IB}$ for example, which describes a transformation from the B-system into the I-system, are the unit vectors of the B-system represented in the I-system. We start with equation 2.22 and differentiate it with respect to time resulting in $\dot{\mathbf{A}}_{IB} \cdot \mathbf{A}_{BI} + \mathbf{A}_{IB} \cdot \dot{\mathbf{A}}_{BI} = 0$ and from that we get

$$ \dot{\mathbf{A}}_{IB} \cdot \mathbf{A}_{BI} = -\mathbf{A}_{IB} \cdot \dot{\mathbf{A}}_{BI} = -(\dot{\mathbf{A}}_{IB} \cdot \mathbf{A}_{BI})^T. \tag{2.29} $$

Equation 2.29 includes the following facts. Obviously the matrix expression $\dot{\mathbf{A}}_{IB} \cdot \mathbf{A}_{BI}$ is skew-symmetric. Additionally, it must represent angular rotational velocities, which results from the rows of the matrix $\mathbf{A}$ being unitvectors in the coordinate system under consideration. Any time derivative of

unit-vectors can only come out with a rotation, because the magnitudes of the vectors themselves do not change.

If we apply the results of equation 2.29 to the transformation matrices $\mathbf{A}_{IB}$ on the basis of the Euler- and Cardan-angles in the equations 2.15 and 2.19, then after some lengthy calculations and rearranging we come out with the well-known Euler-equations for rotational kinematics

$$ {}_I\boldsymbol{\omega} = \begin{pmatrix} 0 & \cos\psi & \sin\psi\sin\theta \\ 0 & \sin\psi & -\cos\psi\sin\theta \\ 1 & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} \dot{\psi} \\ \dot{\theta} \\ \dot{\phi} \end{pmatrix}, \tag{2.30} $$

$$ {}_B\boldsymbol{\omega} = \begin{pmatrix} \sin\theta\sin\phi & \cos\phi & 0 \\ \sin\theta\cos\phi & -\sin\phi & 0 \\ \cos\theta & 0 & 1 \end{pmatrix} \begin{pmatrix} \dot{\psi} \\ \dot{\theta} \\ \dot{\phi} \end{pmatrix}, \tag{2.31} $$

which is the set for the angular velocities using Euler-angles, and

$$ {}_I\boldsymbol{\omega} = \begin{pmatrix} 1 & 0 & \sin\beta \\ 0 & \cos\alpha & -\sin\alpha\cos\beta \\ 0 & \sin\alpha & \cos\alpha\cos\beta \end{pmatrix} \begin{pmatrix} \dot{\alpha} \\ \dot{\beta} \\ \dot{\gamma} \end{pmatrix}, \tag{2.32} $$

$$ {}_B\boldsymbol{\omega} = \begin{pmatrix} \cos\beta\cos\gamma & \sin\beta & 0 \\ -\cos\beta\sin\gamma & \cos\gamma & 0 \\ \sin\beta & 0 & 1 \end{pmatrix} \begin{pmatrix} \dot{\alpha} \\ \dot{\beta} \\ \dot{\gamma} \end{pmatrix}, \tag{2.33} $$

which is the set for the angular velocities using Cardan angles. For later considerations we should keep in mind, that the equations (2.31) and (2.33) represent a linear relationship between the rotational velocities ω and the time derivatives of the orientation angles $\bar{\boldsymbol{\varphi}} \Rightarrow (\psi\theta\phi) \Rightarrow (\alpha\beta\gamma)$ in the form

$$ \boldsymbol{\omega} = \mathbf{H}\dot{\boldsymbol{\varphi}} \quad \Longrightarrow \quad \dot{\boldsymbol{\varphi}} = \mathbf{H}^{*}\boldsymbol{\omega} \tag{2.34} $$

With respect to the time derivatives of the transformation matrices we conclude from the above relations, that the equations 2.29 can only possess the meaning (see [27])

$$ \dot{\mathbf{A}}_{IB}\ \mathbf{A}_{BI} = {}_I\tilde{\boldsymbol{\omega}}, \quad \mathbf{A}_{BI}\ \dot{\mathbf{A}}_{IB} = {}_B\tilde{\boldsymbol{\omega}}, \quad {}_I\tilde{\boldsymbol{\omega}} = \mathbf{A}_{IB}\ {}_B\tilde{\boldsymbol{\omega}}\ \mathbf{A}_{BI} \tag{2.35} $$

with the skew-symmetric $\tilde{\boldsymbol{\omega}}$-matrix

$$ \tilde{\boldsymbol{\omega}} = \begin{pmatrix} 0 & -\omega_z & +\omega_y \\ +\omega_z & 0 & -\omega_x \\ -\omega_y & +\omega_x & 0 \end{pmatrix} \tag{2.36} $$

which includes the following helpful properties

$$ \boldsymbol{\omega} \times \mathbf{r} = \tilde{\boldsymbol{\omega}}\ \mathbf{r}, \qquad \tilde{\boldsymbol{\omega}}^{T} = -\tilde{\boldsymbol{\omega}}, \qquad \tilde{\boldsymbol{\omega}}\ \mathbf{r} = -\tilde{\mathbf{r}}\ \boldsymbol{\omega} \tag{2.37} $$

With these relations in mind we come back to equation 2.28, which gives us the absolute velocity of a point in the inertial frame I. Transforming these

equations from I to B by applying the transformation $\mathbf{A}_{BI}$ we come out with the absolute velocity of the point P (figure2.13) in the body-fixed frame B. We get

$$
\begin{aligned}
{}_B\mathbf{v}_{P,abs} &= \mathbf{A}_{BI}\ {}_I\mathbf{v}_{P,abs} = \mathbf{A}_{BI}\ ({}_I\dot{\mathbf{r}}_{OQ} + \dot{\mathbf{A}}_{IB}\ {}_B\mathbf{r}_{QP} + \mathbf{A}_{IB}\ {}_B\dot{\mathbf{r}}_{QP}) \\
{}_B\mathbf{v}_{P,abs} &= {}_B\mathbf{v}_{Q,abs} + {}_B\tilde{\boldsymbol{\omega}}\ {}_B\mathbf{r} + {}_B\dot{\mathbf{r}}
\end{aligned}
\tag{2.38}
$$

where the abbreviations $\mathbf{r}_{QP} = \mathbf{r}$, ${}_I\mathbf{v}_{P,abs} = {}_I\dot{\mathbf{r}}_{OP}$ and ${}_B\mathbf{v}_{Q,abs} = \mathbf{A}_{BI}\ {}_I\dot{\mathbf{r}}_{OQ}$ have been used. Equation 2.38 is of equal importance as equation 2.28, the first one being written in the I-system, the other one in the B-system. Therefore the terms of the last line of equation 2.38 have the following meaning: the left hand term represents the absolute velocity of the point P defined in the body-fixed system of Figure 2.13, the first right hand term is the absolute velocity of point Q also defined in the B-system, the second right hand term is the velocity of point P resulting from the rotation of the body-fixed B-system with respect to the inertial I-system but given in coordinates of the B-system, and finally ${}_B\dot{\mathbf{r}}$ of the third right hand term is the relative velocity of P with respect to the B-system. Examples might be deformation or the motion of a passenger in a flying airplane. The last line of equation 2.38 is the form usually given in textbooks. It is often addressed to as Coriolis equation.

The accelerations follow in a straightforward manner from the above equations. The acceleration is defined in a similar way as the velocity by

$$
\mathbf{a} = \lim_{\Delta t \to 0} \frac{\mathbf{v}(t + \Delta t) - \mathbf{v}(t)}{\Delta t} = \frac{\mathrm{d}\,\mathbf{v}}{\mathrm{d}\,t} \tag{2.39}
$$

Starting with

$$
{}_I\mathbf{a}_{abs} = \frac{\mathrm{d}\,{}_I\mathbf{v}_{abs}}{\mathrm{d}\,t} = {}_I\ddot{\mathbf{r}}, \qquad {}_B\mathbf{a}_{P,abs} = \mathbf{A}_{BI} \cdot {}_I\mathbf{a}_{P,abs} \tag{2.40}
$$

we get from the above equations the accelerations in the I-system.

$$
{}_I\mathbf{a}_{P,abs} = {}_I\mathbf{a}_{Q,abs} + \mathbf{A}_{IB} \cdot ({}_B\dot{\tilde{\boldsymbol{\omega}}}_B\mathbf{r} + {}_B\tilde{\boldsymbol{\omega}}_B\tilde{\boldsymbol{\omega}}_B\mathbf{r}) + \mathbf{A}_{IB} \cdot (2\,{}_B\tilde{\boldsymbol{\omega}}_B\dot{\mathbf{r}} + {}_B\ddot{\mathbf{r}}). \tag{2.41}
$$

The first term on the right hand side is the absolute acceleration of point Q (Figure 2.13), the second term the angular and the third term the centrifugal acceleration. The first three terms are applied accelerations, the fourth term the Coriolis- and the last term the relative acceleration due to some relative motion within the moving system. Transforming the expression 2.41 into a body-fixed coordinate system we come out with the well-known formula

$$
{}_B\mathbf{a}_{P,abs} = \mathbf{A}_{BI}\ {}_I\mathbf{a}_{Q,abs} + {}_B\dot{\tilde{\boldsymbol{\omega}}}_B\mathbf{r} + {}_B\tilde{\boldsymbol{\omega}}_B\tilde{\boldsymbol{\omega}}_B\mathbf{r} + 2\,{}_B\tilde{\boldsymbol{\omega}}_B\dot{\mathbf{r}} + {}_B\ddot{\mathbf{r}}. \tag{2.42}
$$

The explanations with regard to the individual terms of this expression are the same as above, all these terms are now written in a body-fixed frame.

Coming back again to the point trajectory of figure 2.14 we represent the vector $\mathbf{r}$(t) in cylinder coordinates and derive from equations 2.23 and 2.24

the corresponding accelerations. We get with $(x, y, z)^T = (r\cos\varphi, r\sin\varphi, z)^T$ the following expressions

$$
\begin{aligned}
{}_I\mathbf{v}_P &= {}_I\dot{\mathbf{r}}_P = \begin{pmatrix} \dot{r}\cos\varphi - r\dot{\varphi}\sin\varphi \\ \dot{r}\sin\varphi + r\dot{\varphi}\cos\varphi \\ \dot{z} \end{pmatrix}, \\
{}_I\mathbf{a}_P &= {}_I\ddot{\mathbf{r}}_P = \begin{pmatrix} \ddot{r}\cos\varphi - 2\dot{r}\dot{\varphi}\sin\varphi - r\ddot{\varphi}\sin\varphi - r\dot{\varphi}^2\cos\varphi \\ \ddot{r}\sin\varphi + 2\dot{r}\dot{\varphi}\cos\varphi + r\ddot{\varphi}\cos\varphi - r\dot{\varphi}^2\sin\varphi \\ \ddot{z} \end{pmatrix}.
\end{aligned} \tag{2.43}
$$

For the evaluation of these formulas in a body-fixed frame we have to multiply the last equation with the transformation matrix from I to B, which results in

$$
{}_B\mathbf{a}_P = \mathbf{A}_{BI} \cdot {}_I\dot{\mathbf{r}}_P = \begin{pmatrix} \ddot{r} - r\dot{\varphi}^2 \\ 2\dot{r}\dot{\varphi} + r\ddot{\varphi} \\ \ddot{z} \end{pmatrix}, \quad \text{with } \mathbf{A}_{BI} = \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{2.44}
$$

2.2.4 Transformation Chains and Recurrence Relations

The dynamics of mechanical systems requires as a rule the transformations over a long chain of bodies. In most cases these chains possess a tree-like structure, which is either given by the system structure itself, like in the case of robots, or it can be generated by cutting loops and introducing additional constraints or, if possible, by representing loops by analytical expressions. For developing the kinematics of such systems we establish a recursive algorithm on the basis of the above considerations [208]. In a first step we evaluate the angular velocities of the multibody components. If $\mathbf{\Omega}_{B_{i-1}}$ and $\mathbf{\Omega}_{B_i}$ are coordinate-free representations of the angular velocities of the bodies (i-1) and (i), respectively, then they are connected by the relation (see figure 2.15)

$$
\mathbf{\Omega}_{B_i} = \mathbf{\Omega}_{B_{i-1}} + \mathbf{\Omega}_{B_{i-1}B_i}, \tag{2.45}
$$

where the last term is the relative velocity between the bodies (i-1) and (i). Following the composition of transformations of figure 2.9 we come from the inertial coordinate system to that of the body B_{i-1} and from there to the body B_i using the matrix $\mathbf{A}_{B_{i-1}B_i}$ for the relative rotation between (i-1) and (i). According to equation 2.9 we get:

$$
\mathbf{A}_{IB_i} = \mathbf{A}_{IB_{i-1}} \cdot \mathbf{A}_{B_{i-1}B_i} \qquad \text{and} \quad \mathbf{A}_{B_{i-1}B_i} = \mathbf{T}_i. \tag{2.46}
$$

The last equation defines the transformation concerning the rotation beteen I and B_i, as a convenient abbreviation for the following. With the equations 2.7 and 2.35 we define

$$
{}_{B_i}\tilde{\boldsymbol{\omega}}_{IB_i} = K_{B_i} \circ \mathbf{\Omega}_{B_i} \qquad \text{and} \qquad {}_{B_i}\tilde{\boldsymbol{\omega}}_{IB_i} = \mathbf{A}_{B_iI}\ \dot{\mathbf{A}}_{IB_i}, \tag{2.47}
$$

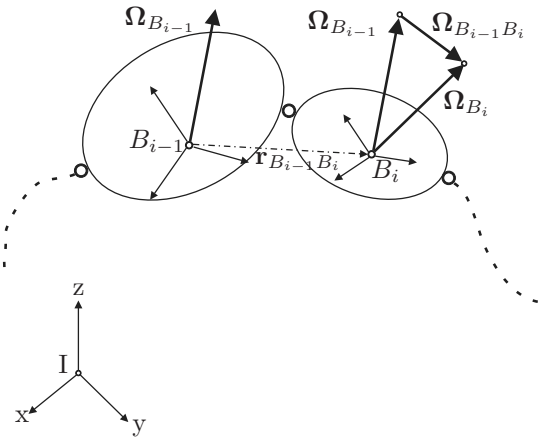

Fig. 2.15: Angular velocities of two successive bodies

which is the angular velocity between I and B_i defined in the B_i-coordinate system. With the equations 2.46 and 2.47 we write

$$\begin{aligned}
{}_{B_i}\tilde{\boldsymbol{\omega}}_{IB_i} &= \mathbf{A}_{B_iB_{i-1}} \ (\mathbf{A}_{B_{i-1}I} \ \dot{\mathbf{A}}_{IB_{i-1}}) \ \mathbf{A}_{B_{i-1}B_i} + (\mathbf{A}_{B_iB_{i-1}} \ \dot{\mathbf{A}}_{B_{i-1}B_i}) \\
&= \mathbf{A}_{B_iB_{i-1}} \ ({}_{B_{i-1}}\tilde{\boldsymbol{\omega}}_{IB_{i-1}}) \ \mathbf{A}_{B_{i-1}B_i} + {}_{B_i}\tilde{\boldsymbol{\omega}}_{B_{i-1}Bi} \\
&= \mathbf{T}_i^T \ ({}_{B_{i-1}}\tilde{\boldsymbol{\omega}}_{IB_{i-1}}) \ \mathbf{T}_i + \mathbf{T}_i^T \ \dot{\mathbf{T}}_i
\end{aligned} \tag{2.48}$$

The last equation is a recursive relation for going from the body (i-1) to body (i), which can be also written in the more convenient form

$${}_{B_i}\boldsymbol{\omega}_{IB_i} = \mathbf{A}_{B_iB_{i-1}} \ ({}_{B_{i-1}}\boldsymbol{\omega}_{IB_{i-1}}) + {}_{B_i}\boldsymbol{\omega}_{B_{i-1}B_i} \tag{2.49}$$

In a similar way we can derive such a recurrence relation for the vectors between the bodies. We start with figure 2.16 and consider the coordinate-free relation

$$\mathbf{r}_{IB_i} = \mathbf{r}_{IB_{i-1}} + \mathbf{r}_{B_{i-1}B_i}, \tag{2.50}$$

which we might define in any coordinate frame according to the equations 2.7. For a description in an I-system we get

$$\begin{aligned}
{}_I\mathbf{r}_{IB_i} &= {}_I\mathbf{r}_{IB_{i-1}} + {}_I\mathbf{r}_{B_{i-1}B_i} \\
&= {}_I\mathbf{r}_{IB_{i-1}} + \mathbf{A}_{IB_i} \ {}_{B_i}\mathbf{r}_{B_{i-1}B_i}
\end{aligned} \tag{2.51}$$

Differentiating this expression with respect to time and transforming the result into a body-fixed frame B_i yields after some manipulations the absolute velocities in the form (I- and B_i-system)

$$\begin{aligned}
{}_I\dot{\mathbf{r}}_{IB_i} &= {}_I\dot{\mathbf{r}}_{IB_{i-1}} + \mathbf{A}_{IB_i} \ ({}_{B_i}\tilde{\boldsymbol{\omega}}_{IB_i} \ {}_{B_i}\mathbf{r}_{B_{i-1}B_i} + {}_{B_i}\dot{\mathbf{r}}_{B_{i-1}B_i}), \\
{}_{B_i}\dot{\mathbf{r}}_{IB_i} &= \mathbf{A}_{B_iB_{i-1}}[({}_{B_{i-1}}\dot{\mathbf{r}}_{IB_{i-1}} + {}_{B_{i-1}}\dot{\mathbf{r}}_{B_{i-1}B_i}) + \\
&\qquad + ({}_{B_{i-1}}\tilde{\boldsymbol{\omega}}_{IB_{i-1}} + {}_{B_{i-1}}\tilde{\boldsymbol{\omega}}_{B_{i-1}B_i}) \ {}_{B_{i-1}}\mathbf{r}_{B_{i-1}B_i}],
\end{aligned} \tag{2.52}$$

Equation 2.52 is in combination with equation 2.48 also a recurrence relation, which can be used to build up the vector equations for a chain of bodies.

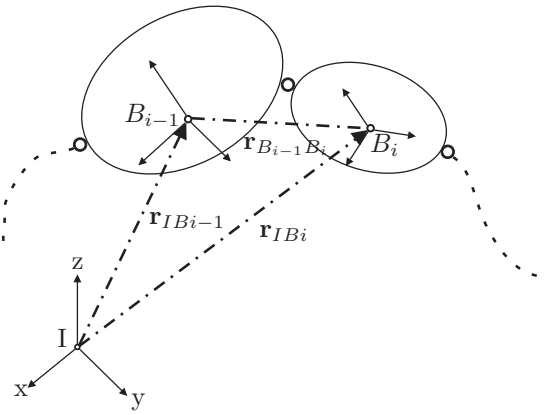

Fig. 2.16: Vectors of two successive bodies

For the equations of motion we need the accelerations. Considering a chain of bodies (Figure 2.16) we define for the body (i)

$$\begin{aligned} p(i) &= i-1 \quad \Longrightarrow \quad \text{predecessor body,} \\ s(i) &= i+1 \quad \Longrightarrow \quad \text{successor body,} \\ i &= 0 \quad \Longrightarrow \quad \text{inertial system (base body),} \\ i &= n \quad \Longrightarrow \quad \text{end body.} \end{aligned}$$

An alternative form of equation (2.51) is given by

$$ {}_{B_i}\mathbf{r}_{IB_i} = \sum_{j=1}^{i} \mathbf{A}_{B_iB_j} ({}_{B_j}\mathbf{r}_{B_{j-1}B_j}), \tag{2.53}$$

where we compose the radius vector ${}_{B_i}\mathbf{r}_{IB_i}$ by the incremental vectors between the bodies, for example from (j-1) to (j). These inremental vectors $({}_{B_j}\mathbf{r}_{B_{j-1}B_j})$ are defined in the coordinate frame B_j and must consequently transformed to the system B_i. In the same way we can determine the rotational velocity by (see equation (2.49) and Figure (2.15))

$$ {}_{B_i}\boldsymbol{\omega}_{IB_i} = \sum_{j=1}^{i} \mathbf{A}_{B_iB_j} ({}_{B_j}\boldsymbol{\omega}_{B_{j-1}B_j}). \tag{2.54}$$

The corresponding translational velocity can be derived by a standard process: Transform equation (2.53) into the inertial system by the matrix $\mathbf{A}_{IB_i}$, differentiate quite formally the resulting equation with respect to time and finally transform the result back to the body-fixed coordinates B_i. We get

$$ {}_{B_i}\dot{\mathbf{r}}_{IB_i} = \sum_{j=1}^{i} \mathbf{A}_{B_iB_j}[({}_{B_j}\dot{\mathbf{r}}_{B_{j-1}B_j}) + {}_{B_j}\tilde{\boldsymbol{\omega}}_{IB_j} \cdot ({}_{B_j}\mathbf{r}_{B_{j-1}B_j})]. \tag{2.55} $$

Before going to the accelerations we recall the indexing. An index on the left side of a magnitude indicates the coordinate system, in which the components of this magnitude are defined, for example the coordinates in B_i for the radius vector $\dot{\mathbf{r}}$ in the equation above. The right side indices indicate positions or orientations (from - to) or (between), for example and again in the above equation $\dot{\mathbf{r}}_{IB_i}$ means the velocity between the inertial system I and the body-fixed system B_i, or ω_{IB_j} means the angular velocity between the inertial system I and the body-fixed frame B_j. On the other side we read the transformations from the left side to the right one. For example $\mathbf{A}_{B_iB_j}$ means a transformation from the coordinates B_j into the coordinates B_i, which makes sense, because the second right index of these transformation matrices must match with the left index of those magnitudes to be transformed, see again the above equation.

The acceleration will be achieved by the same time-derivation process as discussed above. For the translational and rotational accelerations we come out with

$$ \begin{aligned} {}_{B_i}\ddot{\mathbf{r}}_{IB_i} = \sum_{j=1}^{i} \mathbf{A}_{B_iB_j}[&({}_{B_j}\ddot{\mathbf{r}}_{B_{j-1}B_j}) + {}_{B_j}\dot{\tilde{\boldsymbol{\omega}}}_{IB_j} \ ({}_{B_j}\mathbf{r}_{B_{j-1}B_j}) + \\ &+ {}_{B_j}\tilde{\boldsymbol{\omega}}_{IB_j} \ {}_{B_j}\tilde{\boldsymbol{\omega}}_{IB_j} \ ({}_{B_j}\mathbf{r}_{B_{j-1}B_j}) + \\ &+ 2{}_{B_j}\tilde{\boldsymbol{\omega}}_{IB_j} \cdot ({}_{B_j}\dot{\mathbf{r}}_{B_{j-1}B_j})] \end{aligned} \tag{2.56} $$

$$ {}_{B_i}\dot{\boldsymbol{\omega}}_{IB_i} = \sum_{j=1}^{i} \mathbf{A}_{B_iB_j}[({}_{B_j}\dot{\boldsymbol{\omega}}_{B_{j-1}B_j}) + {}_{B_j}\tilde{\boldsymbol{\omega}}_{IB_j} \ ({}_{B_j}\boldsymbol{\omega}_{B_{j-1}B_j})]. \tag{2.57} $$

Formally we may replace the summation of these formulas by a vector-matrix notation, which writes

$$ {}_{B_i}\ddot{\mathbf{r}}_{IB_i} = \mathbf{A}_i(\Delta\ddot{\mathbf{r}}_i + \mathrm{diag}(\Delta\tilde{\mathbf{r}}_i^T)\dot{\omega}_i) + \mathbf{a}_{Ti}, \qquad {}_{B_i}\dot{\omega}_{IB_i} = \mathbf{A}_i\Delta\dot{\omega}_i + \mathbf{a}_{Ri}, \tag{2.58} $$

$$ \begin{aligned} \Delta\mathbf{r}_i &= ({}_1\mathbf{r}_{01}{}^T, {}_2\mathbf{r}_{12}{}^T, {}_3\mathbf{r}_{23}{}^T, \cdots {}_i\mathbf{r}_{i-1,i}{}^T)^T = (\ldots [{}_{B_j}\mathbf{r}_{B_{j-1}B_j}]^T, \ldots)^T, \\ \Delta\omega_i &= ({}_1\omega_{01}{}^T, {}_2\omega_{12}{}^T, {}_3\omega_{23}{}^T, \cdots {}_i\omega_{i-1,i}{}^T)^T = (\ldots [{}_{B_j}\omega_{B_{j-1}B_j}]^T, \ldots)^T, \\ \omega_i &= ({}_1\omega_{01}{}^T, {}_2\omega_{02}{}^T, {}_3\omega_{03}{}^T, \cdots {}_i\omega_{0i}{}^T)^T = (\ldots [{}_{B_j}\omega_{IB_j}]^T, \ldots)^T, \\ \mathbf{a}_{Ti} &= (\mathbf{a}_{T1}^T, \mathbf{a}_{T2}^T, \mathbf{a}_{T3}^T, \ldots \mathbf{a}_{Ti}^T)^T, \\ \mathbf{a}_{Ri} &= (\mathbf{a}_{R1}^T, \mathbf{a}_{R2}^T, \mathbf{a}_{R3}^T, \ldots \mathbf{a}_{Ri}^T)^T, \\ \mathbf{a}_{Tj} &= \mathbf{A}_{B_iB_j}[{}_{B_j}\tilde{\omega}_{IB_j} \cdot {}_{B_j}\tilde{\omega}_{IB_j} \cdot ({}_{B_j}\mathbf{r}_{B_{j-1}B_j}) + 2\,{}_{B_j}\tilde{\omega}_{IB_j} \cdot ({}_{B_j}\dot{\mathbf{r}}_{B_{j-1}B_j})], \\ \mathbf{a}_{Rj} &= \mathbf{A}_{B_iB_j}[{}_{B_j}\tilde{\omega}_{IB_j} \cdot ({}_{B_j}\omega_{B_{j-1}B_j})], \end{aligned} \tag{2.59} $$

$$ \mathbf{A}_i = \begin{pmatrix} \mathbf{E} & \mathbf{0} & \ldots & \ldots \\ \mathbf{A}_{21} & \mathbf{E} & \mathbf{0} & \ldots \\ \mathbf{A}_{31} & \mathbf{A}_{32} & \mathbf{E} & \mathbf{0} \ \ldots \\ \vdots & \vdots & \vdots & \ddots \\ \mathbf{A}_{i,1} & \mathbf{A}_{i,2} & \mathbf{A}_{i,3} & \ldots \ \mathbf{E} \end{pmatrix}. \tag{2.60} $$

We can combine the above relations into one equation of the form

$$ \ddot{\mathbf{z}}_i = \bar{\mathbf{A}}_i\Delta\ddot{\mathbf{z}}_i + \mathbf{a}_i, \tag{2.61} $$

$$ \begin{aligned} \mathbf{z}_i &= (\mathbf{z}_1^T, \mathbf{z}_2^T, \mathbf{z}_3^T, \ldots \mathbf{z}_i^T)^T, \qquad \Delta\mathbf{z}_i = (\Delta\mathbf{z}_1^T, \Delta\mathbf{z}_2^T, \Delta\mathbf{z}_3^T, \ldots \Delta\mathbf{z}_i^T)^T, \\ \mathbf{a}_i &= (\mathbf{a}_1^T, \mathbf{a}_2^T, \mathbf{a}_3^T, \ldots \mathbf{a}_i^T)^T, \\ \bar{\mathbf{A}}_i &= \begin{pmatrix} \mathbf{A}_i & [\mathbf{A}_i(\Delta\tilde{\mathbf{r}}_i^T)\mathbf{A}_i] \\ \mathbf{0} & \mathbf{A}_i \end{pmatrix}, \qquad \ddot{\mathbf{z}}_j = \begin{pmatrix} {}_{B_j}\ddot{\mathbf{r}}_{IB_j} \\ {}_{B_j}\dot{\omega}_{IB_j} \end{pmatrix}_{j=1,2,\ldots i}, \\ \Delta\ddot{\mathbf{z}}_j &= \begin{pmatrix} {}_{B_j}\Delta\ddot{\mathbf{r}}_{B_{j-1}B_j} \\ {}_{B_j}\Delta\dot{\omega}_{B_{j-1}B_j} \end{pmatrix}_{j=1,2,\ldots i}, \qquad \mathbf{a}_j = \begin{pmatrix} \mathbf{a}_{Tj} \\ \mathbf{a}_{Rj} \end{pmatrix}_{j=1,2,\ldots i}. \end{aligned} \tag{2.62} $$

The matrices $\mathbf{A}_i$ and $\bar{\mathbf{A}}_i$ are both triangular matrices, which offers the possibilty to solve the equations (2.58) or (2.61) in an iterative way starting with the matrix row containing one element only and proceeding step by step until the equation has been solved. This represents a second type of recursion. We shall come back to it.

2.2.5 Kinematics of Systems

We go back to chapter 2.2.1 and recall the definitions of coordinates given there. We consider system coordinates $\mathbf{z} \in \mathbb{R}^{6n}$ or $\mathbf{r} \in \mathbb{R}^{6n}$, which in a constrained system do not correspond to the degrees of freedom but are more a representation of the system design and configuration. If we are able to eliminate all m constraints (for example equation 2.3), we come out with a set of

generalized coordinates $\mathbf{q} \in \mathbb{R}^f$ with f = 6n - m. If we cannot fulfill all but only a few of the given constraints we remain with a rest of these constraints representing together with the differential equations a set of differential-algebraic equations for the coordinates $\mathbf{q}_{min} \in \mathbb{R}^{f_{min}}$, which is the minimum achievable set of generalized coordinates for the system under consideration.

Anyway, the system coordinates $\mathbf{z}$ are functions of the generalized coordinates $\mathbf{q}$, namely $\mathbf{z} = \mathbf{z}(\mathbf{q}, t)$, and therefore we have the time derivative

$$\dot{\mathbf{z}} = (\frac{\partial \mathbf{z}}{\partial \mathbf{q}}) \cdot \dot{\mathbf{q}} + \frac{\partial \mathbf{z}}{\partial t}, \qquad (\frac{\partial \mathbf{z}}{\partial \mathbf{q}}) \in \mathbb{R}^{6n,f}. \tag{2.63}$$

Depending on the above definitions we call the velocities $\dot{\mathbf{q}}$ the "generalized velocities". From the equation 2.63 we get a very useful and important relationship by differentiating it partially with respect to $\dot{\mathbf{q}}$

$$\frac{\partial \mathbf{z}}{\partial \mathbf{q}} = \frac{\partial \dot{\mathbf{z}}}{\partial \dot{\mathbf{q}}}, \tag{2.64}$$

where the unknown Jacobian $\frac{\partial \dot{\mathbf{z}}}{\partial \dot{\mathbf{q}}}$ can be evaluated from the constraint equations. Going one step further to the acceleration $\ddot{\mathbf{z}}$, we have to differentiate again equation (2.63) with respect to time and receive

$$\ddot{\mathbf{z}} = (\frac{\partial \mathbf{z}}{\partial \mathbf{q}}) \cdot \ddot{\mathbf{q}} + \frac{\partial}{\partial \mathbf{q}}[(\frac{\partial \mathbf{z}}{\partial \mathbf{q}})\dot{\mathbf{q}}]\dot{\mathbf{q}} + 2(\frac{\partial^2 \mathbf{z}}{\partial \mathbf{q} \partial t})\dot{\mathbf{q}} + (\frac{\partial^2 \mathbf{z}}{\partial^2 t}), \tag{2.65}$$

where in most cases of practical relevancy $\mathbf{z} = \mathbf{z}(\mathbf{q})$ and not $\mathbf{z} = \mathbf{z}(\mathbf{q}, t)$. For $\mathbf{z}$ not dependent on the time t the last two terms in equation (2.65) vanish.

Differentiating the constraint equation 2.3 with respect to time we get(see [27])

$$\dot{\mathbf{\Phi}} = (\frac{\partial \mathbf{\Phi}}{\partial \mathbf{z}}) \cdot \dot{\mathbf{z}} + \frac{\partial \mathbf{\Phi}}{\partial t} = \mathbf{0}, \qquad \{= (\frac{\partial \mathbf{\Phi}}{\partial \mathbf{z}})[(\frac{\partial \mathbf{z}}{\partial \mathbf{q}}) \cdot \dot{\mathbf{q}} + \frac{\partial \mathbf{z}}{\partial t}] + \frac{\partial \mathbf{\Phi}}{\partial t}\}, \tag{2.66}$$

which we again differentiate partially with respect to $\dot{\mathbf{q}}$ resulting in

$$(\frac{\partial \mathbf{\Phi}}{\partial \mathbf{z}}) \cdot (\frac{\partial \dot{\mathbf{z}}}{\partial \dot{\mathbf{q}}}) = \mathbf{0} \qquad \text{with} \qquad (\frac{\partial \mathbf{\Phi}}{\partial \mathbf{z}}) \in \mathbb{R}^{m,6n}, \qquad (\frac{\partial \dot{\mathbf{z}}}{\partial \dot{\mathbf{q}}}) \in \mathbb{R}^{6n,f}. \tag{2.67}$$

Equation 2.67 tells us, that the rows of the Jacobian $(\frac{\partial \mathbf{\Phi}}{\partial \mathbf{z}})$ are orthogonal to the columns of the Jacobian $(\frac{\partial \dot{\mathbf{z}}}{\partial \dot{\mathbf{q}}})$. The Jacobian $(\frac{\partial \mathbf{\Phi}}{\partial \mathbf{z}})$ is regular, if we define the constraints in an unambiguous way, which is always possible. In this case we can solve equation 2.67 in a form, which represents a linear relationship of the velocities $\dot{\mathbf{z}}$ and $\dot{\mathbf{q}}$ (see [27])

$$\dot{\mathbf{q}} = (\frac{\partial \dot{\mathbf{q}}}{\partial \dot{\mathbf{z}}}) \cdot \dot{\mathbf{z}} = \{[(\frac{\partial \dot{\mathbf{z}}}{\partial \dot{\mathbf{q}}})^T (\frac{\partial \dot{\mathbf{z}}}{\partial \dot{\mathbf{q}}})]^{-1} (\frac{\partial \dot{\mathbf{z}}}{\partial \dot{\mathbf{q}}})^T\} \cdot \dot{\mathbf{z}} = (\frac{\partial \dot{\mathbf{z}}}{\partial \dot{\mathbf{q}}})^+ \cdot \dot{\mathbf{z}}, \tag{2.68}$$

where the term $(\frac{\partial \dot{\mathbf{z}}}{\partial \dot{\mathbf{q}}})^+$ represents the pseudo-inverse. The equations (2.63), (2.64) and (2.67) are essential concerning multibody theory. We remind of the relation (2.34) and shall come back to them in a later stage.

One of the key points in considering systems consists in the fact, that according to equation 2.63 we always have a linear relationship of the two velocities $\dot{\mathbf{z}}$ and $\dot{\mathbf{q}}$ of the form

$$\dot{\mathbf{z}} = \mathbf{J}(\mathbf{q}, t) \cdot \dot{\mathbf{q}} + \mathbf{j}(\mathbf{q}, t) \qquad \text{with} \qquad \mathbf{J}(\mathbf{q}, t) = \frac{\partial \mathbf{z}}{\partial \mathbf{q}} = \frac{\partial \dot{\mathbf{z}}}{\partial \dot{\mathbf{q}}} \in \mathbb{R}^{6n,f}. \tag{2.69}$$

The term $\mathbf{j}(\mathbf{q}, t)$ comes from external excitation sources. Depending on the cuts in our system it is given by the input or output loads with respect to these cuts (chapter 2.1.2).

The linear relationship of the two velocities $\dot{\mathbf{z}}$ and $\dot{\mathbf{q}}$ also holds for the constraints independent of their type. According to equation 2.66 we have

$$\mathbf{W}^T(\mathbf{z}, t) \;\; \dot{\mathbf{z}} + \mathbf{w}(\mathbf{z}, t) = \mathbf{0} \quad \text{with} \quad \mathbf{W}^T(\mathbf{z}, t) = \frac{\partial \mathbf{\Phi}}{\partial \mathbf{z}} = \frac{\partial \dot{\mathbf{\Phi}}}{\partial \dot{\mathbf{z}}}, \quad \mathbf{w}(\mathbf{z}, t) = \frac{\partial \mathbf{\Phi}}{\partial t}. \tag{2.70}$$

It is necessary to discuss a bit more in detail the above statement concerning the linear relation of velocities in the constraints. Taking into account first holonomic constraints of the form $\mathbf{\Phi}(\mathbf{z}, t) = \mathbf{0} \in \mathbb{R}^m$ (eq. 2.3), which usually represent a set of nonlinear algebraic functions of $(\mathbf{z}(t), t)$, we might differentiate this set as performed in equation (2.70) and receive linear equations in $\dot{\mathbf{z}}$. They hold in an exact way for all holonomic constraints.

But if we have non-holonomic constraints, which are really non-holonomic and not reducible to a position and orientation level, the corresponding constraint equations are usually given in the form

$$\mathbf{\Phi}[\mathbf{z}(t), \dot{\mathbf{z}}(t), t] = \mathbf{0} \quad \in \mathbb{R}^m, \tag{2.71}$$

which again are nonlinear equations in the arguments $(\mathbf{z}(t), \dot{\mathbf{z}}(t), t)$. It means at least formally that the linear structure of the equations (2.70) does not apply to these non-holonomic constraints. But, on the other hand, practical experience in all areas of technology indicates, that there are no non-holonomic constraints being nonlinear in the velocities $\dot{\mathbf{z}}$. Therefore the linear equations (2.70) possess the quality of an axiom in a physical, in a mechanical sense, at least approximately and as long as there will come no contradiction. The matrix $\mathbf{W}(\mathbf{z}, t)$ will be of course completely different for the two constraint types, a Jacobian derived above for the holonomic case, and a Jacobian given by the system configuration for the non-holonomic case.

2.2.6 Parameterized Coordinates

The upcoming fields of robotics and of contact dynamics are accompanied by more frequent applications of curvilinear coordinate systems either for robot trajectories or for contact surfaces ([208],[152]). Let us first discuss the idea to project a mechanism with many degrees of freedom like a manipulator

or a gear-mechanism onto one suitable degree of freedom connected with a prescribed trajectory or the like. Considering figure 2.17 and borrowing from the theory of spatial curves (see [51]) the most important relationships we introduce the path coordinate s = s(t) as a parameter and write

$$\mathbf{r}(t) = \mathbf{r}[s(t)] \qquad \text{with} \qquad \mathbf{v}(t) = \dot{\mathbf{r}}(t) = \frac{\mathrm{d}\,\mathbf{r}}{\mathrm{d}\,s} \cdot \frac{\mathrm{d}\,s}{\mathrm{d}\,t} = \mathbf{r}' \cdot \dot{s}. \tag{2.72}$$

The derivation $\dot{\mathbf{r}}$ of the radius vector $\mathbf{r}(t)$ with respect to time divided by the path velocity $\dot{s}$ represents the tangent unit vector $\mathbf{t}$ to the point trajectory:

$$\mathbf{t}(t) = \frac{\dot{\mathbf{r}}}{\dot{s}} = \mathbf{r}' = \frac{\mathrm{d}\,\mathbf{r}}{\mathrm{d}\,s}. \tag{2.73}$$

From the velocity vector of equation 2.72 we derive the acceleration vector by a differentiation with respect to time and get

$$\mathbf{a}(t) = \ddot{\mathbf{r}}(t) = \frac{\mathrm{d}\,(\mathbf{r}' \cdot \dot{s})}{\mathrm{d}\,t} = \frac{\mathrm{d}\,(\mathbf{r}' \cdot \dot{s})}{\mathrm{d}\,s} \cdot \frac{\mathrm{d}\,s}{\mathrm{d}\,t} = \mathbf{r}'' \cdot \dot{s}^2 + \mathbf{r}' \cdot \ddot{s}, \tag{2.74}$$

the first part of this equation being the centripetal and the second part the tangential acceleration. The vector $\mathbf{r}''$ corresponds to the derivation of equation 2.73 and represents thus the derivation of the tangent vector to a certain point of the path, $\mathbf{t}'(t) = \mathbf{r}''$. As $\mathbf{t}(t)$ is a unit vector, its derivative $\mathbf{t}'(t)$ is perpendicular to it. It should be noted that this parameterization of spatial curves must not be carried out by using the time t. Instead we may take any general parameter u, which of course must be selected according to the requirements coming from the dynamical system.

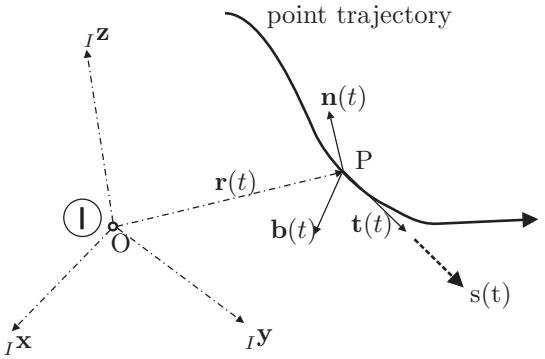

Fig. 2.17: Moving trihedron

For certain applications it makes sense to supplement the path coordinate s(t) and its corresponding unit vector $\mathbf{t}(t)$ by the normal and binormal vectors resulting in an orthogonal trihedron connected with the spatial path

curve (see figure 2.17). As pointed out above the principal normal vector $\mathbf{n}(t)$ is perpendicular to the tangent vector $\mathbf{t}(t)$, these two vectors forming the osculating plane for the spatial curve at the point P, and the bi-normal vector is perpendicular to this plane, which means $\mathbf{b}(t) = [\mathbf{t}(t)] \times [\mathbf{n}(t)]$. These properties can be used to evaluate the well-known Frenet-equations, which are a basis for all applications connected with spatial trajectories. They write [51]

$$\begin{aligned}
\frac{\mathrm{d}\,\mathbf{t}}{\mathrm{d}\,s} &= +\kappa \cdot \mathbf{n}, \\
\frac{\mathrm{d}\,\mathbf{n}}{\mathrm{d}\,s} &= -\kappa \cdot \mathbf{t} + \tau \cdot \mathbf{b}, \\
\frac{\mathrm{d}\,\mathbf{b}}{\mathrm{d}\,s} &= -\tau \cdot \mathbf{b}.
\end{aligned} \tag{2.75}$$

The magnitudes κ and τ are the curvature and the torsion of the spatial trajectory, respectively. It might be helpful to repeat the behavior of the moving trihedron (see [132]). The tangent rotates about the instantaneous binormal direction at the positive angular rate κ (curvature at point P). The binormal rotates about the instantaneous tangent direction at the angular rate τ (torsion at point P). The entire moving trihedron rotates about the instantaneous direction of the Darboux vector $\mathbf{\Omega} = \tau\mathbf{t} + \kappa\mathbf{b}$ at the positive angular rate $\|\mathbf{\Omega}\| = \sqrt{\tau^2 + \kappa^2}$ (total curvature at point P).

For certain robot- or mechanism-problems the advantages are obvious. A system with many degrees of freedom can be projected onto the coordinates of a given path coming out with three degrees of freedom or if we do not consider disturbances around such a path, only with one degree of freedom. We shall discuss an example in one of the applications chapters.

Another and a more demanding type of parameterization is connected with contact problems, where the contacts themselves take place between the surfaces of bodies. Especially if two bodies are approaching it is convenient to express the corresponding kinematics not in minimum but in surface coordinates, which requires some differential geometric transformations for going from the orthogonal world coordinates to the curvilinear surface coordinates (see for example [51], [152], [200] and Figure 2.6). We start with Figure 2.6 and equation 2.4, where a radius vector $\mathbf{r}$ depends on the curvilinear surface coordinates (u,v) by $\mathbf{r}(u,v) = [x(u,v), y(u,v), z(u,v)]^T$. In many cases it is possible to choose the surface coordinates in such a way, that they are mutually orthogonal, which brings some advantages for the evaluation. A partial derivation with respect to these curved coordinates comes out with the tangential vectors in the surface point under consideration.

$$\mathbf{e}_1 = \mathbf{u} = \frac{\partial \mathbf{r}}{\partial u}, \qquad \mathbf{e}_2 = \mathbf{v} = \frac{\partial \mathbf{r}}{\partial v}. \tag{2.76}$$

The vectors $\mathbf{u}, \mathbf{v}$ span the tangential plane in the point P. The normal unit vector follows from the vector product of $\mathbf{u}$ and $\mathbf{v}$ in the form

$$\mathbf{n} = \frac{\mathbf{u} \times \mathbf{v}}{\|\mathbf{u} \times \mathbf{v}\|}. \tag{2.77}$$

With these vectors,two tangential and one normal, we are able to evaluate the elements of the first fundamental form of a surface, which writes:

$$E = \mathbf{u}^T \cdot \mathbf{u}, \qquad F = \mathbf{u}^T \cdot \mathbf{v}, \qquad G = \mathbf{v}^T \cdot \mathbf{v}. \tag{2.78}$$

If the parameter coordinates u and v are perpendicular to each other, then we get F = 0 for $\mathbf{u} \perp \mathbf{v}$. To determine the elements of the second fundamental form of a surface we need the second derivatives of the vector $\mathbf{r}(u, v)$. They characterize the curvature and torsion properties of the surface. We define

$$L = \mathbf{n}^T \cdot \frac{\partial^2 \mathbf{r}}{\partial u^2}, \qquad M = \mathbf{n}^T \cdot \frac{\partial^2 \mathbf{r}}{\partial u \partial v}, \qquad N = \mathbf{n}^T \cdot \frac{\partial^2 \mathbf{r}}{\partial v^2}. \tag{2.79}$$

Applying the equations 2.76 we get an equivalent formulation

$$L = \mathbf{n}^T \cdot \frac{\partial \mathbf{u}}{\partial u}, \qquad M = \mathbf{n}^T \cdot \frac{\partial \mathbf{u}}{\partial v} = \mathbf{n}^T \cdot \frac{\partial \mathbf{v}}{\partial u}, \qquad N = \mathbf{n}^T \cdot \frac{\partial \mathbf{v}}{\partial v}. \tag{2.80}$$

Again, for parameter coordinates (u,v) being mutually orthogonal we have M = 0 for $\mathbf{u} \perp \mathbf{v}$. Establishing a general theory of contacts requires additionally some further second derivatives and the derivation of the normal vector as defined in equation 2.77, which will be achieved by the formulas of Weingarten and Gauss in the form (see [51],[132],[152])

$$\begin{aligned} \frac{\partial \mathbf{n}}{\partial u^\alpha} &= -g^{\sigma\gamma} \cdot b_{\gamma\alpha} \cdot \mathbf{e}_\sigma && \text{(Weingarten)}, \\ \frac{\partial^2 \mathbf{r}}{\partial u^\alpha u^\beta} &= \frac{\partial \mathbf{e}_\alpha}{\partial u^\beta} = \Gamma^\sigma_{\alpha\beta} \cdot \mathbf{e}_\sigma + b_{\alpha\beta} \cdot \mathbf{n} && \text{(Gauss)}, \end{aligned} \tag{2.81}$$

with the following notations: The magnitudes α, β, γ and σ take on the values 1,2 and follow the summation convention of Einstein. From this u^α means u = u^1 and v = u^2. The vector $\mathbf{e}_\alpha = \frac{\partial \mathbf{r}}{\partial u^\alpha}$ is defined in equation 2.76. The Christoffel three-index symbol $\Gamma^\sigma_{\alpha\beta} \cdot \mathbf{e}_\sigma$ writes

$$\Gamma^\sigma_{\alpha\beta} \cdot \mathbf{e}_\sigma = \frac{1}{2} g^{\sigma\delta} \left(\frac{\partial g_{\alpha\delta}}{\partial u^\beta} + \frac{\partial g_{\beta\delta}}{\partial u^\alpha} - \frac{\partial g_{\alpha\beta}}{\partial u^\delta} \right). \tag{2.82}$$

Its properties may be seen from [51] and [132]. The elements $g^{\sigma\delta}$ and $g_{\alpha\beta}$ are the fundamental tensor components of a Riemann space. Within the context of surfaces these measure tensors depend on the elements of the first and second fundamental form of a surface (equations 2.78 and 2.80). We get

$$\begin{aligned} &g_{11} = E, && g_{12} = g_{21} = F, && g_{22} = G, \\ &g^{11} = \frac{G}{EG - F^2}, && g^{12} = g^{21} = \frac{-F}{EG - F^2}, && g^{22} = \frac{E}{EG - F^2}, \\ &b_{11} = L, && b_{12} = b_{21} = M, && b_{22} = N. \end{aligned} \tag{2.83}$$

The evaluation of the equations 2.81, 2.82 and 2.83 is straightforward but very lengthy and tedious. It can be found in all details in [152]. As a final result we get

$$\begin{aligned}
\frac{\partial \mathbf{u}}{\partial u} &= \Gamma_{11}^1 \mathbf{u} + \Gamma_{11}^2 \mathbf{v} + L\mathbf{n}, \qquad \frac{\partial \mathbf{u}}{\partial v} = \Gamma_{12}^1 \mathbf{u} + \Gamma_{12}^2 \mathbf{v} + M\mathbf{n}, \\
\frac{\partial \mathbf{v}}{\partial u} &= \Gamma_{12}^1 \mathbf{u} + \Gamma_{12}^2 \mathbf{v} + M\mathbf{n}, \qquad \frac{\partial \mathbf{v}}{\partial v} = \Gamma_{22}^1 \mathbf{u} + \Gamma_{22}^2 \mathbf{v} + N\mathbf{n}, \\
\frac{\partial \mathbf{n}}{\partial u} &= \frac{FM - GL}{EG - F^2}\mathbf{u} + \frac{FL - EM}{EG - F^2}\mathbf{v}, \\
\frac{\partial \mathbf{n}}{\partial v} &= \frac{FN - GM}{EG - F^2}\mathbf{u} + \frac{FM - EN}{EG - F^2}\mathbf{v}.
\end{aligned} \tag{2.84}$$

Obviously $\frac{\partial \mathbf{u}}{\partial v} = \frac{\partial \mathbf{v}}{\partial u}$. The Christoffel symbols can be evaluated in the form [152]

$$\begin{aligned}
\Gamma_{11}^1 &= \frac{1}{2(EG - F^2)} \cdot \left(+G\frac{\partial E}{\partial u} - 2F\frac{\partial F}{\partial u} + F\frac{\partial E}{\partial v} \right), \\
\Gamma_{11}^2 &= \frac{1}{2(EG - F^2)} \cdot \left(-F\frac{\partial E}{\partial u} + 2E\frac{\partial F}{\partial u} - E\frac{\partial E}{\partial v} \right), \\
\Gamma_{12}^1 &= \frac{1}{2(EG - F^2)} \cdot \left(+G\frac{\partial E}{\partial v} - F\frac{\partial G}{\partial u} \right), \\
\Gamma_{12}^2 &= \frac{1}{2(EG - F^2)} \cdot \left(-F\frac{\partial E}{\partial v} + E\frac{\partial G}{\partial u} \right), \\
\Gamma_{22}^1 &= \frac{1}{2(EG - F^2)} \cdot \left(-G\frac{\partial G}{\partial u} - 2G\frac{\partial F}{\partial v} - F\frac{\partial G}{\partial v} \right), \\
\Gamma_{22}^2 &= \frac{1}{2(EG - F^2)} \cdot \left(+F\frac{\partial G}{\partial u} - 2F\frac{\partial F}{\partial v} + E\frac{\partial G}{\partial v} \right).
\end{aligned} \tag{2.85}$$

The above equations represent a general set for the description of arbitrary surfaces. They are also a necessary set for the evaluation of the relative kinematics of contacts between arbitrary bodies, where especially the moving trihedron with its rectangular axes of the principal normal, the tangent vector and the binormal is an indispensable requirement for analyzing contact dynamics. The rather costly evaluation for the general case can of course be reduced considerably for special geometries like bodies with rotational symmetry or with at least partly plane surfaces. The combination of bodies with various shapes like a cube on a sphere must be analyzed by considering all possible cases of contact. This is for example a typical problem of assembly processes (see [271], [152]).

As an example we consider bodies with rotational symmetry, which are used quite often in various technologies. Figure 2.18 illustrates the principal situation. The coordinate u is the circumferential coordinate with $u \in (0, 2\Pi)$, and v is the generatrix coordinate. The radius of the body at the position v is f(v). With these definitions we get $((\cdot)' = \frac{\mathrm{d}(\cdot)}{\mathrm{d}v})$

$$\mathbf{r} = \begin{pmatrix} f(v)\cdot\cos u \\ f(v)\cdot\sin u \\ g(v) \end{pmatrix}, \quad \mathbf{u} = \begin{pmatrix} -f(v)\cdot\sin u \\ f(v)\cdot\cos u \\ 0 \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} f'(v)\cdot\cos u \\ f'(v)\cdot\sin u \\ g'(v) \end{pmatrix}. \tag{2.86}$$

The principal normal $\mathbf{n}$ and the elements E, F, G of the first order funda-

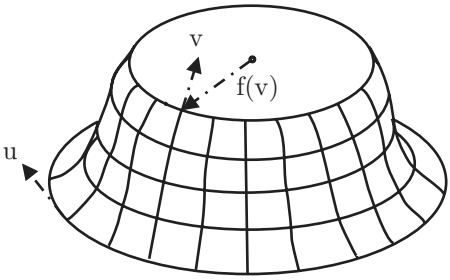

Fig. 2.18: Typical body with rotational symmetry

mental form are then derived by

$$\mathbf{n} = \frac{1}{\sqrt{f'^2(v)+g'^2(v)}}\cdot\begin{pmatrix} g'(v)\cdot\cos u \\ g'(v)\cdot\sin u \\ f'(v) \end{pmatrix}, \qquad \begin{aligned} E &= f^2(v), \\ F &= 0, \\ G &= f'^2(v)+g'^2(v). \end{aligned} \tag{2.87}$$

The coordinates $\mathbf{u}$ and $\mathbf{v}$ are perpendicular and therefore F=0 and M=0. The elements of the second fundamental form write

$$L = -\frac{f(v)\cdot g'(v)}{\sqrt{f'^2(v)+g'^2(v)}}, \qquad N = \frac{f''(v)\cdot g'(v) - f'(v)\cdot g''(v)}{\sqrt{f'^2(v)+g'^2(v)}}. \tag{2.88}$$

For the specific case of rotational symmetry the Christoffel elements Γ^1_{11}, Γ^1_{12} and Γ^1_{22} also become zero. A collection of different contact forms is given in [152] and in [277].

2.2.7 Relative Contact Kinematics

2.2.7.1 Plane Case

In order to derive the kinematic contact equations of a whole system we consider first the geometry of a single plane body. Figure 2.19 shows such a body, which may have the rotational velocity $\boldsymbol{\Omega}$ and the rotational acceleration $\dot{\boldsymbol{\Omega}}$. The body-fixed point P moves with a velocity $\mathbf{v}_P$ and has the acceleration $\mathbf{a}_P$. The smooth and planar contour Σ is supposed to be strictly convex and can be described in a parametric form by the vector ${}_B\dot{\mathbf{r}}_{P\Sigma}(s)$ in the body-fixed frame B. The parameter s corresponds to the arc length of the body's

contour. With these definitions and the formulas of the preceding chapter we can state the moving trihedral $(\mathbf{t}, \mathbf{n}, \mathbf{b})$ in a body-fixed coordinate system B ([86],[200] and [212]).

$$
\begin{aligned}
&{}_B\mathbf{t} = {}_B\mathbf{r}'_{P\Sigma}; \quad \kappa \cdot {}_B\mathbf{n} = {}_B\mathbf{r}''_{P\Sigma}; \quad (\cdot)' = \frac{\mathrm{d}}{\mathrm{d}\,s};\\
&{}_B\mathbf{n} = {}_B\mathbf{b} \times {}_B\mathbf{t}; \quad {}_B\mathbf{b} = {}_B\mathbf{t} \times {}_B\mathbf{n}; \quad {}_B\mathbf{t} = {}_B\mathbf{n} \times {}_B\mathbf{b}.\\
&{}_B\mathbf{n}' = {}_B\mathbf{b} \times {}_B\mathbf{t}' = {}_B\mathbf{b} \times (\kappa \cdot {}_B\mathbf{n}) = -\kappa \cdot {}_B\mathbf{t}; \qquad {}_B\mathbf{t}' = \kappa \cdot {}_B\mathbf{n}.
\end{aligned} \tag{2.89}
$$

where κ denotes the curvature of our convex contour at point s, and the normal vector $\mathbf{n}$ is pointing inward. The binormal ${}_B\mathbf{b}$ is constant for the plane case and thus independent of s. Now we imagine a point Σ moving along the contour with the velocity $\dot{s}$. As a consequence the normal and the tangent vectors change their direction with respect to the body-fixed frame B. We describe this effect by differentiating the tangential and the normal vector with respect to time with $\frac{\mathrm{d}}{\mathrm{d}\,t} = \frac{\mathrm{d}}{\mathrm{d}\,s} \cdot \frac{\mathrm{d}\,s}{\mathrm{d}\,t} = \frac{\mathrm{d}}{\mathrm{d}\,s} \cdot \dot{s}$ we come out with

$$
{}_B\dot{\mathbf{n}} = {}_B\mathbf{n}' \cdot \dot{s} = -\kappa\dot{s} \cdot {}_B\mathbf{t}, \qquad {}_B\dot{\mathbf{t}} = {}_B\mathbf{t}' \cdot \dot{s} = +\kappa\dot{s} \cdot {}_B\mathbf{n}. \tag{2.90}
$$

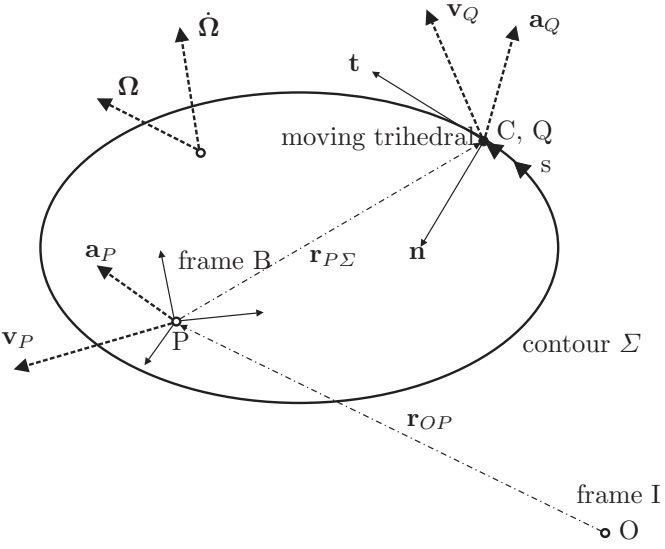

Fig. 2.19: Planar Contour Geometry

On the other hand, the absolute changes of the vectors ${}_B\mathbf{n}$ and ${}_B\mathbf{t}$ with respect to time follow from the Coriolis equations 2.38 with ${}_B\dot{\mathbf{r}} = 0$

$$
{}_B(\dot{\mathbf{n}}) = {}_B\dot{\mathbf{n}} + {}_B\tilde{\boldsymbol{\Omega}} \cdot {}_B\mathbf{n}, \qquad {}_B(\dot{\mathbf{t}}) = {}_B\dot{\mathbf{t}} + {}_B\tilde{\boldsymbol{\Omega}} \cdot {}_B\mathbf{t}, \tag{2.91}
$$

where we have to keep in mind that ${}_B\boldsymbol{\omega}_{IB} = {}_B\boldsymbol{\Omega}$ for a body-fixed coordinate system expressing the property of rotation of the body-fixed frame with respect to the inertial frame. Combining the equations 2.90 and 2.91 we come out with a coordinate-free representation of the overall changes of the contour normal and tangential vectors

$$\dot{\mathbf{n}} = \tilde{\boldsymbol{\Omega}}\mathbf{n} - \kappa\dot{s}\cdot\mathbf{t}, \qquad \dot{\mathbf{t}} = \tilde{\boldsymbol{\Omega}}\mathbf{t} + \kappa\dot{s}\cdot\mathbf{n}, \tag{2.92}$$

which we might evaluate in any basis. From this the main advantage of equation 2.92 consists in this coordinate-free form and in avoiding the determination of the frame-dependent time-derivatives ${}_B\dot{\mathbf{n}}$ and ${}_B\dot{\mathbf{t}}$. In the same way we proceed with the contour vector $\mathbf{r}_{P\Sigma}$. Following the equations 2.90 and 2.91 we write

$${}_B\dot{\mathbf{r}}_{P\Sigma} = {}_B\mathbf{r}'_{P\Sigma}\cdot\dot{s} = \dot{s}\cdot{}_B\mathbf{t}, \qquad {}_B(\dot{\mathbf{r}}_{P\Sigma}) = {}_B\dot{\mathbf{r}}_{P\Sigma} + {}_B\tilde{\boldsymbol{\Omega}}\cdot{}_B\mathbf{r}_{P\Sigma}, \tag{2.93}$$

and we eliminate ${}_B\dot{\mathbf{r}}_{P\Sigma}$ resulting in the absolute changes of $\mathbf{r}_{P\Sigma}$ with time

$$\dot{\mathbf{r}}_{P\Sigma} = \tilde{\boldsymbol{\Omega}}\cdot\mathbf{r}_{P\Sigma} + \dot{s}\mathbf{t}. \tag{2.94}$$

Due to Figure 2.19 we have $\mathbf{v}_\Sigma = \mathbf{v}_P + \dot{\mathbf{r}}_{P\Sigma}$, and therefore the absolute velocity of a moving contour point Σ is given by

$$\mathbf{v}_\Sigma = \mathbf{v}_P + \tilde{\boldsymbol{\Omega}}\cdot\mathbf{r}_{P\Sigma} + \dot{s}\mathbf{t} = \mathbf{v}_C + \dot{s}\mathbf{t}, \quad \text{with} \quad \mathbf{v}_C = \mathbf{v}_P + \tilde{\boldsymbol{\Omega}}\cdot\mathbf{r}_{P\Sigma}. \tag{2.95}$$

The velocity $\mathbf{v}_C$ results from rigid body kinematics and corresponds to the velocity of a body-fixed point C at the contour, it is the applied velocity in a classical sense.

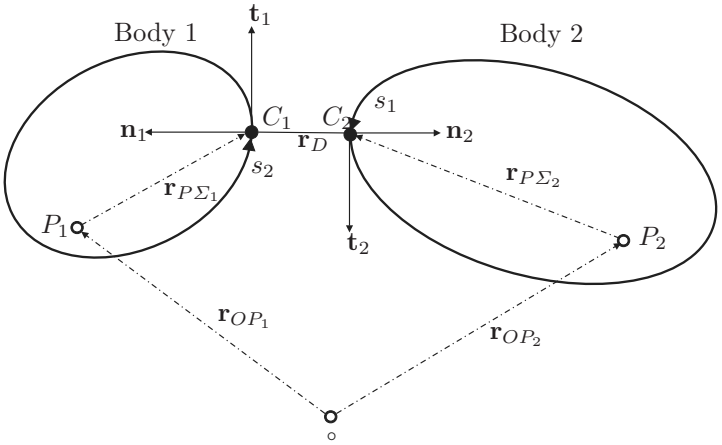

Fig. 2.20: Orientation of two Bodies

In a next step we must evaluate the absolute acceleration of the point C of the body contour. Differentiating equation 2.95 with respect to time we get

$$\dot{\mathbf{v}}_C = \dot{\mathbf{v}}_P + \dot{\tilde{\boldsymbol{\Omega}}} \cdot \mathbf{r}_{P\Sigma} + \tilde{\boldsymbol{\Omega}} \cdot \dot{\mathbf{r}}_{P\Sigma}. \tag{2.96}$$

With the abbreviations $\dot{\mathbf{v}}_C = \mathbf{a}_C$, $\dot{\mathbf{v}}_P = \mathbf{a}_P$ and with $\dot{\mathbf{r}}_{P\Sigma}$ from equation 2.93 we continue with the expressions

$$\begin{aligned}\mathbf{a}_C =& \mathbf{a}_P + \dot{\tilde{\boldsymbol{\Omega}}} \cdot \mathbf{r}_{P\Sigma} + \tilde{\boldsymbol{\Omega}}\tilde{\boldsymbol{\Omega}} \cdot \mathbf{r}_{P\Sigma} + \tilde{\boldsymbol{\Omega}}\mathbf{t}\dot{s} = \mathbf{a}_Q + \tilde{\boldsymbol{\Omega}}\mathbf{t}\dot{s} \qquad \text{with} \\ \mathbf{a}_Q =& \mathbf{a}_P + \dot{\tilde{\boldsymbol{\Omega}}} \cdot \mathbf{r}_{P\Sigma} + \tilde{\boldsymbol{\Omega}}\tilde{\boldsymbol{\Omega}} \cdot \mathbf{r}_{P\Sigma}\end{aligned} \tag{2.97}$$

The acceleration $\mathbf{a}_Q$ results from rigid body kinematics and corresponds to the acceleration of a body-fixed point Q, it is an applied acceleration in analogy to equation 2.95. For later evaluations we need the relative velocities and their time derivatives in normal and tangential directions. For that purpose we consider the corresponding velocities in the form

$$v_n = \mathbf{n}^T \mathbf{v}_C, \qquad v_t = \mathbf{t}^T \mathbf{v}_C, \tag{2.98}$$

with their time derivatives

$$\dot{v}_n = \dot{\mathbf{n}}^T \mathbf{v}_C + \mathbf{n}^T \dot{\mathbf{v}}_C, \qquad \dot{v}_t = \dot{\mathbf{t}}^T \mathbf{v}_C + \mathbf{t}^T \dot{\mathbf{v}}_C. \tag{2.99}$$

With $\dot{\mathbf{n}}, \dot{\mathbf{t}}$ from equation 2.92, $\dot{\mathbf{v}}_C = \mathbf{a}_C$ from equation 2.97 and noting the relations $\mathbf{n}^T\tilde{\boldsymbol{\Omega}}\mathbf{t} = \mathbf{b}^T\boldsymbol{\Omega}$ and $\mathbf{t}^T\tilde{\boldsymbol{\Omega}}\mathbf{t} = 0$ we derive

$$\dot{v}_n = \mathbf{n}^T(\mathbf{a}_Q - \tilde{\boldsymbol{\Omega}}\mathbf{v}_C) - \kappa\dot{s}\mathbf{t}^T\mathbf{v}_C + \dot{s}\mathbf{b}^T\Omega, \tag{2.100}$$

$$\dot{v}_t = \mathbf{t}^T(\mathbf{a}_Q - \tilde{\boldsymbol{\Omega}}\mathbf{v}_C) + \kappa\dot{s}\mathbf{n}^T\mathbf{v}_C. \tag{2.101}$$

With these fundamental equations for one body we are able to consider in a next step two contacting bodies. We are still in a plane. Figure 2.20 gives the nomenclature and the directions. The sense of the contour parameters s_1 and s_2 are chosen in such a way that the binormals of both moving trihedrals are the same, $\mathbf{b}_1 = \mathbf{b}_2$. The origins of the two trihedrals are connected with the relative distance vector $\mathbf{r}_D$. To determine this distance vector we orient the trihedrals in such a manner, that they are mutually perpendicular, which is always possible and gives the conditions:

$$\mathbf{n}_1^T(s_1) \cdot \mathbf{t}_2(s_2) = 0, \qquad \Leftrightarrow \qquad \mathbf{n}_2^T(s_2) \cdot \mathbf{t}_1(s_1) = 0. \tag{2.102}$$

These conditions require that both, normals and tangents, are parallel (see Figure 2.20). For an evaluation of these equations we need of course only one, because the two are equivalent. The next requirement consists in putting the relative distance $\mathbf{r}_D$ unidirectional with the two normals and thus perpendicular to the two tangent vectors. This gives

$$\mathbf{r}_D^T(s_1, s_2) \cdot \mathbf{t}_1(s_1) = 0, \qquad \Leftrightarrow \qquad \mathbf{r}_D^T(s_1, s_2) \cdot \mathbf{t}_2(s_2) = 0. \tag{2.103}$$

From the four equations 2.102 and 2.103 we need only two, for example the first ones. The solution (s_1, s_2) of these two conditions as nonlinear functions

of (s_1, s_2) results in a configuration indicated in Figure 2.20: normal and tangent vectors are antiparallel to each other, and the relative distance vector $\mathbf{r}_D$ is perpendicular to the two surfaces, which at the same time is the shortest possible distance between the two bodies. The values (s_1, s_2) are the "contact parameters" of our problem, and the accompanying points (C_1, C_2) the "contact points". The axes of the two trihedrals are given by

$$\mathbf{n}_1 = -\mathbf{n}_2, \qquad \mathbf{t}_1 = -\mathbf{t}_2, \qquad \mathbf{b}_{12} = \mathbf{b}_1 = \mathbf{b}_2. \tag{2.104}$$

From this we get easily the distance between the two bodies

$$g_N(\mathbf{q}, t) = \mathbf{r}_D^T \mathbf{n}_2 = -\mathbf{r}_D^T \mathbf{n}_1. \tag{2.105}$$

Since the normal vector always points inwards, g_N is positive for separation and negative for overlapping. Thus, a change of sign from positive to negative values indicates a transition from initially separated bodies to contact.

Relative kinematics plays a key role in detecting a change of contact situations, for example such transitions like detachment-contact or stick-slip and vice versa. For impacts we have in addition a non-continuous change of the relative velocities, and if we want to combine these inequality constraints with the equations of motion we need also the relative accelerations. Therefore relative kinematics of contacts include the whole set of position and orientation, of velocities and of accelerations. Some of these magnitudes we get by differentiation. In that case we should not forget the original state of the relative kinematic magnitudes for the special contact event under consideration, for example, in normal direction of a contact the event contact is indicated by the relative distance becoming then a constraint. In tangential direction the relative tangential velocity indicates the stick or slip situation. We shall come back to these properties later.

We consider Figure 2.20 and assume for a while that the equations 2.102 and 2.103 have not yet been fulfilled. The relative distance $\mathbf{r}_D$ is then not perpendicular to the two surfaces Σ_1 and Σ_2, but it represents some straight connection between the future contact points C_1 and C_2. The absolute change of $\mathbf{r}_D$ with time writes

$$\dot{\mathbf{r}}_D = \mathbf{v}_{\Sigma 2} - \mathbf{v}_{\Sigma 1}, \tag{2.106}$$

where the contour velocities $\mathbf{v}_\Sigma$ come from the equations 2.95. For evaluating the relative velocities we need in a further step the contour velocities $\dot{s}$. For this purpose we differentiate the equations 2.102 and 2.103 resulting in

$$\begin{aligned}
(\mathbf{r}_D^T \cdot \mathbf{t}_1)^\cdot &= \dot{\mathbf{r}}_D^T \cdot \mathbf{t}_1 + \mathbf{r}_D^T \cdot \dot{\mathbf{t}}_1, \\
(\mathbf{r}_D^T \cdot \mathbf{n}_1)^\cdot &= \dot{\mathbf{r}}_D^T \cdot \mathbf{n}_1 + \mathbf{r}_D^T \cdot \dot{\mathbf{n}}_1, \\
(\mathbf{n}_1^T \cdot \mathbf{t}_2)^\cdot &= \dot{\mathbf{n}}_1^T \cdot \mathbf{t}_2 + \mathbf{n}_1^T \cdot \dot{\mathbf{t}}_2,
\end{aligned} \tag{2.107}$$

where $\dot{\mathbf{r}}_D$ is given with the equation 2.106 and the time-derivatives of the unit vectors $\mathbf{n}$ and $\mathbf{t}$ come from the equations 2.92. Together with equation 2.95 we finally get

$$
\begin{aligned}
(\mathbf{r}_D^T \cdot \mathbf{t}_1)^{\cdot} =& \mathbf{t}_1^T \cdot (\mathbf{v}_{C2} - \mathbf{v}_{C1} - \boldsymbol{\Omega}_1 \times \mathbf{r}_D) + \mathbf{t}_1^T \mathbf{t}_2 \dot{s}_2 - \mathbf{t}_1^T \mathbf{t}_1 \dot{s}_1 + \mathbf{r}_D^T \mathbf{n}_1 \kappa_1 \dot{s}_1, \\
(\mathbf{r}_D^T \cdot \mathbf{n}_1)^{\cdot} =& \mathbf{n}_1^T \cdot (\mathbf{v}_{C2} - \mathbf{v}_{C1} - \boldsymbol{\Omega}_1 \times \mathbf{r}_D) + \mathbf{n}_1^T \mathbf{t}_2 \dot{s}_2 - \mathbf{n}_1^T \mathbf{t}_1 \dot{s}_1 + \mathbf{r}_D^T \mathbf{t}_1 \kappa_1 \dot{s}_1, \\
(\mathbf{n}_1^T \cdot \mathbf{t}_2)^{\cdot} =& -\mathbf{t}_1^T \mathbf{t}_2 \kappa_1 \dot{s}_1 + \mathbf{n}_1^T \mathbf{n}_2 \kappa_2 \dot{s}_2 + (\mathbf{t}_2 \times \mathbf{n}_1)^T \cdot (\boldsymbol{\Omega}_2 - \boldsymbol{\Omega}_1). \qquad (2.108)
\end{aligned}
$$

These expressions hold for the general case, where the trihedrals are not yet oriented. They simplify by applying the conditions 2.102 and 2.103, and they can then be taken to evaluate the contour velocities $\dot{s}_1$ and $\dot{s}_2$. After some elementary calculations (see [200]) we come out with

$$
\begin{aligned}
\dot{s}_1 =& \frac{\kappa_2 \mathbf{t}_1^T (\mathbf{v}_{C2} - \mathbf{v}_{C1}) - \kappa_2 g_N \mathbf{b}_{12}^T \boldsymbol{\Omega}_1 + \mathbf{b}_{12}^T (\boldsymbol{\Omega}_2 - \boldsymbol{\Omega}_1)}{\kappa_1 + \kappa_2 + g_N \kappa_1 \kappa_2}, \\
\dot{s}_2 =& \frac{\kappa_1 \mathbf{t}_1^T (\mathbf{v}_{C2} - \mathbf{v}_{C1}) - \kappa_1 g_N \mathbf{b}_{12}^T \boldsymbol{\Omega}_2 - \mathbf{b}_{12}^T (\boldsymbol{\Omega}_2 - \boldsymbol{\Omega}_1)}{\kappa_1 + \kappa_2 + g_N \kappa_1 \kappa_2}, \qquad (2.109)
\end{aligned}
$$

with the "binormal" $\mathbf{b}_{12} = -\mathbf{t}_2 \times \mathbf{n}_1$ to the vectors $\mathbf{n}_1$ and $\mathbf{t}_2$. The relative distance g_N is defined by equation 2.105.

With the above equations we can now evaluate the relative velocities and the relative accelerations of two bodies coming into contact or sliding on each other. Starting with the relative velocities we write (see Figure 2.20 and [200])

$$
\dot{g}_N = \mathbf{n}_1^T \mathbf{v}_{C1} + \mathbf{n}_2^T \mathbf{v}_{C2}, \qquad \dot{g}_T = \mathbf{t}_1^T \mathbf{v}_{C1} + \mathbf{t}_2^T \mathbf{v}_{C2}, \qquad (2.110)
$$

where $\mathbf{v}_{C1}$ and $\mathbf{v}_{C2}$ are the absolute velocities of the potential contact points C_1 and C_2. These velocities might be expressed by the generalized, or minimal, velocities $\dot{\mathbf{q}}$ using the Jacobians $\mathbf{J}_{C1}$ and $\mathbf{J}_{C2}$ (see [27], [200])

$$
\mathbf{v}_{C1} = \mathbf{J}_{C1} \dot{\mathbf{q}} + \tilde{\mathbf{j}}_{C1}, \qquad \mathbf{v}_{C2} = \mathbf{J}_{C2} \dot{\mathbf{q}} + \tilde{\mathbf{j}}_{C2}. \qquad (2.111)
$$

Combining the last two equations results in

$$
\dot{g}_N = \mathbf{w}_N^T \dot{\mathbf{q}} + \tilde{w}_N, \qquad \dot{g}_T = \mathbf{w}_T^T \dot{\mathbf{q}} + \tilde{w}_T, \qquad (2.112)
$$

with

$$
\begin{aligned}
\mathbf{w}_N =& \mathbf{J}_{C1}^T \mathbf{n}_1 + \mathbf{J}_{C2}^T \mathbf{n}_2, \qquad \mathbf{w}_T = \mathbf{J}_{C1}^T \mathbf{t}_1 + \mathbf{J}_{C2}^T \mathbf{t}_2, \\
\tilde{w}_N =& \tilde{\mathbf{j}}_{C1}^T \mathbf{n}_1 + \tilde{\mathbf{j}}_{C2}^T \mathbf{n}_2, \qquad \tilde{w}_T = \tilde{\mathbf{j}}_{C1}^T \mathbf{t}_1 + \tilde{\mathbf{j}}_{C2}^T \mathbf{t}_2, \qquad (2.113)
\end{aligned}
$$

which will be used in the sequel as a representation of the relative velocities between neighboring bodies within a whole system of bodies. It should be noticed that a negative value of the relative normal velocity $\dot{g}_N$ corresponds to an approaching process of the neighboring bodies and coincides at vanishing distance $g_N = 0$ with the relative velocity in normal contact direction shortly before a contact, which will be in most cases an impact. In the case of a continual contact with $g_N = 0, \dot{g}_N = 0$ the relative tangential velocity indicates sliding between the two bodies, which we need to determine the tangential transition event from sliding with $\dot{g}_T \neq 0$ to sticking with $\dot{g}_T = 0$.

The relative accelerations are derived by differentiation of the velocity equations 2.112. We get the same structure of the velocity equations

$$\ddot{g}_N = \mathbf{w}_N^T \ddot{\mathbf{q}} + \bar{w}_N, \qquad \ddot{g}_T = \mathbf{w}_T^T \ddot{\mathbf{q}} + \bar{w}_T, \tag{2.114}$$

where the constraint vectors $\mathbf{w}_N, \mathbf{w}_T$ are known from the equations 2.113 and the nonlinear functions $\bar{w}_N, \bar{w}_T$ follow from

$$\begin{aligned} \bar{w}_N = \quad & \mathbf{n}_1^T(\bar{\mathbf{j}}_{Q1} - \tilde{\boldsymbol{\Omega}}_1 \mathbf{v}_{C1}) - \kappa_1 \dot{s}_1 \mathbf{t}_1^T \mathbf{v}_{C1} + \dot{s}_1 \mathbf{b}_{12}^T \boldsymbol{\Omega}_1 + \\ & \mathbf{n}_2^T(\bar{\mathbf{j}}_{Q2} - \tilde{\boldsymbol{\Omega}}_2 \mathbf{v}_{C2}) - \kappa_2 \dot{s}_2 \mathbf{t}_2^T \mathbf{v}_{C2} + \dot{s}_2 \mathbf{b}_{12}^T \boldsymbol{\Omega}_2, \\ \bar{w}_T = \quad & \mathbf{t}_1^T(\bar{\mathbf{j}}_{Q1} - \tilde{\boldsymbol{\Omega}}_1 \mathbf{v}_{C1}) + \kappa_1 \dot{s}_1 \mathbf{n}_1^T \mathbf{v}_{C1} + \\ & \mathbf{t}_2^T(\bar{\mathbf{j}}_{Q2} - \tilde{\boldsymbol{\Omega}}_2 \mathbf{v}_{C2}) + \kappa_2 \dot{s}_2 \mathbf{n}_2^T \mathbf{v}_{C2}. \end{aligned} \tag{2.115}$$

All magnitudes of these equations are defined in the preceding equations. For a more detailed description see the literature ([200], [212] and [86]).

2.2.7.2 Spatial Case

Spatial contacts require a more complex description. We assume as above that the two approaching bodies are convex at least in the neighborhood of those points, where contacts might occur (Figure 2.21). The two bodies with their coordinate bases B_1, B_2 as part of a whole system of bodies move with the velocities $\mathbf{v}_i, \boldsymbol{\Omega}_i$ (i=1,2) and have a certain position and orientation with respect to an inertial frame and with respect to each other. They possess a contour Σ in two dimensions, which are parameterized by the curvilinear coordinates s and t. The two bodies may come into contact or they may be already in contact, for example sliding on each other. We assume, that there is only one contact point, but in many technical applications bodies might have several contacts, for example in the fields of power transmission, gears and chains. However, each of these multiple contacts follows the same theory as given below. What we have to do, is some additional indexing.

What we are looking for, is a representation of the relative kinematics in terms of the surface coordinates and their derivatives as a first step, and as a second step we want to express these relative kinematic magnitudes in terms of the body velocities finally arriving at the relative velocities and accelerations depending on the corresponding generalized magnitudes. As a base for all considerations to follow we use chapter 2.2.6. Let us start with a point of the contours Σ_1 and Σ_2 with the coordinates s_i, t_i (i=1,2) and the surface unit vectors in these points $\mathbf{n}_i, \mathbf{s}_i, \mathbf{t}_i$ (i=1,2). The unit vectors $\mathbf{s}_i, \mathbf{t}_i$ (i=1,2) span the tangent plane in the potential points of contact, on both sides. These unit vectors are defined by

$$\mathbf{s} = \frac{\partial \mathbf{r}_\Sigma}{\partial s}, \qquad \mathbf{t} = \frac{\partial \mathbf{r}_\Sigma}{\partial t}. \tag{2.116}$$

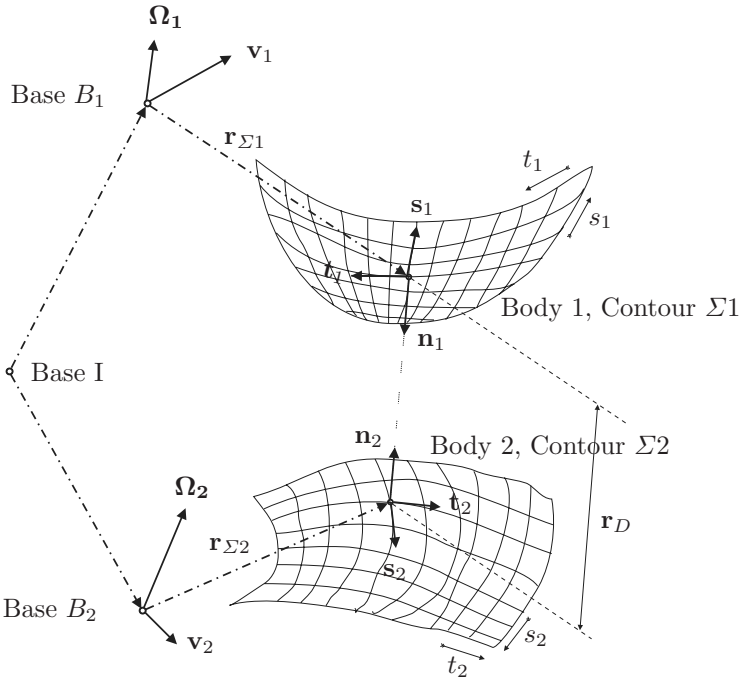

Fig. 2.21: Contact Zones in the Spatial Case

From these basic vectors we are able to determine the fundamental magnitudes of first order

$$E = \mathbf{s}^T\mathbf{s}, \qquad F = \mathbf{s}^T\mathbf{t}, \qquad G = \mathbf{t}^T\mathbf{t}. \tag{2.117}$$

The normal unit vector $\mathbf{n}$ is perpendicular to the surface tangent plane pointing outwardly, therefore

$$\mathbf{n} = \frac{\mathbf{s} \times \mathbf{t}}{\sqrt{EG - F^2}}. \tag{2.118}$$

The fundamental magnitudes of second order follow from the equations 2.79

$$L = \mathbf{n}^T \cdot \frac{\partial^2 \mathbf{r}_\Sigma}{\partial s^2}, \qquad M = \mathbf{n}^T \cdot \frac{\partial^2 \mathbf{r}_\Sigma}{\partial s \partial t}, \qquad N = \mathbf{n}^T \cdot \frac{\partial^2 \mathbf{r}_\Sigma}{\partial t^2} \tag{2.119}$$

For a potential contact point we must achieve some requirements with respect to the directions of the surface unit vectors and the distance vector $\mathbf{r}_D$. From several possibilities we take the following four ones

$$\mathbf{n}_1^T\mathbf{s}_2 = 0, \qquad \mathbf{n}_1^T\mathbf{t}_2 = 0, \qquad \mathbf{r}_D^T\mathbf{s}_2 = 0, \qquad \mathbf{r}_D^T\mathbf{t}_2 = 0. \tag{2.120}$$

This nonlinear problem has to be solved at every time step of the numerical integration by appropriate algorithms, analytical solutions are very unlikely.

Once the solution has been achieved we are able to evaluate the relative distance g_N between the two bodies by

$$g_N = \mathbf{n}_1^T \mathbf{r}_D = -\mathbf{n}_2^T \mathbf{r}_D \tag{2.121}$$

The relative distance g_N is one of the important contact magnitudes indicating the state "no contact" or the state "contact". It is positive for the first state and negative for the second one in the case of penetration.

For a further definition of the unilateral constraints we need the relative velocities and accelerations. Regarding the spatial case as indicated in Figure 2.21 we have one relative velocity in normal and two relative velocities in tangential directions

$$\begin{aligned}
\dot{g}_N(\mathbf{q}, \dot{\mathbf{q}}, t) =& \mathbf{n}_1^T \cdot (\mathbf{v}_{\Sigma 2} - \mathbf{v}_{\Sigma 1}), \\
\dot{g}_S(\mathbf{q}, \dot{\mathbf{q}}, t) =& \mathbf{s}_1^T \cdot (\mathbf{v}_{\Sigma 2} - \mathbf{v}_{\Sigma 1}), \\
\dot{g}_T(\mathbf{q}, \dot{\mathbf{q}}, t) =& \mathbf{t}_1^T \cdot (\mathbf{v}_{\Sigma 2} - \mathbf{v}_{\Sigma 1}),
\end{aligned} \tag{2.122}$$

with $\mathbf{v}_{\Sigma 1}$ and $\mathbf{v}_{\Sigma 2}$ defined in the following way: In the plane case we have considered the potential contact points C_1 and C_2 and expressed their velocities $\mathbf{v}_{C1}$ and $\mathbf{v}_{C2}$ by the generalized or minimal velocities $\dot{\mathbf{q}}$ using some Jacobians for these contact points (see equations (2.95) and (2.111)). The same equations can be used for the points Σ_1 and Σ_2. That means the velocities $\mathbf{v}_{\Sigma 1}$ and $\mathbf{v}_{\Sigma 2}$ in the spatial case correspond to the velocities $\mathbf{v}_{C1}$ and $\mathbf{v}_{C2}$ in the plane case. The velocities $\mathbf{v}_{C1}, \mathbf{v}_{C2}$ in the plane and $\mathbf{v}_{\Sigma 1}, \mathbf{v}_{\Sigma 2}$ in the spatial case are those of body-fixed contour points, which momentarily coincide with the potential contact point.

Differentiating the equations 2.122 with respect to time we get the relative accelerations by

$$\begin{aligned}
\ddot{g}_N(\mathbf{q}, \dot{\mathbf{q}}, t) =& \mathbf{n}_1^T \cdot (\dot{\mathbf{v}}_{\Sigma 2} - \dot{\mathbf{v}}_{\Sigma 1}) + \dot{\mathbf{n}}_1^T \cdot (\mathbf{v}_{\Sigma 2} - \mathbf{v}_{\Sigma 1}), \\
\ddot{g}_S(\mathbf{q}, \dot{\mathbf{q}}, t) =& \mathbf{s}_1^T \cdot (\dot{\mathbf{v}}_{\Sigma 2} - \dot{\mathbf{v}}_{\Sigma 1}) + \dot{\mathbf{s}}_1^T \cdot (\mathbf{v}_{\Sigma 2} - \mathbf{v}_{\Sigma 1}), \\
\ddot{g}_T(\mathbf{q}, \dot{\mathbf{q}}, t) =& \mathbf{t}_1^T \cdot (\dot{\mathbf{v}}_{\Sigma 2} - \dot{\mathbf{v}}_{\Sigma 1}) + \dot{\mathbf{t}}_1^T \cdot (\mathbf{v}_{\Sigma 2} - \mathbf{v}_{\Sigma 1}),
\end{aligned} \tag{2.123}$$

The velocities and accelerations of the contact points follow from the above considerations and by differentiation with respect to time as

$$\begin{aligned}
\mathbf{v}_{\Sigma 1} =& \mathbf{J}_{\Sigma 1}(\mathbf{q}, t)\dot{\mathbf{q}} + \tilde{\mathbf{j}}_{\Sigma 1}(\mathbf{q}, \dot{\mathbf{q}}, t), \qquad & \mathbf{v}_{\Sigma 2} = \mathbf{J}_{\Sigma 2}(\mathbf{q}, t)\dot{\mathbf{q}} + \tilde{\mathbf{j}}_{\Sigma 2}(\mathbf{q}, \dot{\mathbf{q}}, t), \\
\dot{\mathbf{v}}_{\Sigma 1} =& \mathbf{J}_{\Sigma 1}(\mathbf{q}, t)\ddot{\mathbf{q}} + \bar{\mathbf{j}}_{\Sigma 1}(\mathbf{q}, \dot{\mathbf{q}}, t), \qquad & \dot{\mathbf{v}}_{\Sigma 2} = \mathbf{J}_{\Sigma 2}(\mathbf{q}, t)\ddot{\mathbf{q}} + \bar{\mathbf{j}}_{\Sigma 2}(\mathbf{q}, \dot{\mathbf{q}}, t).
\end{aligned} \tag{2.124}$$

The surface vectors $\dot{\mathbf{n}}_1$, $\dot{\mathbf{s}}_1$ and $\dot{\mathbf{t}}_1$ can be determined by the formulas of Weingarten and Gauss (equations 2.81), which results in:

$$\dot{\mathbf{n}}_1 = \boldsymbol{\Omega}_1 \times \mathbf{n}_1 + \frac{\partial \mathbf{n}_1}{\partial s_1}\dot{s}_1 + \frac{\partial \mathbf{n}_1}{\partial t_1}\dot{t}_1,$$

$$\begin{aligned}
\frac{\partial \mathbf{n}_1}{\partial s_1} &= \underbrace{\frac{M_1F_1 - L_1G_1}{E_1G_1 - F_1^2}\mathbf{s}_1}_{\alpha_1} + \underbrace{\frac{L_1F_1 - M_1E_1}{E_1G_1 - F_1^2}\mathbf{t}_1}_{\beta_1}, \\
\frac{\partial \mathbf{n}_1}{\partial t_1} &= \underbrace{\frac{N_1F_1 - M_1G_1}{E_1G_1 - F_1^2}\mathbf{s}_1}_{\alpha_1'} + \underbrace{\frac{M_1F_1 - N_1E_1}{E_1G_1 - F_1^2}\mathbf{t}_1}_{\beta_1'},
\end{aligned} \tag{2.125}$$

$$\dot{\mathbf{s}}_1 = \boldsymbol{\Omega}_1 \times \mathbf{s}_1 + \frac{\partial \mathbf{s}_1}{\partial s_1}\dot{s}_1 + \frac{\partial \mathbf{s}_1}{\partial t_1}\dot{t}_1,$$

$$\begin{aligned}
\frac{\partial \mathbf{s}_1}{\partial s_1} &= (\Gamma_{11}^1)_1\mathbf{s}_1 + (\Gamma_{11}^2)_1\mathbf{t}_1 + L_1\mathbf{n}_1, \\
\frac{\partial \mathbf{s}_1}{\partial t_1} &= (\Gamma_{12}^1)_1\mathbf{s}_1 + (\Gamma_{12}^2)_1\mathbf{t}_1 + M_1\mathbf{n}_1,
\end{aligned} \tag{2.126}$$

$$\dot{\mathbf{t}}_1 = \boldsymbol{\Omega}_1 \times \mathbf{t}_1 + \frac{\partial \mathbf{t}_1}{\partial s_1}\dot{s}_1 + \frac{\partial \mathbf{t}_1}{\partial t_1}\dot{t}_1,$$

$$\begin{aligned}
\frac{\partial \mathbf{t}_1}{\partial s_1} &= (\Gamma_{12}^1)_1\mathbf{s}_1 + (\Gamma_{12}^2)_1\mathbf{t}_1 + M_1\mathbf{n}_1, \\
\frac{\partial \mathbf{t}_1}{\partial t_1} &= (\Gamma_{22}^1)_1\mathbf{s}_1 + (\Gamma_{22}^2)_1\mathbf{t}_1 + N_1\mathbf{n}_1.
\end{aligned} \tag{2.127}$$

The Christoffel symbols $\Gamma_{\alpha\beta}^{\sigma}$ with ($\alpha\beta$=1,2) are defined in equation 2.82 (see also [51]). Inserting the above relations for the surface vectors ($\mathbf{n}_1$, $\mathbf{s}_1$, $\mathbf{t}_1$) and ($\dot{\mathbf{n}}_1$, $\dot{\mathbf{s}}_1$, $\dot{\mathbf{t}}_1$) into the equations 2.123 we come out with

$$
\begin{aligned}
\ddot{g}_N =& \mathbf{n}_1^T[(\mathbf{J}_{\Sigma 2} - \mathbf{J}_{\Sigma 1})\ddot{\mathbf{q}} + (\bar{\mathbf{j}}_{\Sigma 2} - \bar{\mathbf{j}}_{\Sigma 1})] + \\
& (\mathbf{v}_{\Sigma 2} - \mathbf{v}_{\Sigma 1})^T \cdot [(\boldsymbol{\Omega}_1 \times \mathbf{n}_1) + ((\alpha_1 \mathbf{s}_1 + \beta_1 \mathbf{t}_1)\dot{s}_1 + (\alpha_1' \mathbf{s}_1 + \beta_1' \mathbf{t}_1)\dot{t}_1)], \\
\\
\ddot{g}_S =& \mathbf{s}_1^T[(\mathbf{J}_{\Sigma 2} - \mathbf{J}_{\Sigma 1})\ddot{\mathbf{q}} + (\bar{\mathbf{j}}_{\Sigma 2} - \bar{\mathbf{j}}_{\Sigma 1})] + \\
& (\mathbf{v}_{\Sigma 2} - \mathbf{v}_{\Sigma 1})^T \cdot [(\boldsymbol{\Omega}_1 \times \mathbf{s}_1) + ((\Gamma_{11}^1)_1 \mathbf{s}_1 + (\Gamma_{11}^2)_1 \mathbf{t}_1 + L_1 \mathbf{n}_1)\dot{s}_1 + \\
& \qquad ((\Gamma_{12}^1)_1 \mathbf{s}_1 + (\Gamma_{12}^2)_1 \mathbf{t}_1 + M_1 \mathbf{n}_1)\dot{t}_1], \\
\\
\ddot{g}_T =& \mathbf{t}_1^T[(\mathbf{J}_{\Sigma 2} - \mathbf{J}_{\Sigma 1})\ddot{\mathbf{q}} + (\bar{\mathbf{j}}_{\Sigma 2} - \bar{\mathbf{j}}_{\Sigma 1})] + \\
& (\mathbf{v}_{\Sigma 2} - \mathbf{v}_{\Sigma 1})^T \cdot [(\boldsymbol{\Omega}_1 \times \mathbf{s}_1) + ((\Gamma_{12}^1)_1 \mathbf{s}_1 + (\Gamma_{12}^2)_1 \mathbf{t}_1 + M_1 \mathbf{n}_1)\dot{s}_1 + \\
& \qquad ((\Gamma_{22}^1)_1 \mathbf{s}_1 + (\Gamma_{22}^2)_1 \mathbf{t}_1 + N_1 \mathbf{n}_1)\dot{t}_1]. \qquad (2.128)
\end{aligned}
$$

The Jacobian matrices $\mathbf{J}_{\Sigma 1}$ and $\mathbf{J}_{\Sigma 2}$ are known from the elastic or rigid body kinematics as discussed in connection with the equations 2.69 and 2.111. The time derivatives $(\dot{s}_1, \dot{t}_1, \dot{s}_2, \dot{t}_2)$ of the surface unit vectors can be evaluated from the surface geometry and its curvilinear coordinates, in a similar way as in the plane case with the equations (2.106) to (2.109). Differentiating the equations (2.120) with respect to time

$$
\begin{aligned}
(\mathbf{n}_1^T \mathbf{s}_2)^{\cdot} =& 0, \qquad (\mathbf{n}_1^T \mathbf{t}_2)^{\cdot} = 0, \\
(\mathbf{r}_D^T \mathbf{s}_2)^{\cdot} =& 0, \qquad (\mathbf{r}_D^T \mathbf{t}_2)^{\cdot} = 0, \qquad (2.129)
\end{aligned}
$$

which states, that the contact conditions remain the same while the bodies are moving, and which result in a linear system for the unknown contour derivatives $(\dot{s}_1, \dot{t}_1, \dot{s}_2, \dot{t}_2)$

$$
\mathbf{A}_C \cdot \mathbf{x}_C = \mathbf{b}_C \qquad \text{with}
$$

$$
\mathbf{A}_C = \begin{pmatrix}
\mathbf{s}_2^T(\alpha_1 \mathbf{s}_1 + \beta_1 \mathbf{t}_1) & \mathbf{s}_2^T(\alpha_1' \mathbf{s}_1 + \beta_1' \mathbf{t}_1) & L_2 & M_2 \\
\mathbf{t}_2^T(\alpha_1 \mathbf{s}_1 + \beta_1 \mathbf{t}_1) & \mathbf{t}_2^T(\alpha_1' \mathbf{s}_1 + \beta_1' \mathbf{t}_1) & M_2 & N_2 \\
-\mathbf{s}_1^T \mathbf{s}_2 & -\mathbf{s}_1^T \mathbf{s}_2 & \mathbf{s}_2^T \mathbf{s}_2 & \mathbf{s}_2^T \mathbf{t}_2 \\
-\mathbf{s}_1^T \mathbf{t}_2 & -\mathbf{s}_1^T \mathbf{t}_2 & \mathbf{s}_2^T \mathbf{t}_2 & \mathbf{t}_2^T \mathbf{t}_2
\end{pmatrix}
$$

$$
\mathbf{b}_C = \begin{pmatrix}
(\mathbf{s}_2 \times \mathbf{n}_2)^T \cdot (\boldsymbol{\Omega}_2 - \boldsymbol{\Omega}_1) \\
(\mathbf{t}_2 \times \mathbf{n}_2)^T \cdot (\boldsymbol{\Omega}_2 - \boldsymbol{\Omega}_1) \\
\mathbf{s}_2^T(\mathbf{v}_{\Sigma 2} - \mathbf{v}_{\Sigma 1}) \\
\mathbf{t}_2^T(\mathbf{v}_{\Sigma 2} - \mathbf{v}_{\Sigma 1})
\end{pmatrix} \qquad \mathbf{x}_C = \begin{pmatrix}
\dot{s}_1 \\ \dot{t}_1 \\ \dot{s}_2 \\ \dot{t}_2
\end{pmatrix} \qquad (2.130)
$$

These linear equations have to be solved at every time step of a numerical simulation. For further developments of the dynamics equations we abbreviate the equations (2.128) in the form

$$
\ddot{g}_N = \mathbf{w}_N^T \ddot{\mathbf{q}} + \bar{w}_N, \qquad \ddot{g}_S = \mathbf{w}_S^T \ddot{\mathbf{q}} + \bar{w}_S, \qquad \ddot{g}_T = \mathbf{w}_T^T \ddot{\mathbf{q}} + \bar{w}_T. \qquad (2.131)
$$

Combining these equations for a contact (i) of a multibody system results in a form more convenient for the later evaluation of the equations of motion. We get

$$\ddot{g}_{Ni} = \mathbf{w}_{Ni}^T \ddot{\mathbf{q}} + \bar{w}_{Ni}, \qquad \ddot{\mathbf{g}}_{Ti} = \mathbf{W}_{Ti}^T \ddot{\mathbf{q}} + \bar{\mathbf{w}}_{Ti},$$
$$\ddot{\mathbf{g}}_{Ti} = \begin{pmatrix} \ddot{g}_S \\ \ddot{g}_T \end{pmatrix}, \qquad \mathbf{W}_{Ti}^T = \begin{pmatrix} \mathbf{w}_S^T \\ \mathbf{w}_T^T \end{pmatrix}, \qquad \bar{\mathbf{w}}_{Ti} = \begin{pmatrix} \bar{w}_S \\ \bar{w}_T \end{pmatrix}. \tag{2.132}$$

The vectors $\mathbf{w}_k$ and the scalars $\bar{w}_k$ with (k=N,S,T) follow by comparison with the equations (2.128). The scalar terms usually excitations from external sources.

2.2.8 Influence of Elasticity

Machines, mechanisms and structures are always elastic. How "much elastic" depends on the eigenfrequencies of the overall system and of the components. If one of these frequencies is low enough to produce a significant influence on the dynamics within the frequency range we want to consider, then such a component must be modeled elastically, otherwise rigidly. In most of this practical cases we have linear elasticity, which means, that we have approximately a rigid body motion of the multibody system superimposed by small elastic deformations usually in the form of elastic vibrations. Of course we then get a mutual influence of system dynamics and elastic deformations. As many practical systems belong to this class, we shall restrict ourselves to the influence of small elastic deformations of some components in the system. A very good description of linear and nonlinear influences of elasticity in multibody dynamics is given by Bremer [28] and Shabana [242].

We consider small deformations of the body (i) as part of a system of rigid and elastic bodies. These deformations result in a displacement and in a rotation of every mass-element dm of the body (i) (see Figure 2.22) with the base B_i. The vector chain from the Base I to the Base B_{ei} includes three vectors, the vector $\mathbf{r}_{IB_i}$ from I to B_i, the vector $\mathbf{r}_{B_iB_{ri}}$ from B_i to B_{ri} and the vector $\mathbf{r}_{B_{ri}B_{ei}}$ from the undeformed reference B_{ri} to the deformed reference B_{ei}. The three bases with the accompanying coordinate systems may be seen from Figure 2.22. Without touching the principal considerations we could have chosen any other chain of vectors as indicated by the dashed lines in Figure (2.22), where we have depicted specific points for additional coordinates systems. According to the definitions with regard to the coordinates we write the vector chain relation in the coordinates of the body (i):

$${}_{B_i}\mathbf{r}_{IB_{ei}} = {}_{B_i}\mathbf{r}_{IB_i} + {}_{B_i}\mathbf{r}_{B_iB_{ri}} + \mathbf{A}_{B_iB_{ei}} \cdot ({}_{B_{ei}}\mathbf{r}_{B_{ri}B_{ei}}), \qquad \text{with}$$

$${}_{B_i}\mathbf{r}_{B_iB_{ei}} = {}_{B_i}\mathbf{r}_{B_iB_{ri}} + \mathbf{A}_{B_iB_{ei}} \cdot ({}_{B_{ei}}\mathbf{r}_{B_{ri}B_{ei}})$$
$$\mathbf{A}_{B_iB_{ei}} = \mathbf{A}_{B_{ri}B_{ei}}. \tag{2.133}$$

The last two terms of the first equation include all influences of elasticity,

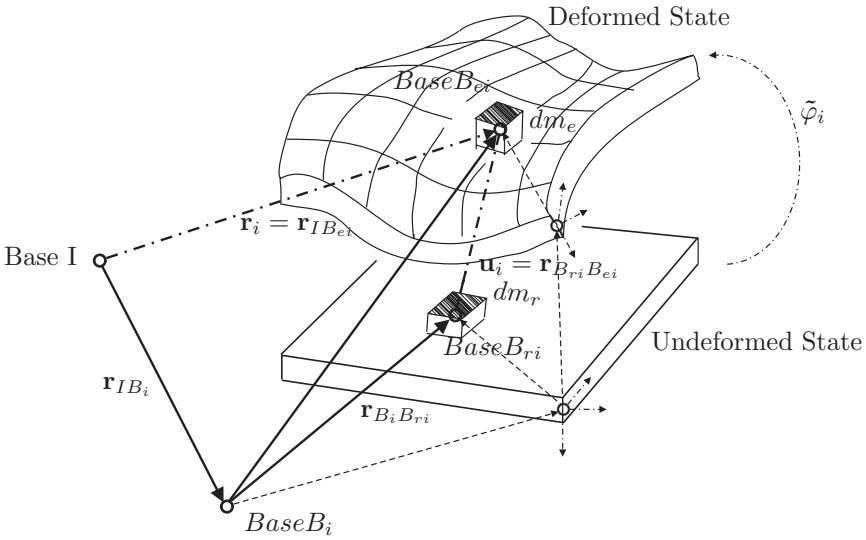

Fig. 2.22: Deformation of a Multibody Component i

the deformation vector $\mathbf{u}_i = \mathbf{r}_{B_{ri}B_{ei}}$ directly, and the rotation matrix $\mathbf{A}_{B_iB_{ei}}$ represents the rotation between the base B_{ei} and the base B_i or B_{ri} due to elasticity. This "elastic rotation" is of the same magnitude for the base B_{ri} and for the base B_i. For a rigid body we get $\mathbf{r}_{B_{ri}B_{ei}} = \mathbf{0}$ and $\mathbf{A}_{B_iB_{ei}} = \mathbf{E}$. Assuming small elastic deformations and rotations and using Cardan angles we can evaluate the rotation matrix $\mathbf{A}_{B_iB_{ei}}$ from the formula (2.19) by a Taylor expansion. We get (hot = higher order terms)

$$\mathbf{A}_{B_iB_{ei}} \approx \begin{pmatrix} 1 & -\gamma & +\beta \\ +\gamma & 1 & -\alpha \\ -\beta & +\alpha & 1 \end{pmatrix} + hot = \mathbf{E} + \tilde{\varphi} + hot, \qquad [\varphi = (\alpha, \beta, \gamma)^T]. \quad (2.134)$$

For calculating the absolute velocity in a body-fixed coordinate system we transform the relation (2.133) into inertial coordinates, differentiate with respect to time, and transform the resulting velocity back into body-fixed coordinates. This yields

$$\begin{aligned} {}_{B_i}\dot{\mathbf{r}}_{IB_{ei}} =& {}_{B_i}\dot{\mathbf{r}}_{IB_i} + {}_{B_i}\dot{\mathbf{r}}_{B_iB_{ri}} + {}_{B_i}\tilde{\omega}_{IB_i} \cdot [{}_{B_i}\mathbf{r}_{B_iB_{ri}} + \mathbf{A}_{B_iB_{ei}}({}_{B_{ei}}\mathbf{u})] + \\ & + \mathbf{A}_{B_iB_{ei}}[{}_{B_{ei}}\dot{\mathbf{u}} + {}_{B_{ei}}\tilde{\omega}_{B_{ri}B_{ei}}({}_{B_{ei}}\mathbf{u})], \end{aligned}$$

$$\begin{aligned} {}_{B_i}\dot{\mathbf{r}}_{IB_{ei}} =& {}_{B_i}\dot{\mathbf{r}}_{IB_i} + {}_{B_i}\dot{\mathbf{r}}_{B_iB_{ri}} + {}_{B_i}\tilde{\omega}_{IB_i} \cdot [{}_{B_i}\mathbf{r}_{IB_i} + {}_{B_i}\mathbf{r}_{B_iB_{ri}}] + \\ & + \mathbf{A}_{B_iB_{ei}}[{}_{B_{ei}}\dot{\mathbf{u}} + {}_{B_{ei}}\tilde{\omega}_{IB_{ei}}({}_{B_{ei}}\mathbf{u})], \end{aligned}$$

$${}_{B_i}\tilde{\omega}_{IB_{ei}} = {}_{B_i}\tilde{\omega}_{IB_i} + {}_{B_i}\tilde{\omega}_{B_{ri}B_{ei}} = {}_{B_i}\tilde{\omega}_{IB_i} + \mathbf{A}_{B_iB_{ei}}({}_{B_{ei}}\tilde{\omega}_{B_{ri}B_{ei}})\mathbf{A}_{B_{ei}B_i} \quad (2.135)$$

In addition to the rigid body motion we get in the same structural way the influence of elasticity. It should be kept in mind, that for the evaluation of

the Jacobians as part of the equations of motion we have to develop the kinematic expressions for position and orientation, for translational and rotational velocities and accelerations up to terms of second order with respect to the elastic coordinates. For large systems this is better done either by symbolic or by numerical computer codes.

In a next step we must express the deformation vectors by an elastic model, where we refer mainly to the books of Becker [14], Betten [19], Bremer [28] and Wriggers [278]. We go from some reference configuration of a body $\mathcal{B}$ to a deformed configuration $\varphi(\mathcal{B})$, and we shall use the vectors $\mathbf{X} \in \mathbb{R}^3$ from the origin to the reference state, $\mathbf{x} \in \mathbb{R}^3$ from the origin to the deformed state and the displacement vector $\mathbf{u} \in \mathbb{R}^3$ from the reference to the deformed state (see [278] and Figure 2.23). Any element d$\mathbf{x}$ can then be expressed by a reference element d$\mathbf{X}$ in the form

$$d\mathbf{x} = (\frac{\partial \mathbf{x}}{\partial \mathbf{X}}) \cdot d\mathbf{X} = \mathbf{F} \cdot d\mathbf{X} \qquad \text{with} \qquad \mathbf{F} = \frac{\partial \mathbf{x}}{\partial \mathbf{X}} = grad_{\mathbf{X}}(\mathbf{x}) \in \mathbb{R}^{3,3}. \quad (2.136)$$

The deformation gradient $\mathbf{F}$ is one of the fundamental magnitudes of continuum mechanics. It can be expressed as a gradient of the vector $\mathbf{x}$ with respect to $\mathbf{X}$. It is well known that the deformation gradient $\mathbf{F}$ can never be singular and that it can be decomposed by the polar decomposition theorem into a stretching and into a rotational part

$$\mathbf{F} = \mathbf{RU} = \mathbf{VR}, \quad (2.137)$$

where $\mathbf{U}$ is the right stretch or the right Cauchy Green tensor, and $\mathbf{V}$ is the left stretch or the left Cauchy Green tensor. They are defined in the reference or the current configuration, respectively. The rotation tensor $\mathbf{R}$ is orthogonal with $\det(\mathbf{R}) = +1$, the tensors $\mathbf{U}$ and $\mathbf{V}$ are positive definite and possess the same eigenvalues $\lambda_i, (i = 1, 2, 3)$. This is important, because the stretching effects are proportional to these eigenvalues, and they must be independent from the sequences stretching/rotation or rotation/stretching. Applying therefore $d\mathbf{x} = \mathbf{RU} \cdot d\mathbf{X}$ to a mass element results in a stretching of the element with a following rotation, whereas $d\mathbf{x} = \mathbf{VR} \cdot d\mathbf{X}$ comes out with a rotation in a first and a stretching in a second step ([14]).

The usual way to derive the strain tensor consists in considering the difference of the squared elements $(d\mathbf{x}^T d\mathbf{x} - d\mathbf{X}^T d\mathbf{X}) = d\mathbf{X}^T(\mathbf{F}^T\mathbf{F} - \mathbf{E})d\mathbf{X} = 2d\mathbf{X}^T\mathbf{G}d\mathbf{X}$, which represents a suitable deformation measure. The resulting strain tensor $\mathbf{G}$ is then defined as

$$\mathbf{G} = \frac{1}{2}(\mathbf{F}^T\mathbf{F} - \mathbf{E}) \in \mathbb{R}^{3,3}, \quad (2.138)$$

where $\mathbf{E}$ is the unit matrix. The tensor $\mathbf{G}$ is called the Cauchy-Lagrangian strain tensor. It refers to the initial configuration $\mathcal{B}$. From Figure 2.23 we have the property $\mathbf{x} = \mathbf{X} + \mathbf{u}$ and with this the deformation gradient $\mathbf{F} = \mathbf{E} + \frac{\partial \mathbf{u}}{\partial \mathbf{X}}$. With these relations the equation (2.138) writes

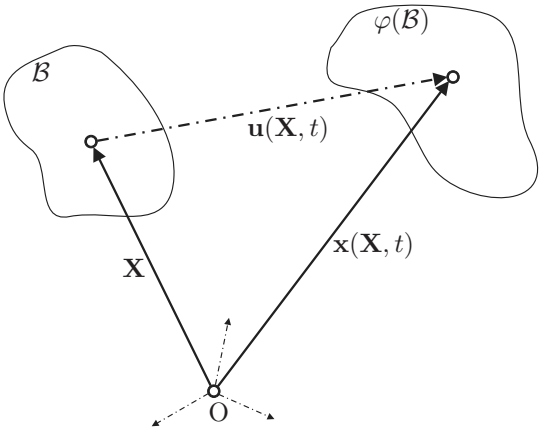

Fig. 2.23: Reference and Deformed States [278]

$$\mathbf{G} = \frac{1}{2}\{(\frac{\partial \mathbf{u}}{\partial \mathbf{X}}) + (\frac{\partial \mathbf{u}}{\partial \mathbf{X}})^T + (\frac{\partial \mathbf{u}}{\partial \mathbf{X}})^T \cdot (\frac{\partial \mathbf{u}}{\partial \mathbf{X}})\}. \tag{2.139}$$

The Green-Lagrangian strain tensor is symmetric and thus a measure for the strain alone, excluding rotations. The nonlinear terms in equation 2.139 are called geometric or kinematic nonlinearities. They can be neglected for very small strains resulting in the so-called kinematic or geometric linearization. Within the frame-work of system dynamics the strain tensor will be needed for defining the potential energy of elastic parts. Some problems do not allow a linearization due to effects of geometrical stiffnesses, which is connected with cases like buckling or tilting of bars or the well-known problem of rotating bars stiffened by centrifugal forces. For these or related cases a linearization comes out with wrong results. Kane and his school called it "premature linearization" ([122], [11], [10]). Anyway, some prudence will be necessary. We shall come back to this matter in later chapters.

Two main ideas characterize the combination of multibody system concepts with the continuum mechanics concepts. The first important point has been considered with the evaluation of the strain tensor $\mathbf{G}$, which usually is defined by the symbol ϵ. The accompanying matrix is given by

$$\epsilon = \mathbf{G} = \begin{pmatrix} \epsilon_{xx} & \epsilon_{xy} & \epsilon_{xz} \\ \epsilon_{yx} & \epsilon_{yy} & \epsilon_{yz} \\ \epsilon_{zx} & \epsilon_{zy} & \epsilon_{zz} \end{pmatrix}. \tag{2.140}$$

From the equations (2.139) and (2.140) as well as from the definitions $\mathbf{X} = (\xi, \eta, \zeta)^T$ and $\mathbf{u} = (u, v, w)^T$ we get the components of the strain tensor in the following form

$$\epsilon_{xx} = \frac{\partial u}{\partial \xi} + \frac{1}{2}[(\frac{\partial u}{\partial \xi})^2 + (\frac{\partial v}{\partial \xi})^2 + (\frac{\partial w}{\partial \xi})^2],$$
$$\epsilon_{yy} = \frac{\partial v}{\partial \eta} + \frac{1}{2}[(\frac{\partial u}{\partial \eta})^2 + (\frac{\partial v}{\partial \eta})^2 + (\frac{\partial w}{\partial \eta})^2],$$
$$\epsilon_{zz} = \frac{\partial w}{\partial \zeta} + \frac{1}{2}[(\frac{\partial u}{\partial \zeta})^2 + (\frac{\partial v}{\partial \zeta})^2 + (\frac{\partial w}{\partial \zeta})^2],$$

$$\epsilon_{xy} = \epsilon_{yx} = \frac{1}{2}\{\frac{\partial u}{\partial \eta} + \frac{\partial v}{\partial \xi} + [\frac{\partial u}{\partial \xi}\frac{\partial u}{\partial \eta} + \frac{\partial v}{\partial \xi}\frac{\partial v}{\partial \eta} + \frac{\partial w}{\partial \xi}\frac{\partial w}{\partial \eta}]\},$$
$$\epsilon_{yz} = \epsilon_{zy} = \frac{1}{2}\{\frac{\partial v}{\partial \eta} + \frac{\partial w}{\partial \xi} + [\frac{\partial u}{\partial \eta}\frac{\partial u}{\partial \zeta} + \frac{\partial v}{\partial \eta}\frac{\partial v}{\partial \zeta} + \frac{\partial w}{\partial \eta}\frac{\partial w}{\partial \zeta}]\},$$
$$\epsilon_{zx} = \epsilon_{xz} = \frac{1}{2}\{\frac{\partial w}{\partial \eta} + \frac{\partial u}{\partial \xi} + [\frac{\partial u}{\partial \zeta}\frac{\partial u}{\partial \xi} + \frac{\partial v}{\partial \zeta}\frac{\partial v}{\partial \xi} + \frac{\partial w}{\partial \zeta}\frac{\partial w}{\partial \xi}]\}, \tag{2.141}$$

which can be easily adapted to the coordinates chosen for the case under consideration. If we are able to really linearize these expressions, for example for problems without large overall motion, then all the nonlinear terms of the equations (2.141) become approximately zero.

The second important connection with the multibody concept concerns the rotation of the mass elements as a consequence of the deformation. We have seen by equation (2.137), that the deformation gradient $\mathbf{F}$ can be split into a stretching and a rotation part, where the rotation tensor $\mathbf{R}$ is orthogonal and its determinant $\det(\mathbf{R})=+1$. The effect of this tensor consists in a rigid body rotation. To evaluate this rotation for our multibody purposes we come back to the deformation gradient $\mathbf{F}$ and consider its symmetric and its skew-symmetric part by the decomposition (see equation 2.136)

$$\mathbf{F} = (\frac{\partial \mathbf{x}}{\partial \mathbf{X}}) = \frac{1}{2}\Big[(\frac{\partial \mathbf{x}}{\partial \mathbf{X}}) + (\frac{\partial \mathbf{x}}{\partial \mathbf{X}})^T\Big] + \frac{1}{2}\Big[(\frac{\partial \mathbf{x}}{\partial \mathbf{X}}) - (\frac{\partial \mathbf{x}}{\partial \mathbf{X}})^T\Big] = \mathbf{F}_{sym} + \mathbf{F}_{skew}$$

$$\mathbf{F}_{sym} = \frac{1}{2}\Big[(\frac{\partial \mathbf{x}}{\partial \mathbf{X}}) + (\frac{\partial \mathbf{x}}{\partial \mathbf{X}})^T\Big], \qquad \mathbf{F}_{skew} = \frac{1}{2}\Big[(\frac{\partial \mathbf{x}}{\partial \mathbf{X}}) - (\frac{\partial \mathbf{x}}{\partial \mathbf{X}})^T\Big]. \tag{2.142}$$

Remembering that according to Figure (2.23) the deformation gradient can also be expressed by $\mathbf{F} = \mathbf{E} + \frac{\partial \mathbf{u}}{\partial \mathbf{X}}$, we can write

$$\mathbf{F}_{skew} = \frac{1}{2}\Big[(\frac{\partial \mathbf{u}}{\partial \mathbf{X}}) - (\frac{\partial \mathbf{u}}{\partial \mathbf{X}})^T\Big] = \frac{1}{2}(\nabla \mathbf{u} - \nabla \mathbf{u}^T). \tag{2.143}$$

This skew-symmetric part of the deformation gradient represents the rotation field (see [264]). The rotation vector itself can be written as

$$\varphi = \frac{1}{2}\mathrm{curl}(\mathbf{u}) \qquad \text{with} \qquad \tilde{\varphi} = \mathbf{F}_{skew}. \tag{2.144}$$

Returning to our assumption of small deformations superimposed on large rigid body motion we can compare the equations (2.143), (2.144) and (2.134) and come out with the rotation vector

$$\varphi = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \frac{\partial w}{\partial \eta} - \frac{\partial v}{\partial \zeta} \\ \frac{\partial u}{\partial \zeta} - \frac{\partial w}{\partial \xi} \\ \frac{\partial v}{\partial \xi} - \frac{\partial u}{\partial \eta} \end{pmatrix} \tag{2.145}$$

With these relations we have established a correlation between the multibody concept of equation (2.134) and the rotational effect of small elastic deformations. With respect to large deformations we recommend the appropriate literature. An explanation for the above formulas is a matter of many undergraduate textbooks. Referring to Figure (2.24) we immediately realize, that the sum of the two angles $\frac{\partial u}{\partial \eta}$ and $\frac{\partial v}{\partial \xi}$ represents the stretch of the element and that the difference of the same two angles represents the rotation of the element, here depicted for the plane ξ, η.

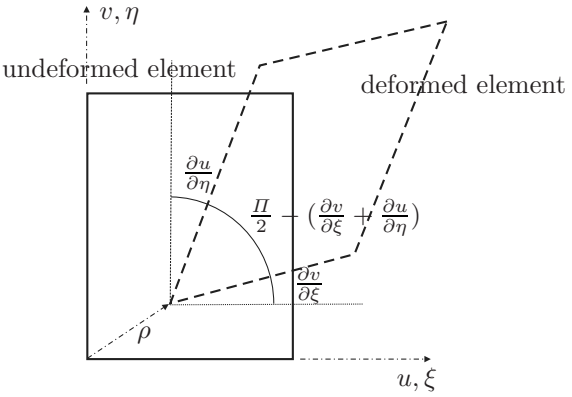

Fig. 2.24: Stretching and Rotating of an Elastic Element

2.3 Momentum and Moment of Momentum

2.3.1 Definitions and Axioms

Mutual interactions of forces with masses are the basic concept of all mechanical sciences. The result of these interactions may be very different depending on the system considered, it may be motion, deformation or just, as a limiting case, a system at rest, statically in an equilibrium state. The early days of mechanics as a science were therefore characterized by the search for some kind of relationships for force, mass and a kinematic magnitude like acceleration or velocity. The great achievements of Newton [169] must also be seen before the background of his time, where all transportation took place in coaches and carts with wooden wheels and primitive bearings, drawn by horses, giving more the impression, that velocity is much more proportional to forces than acceleration. Newton's laws overcame these popular ideas.

An equally great achievement was the finding of Euler [58], that in addition to Newton's ideas of momentum the laws for the moment of momentum represent independent mechanical statements and cannot be "derived" from the momentum equations, which is sometimes done in older textbooks of mechanics. In the meantime we know, that the moment of momentum equations as usually applied depend on Boltzmann's axiom or the symmetry of Cauchy's stress tensor. For polar materials for example these moment of momentum equations must be supplemented by some expressions including the tensor of moment stresses.

We consider some rigid or elastic body under the influence of active and passive forces (Figure 2.25), where the active forces contribute to the motion and the passive forces not. We know, that the moment equation of Newton and the moment of momentum equation of Euler are independent laws not derivable from each other. We furtheron assume, that there will exist an inertial coordinate system, where these equations become valid. Therefore and following an idea of [63] we define these two equations in the form of two axioms and write

$$\int_{\mathcal{B}} (\ddot{\mathbf{r}}\mathrm{dm} - \mathrm{d}\mathbf{F}_a) = \mathbf{0}, \qquad \int_{\mathcal{B}} \mathbf{r} \times (\ddot{\mathbf{r}}\mathrm{dm} - \mathrm{d}\mathbf{F}_a) = \mathbf{0}, \tag{2.146}$$

where $\mathbf{r}$ is a vector in an inertial frame I to a mass element dm of the body B, and $(\mathrm{d}\mathbf{F}_a)$ are active forces. With respect to Figure 2.25 we should note, that the passive forces $\mathrm{d}\mathbf{F}_p$ indicated in that Figure are only passive without internal deformations.

We have used the definitions "a" for active and "p" for passive. The idea of active and passive forces being used in continuum mechanics for quite a time is more adequate for our considerations than external and internal forces, though in some cases it means the same. But for unilateral problems the features "active" and "passive" change during the motion, and therefore the

definitions “external” and “internal” do not help so much. As a reminder: active forces can be shifted along their lines of action, passive forces cannot. From this we state, that active forces produce work and power, passive forces not.

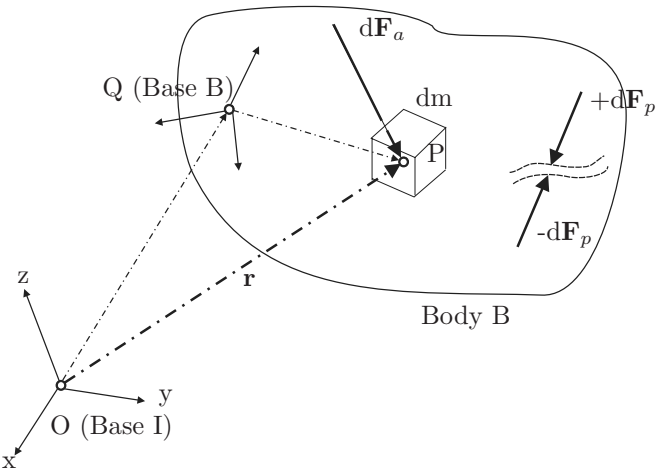

Fig. 2.25: Moment and Moment of Momentum

2.3.2 Momentum

According to the equations 2.146 (see also Figure 2.25) we define the momentum by

$$\mathbf{p} = \int_{\mathcal{B}} \dot{\mathbf{r}} \mathrm{dm}, \tag{2.147}$$

which is a coordinate-free representation. The velocity $\dot{\mathbf{r}}$ is an absolute velocity, and as always, derivations with respect to time have to be performed in an inertial system. On the other hand it is of course possible to transform these equations into any other coordinate system, for example into a body-fixed frame. We come back to this point in chapter 2.3.4.

The fundamental laws considering momentum are the famous three laws of Newton, which possess the quality of axioms. We shall be not so ambitious to give his statements in the original form (for this purpose see [169]). The first axiom writes [175]

Axiom 1. *A body at rest remains at rest and a body in motion moves in a straight line with unchanging velocity, unless some external force acts on it.*

To illustrate this basic law, which we find already in the statements of Galilei [259], we shall use the notation introduced by Euler for the momentum and moment of momentum laws. Referring to Axiom 1 we have no external, thus no active forces, which means $\int_{\mathcal{B}} \mathrm{d}\mathbf{F}_a = \mathbf{0}$ and therefore $\int_{\mathcal{B}} \ddot{\mathbf{r}}\mathrm{dm} = \mathbf{0}$ resulting in

$$\mathbf{p} = \int_{\mathcal{B}} \dot{\mathbf{r}}\mathrm{dm} = \text{constant}, \tag{2.148}$$

which represents the law of conservation of momentum. Considering the mass center of a body we get

$$\mathbf{p}_C = \mathbf{p}_{C0} = \dot{\mathbf{r}_C}\mathrm{m} \qquad \text{with} \qquad \mathbf{r}_C\mathrm{m} = \int_{\mathcal{B}} \mathbf{r}\mathrm{dm} \tag{2.149}$$

Axiom 2. *The rate of change of the momentum of a body is proportional to the resultant external force that acts on the body.*

For the mass element of Figure 2.25 we get from the first equation 2.146

$$\ddot{\mathbf{r}}\mathrm{dm} - \mathrm{d}\mathbf{F}_a = \mathbf{0} \tag{2.150}$$

which represents also the momentum budget for a point mass. The time derivative of the momentum is mass times acceleration if we are dealing with a constant mass, as in the above equation. For not constant masses the time derivative of the mass must be considered in addition. In terms of our definitions we may write

$$\frac{\mathrm{d}\,\mathbf{p}}{\mathrm{d}\,t} = \mathbf{F}, \qquad \text{with} \qquad \mathbf{p} = \int_{\mathcal{B}} \dot{\mathbf{r}}\mathrm{dm}, \qquad \text{and} \qquad \mathbf{F} = \int_{\mathcal{B}} \mathrm{d}\mathbf{F}_a. \tag{2.151}$$

Taking again the center of mass of the body we come out with

$$\mathrm{m}\left(\frac{\mathrm{d}\,\mathbf{v}_C}{\mathrm{d}\,t}\right) = \mathbf{F}_C. \tag{2.152}$$

The velocity $\mathbf{v}_C$ is defined with respect to an inertial system. It is an absolute velocity. The force vector $\mathbf{F}_C$ is the vector sum of all forces which act on the body. Generally this vector sum does not pass through the center of mass resulting in an additional torque, which has to be regarded in the moment of momentum equation.

Newton's third law writes

Axiom 3. *Action and reaction are equal and opposite.*

At the times of Newton this finding was new. But it is very obvious from experience. Wherever any force acts on a body or on the environment we get

as a reaction the same force with opposite sign. My feet transfer my weight to the ground, as a reaction the ground is loaded with my weight force in the opposite direction. There is no mechanical interaction without this basic property.

The forces acting on a body might be applied forces, elastic forces or single- and set-valued forces. We shall consider all of them, but concentrate here on forces due to elastic influences. Equation (2.146) then writes

$$\int_{\mathcal{B}} \ddot{\mathbf{r}} \mathrm{d}m = \int_{\partial\mathcal{B}} \mathbf{p}_\sigma \mathrm{d}\mathbf{A} + \int_{\mathcal{B}} \mathbf{f}_a \mathrm{d}m, \tag{2.153}$$

where the first integral on the right hand side is a surface force due to elasticity and the second integral a volume force of some given type. The stress vector $\mathbf{p}_\sigma$ acts on the surface $\partial\mathcal{B}$ with the area vector $d\mathbf{A}$. As is well known we can express the stress vector by

$$\mathbf{p}_\sigma = \sigma\mathbf{n}, \tag{2.154}$$

which indicates, that the stress vector $\mathbf{p}_\sigma$ comes from the surface normal vector $\mathbf{n}$ by a homogenuous and linear transformation with the Cauchy stress tensor σ [14]. The Cauchy stress tensor itself is symmetric and describes the stress situation for a homogenuous and isotropic body (Figure (2.26) and ([228], [147])).

$$\sigma = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix}, \qquad \text{with} \quad \sigma_{ij} = \sigma_{ji}, \quad (i,j = x,y,z). \tag{2.155}$$

Combining these equations and regarding in addition the Gauss theorem for surface and volume integrals results in

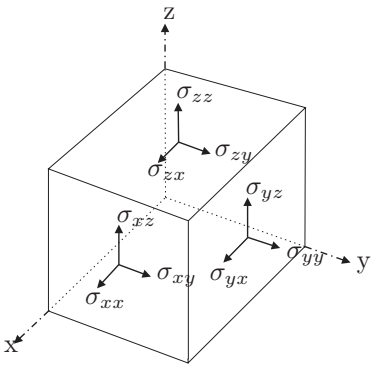

Fig. 2.26: Stresses

$$\int_{\mathcal{B}} \ddot{\mathbf{r}} \mathrm{dm} = \int_{\mathcal{B}} \mathrm{div}(\sigma) dV + \int_{\mathcal{B}} \mathbf{f}_a \mathrm{dm}, \quad \text{with} \quad \mathrm{div}(\sigma) = \begin{pmatrix} \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} \\ \frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} \\ \frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{zy}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} \end{pmatrix}. \tag{2.156}$$

For our purposes and considering only linearly elastic bodies we introduce for a consitutive law the simple material law of Hooke, which writes

$$\sigma = \mathbf{E}\epsilon \tag{2.157}$$

with a constant module of elasticity matrix $\mathbf{E}$ (Young's modulus). Therefore equation (2.156) writes

$$\int_{\mathcal{B}} \ddot{\mathbf{r}} \mathrm{dm} = \mathbf{E} \int_{\mathcal{B}} \mathrm{div}(\mathbf{G}) dV + \int_{\mathcal{B}} \mathbf{f}_a \mathrm{dm}, \tag{2.158}$$

with the definition of the strain tensor ($\epsilon = \mathbf{G}$) by the equations (2.140) and (2.141), which we shall not evaluate here in combining them with the above equation. For simple structures like bars or plates this can be performed straightforwardly (see [28]).

2.3.3 Moment of Momentum

Euler has been the first one to understand the law of moment of momentum as a basic independent law of mechanics. It cannot be "derived" from the second axiom of Newton just by performing the cross-product. We refer to the literature ([27], [180]). From the equations 2.146 we define the moment of momentum by

$${}_I\mathbf{L} = \int_{\mathcal{B}} {}_I\mathbf{r} \times {}_I\dot{\mathbf{r}} \mathrm{dm}. \tag{2.159}$$

The second axiom of the equations (2.146) may be written in the form

$$\frac{\mathrm{d}\,\mathbf{L}}{\mathrm{d}\,t} = \mathbf{M} \qquad \text{with} \qquad \mathbf{M} = \int_{\mathcal{B}} \mathbf{r} \times \mathrm{d}\mathbf{F}_a \tag{2.160}$$

with all magnitudes represented in an inertial system and the time derivation also performed in an inertial frame. For missing active torques $\mathbf{M}$ the equation (2.160) is $\frac{d\mathbf{L}}{dt} = \mathbf{0}$ and therefore $\mathbf{L} = \mathit{constant}$. This conservation of moment of momentum writes

$$\frac{\mathrm{d}\,\mathbf{L}}{\mathrm{d}\,t} = \mathbf{0}, \qquad \mathbf{L} = \mathbf{L}_0 = \int_{\mathcal{B}} \mathbf{r} \times \dot{\mathbf{r}} \mathrm{dm}. \tag{2.161}$$

We may go two ways. We might assume the equations (2.146) as axioms, which then allows the confirmation of the symmetry of the Cauchy stress tensor from these axioms. Or we might go the way in assuming the symmetry beforehand , that means Boltzmann's axiom, which then comes out with the moment of momentum equation as we know them. We shall pursue the first possibility. Assuming an elastic body as discussed in the chapter (2.3.2) before and evaluating for this body the moment of momentum budget we get the equations (see [14])

$$\frac{\mathrm{d}}{\mathrm{d}\,t}\int\limits_{\mathcal{B}}(\mathbf{r}\times\dot{\mathbf{r}})\rho\mathrm{dV}=\int\limits_{\partial\mathcal{B}}(\mathbf{r}\times\mathbf{p}_\sigma)\mathrm{dA}+\int\limits_{\mathcal{B}}(\mathbf{r}\times\mathbf{f}_a)\rho\mathrm{dV}, \tag{2.162}$$

where we have replaced the mass element dm by (ρdV) with the density ρ, and where we have used the formula (2.153). The stress vector $\mathbf{p}$ acts on the surface $\partial\mathcal{B}$ of a volume element. The applied force $\mathbf{f}_a$ is some given volume force. We assume, that the volume element $\mathcal{B}$ has a constant volume dV and a constant surface dA on $\partial\mathcal{B}$. The stress vector $\mathbf{p}_\sigma$ was already defined with equation (2.154), namly $\mathbf{p}_\sigma=\sigma\mathbf{n}$. The cross-product can be replaced by a tensor product, $(\mathbf{r}\times\mathbf{p}_\sigma)=(\tilde{\mathbf{r}}\cdot(\mathbf{p}_\sigma))$, allowing us to write

$$\int\limits_{\partial\mathcal{B}}(\mathbf{r}\times\sigma\mathbf{n})\mathrm{dA}=\int\limits_{\partial\mathcal{B}}\tilde{\mathbf{r}}\sigma\mathbf{n}\mathrm{dA},\qquad\mathbf{n}=\begin{pmatrix}\mathbf{n}_x\\ \mathbf{n}_y\\ \mathbf{n}_z\end{pmatrix},\qquad\tilde{\mathbf{r}}=\begin{pmatrix}0 & -z & +y\\ +z & 0 & -x\\ -y & +x & 0\end{pmatrix}. \tag{2.163}$$

The first two magnitudes in the above equations can be easily evaluated to give

$$\tilde{\mathbf{r}}\sigma=\frac{1}{2}\begin{pmatrix}(-z\sigma_{yx}+y\sigma_{zx}) & (-2z\sigma_{yy}+y\sigma_{zy}) & (-z\sigma_{yz}+2y\sigma_{zz})\\ (+2z\sigma_{xx}-x\sigma_{zx}) & (+z\sigma_{xy}-x\sigma_{zy}) & (+z\sigma_{xz}-2x\sigma_{zz})\\ (-2y\sigma_{xx}+x\sigma_{yx}) & (-y\sigma_{xy}+2x\sigma_{yy}) & (-y\sigma_{xz}+x\sigma_{yz})\end{pmatrix}. \tag{2.164}$$

Performing some manipulations of the above relations [14] and applying the Gauss theorem we furtheron can write

$$\int\limits_{\partial\mathcal{B}}(\mathbf{r}\times\sigma\mathbf{n})\mathrm{dA}=\int\limits_{\partial\mathcal{B}}\tilde{\mathbf{r}}\sigma\mathbf{n}\mathrm{dA}=\int\limits_{\mathcal{B}}[\mathrm{div}(\tilde{\mathbf{r}}\sigma)+\Delta\mathbf{g}]\mathrm{dV} \tag{2.165}$$

where the term $\Delta\mathbf{g}$ represents a remaining term arising from the evaluation of equation (2.163). Going back to the moment of momentum budget of equation (2.162) and inserting there the momentum balance $\rho\frac{\mathrm{d}\,\dot{\mathbf{r}}}{\mathrm{d}\,t}=\rho\mathbf{f}+\mathrm{div}\sigma$ for the elastic element we get

$$\int\limits_{\mathcal{B}}\tilde{\mathbf{r}}(\mathrm{div}\sigma+\rho\mathbf{f})\mathrm{dV}=\int\limits_{\mathcal{B}}\tilde{\mathbf{r}}(\mathrm{div}\sigma+\rho\mathbf{f})\mathrm{dV}+\int\limits_{\mathcal{B}}\Delta\mathbf{g}\mathrm{dV}. \tag{2.166}$$

We see [14], that all terms cancel out with one exception, namely the volume integral including $\Delta\mathbf{g}$. The vector $\Delta\mathbf{g}$ writes

$$\Delta\mathbf{g} = \begin{pmatrix} +\sigma_{zy} - \sigma_{yz} \\ -\sigma_{zx} + \sigma_{xz} \\ +\sigma_{yx} - \sigma_{xy}. \end{pmatrix} = \mathbf{0}. \tag{2.167}$$

It vanishes for a symmetric Cauchy stress tensor σ. Only then we are allowed to use the moment of momentum equation for rigid bodies in the classical form. A vanishing $\Delta\mathbf{g}$ confirms that internal forces counterbalance and do not contribute to the motion. For very many materials this is true, but for some classes of materials like polar materials this is not true, and then we have to look for it.

2.3.4 Transformations

The above defined expressions for momentum and moment of momentum are to be given in an inertial frame. Also all derivations with respect to time have to be performed in such a coordinate system. But of course it is not forbidden to transform the resulting equations into any other coordinate base, which especially makes sense before carrying out some time derivations. In the following we shall give some formulas transformed from the inertial system into a body-fixed one or vice versa applying the relations developed in the kinematics chapter 2.2.

The momentum equation is defined with (2.147). Considering an additional body-fixed coordinate system on body B and taking into regard the equations (2.28) and (2.38) together with the Figure (2.13) we also can write the momentum definition in the forms

$${}_I\mathbf{p} = \int_{\mathcal{B}} ({}_I\dot{\mathbf{r}}_{OP})\mathrm{dm} = \int_{\mathcal{B}} ({}_I\dot{\mathbf{r}}_{OQ} + \dot{\mathbf{A}}_{IB} \cdot {}_B\mathbf{r}_{QP} + \mathbf{A}_{IB} \cdot {}_B\dot{\mathbf{r}}_{QP})\mathrm{dm},$$

$${}_I\mathbf{p} = \int_{\mathcal{B}} ({}_I\dot{\mathbf{r}}_{OP})\mathrm{dm} = \int_{\mathcal{B}} \mathbf{A}_{IB} \cdot ({}_B\mathbf{v}_{Q,abs} + {}_B\tilde{\boldsymbol{\omega}} \cdot {}_B\mathbf{r} + {}_B\dot{\mathbf{r}})\mathrm{dm}. \tag{2.168}$$

The relative velocity ${}_B\dot{\mathbf{r}}$ is the point, where possible elastic effects might enter the system, or where some particle motion exists. It vanishes for rigid bodies and for bodies with no relatively moved masses.

In a similar way we express the moment of momentum in body-fixed coordinates. Starting with the equations (2.159, 2.28, 2.38) and taking from Figure (2.25) the relation $\mathbf{r}_{OP} = \mathbf{r}_{OQ} + \mathbf{r}_{QP}$ we get

$${}_I\mathbf{L} = \int_{\mathcal{B}} ({}_I\mathbf{r}_{OQ} + {}_I\mathbf{r}_{QP}) \times ({}_I\dot{\mathbf{r}}_{OQ} + {}_I\dot{\mathbf{r}}_{QP})\mathrm{dm} = \mathbf{A}_{IB} \ {}_B\mathbf{L}. \tag{2.169}$$

The determination of ${}_B\mathbf{L}$ is straightforward, if we define ${}_B\mathbf{v}_{P,abs} = ({}_B\mathbf{v}_{Q,abs} + {}_B\tilde{\boldsymbol{\omega}} \cdot {}_B\mathbf{r} + {}_B\dot{\mathbf{r}})$ with the absolute velocities ${}_I\mathbf{v}_{Q,abs} = {}_I\dot{\mathbf{r}}_{OQ}$, ${}_B\mathbf{v}_{Q,abs} = \mathbf{A}_{BI}\,{}_I\dot{\mathbf{r}}_{OQ}$ and the abbreviation ${}_B\mathbf{r}_{QP} = {}_B\mathbf{r}$. We come out with

$$
\begin{aligned}
{}_B\mathbf{L} &= \int_{\mathcal{B}} ({}_B\tilde{\mathbf{r}}_{OQ} + {}_B\tilde{\mathbf{r}}) \cdot ({}_B\mathbf{v}_{Q,abs} + {}_B\tilde{\boldsymbol{\omega}} \cdot {}_B\mathbf{r} + {}_B\dot{\mathbf{r}})\mathrm{dm}, \\
&= \int_{\mathcal{B}} {}_B\tilde{\mathbf{r}}_{OQ} \cdot {}_B\mathbf{v}_{P,abs}\mathrm{dm} - {}_B\tilde{\mathbf{v}}_{Q,abs} \int_B {}_B\mathbf{r}\mathrm{dm} + (-\int_B {}_B\tilde{\mathbf{r}}_B\tilde{\mathbf{r}}\mathrm{dm})_B\boldsymbol{\omega} \\
&+ \int_B {}_B\tilde{\mathbf{r}}_B\dot{\mathbf{r}}\mathrm{dm},
\end{aligned}
$$

$$
{}_B\mathbf{L} = \int_{\mathcal{B}} {}_B\tilde{\mathbf{r}}_{OQ} \cdot ({}_B\mathbf{v}_{P,abs}\mathrm{dm}) + {}_B\tilde{\mathbf{r}}_{QS}(\mathrm{m}_B\mathbf{v}_{Q,abs}) + {}_B\mathbf{I}_B\boldsymbol{\omega} + {}_B\mathbf{L}_Q, \tag{2.170}
$$

where we have introduced the following abbreviations

$$
\int_{\mathcal{B}} {}_B\mathbf{r}\mathrm{dm} = \mathrm{m}_B\mathbf{r}_{QS}, \qquad -\int_{\mathcal{B}} {}_B\tilde{\mathbf{r}}_B\tilde{\mathbf{r}}\mathrm{dm} = {}_B\mathbf{I}, \qquad \int_{\mathcal{B}} {}_B\tilde{\mathbf{r}}_B\dot{\mathbf{r}}\mathrm{dm} = {}_B\mathbf{L}_Q. \tag{2.171}
$$

The meaning is clear. The first term in the last equation (2.170) represents the moment with respect to Q of the absolute momentum $({}_B\mathbf{v}_{P,abs}\mathrm{dm})$ in point P, the second term the moment with respect to the center of mass S of the momentum $(\mathrm{m}_B\mathbf{v}_{Q,abs})$ in point Q, the third term is the classical moment of momentum expression for rigid bodies without relative velocities ${}_B\dot{\mathbf{r}}$ and with point Q fixed and finally the last term is a kind of relative moment of momentum for a body with moving parts. If we choose the center of mass S as a reference point instead of Q, we get ${}_B\mathbf{r}_{QS} = \mathbf{0}$, and the second term becomes zero. Though well known from every undergraduate textbook of mechanics we shall repeat here the components of the inertia tensor ${}_B\mathbf{I}$.

$$
\mathbf{I} = -\int_{\mathcal{B}} \tilde{\mathbf{r}}\tilde{\mathbf{r}}\mathrm{dm} = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix}
$$

$$
\begin{aligned}
I_{xx} &= A = \int_{\mathcal{B}} (y^2 + z^2)\mathrm{dm}, & I_{xy} &= I_{yx} = -F = \int_{\mathcal{B}} xy\mathrm{dm}, \\
I_{yy} &= B = \int_{\mathcal{B}} (z^2 + x^2)\mathrm{dm}, & I_{xz} &= I_{zx} = -E = \int_{\mathcal{B}} zx\mathrm{dm}, \\
I_{zz} &= C = \int_{\mathcal{B}} (x^2 + y^2)\mathrm{dm}, & I_{yz} &= I_{zy} = -D = \int_{\mathcal{B}} yz\mathrm{dm}.
\end{aligned} \tag{2.172}
$$

In classical textbooks we find the notation as above, but for the inertia tensor also quite often $\boldsymbol{\Theta}$ instead of $\mathbf{I}$.

Some remarks to the above transformations, which are of course only specific examples for that, what has to be done in building a set of equations of motion and a complete set for all constraints. We establish the kinematics of a system in a first step, performing all relevant transformations with respect to body-fixed and inertial coordinates, we then establish in a second step the kinetic equations also including all necessary coordinate transformations, in a further step we must consider the constraints to finally end with a complete set of the system dynamics. The methods and the style of formulating things turned out to be very helpful also with respect to very large problems. A pars pro toto example might be the introduction of the tilde operation $\tilde{\mathbf{a}} \cdot \mathbf{b} = \mathbf{a} \times \mathbf{b}$, which simplifies all matrix-vector manipulations considerably.

2.4 Energy

2.4.1 Introduction

If we move a mass element dm under the influence of an active force $d\mathbf{F}_a$ from a point 1 to a point 2 along some arbitrary path (Figure 2.27), then the following work is done

$$\mathrm{dW} = \int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathrm{d}\mathbf{F}_a^T \mathrm{d}\mathbf{r} = \mathrm{dm} \int_{\mathbf{r}_1}^{\mathbf{r}_2} \ddot{\mathbf{r}}^T \mathrm{d}\mathbf{r} \tag{2.173}$$

Applying some manipulations to the second term of the above equation we get

$$\mathrm{dm} \int_{\mathbf{r}_1}^{\mathbf{r}_2} \ddot{\mathbf{r}}^T \mathrm{d}\mathbf{r} = \mathrm{dm} \int_{\mathbf{r}_1}^{\mathbf{r}_2} \frac{\mathrm{d}\,\dot{\mathbf{r}}^T}{\mathrm{d}\,t} \mathrm{d}\mathbf{r} = \frac{1}{2} \mathrm{dm} \int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathrm{d}(\dot{\mathbf{r}}^T \dot{\mathbf{r}}) = \frac{1}{2} \mathrm{dm}(\dot{\mathbf{r}}_2^2 - \dot{\mathbf{r}}_1^2) = \mathrm{dT}_2 - \mathrm{dT}_1. \tag{2.174}$$

From this the work done by shifting dm from point 1 to point 2 is given by

$$\mathrm{dW} = \mathrm{dT}_2 - \mathrm{dT}_1 = \int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathrm{d}\mathbf{F}_a^T \cdot \mathrm{d}\mathbf{r} \tag{2.175}$$

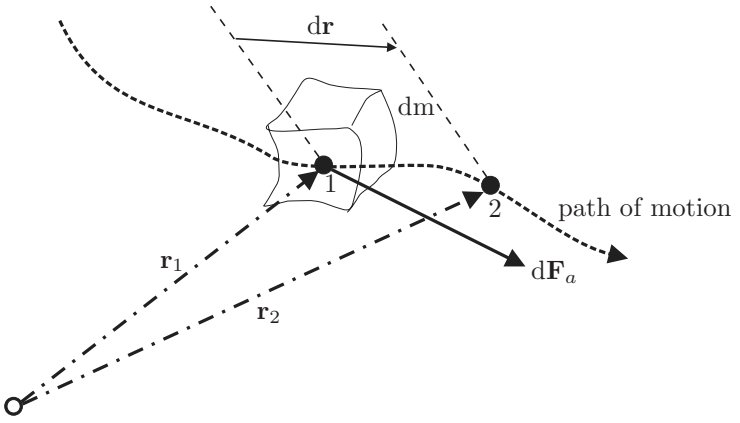

Fig. 2.27: Work and Energy

The work done by the active force is equal to the difference of the kinetic energies. If we move in a force field, where along a closed trajectory s no work

is produced, then we call the corresponding system a conservative system with the property

$$\oint_s \mathrm{d}\mathbf{F}_a^T \cdot \mathrm{d}\mathbf{r} = 0. \tag{2.176}$$

Systems of that kind do not dissipate energy, and they are not supplied with energy. Applying Stokes theorem (see [283]) we can write

$$\oint_s \mathrm{d}\mathbf{F}_a^T \cdot \mathrm{d}\mathbf{r} = \int_{\mathcal{A}} (\mathrm{curl}(\mathrm{d}\mathbf{F}_a))\mathrm{dA}, \tag{2.177}$$

from which we follow that

$$\mathrm{curl}(\mathrm{d}\mathbf{F}_a) = 0 \quad \Leftrightarrow \quad \mathrm{d}\mathbf{F}_a = -\mathrm{grad}(\mathrm{dV}) \tag{2.178}$$

The force field can be derived from a potential V. Therefore we get further on

$$\int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathrm{d}\mathbf{F}_a^T \cdot \mathrm{d}\mathbf{r} = -\int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathrm{grad}(\mathrm{dV})\mathrm{d}\mathbf{r} = -\int_{\mathbf{r}_1}^{\mathbf{r}_2} \frac{\partial V}{\partial \mathbf{r}}\mathrm{d}\mathbf{r} = -(\mathrm{dV}_2 - \mathrm{dV}_1). \tag{2.179}$$

The work done is proportional to the negative difference of the potential function dV between the points 1 and 2 and for a conservative force field as defined by the equations (2.178). Together with equation (2.175) we get

$$\mathrm{dW} = -(\mathrm{dV}_2 - \mathrm{dV}_1) = (\mathrm{dT}_2 - \mathrm{dT}_1) \quad \Leftrightarrow \quad \mathrm{dT}_1 + \mathrm{dV}_1 = \mathrm{dT}_2 + \mathrm{dV}_2 = \mathrm{dT} + \mathrm{dV}, \tag{2.180}$$

or by integration over a whole body

$$\mathrm{T} + \mathrm{V} = \mathrm{E}_0 = \text{constant}. \tag{2.181}$$

In conservative systems there will be no energy losses, and the energy conservation gives a first integral of motion sometimes useful for applications. In the presence of friction the total energy decreases, and equation (2.176) does not apply. But, on the other hand, a force field including also non-smooth force laws might still be conservative as long as no frictional energy losses occur.

2.4.2 Kinetic Energy

Considering a rigid mass-element dm and applying the relevant expressions for the absolute velocity written in a body-fixed frame (equation 2.38) we can express the kinetic energy by

$$
\begin{aligned}
\mathrm{T} =& \frac{1}{2}\int_m \mathbf{v}^T\mathbf{v}\mathrm{dm} \\
=& \frac{1}{2}\int_m ({}_B\mathbf{v}_Q + {}_B\tilde{\boldsymbol{\omega}}\cdot {}_B\mathbf{r})^T\cdot({}_B\mathbf{v}_Q + {}_B\tilde{\boldsymbol{\omega}}\cdot {}_B\mathbf{r})\mathrm{dm} \\
=& \frac{1}{2}\int_m [(\mathbf{E}\ \tilde{\mathbf{r}}^T)\begin{pmatrix}\mathbf{v}_Q\\ \boldsymbol{\omega}\end{pmatrix}]^T\cdot[(\mathbf{E}\ \tilde{\mathbf{r}}^T)\begin{pmatrix}\mathbf{v}_Q\\ \boldsymbol{\omega}\end{pmatrix}]\mathrm{dm}
\end{aligned}
\tag{2.182}
$$

The structure of the last equation will be the same in each coordinate system, because we can transform the velocity equation (2.38) into every other base. Continuing in a body-fixed frame the second equation results in

$$
\begin{aligned}
\mathrm{T} =& \frac{1}{2}\int_m ({}_B\mathbf{v}_Q + {}_B\tilde{\boldsymbol{\omega}}\cdot {}_B\mathbf{r})^T\cdot({}_B\mathbf{v}_Q + {}_B\tilde{\boldsymbol{\omega}}\cdot {}_B\mathbf{r})\mathrm{dm} \\
=& \frac{1}{2}\int_m ({}_B\mathbf{v}_Q{}^T\,{}_B\mathbf{v}_Q + 2\,{}_B\mathbf{r}^T\,{}_B\tilde{\boldsymbol{\omega}}^T\,{}_B\mathbf{v}_Q + {}_B\boldsymbol{\omega}^T(-\tilde{\mathbf{r}}\tilde{\mathbf{r}})\,{}_B\boldsymbol{\omega})\mathrm{dm} \\
=& \frac{1}{2}\mathrm{m}({}_B\mathbf{v}_Q{}^T\,{}_B\mathbf{v}_Q) + \mathrm{m}\,{}_B\mathbf{r}_S{}^T\,{}_B\tilde{\boldsymbol{\omega}}^T\,{}_B\mathbf{v}_Q + \frac{1}{2}\,{}_B\boldsymbol{\omega}^T\,{}_B\mathbf{I}_B\boldsymbol{\omega},
\end{aligned}
\tag{2.183}
$$

with the already mentioned relations

$$
\mathrm{m}\,{}_B\mathbf{r}_S = \int_m {}_B\mathbf{r}\mathrm{dm}, \qquad {}_B\mathbf{I} = -\int_m {}_B\tilde{\mathbf{r}}\,{}_B\tilde{\mathbf{r}}\mathrm{dm}. \tag{2.184}
$$

The last equation of (2.183) contains the well-known terms for a pure translation, a pure rotation and a mixed term, which disappears for ${}_B\mathbf{r}_S = \mathbf{0}$ in the case of choosing the mass center S as a reference point. In a body-fixed frame the magnitudes ${}_B\mathbf{r}_S$ and ${}_B\mathbf{I}$ are constant, which underlines the necessity of using such body-fixed coordinates, but in an inertial frame for example these magnitudes depend on time, because we look at the system so-to-say from outside, from an external point of view. For large multibody systems we must travel through a large number of coordinate systems using the possibilities of chapter (2.2.4), which finally results in structurally similar expressions but multiply augmented by each additional coordinate system and reasonably to be evaluated only by a computer.

To give an idea of the influence of elastic parts on the energy we go back to chapter (2.2.8), consider again the equations (2.133) and Figure (2.22). The absolute velocity of a deformed body point can only be achieved by transforming these equations into an inertial frame and after differentiation by transforming them back into a body-fixed frame, if necessary. The transformation matrix from B_i to I will be called $\mathbf{A}_{IB_i}$. Then we get

$$
\begin{aligned}
{}_I\mathbf{r}_{IB_{ei}} =& \mathbf{A}_{IB_iB_i}\mathbf{r}_{IB_{ei}} \\
=& \mathbf{A}_{IB_i}[{}_{B_i}\mathbf{r}_{IB_i} + {}_{B_i}\mathbf{r}_{B_iB_{ri}} + \mathbf{A}_{B_iB_{ei}}\cdot({}_{B_{ei}}\mathbf{r}_{B_{ri}B_{ei}})].
\end{aligned}
\tag{2.185}
$$

From the equations (2.134) and (2.145) together with Figure (2.22) we have in addition (assuming small elastic deformations)

$$\mathbf{A}_{B_i B_{ei}} \approx \begin{pmatrix} 1 & -\gamma & +\beta \\ +\gamma & 1 & -\alpha \\ -\beta & +\alpha & 1 \end{pmatrix}_i = \mathbf{E}_i + \tilde{\boldsymbol{\varphi}}_i,$$

$$\boldsymbol{\varphi}_i = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}_i = \frac{1}{2} \begin{pmatrix} \frac{\partial w}{\partial \eta} - \frac{\partial v}{\partial \zeta} \\ \frac{\partial u}{\partial \zeta} - \frac{\partial w}{\partial \xi} \\ \frac{\partial v}{\partial \xi} - \frac{\partial u}{\partial \eta} \end{pmatrix}_i ,$$

$${}_{B_{ei}}\mathbf{r}_{B_{ri}B_{ei}} = {}_{B_{ei}}\mathbf{u}_i \quad \text{with} \quad {}_{B_{ei}}\mathbf{u}_i = (\mathrm{u,v,w})_i^T . \tag{2.186}$$

The absolute velocity of the mass element dm_e of Figure (2.22) is then obtained simply by formal differentiation of ${}_I\mathbf{r}_{IB_{ei}}$. We get

$${}_I\dot{\mathbf{r}}_{IB_{ei}} = \dot{\mathbf{A}}_{IB_i B_i}\mathbf{r}_{IB_{ei}} + \mathbf{A}_{IB_i B_i}\dot{\mathbf{r}}_{IB_{ei}}, \tag{2.187}$$

which we can evaluate a bit further. With $\mathbf{A}_{B_i I}\dot{\mathbf{A}}_{IB_i} = {}_B\tilde{\boldsymbol{\omega}}_{IB}$ and according to the corresponding relations in the kinematics chapter we may write for the absolute velocity in the B_i-frame

$${}_{B_i}\mathbf{v}_{abs} = \mathbf{A}_{B_i I}{}_I\dot{\mathbf{r}}_{IB_{ei}} = {}_{B_i}\dot{\mathbf{r}}_{IB_{ei}} + {}_{B_i}\tilde{\boldsymbol{\omega}}_{IB_i B_i}\mathbf{r}_{IB_{ei}}, \tag{2.188}$$

where ${}_{B_i}\mathbf{r}_{IB_{ei}}$ is given with equation (2.185) and the velocity ${}_{B_i}\dot{\mathbf{r}}_{IB_{ei}}$ follows from

$${}_{B_i}\dot{\mathbf{r}}_{IB_{ei}} = {}_{B_i}\dot{\mathbf{r}}_{IB_i} + {}_{B_i}\dot{\mathbf{r}}_{B_i B_{ri}} + \mathbf{A}_{B_i B_{ei}}({}_{B_{ei}}\dot{\mathbf{u}}_i + {}_{B_{ei}}\tilde{\boldsymbol{\omega}}_{B_i B_{ei} B_{ei}}\mathbf{u}_i) \tag{2.189}$$

with $\dot{\mathbf{A}}_{B_i B_{ei}} = \mathbf{A}_{B_i B_{ei}} \cdot {}_{B_{ei}}\tilde{\boldsymbol{\omega}}_{B_i B_{ei}}$ representing the rotation of the mass element due to the elastic deformation. Setting further

$${}_{B_i}\mathbf{u}_i = \mathbf{A}_{B_i B_{ei} B_{ei}}\mathbf{u}_i, \quad \mathbf{A}_{B_i B_{ei}}({}_{B_{ei}}\tilde{\boldsymbol{\omega}}_{B_i B_{ei}})_{B_{ei}}\mathbf{u}_i = {}_{B_i}\tilde{\boldsymbol{\omega}}_{B_i B_{ei} B_i}\mathbf{u}_i \tag{2.190}$$

we get finally

$$\begin{aligned} {}_{B_i}\mathbf{v}_{abs} = {}_{B_i}\dot{\mathbf{r}}_{IB_i} + {}_{B_i}\dot{\mathbf{r}}_{B_i B_{ri}} + {}_{B_i}\tilde{\boldsymbol{\omega}}_{IB_i}({}_{B_i}\mathbf{r}_{IB_i} + {}_{B_i}\mathbf{r}_{B_i B_{ri}}) \\ + {}_{B_i}\dot{\mathbf{u}}_i + ({}_{B_i}\tilde{\boldsymbol{\omega}}_{IB_i} + {}_{B_i}\tilde{\boldsymbol{\omega}}_{B_i B_{ei}}) \cdot {}_{B_i}\mathbf{u}_i, \end{aligned} \tag{2.191}$$

which confirms the classical approach, that we must add to the velocity the deformation velocity and to the angular velocity the angular velocity due to elasticity [28]. The first line of equation (2.191) represents the translational and rotational motion of the undeformed reference B_i of the body B_i. The second line expresses the influence of the elastic deformations, where for example the complete rotation is composed by the rotation between the inertial

frame and that of the body B_i depicted by ${}_{B_i}\tilde{\boldsymbol{\omega}}_{IB_i}$ plus the rotation between the undeformed body element dm_r and the deformed element dm_e (see Figure 2.22) given with ${}_{B_i}\tilde{\boldsymbol{\omega}}_{B_iB_{ei}}$, both terms written in the body-fixed frame B_i.

All necessary magnitudes are now known from the above relations. The evaluation should of course be done by a computer. The overall kinetic energy of a system writes

$$\mathrm{T} = \sum_i \int_{m_i} {}_I\dot{\mathbf{r}}_{IB_{ei}}{}^T {}_I\dot{\mathbf{r}}_{IB_{ei}} \mathrm{dm}_{ei} = \sum_i \int_{m_i} {}_B\mathbf{v}_{abs}{}^T {}_B\mathbf{v}_{abs} \mathrm{dm}_{ei}, \tag{2.192}$$

where there might be also rigid bodies with ${}_{B_{ei}}\mathbf{u}_i = (\mathrm{u,v,w})_i^T = \mathbf{0}$. It does not change formula (2.192) principally.

Some aspects should be considered in establishing the energy for elastic components: Firstly, one should keep in mind the property, that the transformation $\mathbf{A}_{IB_i}$ from the body B_i to the inertial frame I contains the influences of the elasticities of all bodies between I and B_i. As the velocities are generated by multiplication with the transformation matrices it is sufficient to retain only terms up to the second order in the elastic deformations. They are necessary to come out with an exact linearization of the elastic terms in the equations of motion. Secondly, applying a Ritz- or a Galerkin-approach for small elastic deformations we always can separate the integrals of equation (2.192) into a spatial- and a time-dependent part, where the spatial-dependent part can be evaluated beforehand.

2.4.3 Potential Energy

Considering in a first step the deformation energy we confine ourselves to the case of linear elastic deformations of isotropic materials including the symmetry of the stress tensor. One definition of this stress tensor is the following (see Figure 2.26)

$$\sigma = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix}, \qquad \text{with} \qquad \sigma_{ij} = \sigma_{ji}, \quad (i,j = x,y,z). \tag{2.193}$$

The corresponding strain tensor is given with equation (2.140). Assuming small deformations of the element of Figure (2.26), for example in x-direction an $\epsilon_{xx}\mathrm{dx}$ or in y-direction an angular deformation $\gamma_{xy}\mathrm{dx} = 2\epsilon_{xy}\mathrm{dx}$, we get the work done by these small deformations from the stress forces multiplied by the corresponding strains, in our example $\mathrm{dW}_{\epsilon_{xx}} = (\frac{1}{2}\sigma_{xx}\mathrm{dydz})(\epsilon_{xx}\mathrm{dx})$ and $\mathrm{dW}_{\gamma_{xy}} = (\frac{1}{2}\sigma_{xy}\mathrm{dydz})(\gamma_{xy}\mathrm{dx})$ with $\gamma_{xy}/2 = \epsilon_{xy}$. For all the other directions we come out with similar expressions.

Collecting all these terms and integrating over the total volume of the elastic body we have the well-known relation (see for example [147] or [28])

$$\mathrm{V} = \frac{1}{2}\int_{\mathcal{B}} (\sigma_{xx}\epsilon_{xx}+\sigma_{yy}\epsilon_{yy}+\sigma_{zz}\epsilon_{zz}+\sigma_{xy}\gamma_{xy}+\sigma_{yz}\gamma_{yz}+\sigma_{zx}\gamma_{zx})\mathrm{dx\ dy\ dz}. \quad (2.194)$$

For most engineering problems of dynamics the constitutive relations of Hooke's generalized laws will be sufficient. Confining our considerations to isotropic linearly elastic materials then they write for the strains and the shear strains, respectively,

$$\begin{aligned}
\epsilon_{xx} &= \frac{1}{\mathrm{E}}[\sigma_{xx} - \mu(\sigma_{yy} + \sigma_{zz})],\\
\epsilon_{yy} &= \frac{1}{\mathrm{E}}[\sigma_{yy} - \mu(\sigma_{zz} + \sigma_{xx})],\\
\epsilon_{zz} &= \frac{1}{\mathrm{E}}[\sigma_{zz} - \mu(\sigma_{xx} + \sigma_{yy})],
\end{aligned} \quad (2.195)$$

$$\epsilon_{xy} = \frac{\sigma_{xy}}{2\mathrm{G}}, \qquad \epsilon_{xz} = \frac{\sigma_{xz}}{2\mathrm{G}}, \qquad \epsilon_{yz} = \frac{\sigma_{yz}}{2\mathrm{G}}, \quad (2.196)$$

or in a more general form (see [19] and [228])

$$\epsilon_{ij} = \frac{1}{E}[(1+\mu)\sigma_{ij} - \mu\delta_{ij}\sigma_{nn}], \qquad (i,j = x,y,z), \quad (2.197)$$

where we have to summarize over "nn", and δ_{ij} is the Kronecker-symbol. The material constants have the following meaning: E=Young's modulus, μ=Poisson's ratio and G=shear modulus. The shear modulus can be expressed by $\mathrm{G} = \frac{\mathrm{E}}{2(1+\mu)}$. Combining the equations (2.194) to (2.197) yields the potential energy in the form

$$\begin{aligned}
\mathrm{V} = \int_{\mathcal{B}} [\frac{1}{2\mathrm{E}}(\sigma_{xx}^2 + \sigma_{yy}^2 + \sigma_{zz}^2) - \frac{\mu}{\mathrm{E}}(\sigma_{xx}\sigma_{yy} + \sigma_{yy}\sigma_{zz} + \sigma_{zz}\sigma_{xx})\\
+ \frac{1}{2\mathrm{G}}(\sigma_{xy}^2 + \sigma_{yz}^2 + \sigma_{zx}^2)]\mathrm{dxdydz},
\end{aligned} \quad (2.198)$$

or again in a more general form (see [19] and [228])

$$\mathrm{V} = \int_{\mathcal{B}} \frac{1}{2\mathrm{E}}[(1+\mu)\sigma_{ij}\sigma_{ij} - \mu\sigma_{nn}^2]\mathrm{dxdydz} \quad (2.199)$$

In a second step we only want to mention the different features of other potential energies. Any type of springs with linear, nonlinear or non-smooth characteristics possess potential energy if deformed. Gravity and the attraction or the repulsion of masses are connected with potential energies. The same is true for electrostatic and electrodynamic effects. If special problems require these or other forms of potential energy, it is easy to find models in the various literature.

2.5 On Contacts and Impacts

2.5.1 Phenomena

Impulsive motion takes places for a variety of reasons including the classical contact of two or more bodies, a sudden stop of some fluid flows or a velocity jump due to "dynamic locking" [161]. The last phenomenon was and still is a subject of many discussions and many contributions. A practical example represents the chattering chalk on a blackboard. The contacts and the impacts of two or more bodies will be of the main interest here, and therefore we shall focus on some basic aspects of such collisions.

The interest for understanding impact phenomena was always very large, because impacts possess the possibility to augment considerably the forces for a lot of practical processes like hammering or forcing piles into the ground (in German the "bear"). Therefore all great scientists and engineers worked in the one or other way on impact phenomena. Aristoteles, Galilei, Newton, Marcus Marci, Huygens, Euler, Poisson, Coulomb and many others paved the way to a modern theory of impulsive motion (see [259]).

If two or more bodies collide impulsively and with arbitrary direction, the contact zone will be deformed in normal and in tangential direction thus storing elastic potential energy with respect to these two directions. The deformation in tangential direction depends for a given state before the impact at least to a large extent on the properties of the contacting surfaces, especially on the roughnesses, which are for technical surfaces in the order of magnitude of some micrometers (μ). Under the influence of the relative velocities, friction and the stored energies we get different results.

Firstly and the energy losses being small the bodies might separate again very quickly, where the directions of the separation depend on the velocity before the impact, the frictional features and the impulse storage. We might also get a reversal of the incoming motion depending mainly on the properties in tangential direction [15]. The point of contact, averaged over the deformed contact zone, will be usually different from that point, where the spring forces due to the elastic deformation of the contact zone apply. This has influence on the whole contact process, which becomes significant for pairings of soft materials [15].

2.5.2 Impact Structure

Some typical properties of a single impact are indicated in Figure (2.28). Two bodies will impact if their relative distance $\mathbf{r}_D$ becomes zero. This event is then a starting point for a process, which usually is assumed to have an extremely short duration. Nevertheless, deformation of the two bodies occurs, being composed of compression and expansion phases. The forces governing this deformation depend on the initial dynamics and kinematics of the contacting bodies. The impulsive process ends when the normal force of contact vanishes

and changes sign, because a contact cannot realize tension forces. We do not consider adhesion phenomena. The condition of zero relative distance cannot be used as an indicator for the end of an impact, because it does not necessarily indicate also a vanishing contact force. In the general case of impact with

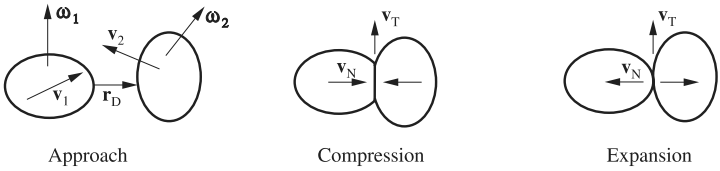

Fig. 2.28: Details of an Impact

friction we must also consider a possible change from sliding to sticking during the impulsive process, or vice versa, which includes frictional aspects as treated later. In the simple case of only normal velocities we sometimes can idealize impacts according to Newton's impact laws, which relate the relative velocity after an impact with that before an impact. Such an idealization can only be performed if the force budget allows it. In the case of impacts by hard loaded bodies we must analyze the deformation in detail. Gear hammering taking place under heavy loads and gear rattling taking place under no load are typical examples [200].

As in all other contact dynamical problems, impacts possess complementarity properties. For ideal classical inelastic impacts either the relative velocity is zero and the accompanying normal constraint impulse is not zero, or vice versa. The scalar product of relative velocity and normal impulse is thus always zero. For the more complicated case of an impact with friction we shall find such a complementarity in each phase of the impact. Friction in one contact only is characterized by a contact condition of vanishing relative distance and by two frictional conditions, either sliding or sticking (see Figure 2.29).

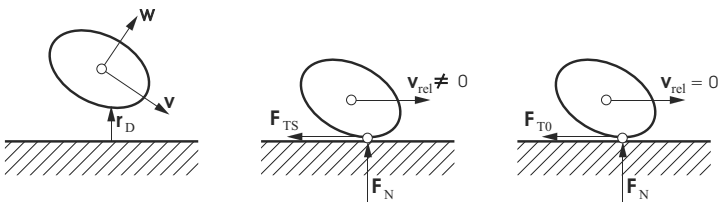

Fig. 2.29: Sliding and Static Friction

A typical property of contacts, whatsoever, is the fact, that kinematic magnitudes indicating the beginning of a contact event become a constraint at that time instant, where a contact becomes "active". For example: a non-zero normal distance between two bodies going to come into contact indicates a "passive" contact state with zero normal constraint force. In the moment it is zero, then the relative distance represents a constraint accompanied by a constraint force, and the contact is "active". In tangential direction it is similar: Non-zero tangential velocity (sliding) means a zero "friction reserve" $\mu_0|\mathbf{F}_N| - |\mathbf{F}_{TC}| = 0$ (see Figure 2.30). This tangential relative velocity becomes zero for stiction and represents then a constraint accompanied by a tangential constraint force. The end of a contact event or better of an active contact state will be always indicated by a constraint force or a combination of constraint forces. The normal constraint force becomes zero indicating a separation, and the friction reserve becomes zero indicating a change from sticking to sliding. In more detail this means:

From the contact constraint $\mathbf{r}_D = \mathbf{0}$ we get a normal constraint force $\mathbf{F}_N$ which, according to Coulomb's laws, is proportional to the friction forces, or better vice versa, the friction forces are proportional to the normal force in the contact. For sliding $\mathbf{F}_{TS} = -\mu \mathbf{F}_N \operatorname{sgn}(\mathbf{v}_{rel})$, and for stiction $\mathbf{F}_{T0} = -\mu_0 \mathbf{F}_N$, where μ and μ_0 are the coefficients of sliding and static friction, respectively. Stiction is indicated by $\mathbf{v}_{rel} = 0$ in tangential direction and by a surplus of the static friction force over the constraint force, $\mu_0|\mathbf{F}_N| - |\mathbf{F}_{TC}| \geq 0$. If this friction reserve becomes zero the stiction situation will end, and sliding will start again with a nonzero relative acceleration $\mathbf{a}_{rel}$ in the tangential direction. Again we find here complementary behavior: Either the relative velocity (acceleration) is zero and the friction reserve (saturation) is not zero, or vice versa. The product of relative acceleration and friction surplus is always zero.

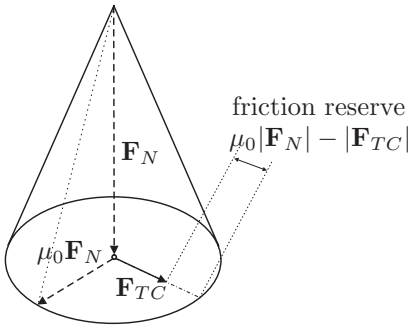

Fig. 2.30: Friction Cone and friction reserve

We may take that in a more classical way. Having stiction we are situated within the friction cone connected with the contact under consideration. The

cone boundary represents the static friction as mentioned above, and the friction reserve or the friction surplus is the distance from the tangential force $\mathbf{F}_{TC}$ to the friction cone given with $\mu_0|\mathbf{F}_N|$, see Figure (2.30). If this distance is used up by the changing dynamics of the overall system, then we are on the friction cone with the possibility of tangential sliding. But this sliding in tangential direction can only start, if the vanishing friction reserve will be accompanied by a non-zero acceleration in one of the tangential directions.

2.5.3 Basic Laws

All modern efforts to establish contact models on a micro-scale down to the molecular surface structures were not very successful, up to now. Therefore we still rely on a few laws developed by Newton, Poisson and Coulomb. Newton's impact law (1687) relates the relative velocity before an impact to the relative velocity after an impact by a coefficient of restitution ϵ, which must be determined experimentally. A value $\epsilon = 1$ represents a completely elastic and $\epsilon = 0$ a completely plastic impact. In the first case we have no energy losses and in the second case a maximum loss. Newton's law is a kinematic law and refers to the normal contact direction only, whereas the law of Poisson (1835) is a kinetic law relating the impulses after and before an impact. It can easily be extended to general impacts with normal and tangential components of velocities and impulses. The coefficients of restitution must also be measured. For our purposes of multiple impacts in multibody systems with multiple contacts Poisson's law is more general and assures correct results for all cases of technical relevancy, Newton's law not. Poisson's law allows an energy transfer between the normal and tangential directions, and vice versa, Newton's law not. Thus, Poisson's law gives a more realistic approach.

Coulomb's idea of applying a very simple relationship for contacts corresponds exactly to that, what we are doing in engineering sciences in cases where detailed and very sophisticated models are hopeless to realize. We go the simple way. It is really fascinating that these simple laws of Coulomb, which he wrote down about 1780, give such a good approximation also in complicated cases of impulsive motion. He assumes the friction forces in a contact to be proportional to the normal force and introduces friction coefficients for static and for sliding friction, which have to be determined from experiments. Coulomb's law represents the basis for the friction cone and from there, in modern non-smooth mechanics, the starting point for convex analysis.

In chapter (2.5.2) we have discussed some structural aspects by considering one contact only. For the forces we used the expressions $\mathbf{F}_N$ and $\mathbf{F}_T$. With respect to multibody systems we shall have in the chapters to come the expressions λ_N and λ_T for all kinds of constraints forces. Therefore we shall use these expressions already in the following considerations. To start with we repeat the well-known and above mentioned laws by considering some contact i, which is involved in impulsive motion. For dry friction we shall apply Coulomb's law in the following form:

$$
\begin{aligned}
|\lambda_{Ti}| < \mu_{0i}|\lambda_{Ni}| \quad &\wedge \quad \dot{g}_{Ti} = 0 \quad \text{sticking,} \\
\lambda_{Ti} = +\mu_{0i}\lambda_{Ni} \quad &\wedge \quad \dot{g}_{Ti} \leq 0 \quad \text{negative sliding,} \\
\lambda_{Ti} = -\mu_{0i}\lambda_{Ni} \quad &\wedge \quad \dot{g}_{Ti} \geq 0 \quad \text{positive sliding,}
\end{aligned} \tag{2.200}
$$

where $\dot{g}_{Ti}$ is the relative velocity in contact i, and $\lambda_{Ni}, \lambda_{Ti}$ are the relevant constraint forces in normal and tangential direction, respectively. Equation 2.200 can be interpreted as a double corner law as shown in Figure 2.31. We are

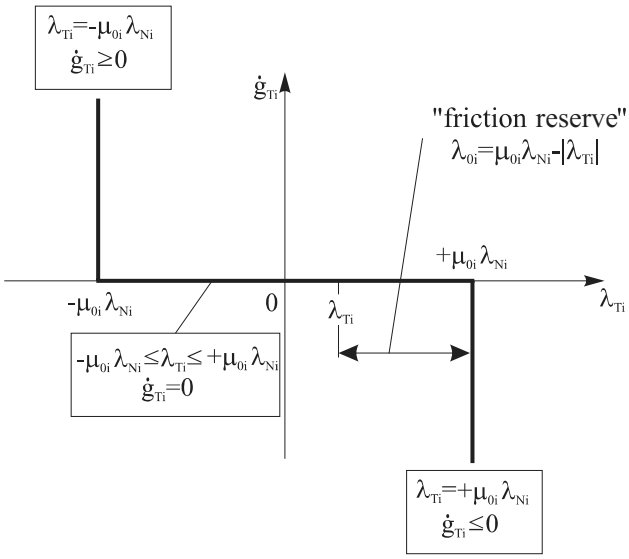

Fig. 2.31: The Friction Law of Coulomb

either within the friction cone with $|\dot{g}_{Ti}| = 0$ and $(-\mu_{0i}\lambda_{Ni} \leq \lambda_{Ti} \leq +\mu_{0i}\lambda_{Ni})$, or we are on the friction cone surface with $|\dot{g}_{Ti}| \neq 0$ and $|\lambda_{Ti}| = \mu_{0i}\lambda_{Ni}$. The friction coefficient μ_{0i} is defined as (see Figure 2.32)

$$
\lim_{\dot{g}_{Ti} \to 0} \mu_i(\dot{g}_{Ti}) = \mu_{0i} \tag{2.201}
$$

In connection with this condition it should be noted, that most of the authors working in the field of non-smooth mechanics apply Coulomb's law with the same friction coefficient μ for sliding and for sticking, which means independent from $\dot{g}_{Ti}$. This is convenient and for many fundamental investigations also sufficient. But with respect to practical contact problems we usually have to consider Stribeck curves (Figure 2.32), which depend on the relative tangential velocity $\dot{g}_{Ti}$. Therefore in applying non-smooth theories the above condition

makes sense, because one important kernel of all non-smooth considerations are the various contact transitions.

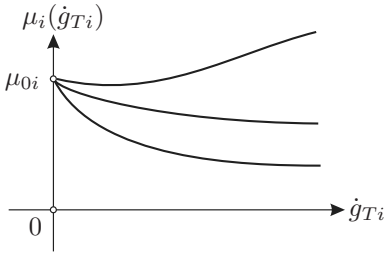

Fig. 2.32: Typical friction characteristics, Stribeck curves

The constraint force λ_{Ni} in normal direction results also from a contact law, which might be characterized by a contact-separation-mechanism. If we have a normal relative distance in contact i designated g_{Ni}, then the interdependency with the corresponding constraint force λ_{Ni} consists in the classical complementarity: Either $g_{Ni} = 0$ and $\lambda_{Ni} \geq 0$ or $g_{Ni} \geq 0$ and $\lambda_{Ni} = 0$, which is depicted in Figure 2.33. From this the product $(g_{Ni} \cdot \lambda_{Ni})$ is always zero. Both contact laws (Figures 2.31, 2.33) include complementary features,

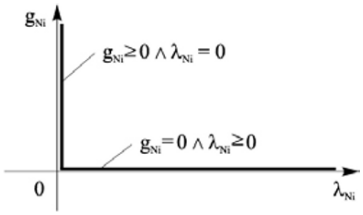

Fig. 2.33: Corner law for normal contacts (Signorini's law)

because also Figure 2.31 can be decomposed in two so-called "unilateral primitives" [87] in form of two simple corners. With the expressions of Figure 2.31 we may say in the case of frictional contacts, that either the relative velocities $|\dot{g}_{Ti}|$ are zero and the expression $\lambda_{0i} = \mu_{0i}\lambda_{Ni} - |\lambda_{Ti}|$ is not zero, or vice versa, the product $(\dot{g}_{Ti}\lambda_{0i})$ is always zero. We shall call that expression either "friction reserve" or "friction surplus", where the first name describes the relevant properties in a better way.

With respect to impact laws we have in classical mechanics two models, Newton's kinematical and Poisson's kinetic relationships. Newton's law connects the relative normal velocity after an impact with that before the impact

stating

$$(\dot{g}_{Ni})^{+} = -\varepsilon_i (\dot{g}_{Ni})^{-} \ , \tag{2.202}$$

where $(+)$ means shortly after and $(-)$ shortly before the impact. Losses are approximated by the coefficient of restitution ε_i. Poisson considers impulses $\Lambda = \int \lambda dt$ and relates them by

$$\Lambda_i^{+} = -\varepsilon_i^{*} \Lambda_i^{-} \ . \tag{2.203}$$

The physical idea behind Poisson's law consists in a storage of impulses during compression and a gain connected with losses during expansion of the impact process. Therefore, Poisson's law can be applied in normal and tangential direction of the contact without generating physical inconsistency [15]. In any case the loss coefficients $\varepsilon_i, \varepsilon_i^{*}$ have to be measured for each specific material pairing.

2.5.4 Impact Models

According to the above discussion we have two possibilities to model impulsive motion. Firstly, we may discretize the contact zone, for example by finite elements or by analytical approximative relations [118], which results in a smooth model usually with the problem of stiff differential equations due to the large stiffnesses in the contacts. As a consequence such models come out with very high frequencies either without physical meaning or very often without the need to know them. Secondly, we may use the above discussed complementarities to establish a non-smooth model not including the drawbacks of discretized models, but for the prize of more mathematics and of some numerical difficulties. For large systems, where we do not need the detailed structures of the contacts, non-smooth models possess definitely more advantages than drawbacks.

It should be pointed out, though, that we may also combine the two methods of modeling. For large systems with unilateral contacts it is often more economical to evaluate a non-smooth model and to go that way. In the case where I want to have in addition some detailed informations of some specific contacts with respect to certain criteria, for example maximum contact forces, I always may establish some post-processing including a discretization for the selected contacts thus yielding detailed results of pressure distributions, deformation distributions and the like. The only requirement to be fulfilled consists in adapting the discretization to the results obtained by the "large" non-smooth simulation. For example the averaged impulses must compare with those of the simulation. Altogether this is a simple and straightforward procedure.

Discretized models work with forces and a finite duration of the impulsive process itself. It is not too difficult to include also wave phenomena. Non-smooth rigid-body-models work with an infinitesimal short duration of the

impulsive process, and it is difficult to include wave processes. The choice of the model with a best fit to the problem depends of course, as always, on the problem itself. Non-smooth rigid-body-models turn out to be an excellent approximation to a large variety of technical problems. They are governed by the following assumptions:

- The duration of the impact is "very short."
- The impact can be divided into two phases: the compression phase and the expansion phase.
- The compression phase starts at time t_A and ends at time t_C. The end of the compression equals the start of the expansion phase. Expansion is finished at time t_E, which is also the end of the impact.
- During the short impact duration all magnitudes of the multibody system for position and orientation as well as all nonimpulsive forces and torques remain constant.
- Wave effects are not taken into account.

In multiple-contact problems there might be one impact only in one of the contacts or several impacts in several contacts simultaneously. The existing theories cover both possibilities (see [200], [87], [135]).

2.6 Damping

2.6.1 Phenomena

From the physical standpoint of view damping is an energy conversion process, in most cases by some kind of friction. From the technological standpoint of view damping is an energy consuming process, in many cases accompanied by useful consequences in form of vibration reduction, for example. All mechanisms of damping require relative motion. Following some systematizing ideas of the German Society of Engineers (VDI - Verein Deutscher Ingenieure, [266]) we have in mechanical systems internal damping like friction in gears or slide ways, external damping by solid-fluid-interactions, material damping by fluid flows or by microplastic deformations of solids and some special forms of damping in connection with electro-dynamical influences, for example. In specific cases of machines and mechanism it is usually not very difficult to localize the damping possibilities, the problem concerns mainly the quantitative evaluation of the damping mechanisms. In spite of very many intelligent theories on damping represented by a huge literature in that area we are mainly concerned with empirical data and thus with experience. It is not by accident, that big companies establish large databases collecting all the experiences available on damping, of course especially with respect to their own products. Nevertheless we shall give some classical and modern results concerning damping and its influence on dynamics of mechanical systems.

Friction is the basis of nearly all types of mechanical damping, friction between solids, between fluids and solids and internal friction of fluids and solids. We can also produce damping by the combination of mechanics and electro-dynamics, like eddy-current brakes, but we shall confine our consideration to mechanical friction effects. Friction between solids might be dry or viscous friction, in the first case governed for example by Coulomb's equations (2.200) and in the second case by some laws usually derived from the boundary layer equations of viscous fluids (see for example [12] and [13]). The viscosity of fluids leads to a large variety of possible representations depending on the environmental parameters like gap size, pressure differences or velocities. In machine dynamics this belongs to the increasing field of fluid film rheology [94]. It should be kept in mind, that fluid films in bearings, between gear teeth or on guide ways have a thickness of some micrometers with usually significantly increasing stiffness with decreasing thickness.

Interactions of solids and fluids generate friction in boundary layers of variable thickness, where the flow might be laminar or turbulent depending on the external conditions. Airplanes, helicopters, buildings and the famous self-excited vibrations of bridges are typical examples. Velocity gradients lead to friction within a fluid, which is of much importance in chemical engineering. For our purposes we need the effects of friction in the form of a linear or nonlinear force law, which we can then implement into the equations of motion. The implementation does of course not solve the data problem, because these

force laws contain as a rule some empirical coefficients, which have to be measured. In the following we shall indicate some possibilities of such force laws.

2.6.2 Linear Damping

Linear damping is characterized by force laws, which give a linear relation between force and relative velocity. Many engineering applications follow approximately this property. We may include such forces in both, linear or nonlinear equations of motion. In many cases the motion can also be linearized. Therefore we shall consider in the following linear equations of motion including linear damping, and we shall give some simple relations with regard to these damping laws.

Linear system dynamics is a very well established area, which might be taken from a large variety of text books, see for example [174], [157], [148] and [187]. For a mechanical system with f degrees of freedom we get the well-known MDGKN-equations of motion, which write:

$$\mathbf{M}\ddot{\mathbf{y}}+(\mathbf{D}+\mathbf{G})\dot{\mathbf{y}}+(\mathbf{K}+\mathbf{N})\mathbf{y}=\mathbf{g}(t) \qquad \text{with} \quad \mathbf{y}\in\mathbb{R}^f, \mathbf{M}\in\mathbb{R}^{f,f}, \text{etc.} \tag{2.204}$$

The matrix $\mathbf{M}$ is the symmetric mass matrix, $\mathbf{D}$ the symmetric damping matrix, $\mathbf{G}$ the skew- symmetric gyroscopic matrix, $\mathbf{K}$ the symmetric stiffness matrix and $\mathbf{N}$ the skew-symmetric matrix of non-conservative forces, due to rotational effects, for example. A Laplace transform of these equations comes out with

$$\mathbf{y}(s)=[\mathbf{M}s^2+(\mathbf{D}+\mathbf{G})s+(\mathbf{K}+\mathbf{N})]^{-1}\mathbf{g}(s), \tag{2.205}$$

which gives immediately the response chart for the linear system (2.204) under the excitation of $\mathbf{g}(t)$. The existence of a linear damping term ($\mathbf{D}s$) results in a reduction of the amplitudes $\mathbf{y}(s)$. If we put $\mathbf{G}=\mathbf{0}$, $\mathbf{N}=\mathbf{0}$, $\mathbf{g}(t)=\mathbf{0}$ and multiply the remaining equation ($\mathbf{M}\ddot{\mathbf{y}}+\mathbf{D}\dot{\mathbf{y}}+\mathbf{K}\mathbf{y}=\mathbf{0}$) from the left side with $\dot{\mathbf{y}}^T$, we get a kind of power equation in the following form

$$\dot{\mathbf{y}}^T\mathbf{M}\ddot{\mathbf{y}}+\dot{\mathbf{y}}^T\mathbf{K}\mathbf{y}=\frac{d(E_k+E_p)}{dt}=-\dot{\mathbf{y}}^T\mathbf{D}\dot{\mathbf{y}}=-2R. \tag{2.206}$$

The left hand side of this equation depicts the time derivatives of the kinetic and the potential energies, and the energy term on the right hand side represents the dissipation power produced by the dissipative forces, and R is the Rayleigh dissipation function [148]. For $R>0$ and thus $\mathbf{D}>\mathbf{0}$ we have *complete damping* reducing the motion energy for any kind of motion. For a positive semidefinite matrix with ($\mathbf{D}\geq\mathbf{0}, det(\mathbf{D})=\mathbf{0}$) we still have damping, which is called *penetrating damping*. In this case we should have a good parameter adaptation to achieve damping for all types of motion.

Sometimes it makes sense to assume, that the damping matrix $\mathbf{D}$ is proportional to the mass- and the stiffness-matrices, which has some advantages

with respect to modal analysis but which on the other hand shifts the data problem to the formula [148]

$$\mathbf{D} = \alpha\mathbf{M} + \beta\mathbf{K}, \tag{2.207}$$

because the coefficients α and β are not known and must be estimated. Instead of choosing the form of equation (2.207) with scalar coefficients we also could choose a coefficient matrix, which of course will enlarge the data problem.

The classical results for an oscillator with one degree of freedom only allows some simple interpretations of damping. The equation (2.204) writes for the case f=1

$$\ddot{y} + 2\delta\dot{y} + \omega_0^2 y = g(t), \qquad \delta = \frac{d}{2m}, \quad \omega_0^2 = \frac{k}{m}. \tag{2.208}$$

For the undamped case ($\delta = 0$) we get from the right hand side of the above equation the sometimes called "undamped natural frequency" $\omega_0 = \sqrt{\frac{k}{m}}$ and for the damped case ($\delta \neq 0$) the "damped natural frequency" $\omega_d = \sqrt{\frac{k}{m} - \frac{d}{2m}^2}$, which indicates the well known reduction of eigenfrequencies with increasing damping. The complete solution possesses two parts, that of the free vibrations, which usually disappear very quickly, and that of the forced vibrations. Assuming a harmonic excitation with ($g(t) = a\cos(\omega t)$) we get as a result for the forced part

$$\begin{aligned} x_{forced} =& R\cos\omega t - \psi, \\ R =& \frac{a}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\delta^2\omega^2}}, \qquad \tan\psi = \frac{2\delta\omega}{\omega_0^2 - \omega^2} \end{aligned} \tag{2.209}$$

The amplitude R of the forced vibration and the phase angle ψ depend both on the damping coefficient. Moreover, for vanishing damping we have no phase shift ψ. The amplitude R increases with decreasing damping, and its maximum is shifted to lower eigenfrequencies with increasing damping. These effects also can be observed in response curves for systems with many degrees of freedom, mainly due to the fact, that near resonances most large systems with many degrees of freedom behave approximately like a one-degree-of-freedom system. For practical applications it is useful to remember a few qualitative tendencies following from the above equations: Eigenfrequencies decrease with increasing masses and with decreasing stiffness. Resonance amplitudes decrease and are shifted to lower frequencies with increasing damping. These are, for a given configuration, the main parameters to influence vibrations and to avoid resonances.

As already discussed, it is very often straightforward to find a model for damping, but it is a difficult task to get the unknown coefficients included in all force laws of damping. For linear damping this concerns the elements of the matrix $\mathbf{D}$ of equation (2.204). Though we never can do without empirical

data, at least with respect to the state of the simulation art at the time being, we shall consider some simple laws for an estimation of damping force laws.

Elementary relations for damping are fluid flow boundary layers of various configurations. The plane laminar flow of a Newtonian fluid between a moving and a not moving wall is governed by a parabolic velocity and a linear shear stress distribution. The shear stress at the moving wall and the volume flow through the gap take on the values ([12], [13], see also Figure (2.34)))

$$\tau_w = \eta\frac{u_w}{h} + \frac{h}{2l}(p_2 - p_1), \qquad \dot{V} = (\frac{u_w h}{2} - \frac{p_2 - p_1}{12\eta l}h^3)b, \tag{2.210}$$

where τ_w and u_w is the shear stress and the velocity of the moving wall,

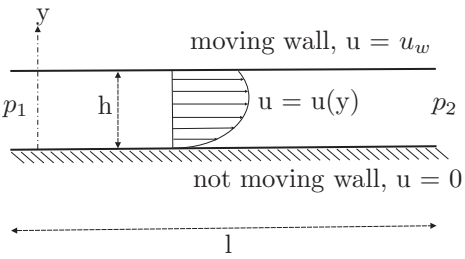

Fig. 2.34: Flow through a Gap

respectively, $\dot{V}$ is the volume flow, h is the gap between the two walls, l the length of the walls and the $(p_i, i = 1, 2)$ are the pressures at the two sides of the gap. The viscosity η is the "dynamic" viscosity. It is assumed that the plane flow has a width b. The first part of the first equation (2.210) is proportional to the velocity, the second part is proportional to the pressure difference $(p_2 - p_1)$. For practical cases various combinations are possible.

We first consider an elementary slide way with the length 2l and the width b, see Figure (2.35). The upper sliding structure does not move $u_w = 0$, and the slide way is supplied by an oil volume $\dot{V}$ in the middle of its length 2l. The gap is very small compared with b and l ($h << l, h << b$), so that we can neglect boundary effects. From the second equation (2.210) we get ($p_2 - p_1 = \frac{12\eta l\dot{V}}{bh^3}$). The pressure distribution in the gap is approximately linear, therefore the averaged pressure difference is $(p_2 - p_1)/2$ acting on the area (2bl). For the normal force F_n in the gap this yields

$$F_n = \frac{12\eta l^2\dot{V}}{h^3} = (\frac{12\eta A l^2}{h^3}) \cdot v_{rel}, \tag{2.211}$$

where A is the cross sectional area of the gap and $v_{rel} = \frac{\dot{V}}{A}$ the relative velocity in the gap. Equation (2.211) describes an interesting result for such oil cushions. The force F_n is proportional to h^{-3} resulting in a sharply increasing

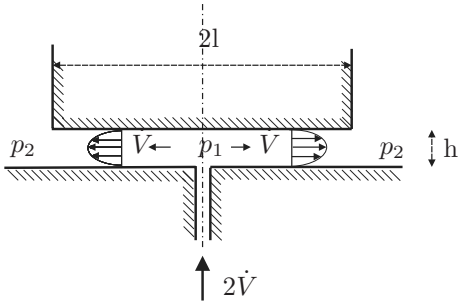

Fig. 2.35: Axial Bearing, Slide Way

pressure difference $(p_2 - p_1)$ without destroying the oil cushion. Thus oil lubrication is maintained even for large loads. We find a similar situation for axial bearings, which is an important component in many machines, for example in power transmissions of ships. Equation (2.211) represents a linear relation of force and velocity.

Another very important application are journal bearings, where the gap between the shaft and the bearing cylinder is very small. Therefore a boundary layer approach gives quite realistic results ([13], [247]). It is well known, that a shaft running exactly symmetrical in a bearing does not develop any force in any direction, because we have around the shaft the same pressure distribution as supplied by some pump, for example (left picture of Figure (2.36)). Putting on a load we get on the upper half of the shaft ($\varphi \in [0, \pi]$) an underpressure with respect to the averaged pressure in the gap, and on the lower half of the shaft ($\varphi \in [\pi, 2\pi]$) an overpressure. The integral over the whole bearing results in a pressure force, exactly vertical upwards, which counterbalances the external load by adapting the "control parameter" $h(\varphi)$ in an appropriate way, see right picture of Figure (2.36). The shaft itself is shifted exactly in a perpendicular direction with respect to the force $\mathbf{F}$.

From the simple Reynolds-Sommerfeld-theory we get the following friction torque ([13], [247]):

$$M = \left(\frac{2(1 + 2\varepsilon^2)}{\sqrt{1 - \varepsilon^2}(1 + \frac{\varepsilon^2}{2})} \right) \cdot \left(\frac{\pi r_0^2 b \eta}{a} \right) \cdot u_w. \tag{2.212}$$

The eccentricity ε describes the offset of the shaft center with respect to the bearing center. For $\varepsilon = 0$ we have the left hand side case of Figure (2.36), that means no eccentricity, and for $\varepsilon = 1$ the other extremum, namely a contact of the shaft at the the left side with $\varphi = 0$. The gap for the load case depends on φ and writes: $h(\varphi) = a(1 - \varepsilon \cos \varphi)$. The expression πr_0^2 is the shaft cross section, and b the bearing length. The gap without load has the value of a, for which we assume $a << r_0$. As above, η is the dynamic viscosity. It should be noticed, that the damping torque for a disturbance Δu_w of the constant

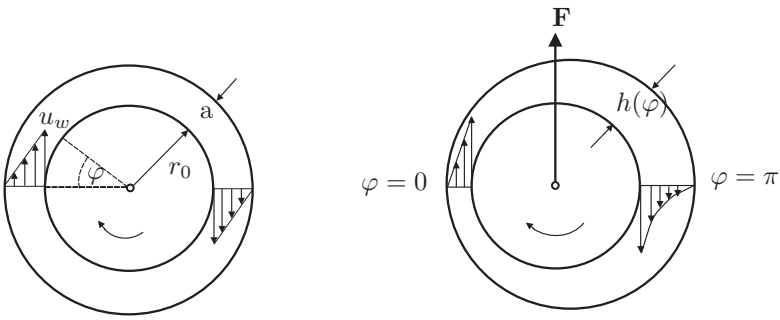

Fig. 2.36: Journal Bearing

shaft speed u_w is given by the same expression as in equation (2.212), we only must replace u_w by Δu_w.

In practice we are faced with deviations from the above theory, and in the meantime a large body of technical literature exists giving very sophisticated models of journal bearing dynamics and of friction in journal bearings. But nevertheless these simple formulas above give a first estimate of friction and damping in bearings in the form of a linear relation between torque and velocity. For many practical application it fits quite well. We shall come back to these problems in the application part of the book.

2.6.3 Nonlinear Damping

Nonlinear damping behavior in machines and mechanisms is very diverse. Material damping, friction in all components having relative motion with respect to other components, damping in all connecting elements like screws or press fits are a few examples. The mounts connecting the motor with the automobile body consist of visco-elastic springs with a progressive characteristic. Surveys of nonlinear damping my be seen from the literature ([47], [266]). A practically reasonable measure of damping is the "relative damping" defined as the friction losses per cycle, which is proportional to the area enclosed by the hysteresis cycle (Figure 2.37).

The mechanical losses due to such a hysteresis behavior is the integral for one cycle

$$W_D = \oint \sigma d\varepsilon, \qquad \text{or} \qquad W_D = \frac{1}{T} \int_t^{t+T} \mathbf{F}_D^T \mathbf{v}_{rel} dt, \tag{2.213}$$

which represents in the first case the work of damping per volume $[Nm/m^3]$ for arbitrary $\sigma - \varepsilon - curves$, in the second case the damping work in [Nm]

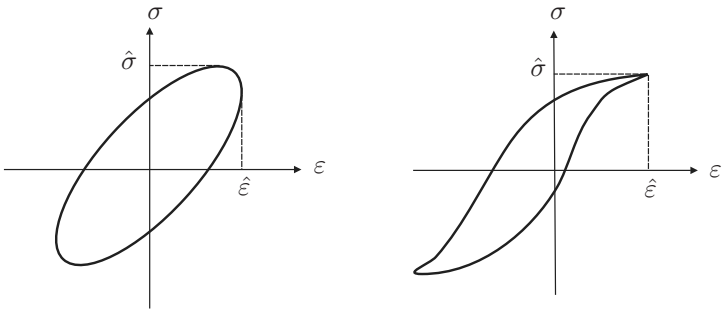

Fig. 2.37: Damping Hysteresis: left-linear material behavior, right-nonlinear material behavior

directly. The first equation is useful only for such cyclic processes with a corresponding friction behavior. The second case is more general by applying the law for the damping force $\mathbf{F}_D$, but we must find a suitable magnitude for the reference time T, usually from simulations.

The equations of motion become nonlinear even in cases, where the dynamics without damping might be linear. Thus we start with the set

$$\mathbf{M}(\mathbf{q},t)-\mathbf{h}(\mathbf{q},\dot{\mathbf{q}},t) = \mathbf{J}_D \cdot \mathbf{F}_D = \mathbf{h}_D(\mathbf{q},\dot{\mathbf{q}},t),$$
$$(\mathbf{q} \in \mathbb{R}^f, \mathbf{M} \in \mathbb{R}^{f,f}, \mathbf{h} \in \mathbb{R}^f, \mathbf{F}_D \in \mathbb{R}^{f_D}, \mathbf{J}_D \in \mathbb{R}^{f,f_D}). \qquad (2.214)$$

The vector $\mathbf{q}$ describes the generalized coordinates, f the degrees of freedom, $\mathbf{M}$ the mass matrix, $\mathbf{h}$ all forces with the exception of the damping forces, and the Jacobian $\mathbf{J}_D$ projects the damping forces into the space of the generalized coordinates. The right hand side of the above equation depends on $(\mathbf{q},\dot{\mathbf{q}},t)$.

Let us first consider a few examples concerning jointing structures, mainly based on the Coulomb friction law in connection with the Stribeck curve (see chapter 2.5). Coulomb's law for sliding and static friction writes

$$\mathbf{F}_{sliding} = -\mu \mathbf{F}_N \operatorname{sgn}(\mathbf{v}_{rel}), \qquad \mathbf{F}_{stiction} = \mu_0 \mathbf{F}_N, \qquad (2.215)$$

where the first equation applies for sliding and the second for stiction. For damping we are interested only in the first case with $\mathbf{F}_{sliding} = \mathbf{F}_S$ in the tangential contact direction, where also $\mathbf{v}_{rel}$ develops between the corresponding surfaces. According to equation (2.201) the validity of Coulomb's law is limited to the area around $g_{Ti}^{\cdot} = 0$, the further characteristic of the relationship $\mu_i(\dot{g}_{Ti})$ depends on the material pairing and must be measured. Modern material pairings, as applied for example in automotive industry, show partly exotic friction behavior often characterized by increasing and not by falling Stribeck curves. They are indicated in Figure (2.38). From this it follows that for the evaluation of system dynamics we must consider set-valued forces near the origin of Figure (2.38) and apart from the origin some type of Stribeck curve in the form of a smooth force law.

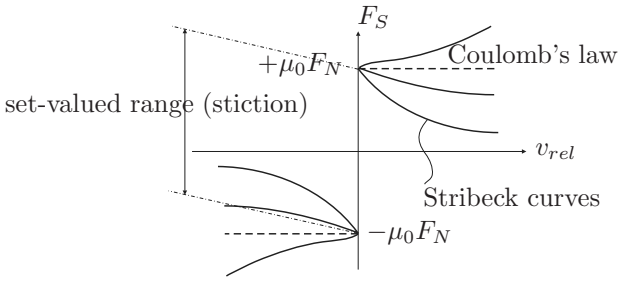

Fig. 2.38: Typical Stribeck Curves

In machine dynamics damping by relative motion in screws, in press fits and in similar contact structures is much larger than material damping, which might be more important in civil engineering. Relative motion in such jointing connections develops as a result of the system dynamics, which can generate loads exceeding the design loads of the connection. It is known from practice, for example, that axial loads on screws oscillating around the mean static value with an amplitude of more than 60% of this mean value will generate sliding within the screws, which must not happen from the standpoint of design, but which results in a damping effect with respect to dynamics. Such damping effects are governed by the nonlinear, non-smooth Coulomb's laws.

Another frequently applied element of machine design are press fits (see [26] and [265]). Such press fits are typically loaded by very large static forces superimposed by vibrations in all directions, spatial bending, torsion and axial vibrations. Most models subdivide the shaft in many elements connected with the hull by a spring-dry-friction- element, which is sometimes called Jenkins element. Figure (2.39) depicts an example from [26], which considers in detail the vibrations of a drive train of a large Diesel-engine with about 4 MW rated power and in this connection especially the influence of press fits on the dynamics of the whole system. These press fits were used to connect several large gear wheels to a shaft. The model of Figure (2.39) was included into the complete system model, and the Jenkins-elements described by the friction laws (2.215).

As a result we get hysteresis curves typical for such machine components. The area circumscribed by these curves increase with increasing load, and the slope of the line connecting the two extremum peaks decreases, both well-known effects from experimental data. The theory was compared with measurements from a laboratory test set-up, correspondence has been very good for all parameter cases. The influence of such press fits on the system dynamics of the complete drive train can be significant. In the case of the 4 MW Diesel engine the reduction of the overall stiffness of the whole train system was nearly 25% with a corresponding modification of the system response and the accompanying resonance frequencies [26].

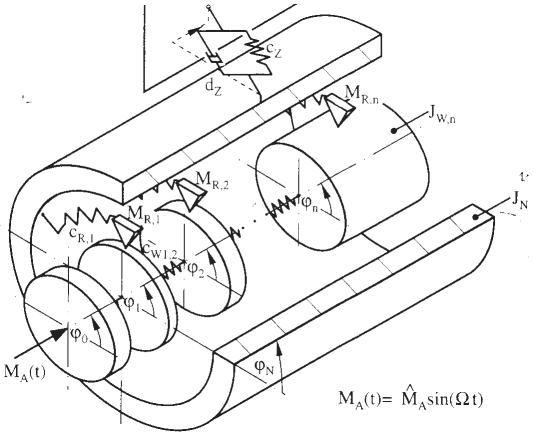

Fig. 2.39: Typical Model of a Press Fit [26]

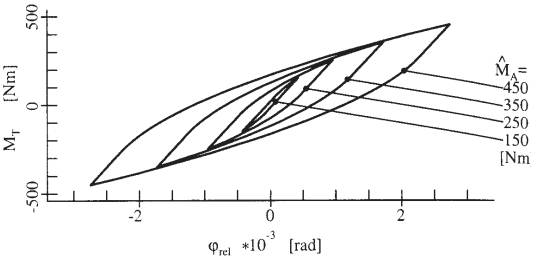

Fig. 2.40: Hyteresis Curves for Figure (2.39), [26]

For some nonlinear systems we might be able to generate a linear approximation by representing the nonlinear hysteresis curves by linear ones. With respect to Figure (2.37) we then must approach the closed hysteresis of the right picture by the ellipse-shaped curve of the left side by watching the condition, that the enclosed area must be the same for both cases, the real nonlinear one and the approximate linear one. Many methods are available to perform such an approximation ([148], [187], [47] or [60]).

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