2

Analytic Expressions of Elementary Trajectories

The basic trajectories are illustrated, classified into three main categories: polynomial, trigonometric, and exponential. Trajectories obtained on the basis of Fourier series expansion are also explained. More complex trajectories, able to satisfy desired constraints on velocity, acceleration and jerk, can be obtained by means of a suitable composition of these elementary functions. The case of a single actuator, or axis of motion, is specifically considered. The discussion is general, and it is therefore valid to define both a trajectory in the joint space and a motion law in the operational space, see Chapter 8 and Chapter 9.

2.1 Polynomial Trajectories

In the most simple case, a motion is defined by assigning the initial and final time instant t_0 and t_1, and conditions on position, velocity and acceleration at t_0 and t_1. From a mathematical point of view, the problem is then to find a function

$$q = q(t), \qquad\qquad t \in [t_0,\ t_1]$$

such that the given conditions are satisfied. This problem can be easily solved by considering a polynomial function

$$q(t) = a_0 + a_1 t + a_2 t^2 + \ldots + a_n t^n$$

where the $n+1$ coefficients a_i are determined so that the initial and final constraints are satisfied. The degree n of the polynomial depends on the number of conditions to be satisfied and on the desired "smoothness" of the resulting motion. Since the number of boundary conditions is usually even, the degree n of the polynomial function is odd, i.e. three, five, seven, and so on.

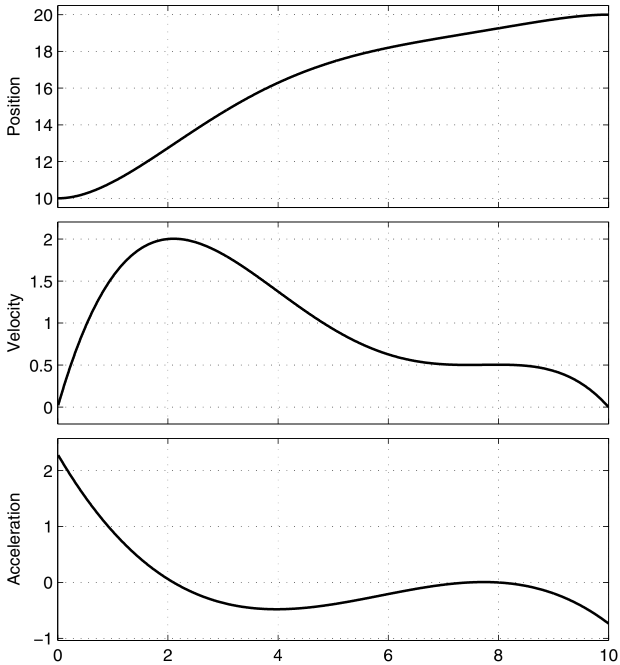


Fig. 2.1. Position, velocity and acceleration profiles of a polynomial trajectory computed by assigning boundary and intermediate conditions (Example 2.1).

In general, besides initial and final conditions on the trajectory, other conditions could be specified concerning its time derivatives (velocity, acceleration, jerk, ...) at generic instants $t_j \in [t_0,\ t_1]$. In other words, one could be interested in determining a polynomial function $q(t)$ whose k-th time-derivative assumes a specific value $q^{(k)}(t_j)$ at a given instant t_j. Mathematically, these conditions can be specified as

$$k!\ a_k + (k+1)!\ a_{k+1}\ t_j + \ldots + \frac{n!}{(n-k)!}\ a_n\ t_j^{n-k} = q^{(k)}(t_j)$$

or, in matrix form, as

$$\boldsymbol{M}\ \boldsymbol{a} = \boldsymbol{b}$$

where $\boldsymbol{M}$ is a known $(n+1)\times(n+1)$ matrix, $\boldsymbol{b}$ collects the given $(n+1)$ conditions to be satisfied, and $\boldsymbol{a} = [a_0,\ a_1,\ \ldots,\ a_n]^T$ is the vector of the unknown parameters to be computed. In principle this equation can be solved simply as

$$\boldsymbol{a} = \boldsymbol{M}^{-1}\ \boldsymbol{b}$$

although, for large values of n, this procedure may lead to numerical problems. These considerations are analyzed in more details in Chapter 4.

Example 2.1 Fig. 2.1 shows the position, velocity and acceleration profiles of a polynomial trajectory computed by assigning the following conditions:

$$\begin{array}{llll} q_0 = 10, & q_1 = 20, & t_0 = 0, & t_1 = 10, \\ \mathtt{v}_0 = 0, & \mathtt{v}_1 = 0, & \mathtt{v}(t=2) = 2, & \mathtt{a}(t=8) = 0. \end{array}$$

There are four boundary conditions (position and velocity at t_0 and t_1) and two intermediate conditions (velocity at $t = 2$ and acceleration at $t = 8$). Note that with six conditions it is necessary to adopt a polynomial at least of degree five. In this case, the coefficients a_i result

$$\begin{array}{lll} a_0 = 10.0000, & a_1 = 0.0000, & a_2 = 1.1462, \\ a_3 = -0.2806, & a_4 = 0.0267, & a_5 = -0.0009. \end{array}$$

□

2.1.1 Linear trajectory (constant velocity)

The most simple trajectory to determine a motion from an initial point q_0 to a final point q_1, is defined as

$$q(t) = a_0 + a_1(t - t_0).$$

Once the initial and final instants t_0, t_1, and positions q_0 and q_1 are specified, the parameters a_0, a_1 can be computed by solving the system

$$\begin{cases} q(t_0) = q_0 = a_0 \\ q(t_1) = q_1 = a_0 + a_1(t_1 - t_0) \end{cases} \qquad \Longrightarrow \qquad \begin{bmatrix} 1 & 0 \\ 1 & T \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} q_0 \\ q_1 \end{bmatrix}$$

where $T = t_1 - t_0$ is the time duration. Therefore

$$\begin{cases} a_0 = q_0 \\ a_1 = \dfrac{q_1 - q_0}{t_1 - t_0} = \dfrac{h}{T} \end{cases}$$

where $h = q_1 - q_0$ is the displacement. The velocity is constant over the interval $[t_0, t_1]$ and its value is

$$\dot{q}(t) = \frac{h}{T} \qquad (= a_1).$$

Obviously, the acceleration is null in the interior of the trajectory and has an impulsive behavior at the extremities.

Example 2.2 Fig. 2.2 reports the position, velocity and acceleration of the linear trajectory with the conditions $t_0 = 0,\ t_1 = 8,\ q_0 = 0,\ q_1 = 10$. Note that at $t = t_0,\ t_1$, the velocity is discontinuous and therefore the acceleration is infinite in these points. For this reason the trajectory in this form is not adopted in the industrial practice. □

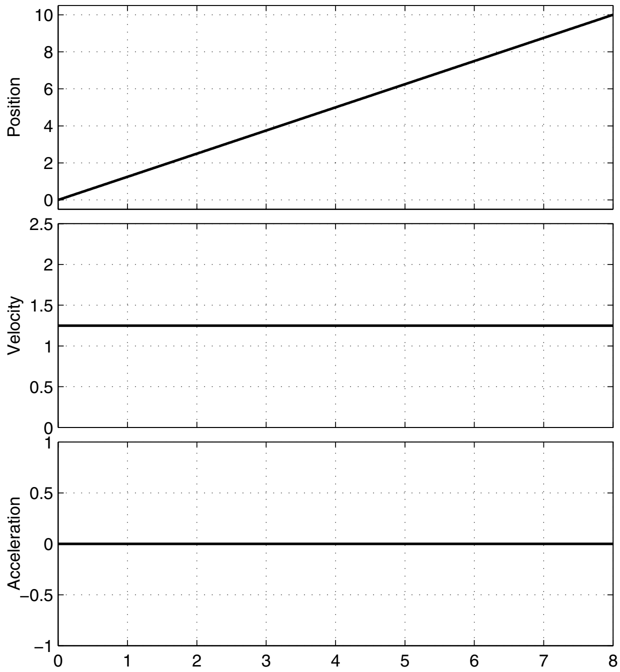


Fig. 2.2. Position, velocity and acceleration of a constant velocity trajectory, with $t_0 = 0,\ t_1 = 8,\ q_0 = 0,\ q_1 = 10$.

2.1.2 Parabolic trajectory (constant acceleration)

This trajectory, also known as *gravitational trajectory* or *with constant acceleration*, is characterized by an acceleration with a constant absolute value and opposite sign in the acceleration/deceleration periods. Analytically, it is the composition of two second degree polynomials, one from t_0 to t_f (the flex point) and the second from t_f to t_1, see Fig. 2.3.

Let us consider now the case of a trajectory symmetric with respect to its middle point, defined by $t_f = \frac{t_0+t_1}{2}$ and $q(t_f) = q_f = \frac{q_0+q_1}{2}$. Note that in this case $T_a = (t_f - t_0) = T/2,\ \ (q_f - q_0) = h/2$.

In the first phase, the "acceleration" phase, the trajectory is defined by

$$q_a(t) = a_0 + a_1 (t - t_0) + a_2 (t - t_0)^2, \qquad t \in [t_0,\ t_f].$$

The parameters a_0, a_1 and a_2 can be computed by imposing the conditions of the trajectory through the points q_0, q_f and the condition on the initial velocity $\mathtt{v}_0$

$$\begin{cases} q_a(t_0) = q_0 = a_0 \\ q_a(t_f) = q_f = a_0 + a_1 (t_f - t_0) + a_2 (t_f - t_0)^2 \\ \dot{q}_a(t_0) = \mathtt{v}_0 = a_1. \end{cases}$$

One obtains

$$a_0 = q_0, \qquad a_1 = \mathtt{v}_0, \qquad a_2 = \frac{2}{T^2}(h - \mathtt{v}_0 T).$$

Therefore, for $t \in [t_0,\ t_f]$, the trajectory is defined as

$$\begin{cases} q_a(t) = q_0 + \mathtt{v}_0(t - t_0) + \dfrac{2}{T^2}(h - \mathtt{v}_0 T)(t - t_0)^2 \\ \dot{q}_a(t) = \mathtt{v}_0 + \dfrac{4}{T^2}(h - \mathtt{v}_0 T)(t - t_0) \\ \ddot{q}_a(t) = \dfrac{4}{T^2}(h - \mathtt{v}_0 T) \qquad \text{(constant).} \end{cases}$$

The velocity at the flex point is

$$\mathtt{v}_{max} = \dot{q}_a(t_f) = 2\frac{h}{T} - \mathtt{v}_0.$$

Note that, if $\mathtt{v}_0 = 0$, the resulting maximum velocity has doubled with respect to the case of the constant velocity trajectory. The jerk is always null except at the flex point, when the acceleration changes its sign and it assumes an infinite value.

In the second part, between the flex and the final point, the trajectory is described by

$$q_b(t) = a_3 + a_4\,(t - t_f) + a_5\,(t - t_f)^2 \qquad t \in [t_f,\ t_1].$$

If the final value of the velocity $\mathtt{v}_1$ is assigned, at $t = t_1$, the parameters a_3, a_4, a_5 can be computed by means of the following equations

$$\begin{cases} q_b(t_f) = q_f = a_3 \\ q_b(t_1) = q_1 = a_3 + a_4\,(t_1 - t_f) + a_5\,(t_1 - t_f)^2 \\ \dot{q}_b(t_1) = \mathtt{v}_1 = a_4\ + 2a_5\,(t_1 - t_f) \end{cases}$$

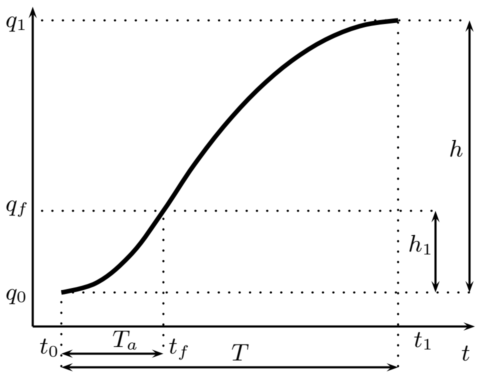


Fig. 2.3. Trajectory with constant acceleration.

from which

$$a_3 = q_f = \frac{q_0 + q_1}{2}, \qquad a_4 = 2\frac{h}{T} - \mathtt{v}_1, \qquad a_5 = \frac{2}{T^2}(\mathtt{v}_1 T - h).$$

The expression of the trajectory for $t \in [t_f,\ t_1]$ is

$$\begin{cases} q_b(t) = q_f + (2\frac{h}{T} - \mathtt{v}_1)(t - t_f) + \frac{2}{T^2}(\mathtt{v}_1 T - h)(t - t_f)^2 \\ \dot{q}_b(t) = 2\frac{h}{T} - \mathtt{v}_1 + \frac{4}{T^2}(\mathtt{v}_1 T - h)(t - t_f) \\ \ddot{q}_b(t) = \frac{4}{T^2}(\mathtt{v}_1 T - h). \end{cases}$$

Note that, if $\mathtt{v}_0 \neq \mathtt{v}_1$, the velocity profile of this trajectory is discontinuous at $t = t_f$.

Example 2.3 Fig. 2.4 reports the position, velocity and acceleration for this trajectory. The conditions $t_0 = 0,\ t_1 = 8,\ q_0 = 0,\ q_1 = 10,\ \mathtt{v}_0 = \mathtt{v}_1 = 0$ have been assigned. □

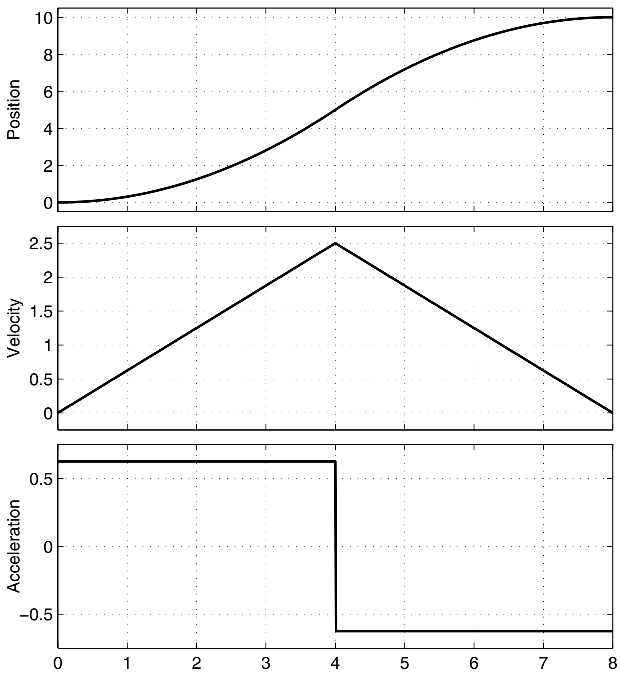


Fig. 2.4. Position, velocity and acceleration of a trajectory with constant acceleration, with $t_0 = 0,\ t_1 = 8,\ q_0 = 0,\ q_1 = 10$.

If the constraint on the position at $t = t_f$ (i.e. $q(t_f) = q_f = \frac{q_0+q_1}{2}$) is not assigned, the six parameters a_i may be determined in order to have a continuous velocity profile, i.e. $\dot{q}_a(t_f) = \dot{q}_b(t_f)$.
As a matter of fact, by imposing the six conditions

$$\begin{cases} q_a(t_0) = a_0 & = q_0 \\ \dot{q}_a(t_0) = a_1 & = \mathtt{v}_0 \\ q_b(t_1) = a_3 + a_4 \dfrac{T}{2} + a_5 \left(\dfrac{T}{2}\right)^2 & = q_1 \\ \dot{q}_b(t_1) = a_4 + 2a_5 \dfrac{T}{2} & = \mathtt{v}_1 \\ q_a(t_f) = a_0 + a_1 \dfrac{T}{2} + a_2 \left(\dfrac{T}{2}\right)^2 & = a_3 \;\; = q_b(t_f) \\ \dot{q}_a(t_f) = a_1 + 2a_2 \dfrac{T}{2} & = a_4 \;\; = \dot{q}_b(t_f) \end{cases}$$

where $T/2 = (t_f - t_0) = (t_1 - t_f)$, one obtains

$$\begin{cases} a_0 = q_0 \\ a_1 = \mathtt{v}_0 \\ a_2 = \dfrac{4h - T(3\mathtt{v}_0 + \mathtt{v}_1)}{2T^2} \\ a_3 = \dfrac{4(q_0 + q_1) + T(\mathtt{v}_0 - \mathtt{v}_1)}{8} \\ a_4 = \dfrac{4h - T(\mathtt{v}_0 + \mathtt{v}_1)}{2T} \\ a_5 = \dfrac{-4h + T(\mathtt{v}_0 + 3\mathtt{v}_1)}{2T^2}. \end{cases}$$

Example 2.4 Fig. 2.5 reports the position, velocity and acceleration for this trajectory. The conditions $t_0 = 0,\ t_1 = 8,\ q_0 = 0,\ q_1 = 10,\ \mathtt{v}_0 = 0.1,\ \mathtt{v}_1 = -1$ have been assigned. □

2.1.3 Trajectory with asymmetric constant acceleration

This trajectory is obtained from the previous one by considering the flex point at a generic instant $t_0 < t_f < t_1$, as shown in Fig. 2.3, and not necessarily at $t = (t_1 + t_0)/2$. The trajectory is described by the two polynomials

$$\begin{aligned} q_a(t) &= a_0 + a_1 (t - t_0) + a_2 (t - t_0)^2, & t_0 \le t < t_f \\ q_b(t) &= a_3 + a_4 (t - t_f) + a_5 (t - t_f)^2, & t_f \le t < t_1 \end{aligned}$$

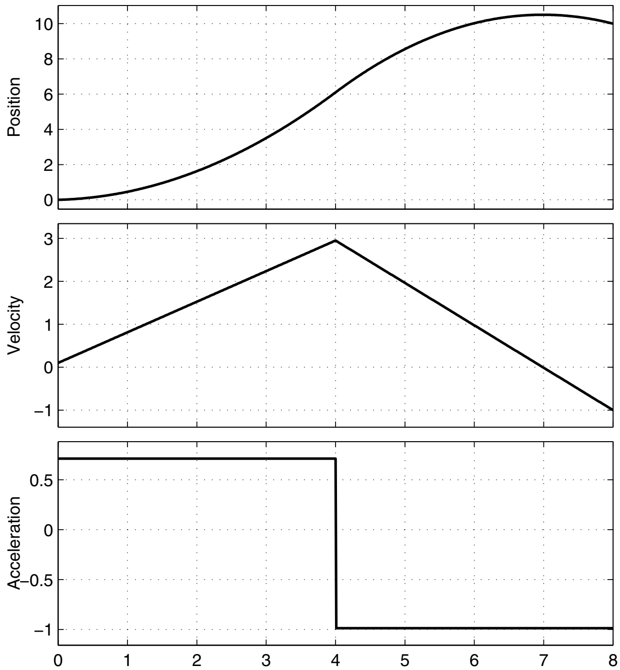


Fig. 2.5. Position, velocity and acceleration of a trajectory with constant acceleration and continuous velocity, with $t_0 = 0$, $t_1 = 8$, $q_0 = 0$, $q_1 = 10$, $\mathtt{v}_0 = 0.1$, $\mathtt{v}_1 = -1$.

where the parameters a_0, a_1, a_2, a_3, a_4 and a_5 are obtained by imposing the four conditions on the position and velocity at t_0, t_1, and the two continuity conditions (position and velocity) at t_f:

$$\begin{cases} q_a(t_0) = a_0 & = q_0 \\ q_b(t_1) = a_3 + a_4(t_1 - t_f) + a_5(t_1 - t_f)^2 & = q_1 \\ \dot{q}_a(t_0) = a_1 & = \mathtt{v}_0 \\ \dot{q}_b(t_1) = a_4 + 2a_5(t_1 - t_f) & = \mathtt{v}_1 \\ q_a(t_f) = a_0 + a_1(t_f - t_0) + a_2(t_f - t_0)^2 & = a_3 \ (= q_b(t_f)) \\ \dot{q}_a(t_f) = a_1 + 2a_2(t_f - t_0) & = a_4 \ (= \dot{q}_b(t_f)). \end{cases}$$

By defining $T_a = (t_f - t_0)$ and $T_d = (t_1 - t_f)$, the resulting parameters are

$$\begin{cases} a_0 = q_0 \\ a_1 = \mathtt{v}_0 \\ a_2 = \dfrac{2h - \mathtt{v}_0(T + T_a) - \mathtt{v}_1 T_d}{2TT_a} \\ a_3 = \dfrac{2q_1 T_a + T_d(2q_0 + T_a(\mathtt{v}_0 - \mathtt{v}_1))}{2T} \\ a_4 = \dfrac{2h - \mathtt{v}_0 T_a - \mathtt{v}_1 T_d}{T} \\ a_5 = -\dfrac{2h - \mathtt{v}_0 T_a - \mathtt{v}_1(T + T_d)}{2TT_d}. \end{cases}$$

Velocity and acceleration for $t_0 \leq t < t_f$ are

$$\dot{q}_a(t) = a_1 + 2a_2(t - t_0) = \mathtt{v}_0 + \frac{2h - \mathtt{v}_0(T + T_a) - \mathtt{v}_1 T_d}{TT_a}(t - t_0)$$

$$\ddot{q}_a(t) = 2a_2 = \frac{2h - \mathtt{v}_0(T + T_a) - \mathtt{v}_1 T_d}{TT_a}$$

while for $t_f \leq t < t_1$ they result

$$\dot{q}_b(t) = a_4 + 2a_5(t - t_f) = \frac{2h - \mathtt{v}_0 T_a - \mathtt{v}_1 T_d}{T} - \frac{2h - \mathtt{v}_0 T_a - \mathtt{v}_1(T + T_d)}{TT_d}(t - t_f)$$

$$\ddot{q}_b(t) = 2a_5 = -\frac{2h - \mathtt{v}_0 T_a - \mathtt{v}_1(T + T_d)}{TT_d}.$$

Note that, in case $\mathtt{v}_0 = \mathtt{v}_1 = 0$, the value of the maximum velocity is the same as in the previous case (symmetric flex point):

$$v_{max} = \dot{q}_a(t_f) = 2\frac{h}{T}.$$

Obviously, if $t_f = \frac{t_0 + t_1}{2}$ the previous trajectory is obtained.

Example 2.5 Fig. 2.6 shows the position, velocity and acceleration for this trajectory with the same conditions as in the Example 2.3. □

2.1.4 Cubic trajectory

In case both position and velocity values are specified at t_0 and t_1 (q_0, q_1, and $\mathtt{v}_0, \mathtt{v}_1$ respectively), there are four conditions to be satisfied. Therefore, a third degree polynomial must be used

$$q(t) = a_0 + a_1(t - t_0) + a_2(t - t_0)^2 + a_3(t - t_0)^3, \qquad t_0 \leq t \leq t_1 \tag{2.1}$$

and, from the given conditions, the four parameters a_0, a_1, a_2, a_3 are

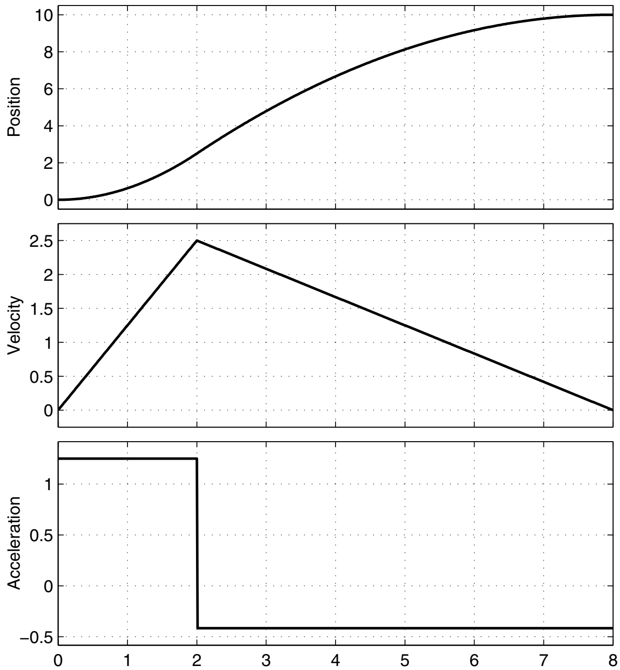
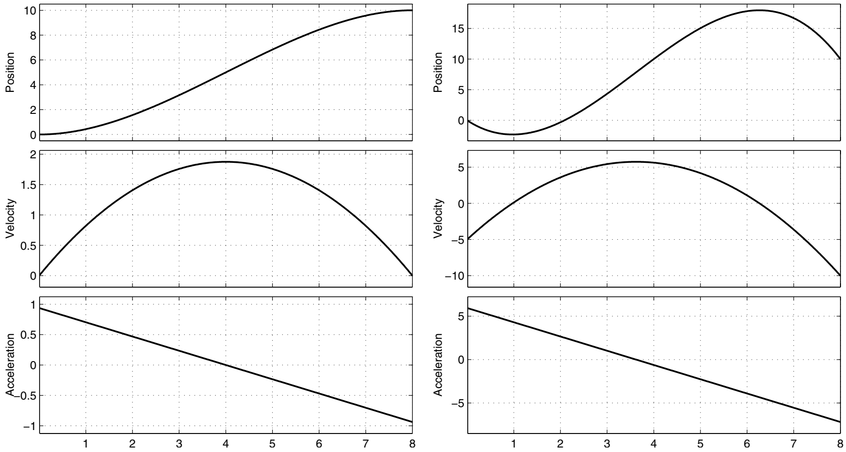


Fig. 2.6. Position, velocity and acceleration od a trajectory with asymmetric constant acceleration and $t_0 = 0, t_1 = 8, t_f = 2, q_0 = 0, q_1 = 10$.

$$\begin{cases} a_0 = q_0 \\ a_1 = \mathrm{v}_0 \\ a_2 = \dfrac{3h - (2\mathrm{v}_0 + \mathrm{v}_1)T}{T^2} \\ a_3 = \dfrac{-2h + (\mathrm{v}_0 + \mathrm{v}_1)T}{T^3}. \end{cases} \tag{2.2}$$

By exploiting this result, it is very simple to compute a trajectory with continuous velocity through a sequence of n points. The overall motion is subdivided into $n-1$ segments. Each of these segments connects the points q_k and q_{k+1} at $t_k,\ t_{k+1}$ and has initial/final velocity $\mathrm{v}_k,\ \mathrm{v}_{k+1}$ respectively. Then, equations (2.2) are used for each of these segments to define the $4(n-1)$ parameters $a_{0k},\ a_{1k},\ a_{2k},\ a_{3k}$.

Example 2.6 Fig. 2.7(a) shows position, velocity and acceleration for this trajectory with $q_0 = 0, q_1 = 10, t_0 = 0, t_1 = 8$ and null initial and final velocities. If these are not null, motion profiles such as those shown in Fig. 2.7(b) are obtained, where the conditions $\mathrm{v}_0 = -5,\ \mathrm{v}_1 = -10$ have been assigned. □



(a) (b)

Fig. 2.7. Position, velocity and acceleration of a third degree polynomial trajectory with the conditions $q_0 = 0,\ q_1 = 10,\ t_0 = 0,\ t_1 = 8$. In (a) the initial and final velocities are null ($\mathtt{v}_0 = \mathtt{v}_1 = 0$), while in (b) the values $\mathtt{v}_0 = -5,\ \mathtt{v}_1 = -10$ have been assigned.

Example 2.7 Fig. 2.8 reports the plots of position, velocity and acceleration for a multipoint trajectory with

$$\begin{array}{lllll} t_0 = 0, & t_1 = 2, & t_2 = 4, & t_3 = 8, & t_4 = 10, \\ q_0 = 10, & q_1 = 20, & q_2 = 0, & q_3 = 30, & q_4 = 40, \\ \mathtt{v}_0 = 0, & \mathtt{v}_1 = -10, & \mathtt{v}_2 = 10, & \mathtt{v}_3 = 3, & \mathtt{v}_4 = 0. \end{array}$$

□

In defining a trajectory through a set of points $q_0, \dots, q_n$, not always the velocities in the intermediate points are specified. In these cases, suitable values for the intermediate velocities may be determined with heuristic rules such as

$$\begin{array}{l} \mathtt{v}_0 \qquad \text{(assigned)} \\ \mathtt{v}_k = \begin{cases} 0 & \text{sign}(d_k) \neq \text{sign}(d_{k+1}) \\ \frac{1}{2}(d_k + d_{k+1}) & \text{sign}(d_k) = \text{sign}(d_{k+1}) \end{cases} \\ \mathtt{v}_n \qquad \text{(assigned)} \end{array} \tag{2.3}$$

where $d_k = (q_k - q_{k-1})/(t_k - t_{k-1})$ is the slope of the line segment between the instants t_{k-1} and t_k, and sign$(\cdot)$ is the sign function.

Example 2.8 The plots obtained with the same sequence of points as in Example 2.7 are reported in Fig. 2.9. In this case, the intermediate velocities are computed with (2.3). □

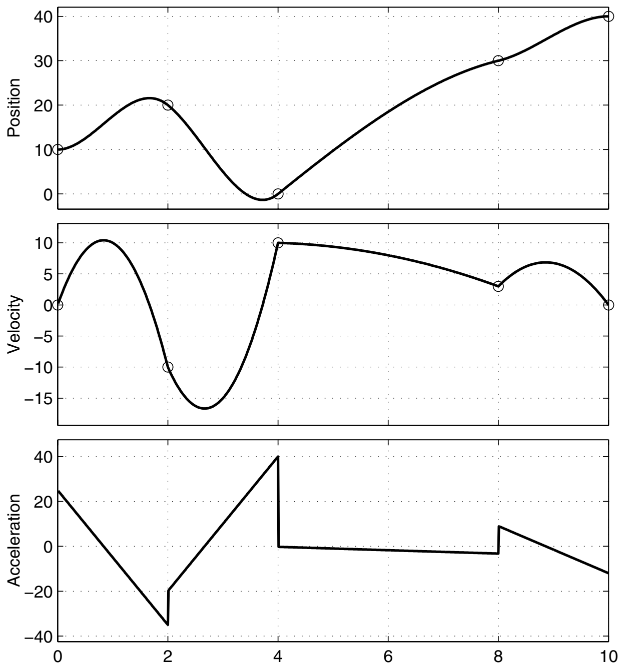


Fig. 2.8. Position, velocity and acceleration for a cubic polynomial through a sequence of points.

2.1.5 Polynomial of degree five

A trajectory through the points $q_0, \ldots, q_n$, based on third degree polynomials, is characterized by continuous position and velocity profiles, while in general the acceleration is discontinuous, see the examples in Fig. 2.8 and Fig. 2.9.

Although this trajectory is in general "smooth" enough, acceleration discontinuities can generate in some applications undesired effects on the kinematic chains and on the inertial loads. This happens in particular when the minimization of time is of concern, and therefore high acceleration (and velocity) values are assigned, or when relevant mechanical elasticities are present in the actuation system. These aspects are discussed with more details in Chapter 7.

In order to obtain trajectories with continuous acceleration, besides conditions on position and velocity it is also necessary to assign suitable initial and final values for the acceleration. Therefore, since there are six boundary conditions (position, velocity, and acceleration), a fifth degree polynomial must be adopted:

$$q(t) = q_0 + a_1(t-t_0) + a_2(t-t_0)^2 + a_3(t-t_0)^3 + a_4(t-t_0)^4 + a_5(t-t_0)^5 \quad (2.4)$$

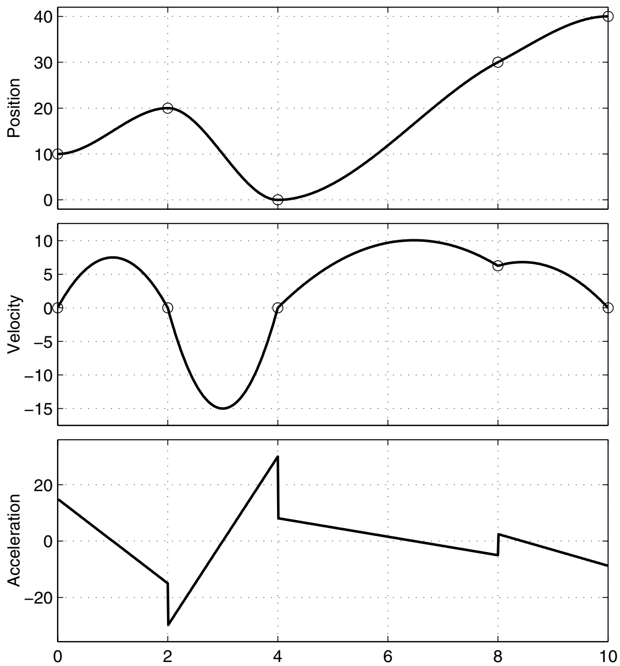


Fig. 2.9. Position, velocity and acceleration of a cubic polynomial trajectory through a sequence of points with the intermediate velocities computed according to (2.3).

with the conditions

$$\begin{aligned} q(t_0) &= q_0, & q(t_1) &= q_1 \\ \dot{q}(t_0) &= \mathtt{v}_0, & \dot{q}(t_1) &= \mathtt{v}_1 \\ \ddot{q}(t_0) &= \mathtt{a}_0, & \ddot{q}(t_1) &= \mathtt{a}_1. \end{aligned}$$

In this case, by defining $T = t_1 - t_0$, the coefficients of the polynomial result

$$\begin{cases} a_0 = q_0 \\ a_1 = \mathtt{v}_0 \\ a_2 = \dfrac{1}{2}\mathtt{a}_0 \\ a_3 = \dfrac{1}{2T^3}[20h - (8\mathtt{v}_1 + 12\mathtt{v}_0)T - (3\mathtt{a}_0 - \mathtt{a}_1)T^2] \\ a_4 = \dfrac{1}{2T^4}[-30h + (14\mathtt{v}_1 + 16\mathtt{v}_0)T + (3\mathtt{a}_0 - 2\mathtt{a}_1)T^2] \\ a_5 = \dfrac{1}{2T^5}[12h - 6(\mathtt{v}_1 + \mathtt{v}_0)T + (\mathtt{a}_1 - \mathtt{a}_0)T^2]. \end{cases} \tag{2.5}$$

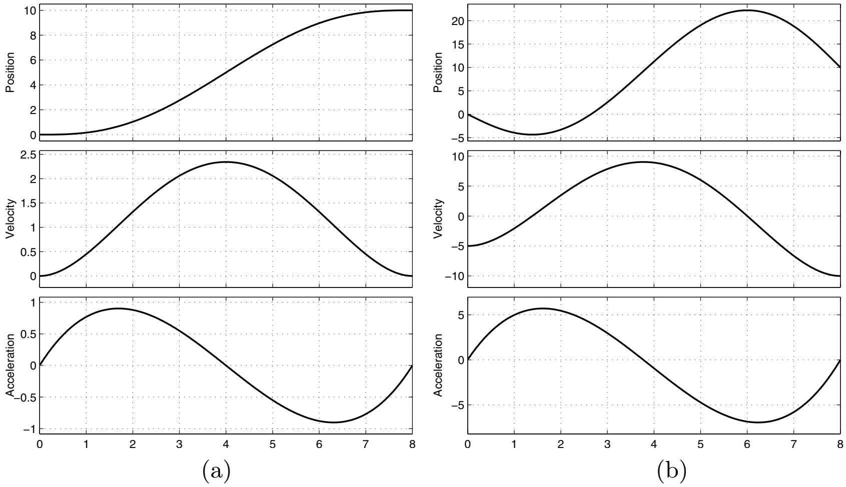


Fig. 2.10. Position, velocity and acceleration of a fifth degree polynomial with $q_0 = 0,\ q_1 = 10,\ \mathtt{v}_0 = \mathtt{v}_1 = 0,\ \mathtt{a}_0 = \mathtt{a}_1 = 0,\ t_0 = 0,\ t_1 = 8$ (a), and $\mathtt{v}_0 = -5,\ \mathtt{v}_1 = -10$ (b).

Example 2.9 A fifth degree trajectory is shown in Fig. 2.10. The initial and final conditions are $q_0 = 0,\ q_1 = 10,\ \mathtt{v}_0 = \mathtt{v}_1 = 0,\ \mathtt{a}_0 = \mathtt{a}_1 = 0,\ t_0 = 0,\ t_1 = 8$ in Fig. 2.10(a), and $\mathtt{v}_0 = -5,\ \mathtt{v}_1 = -10$, in Fig. 2.10(b). Compare these plots with those in Fig. 2.7. Note that, by adopting a cubic polynomial it is not possible to assign boundary values on the acceleration. □

For a motion through a sequence of points, the considerations illustrated for a third degree polynomial can be applied in the same manner, see eq. (2.3).

Example 2.10 Fig. 2.11 reports a fifth degree polynomial, with automatic computation of the intermediate velocities and null intermediate accelerations (compare with Fig. 2.9). Notice the improved "smoothness" in this case. □

2.1.6 Polynomial of degree seven

In particular cases, it might be necessary to define higher degree polynomials in order to obtain smoother profiles. With polynomials of degree seven such as

$$\begin{aligned} q(t) = a_0 + a_1(t-t_0) + a_2(t-t_0)^2 + a_3(t-t_0)^3 + a_4(t-t_0)^4 + \\ + a_5(t-t_0)^5 + a_6(t-t_0)^6 + a_7(t-t_0)^7 \end{aligned} \tag{2.6}$$

it is possible to specify eight boundary conditions

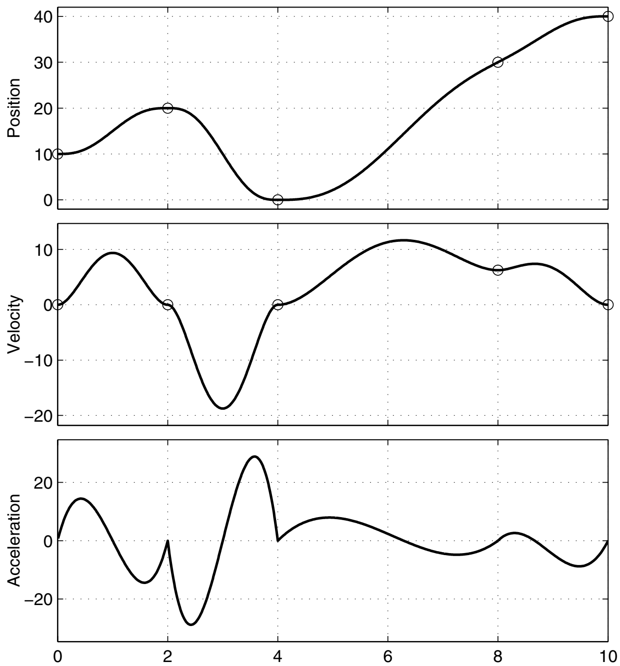


Fig. 2.11. Position, velocity and acceleration with a fifth degree polynomial through a sequence of points (compare with Fig. 2.9).

$$
\begin{array}{llll}
q(t_0) = q_0, & \dot{q}(t_0) = \mathtt{v}_0, & \ddot{q}(t_0) = \mathtt{a}_0, & q^{(3)}(t_0) = \mathtt{j}_0, \\
q(t_1) = q_1, & \dot{q}(t_1) = \mathtt{v}_1, & \ddot{q}(t_1) = \mathtt{a}_1, & q^{(3)}(t_1) = \mathtt{j}_1.
\end{array}
$$

By defining $T = t_1 - t_0$ and $h = q_1 - q_0$, the coefficients $a_i,\ i = 0, \ldots, 7$ are

$$
\begin{cases}
a_0 = q_0 \\
a_1 = \mathtt{v}_0 \\
a_2 = \dfrac{\mathtt{a}_0}{2} \\
a_3 = \dfrac{\mathtt{j}_0}{6} \\
a_4 = \dfrac{210h - T[(30\mathtt{a}_0 - 15\mathtt{a}_1)T + (4\mathtt{j}_0 + \mathtt{j}_1)T^2 + 120\mathtt{v}_0 + 90\mathtt{v}_1]}{6T^4} \\
a_5 = \dfrac{-168h + T[(20\mathtt{a}_0 - 14\mathtt{a}_1)T + (2\mathtt{j}_0 + \mathtt{j}_1)T^2 + 90\mathtt{v}_0 + 78\mathtt{v}_1]}{2T^5} \\
a_6 = \dfrac{420h - T[(45\mathtt{a}_0 - 39\mathtt{a}_1)T + (4\mathtt{j}_0 + 3\mathtt{j}_1)T^2 + 216\mathtt{v}_0 + 204\mathtt{v}_1]}{6T^6} \\
a_7 = \dfrac{-120h + T[(12\mathtt{a}_0 - 12\mathtt{a}_1)T + (\mathtt{j}_0 + \mathtt{j}_1)T^2 + 60\mathtt{v}_0 + 60\mathtt{v}_1]}{6T^7}.
\end{cases}
$$

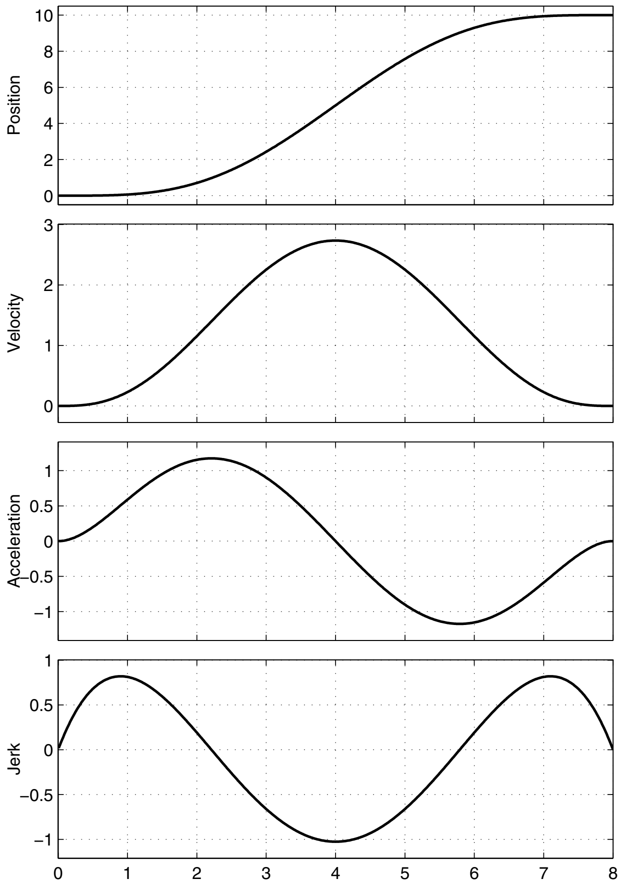


Fig. 2.12. Position, velocity, acceleration and jerk of a seventh degree polynomial (compare with Fig. 2.7 and Fig. 2.10).

Example 2.11 A seventh degree polynomial trajectory is shown in Fig. 2.12, obtained with the boundary conditions $q_0 = 0,\ q_1 = 10,\ \mathtt{v}_0 = \mathtt{v}_1 = 0,\ \mathtt{a}_0 = \mathtt{a}_1 = 0,\ \mathtt{j}_0 = 0,\ \mathtt{j}_1 = 0,\ t_0 = 0,\ t_1 = 8.$ □

Obviously, in case of a desired motion through a sequence of points, the considerations illustrated for third and fifth degree polynomials can be applied.

2.1.7 Polynomials of higher degree

In particular applications it is necessary to adopt polynomials of high degree in order to impose several constraints, such as boundary conditions on velocity, acceleration, jerk, snap and even higher order derivatives or conditions in the intermediate points. In these cases, it may be convenient to express the polynomial function of degree n in normalized form, i.e. as

$$q_N(\tau) = a_0 + a_1\tau + a_2\tau^2 + a_3\tau^3 + \ldots + a_n\tau^n \tag{2.7}$$

with unitary displacement $h = q_1 - q_0 = 1$ and duration $T = \tau_1 - \tau_0 = 1$ (for the sake of simplicity it is also assumed $\tau_0 = 0$).

In order to determine the parameters a_i, it is possible to define an equation of the type

$$\boldsymbol{M}\,\boldsymbol{a} = \boldsymbol{b} \tag{2.8}$$

where $\boldsymbol{a} = [a_0,\, a_1,\, a_2, \ldots,\, a_n]^T$. The vector $\boldsymbol{b}$, containing the boundary conditions on position, velocity, acceleration and so on, is in the form[1]

$$\boldsymbol{b} = [q_0,\, \mathtt{v}_0,\, \mathtt{a}_0,\, \mathtt{j}_0, \ldots,\, q_1,\, \mathtt{v}_1,\, \mathtt{a}_1,\, \mathtt{j}_1,\, \ldots]^T.$$

Finally, matrix $\boldsymbol{M}$ can be easily defined by imposing the boundary conditions on (2.7):

1. $a_0 = 0$: polynomial trough the first point ($q_N(0) = 0$).
2. $a_1 = \mathtt{v}_0,\ a_2 = \mathtt{a}_0,\ a_3 = \mathtt{j}_0,\ \ldots$: initial conditions on velocity, acceleration, ...; in general there are n_{ci} initial conditions on the derivatives of $q_N(\tau)$.
3. $\sum_{i=0}^{n} a_i = 1$: polynomial trough the last point ($q_N(1) = 1$).
4. $\sum_{i=1}^{n} i a_i = \mathtt{v}_1$: final condition on velocity.
5. $\sum_{i=2}^{n} i(i-1)a_i = \mathtt{a}_1$: final condition on acceleration.
6. $\sum_{i=3}^{n} i(i-1)(i-2)a_i = \mathtt{j}_1$: final condition on jerk.
7. $\sum_{i=d}^{n} \frac{i!}{(i-d)!}a_i = c_{d1}$: final condition on the d-th derivative of $q_N(\tau)$ (with n_{cf} final conditions).

The polynomial $q_N(\tau)$, of degree n, has $n+1$ coefficients a_i and therefore matrix $\boldsymbol{M}$ has dimensions $(n+1)\times(n+1)$, where $n+1 = n_{ci} + n_{cf} + 2$. The parameters $\boldsymbol{a}$ are determined from $\boldsymbol{a} = \boldsymbol{M}^{-1}\boldsymbol{b}$. Note that also for relatively

[1] The values of the initial/final velocity, acceleration, ..., ($\mathtt{v}_{N_j}$, $\mathtt{a}_{N_j}, \ldots, j = 1, 0$) are obtained by "normalizing" the corresponding boundary conditions $\mathtt{v}_j$, $\mathtt{a}_j, \ldots$ as $q_{N_j}^{(k)} = \frac{q_j^{(k)}}{h/T^k}$, being $q_j^{(k)}$ the given constraint on the derivative of order k of the desired trajectory $q(t)$ from q_0 to q_1 ($h = q_1 - q_0$) and of duration T. For the sake of simplicity, also the normalized boundary conditions $\mathtt{v}_{N_0}$, $\mathtt{v}_{N_1}, \ldots$ are denoted here as $\mathtt{v}_0, \mathtt{v}_1, \mathtt{a}_0, \mathtt{a}_1, \ldots$.

low values of n (e.g. $n = 18, 19, \ldots$), the computation of $\boldsymbol{M}^{-1}$ may give numerical problems due to bad conditioning.

For this reason, if necessary, it is possible to compute the coefficients a_i with other approaches, more robust from the computational point of view. As a matter of fact, it is possible to exploit the so-called *Bézier/Bernstein form* of polynomials, i.e.

$$q_N(\tau) = \sum_{i=0}^{n} \binom{n}{i} \tau^i (1-\tau)^{n-i} p_i, \qquad 0 \le \tau \le 1 \tag{2.9}$$

where $\binom{n}{i}$ are *binomial coefficients* defined as

$$\binom{n}{i} = \frac{n!}{i!\,(n-i)!}\,,$$

$\binom{n}{i} \tau^i (1-\tau)^{n-i}$ are the Bernstein basis polynomials, and p_i are scalar coefficients called *control points*, see also Sec. B.3. Obviously, the expressions (2.7) and (2.9) are equivalent, and it is possible to express a polynomial in both the forms. Accordingly, the relationship between the coefficients a_i and the parameters p_i is:

$$a_j = \frac{n!}{(n-j)!} \sum_{i=0}^{j} \frac{(-1)^{i+j}}{i!\,(j-i)!}\, p_i, \qquad j = 0, 1, \ldots, n, \tag{2.10}$$

see also (B.22). The parameters p_i in (2.9) can be computed by imposing the boundary conditions on $q_N(\tau)$, i.e

$$\begin{array}{ll} q_N(0) = 0, & q_N(1) = 1 \\ \dot{q}_N(0) = \mathtt{v}_0, & \dot{q}_N(1) = \mathtt{v}_1 \\ \ddot{q}_N(0) = \mathtt{a}_0, & \ddot{q}_N(1) = \mathtt{a}_1 \\ \vdots & \vdots \end{array} \tag{2.11}$$

An interesting property of the expression (2.9) is that it allows to solve independently the two problems tied to the imposition of boundary conditions at the initial and at the final points (these problems must be solved together if eq. (2.8) is used). As a matter of fact, the derivatives of $q_N(\tau)$ in (2.9) for $\tau = 0$ and $\tau = 1$ are

$$\begin{cases} \dot{q}_N(0) &= n(-p_0 + p_1) \\ \ddot{q}_N(0) &= n(n-1)(p_0 - 2p_1 + p_2) \\ &\vdots \\ q_N^{(k)}(0) &= \dfrac{n!}{(n-k)!} \displaystyle\sum_{i=0}^{k} \binom{k}{i} (-1)^{k+i} p_i \end{cases} \tag{2.12}$$

and

$$\begin{cases} \dot{q}_N(1) &= n(p_n - p_{n-1}) \\ \ddot{q}_N(1) &= n(n-1)(p_n - 2p_{n-1} + p_{n-2}) \\ &\vdots \\ q_N^{(k)}(1) &= \dfrac{n!}{(n-k)!} \displaystyle\sum_{i=0}^{k} \binom{k}{i} (-1)^i p_{n-i}. \end{cases} \tag{2.13}$$

As already pointed out, in order to meet all the conditions the degree n of the polynomial must be at least equal to $n_{ci} + n_{cf} + 1$. Note that the problem (2.12) depends only on the value of the first $n_{ci}+1$ control points p_i. Likewise, the problem (2.13) involves only the last $n_{cf} + 1$ control points.
From (2.12) and the obvious condition $q_N(0) = q_0$ (in this case $q_0 = 0$) it is possible to define an equation of the type

$$\boldsymbol{M}_0\, \boldsymbol{p}_0 = \boldsymbol{b}_0 \tag{2.14}$$

with

$$\boldsymbol{M}_0 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \dots 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \dots 0 \\ 1 & -2 & 1 & 0 & 0 & 0 \dots 0 \\ -1 & 3 & -3 & 1 & 0 & 0 \dots 0 \\ 1 & -4 & 6 & -4 & 1 & 0 \dots 0 \\ & & & \dots & & \end{bmatrix}, \qquad \boldsymbol{b}_0 = \begin{bmatrix} 0 \\ \frac{\mathtt{v}_0}{n} \\ \frac{\mathtt{a}_0}{n(n-1)} \\ \frac{\mathtt{j}_0}{n(n-1)(n-2)} \\ \frac{\mathtt{s}_0}{n(n-1)(n-2)(n-3)} \\ \vdots \end{bmatrix}$$

and the vector of the $n_{ci} + 1$ unknowns $\boldsymbol{p}_0 = [p_0,\ p_1,\ p_2,\ \dots,\ p_{n_{ci}}]^T$. Note that matrix $\boldsymbol{M}_0$ has a triangular structure, and therefore the procedure for its inversion, necessary to find the solution $\boldsymbol{p}_0$, results numerically robust. The last $n_{cf} + 1$ control points $\boldsymbol{p}_1 = [p_n,\ p_{n-1},\ p_{n-2},\ \dots,\ p_{n-n_{cf}}]^T$ are the solution of a system of equations similar to (2.14) (in this case the first equation is $q_N(1) = q_1 = 1$):

$$\boldsymbol{M}_1\, \boldsymbol{p}_1 = \boldsymbol{b}_1 \tag{2.15}$$

with

$$\boldsymbol{M}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \dots 0 \\ 1 & -1 & 0 & 0 & 0 & 0 \dots 0 \\ 1 & -2 & 1 & 0 & 0 & 0 \dots 0 \\ 1 & -3 & 3 & -1 & 0 & 0 \dots 0 \\ 1 & -4 & 6 & -4 & 1 & 0 \dots 0 \\ & & & \dots & & \end{bmatrix}, \qquad \boldsymbol{b}_1 = \begin{bmatrix} 1 \\ \frac{\mathtt{v}_1}{n} \\ \frac{\mathtt{a}_1}{n(n-1)} \\ \frac{\mathtt{j}_1}{n(n-1)(n-2)} \\ \frac{\mathtt{s}_1}{n(n-1)(n-2)(n-3)} \\ \vdots \end{bmatrix}.$$

Once all the control points $\boldsymbol{p} = [p_0, p_1, \ldots, p_{n_{ci}}, p_{n-n_{cf}}, \ldots, p_{n-1}, p_n]^T$ in (2.9) are known, it is possible to determine the parameters a_i in (2.7) according to (2.10).

After the computation of the parameters which define the normalized polynomial $q_N(\tau)$ either in the form (2.7) or (2.9), the function describing the motion between the two generic points (t_0, q_0) and (t_1, q_1) is

$$q(t) = q_0 + q_N(\tau)\; h, \quad \text{with} \quad \tau = \frac{t - t_0}{T} \tag{2.16}$$

and the velocity, acceleration, ... profiles are

$$\begin{cases} \dot{q}(t) & = \dot{q}_N(\tau)\, \dfrac{h}{T} \\ \ddot{q}(t) & = \ddot{q}_N(\tau)\, \dfrac{h}{T^2} \\ & \vdots \\ \dfrac{d\, q^d(t)}{d\, t^d} & = \dfrac{d\, q_N^d(\tau)}{d\, \tau^d}\, \dfrac{h}{T^d} \end{cases} \tag{2.17}$$

see also Sec. 5.2.1.

Example 2.12 Let us define a polynomial function with the following conditions

$$\begin{array}{lllll} q_0 = 10, & \mathtt{v}_0 = 5, & \mathtt{a}_0 = 0, & \mathtt{j}_0 = 0, & \mathtt{s}_0 = 0 \\ q_1 = 30, & \mathtt{v}_1 = 0, & \mathtt{a}_1 = 10, & \mathtt{j}_1 = 0, & \mathtt{s}_1 = 0 \end{array}$$

and $t_0 = 1$, $t_1 = 5$. In this case, the boundary conditions on the derivatives of the polynomial are 4 at the initial point and 4 at the final point ($n_{ci} = n_{cf} = 4$). Therefore, the degree n of the polynomial function must be 9. In order to find the coefficients p_i which define the Bézier/Bernstein polynomial, it is necessary to normalize the constraints. With $h = q_1 - q_0 = 20$ and $T = t_1 - t_0 = 4$ the normalized boundary conditions result

$$\begin{array}{lllll} q_0 = 0, & \mathtt{v}_0 = 1, & \mathtt{a}_0 = 0, & \mathtt{j}_0 = 0, & \mathtt{s}_0 = 0 \\ q_1 = 1, & \mathtt{v}_1 = 0, & \mathtt{a}_1 = 8, & \mathtt{j}_1 = 0, & \mathtt{s}_1 = 0. \end{array}$$

Therefore, the matrices $\boldsymbol{M}_j$ and the vectors $\boldsymbol{b}_j$ in (2.14) and (2.15) are respectively

$$\boldsymbol{M}_0 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 \\ -1 & 3 & -3 & 1 & 0 \\ 1 & -4 & 6 & -4 & 1 \end{bmatrix}, \qquad \boldsymbol{b}_0 = \begin{bmatrix} 0 \\ \frac{1}{9} \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

and

$$\boldsymbol{M}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 \\ 1 & -3 & 3 & -1 & 0 \\ 1 & -4 & 6 & -4 & 1 \end{bmatrix}, \qquad \boldsymbol{b}_1 = \begin{bmatrix} 1 \\ 0 \\ \frac{1}{9} \\ 0 \\ 0 \end{bmatrix}.$$

The control points are

$$\boldsymbol{p} = \frac{1}{9}\,[0,\ 1,\ 2,\ 3,\ 4,\ 15,\ 12,\ 10,\ 9,\ 9]^T$$

and the relative normalized trajectory is

$$\begin{aligned} q_N(\tau) = {} & (1-\tau)^8\tau + 8(1-\tau)^7\tau^2 + 28(1-\tau)^6\tau^3 + 56(1-\tau)^5\tau^4 + \\ & +210(1-\tau)^4\tau^5 + 112(1-\tau)^3\tau^6 + 40(1-\tau)^2\tau^7 + 9(1-\tau)\tau^8 + \tau^9. \end{aligned}$$

By exploiting (2.10), this trajectory can be rewritten in the standard polynomial form as

$$q_N(\tau) = \tau + 140\tau^5 - 504\tau^6 + 684\tau^7 - 415\tau^8 + 95\tau^9.$$

The profiles of position, velocity and acceleration of $q_N(\tau)$ are shown in Fig. 2.13(a).

Finally, by adopting (2.16) and (2.17), the expression of the desired trajectory with displacement $h = 20$ and duration $T = 4$ is obtained. The corresponding profiles of position, velocity and acceleration are shown in Fig. 2.13(b).

□

If the standard form (2.7) is assumed, the coefficients of the polynomial $q(t)$ and of its derivatives can be easily deduced from (2.16) and (2.17) as functions of a_i, T, and h. As a matter of fact, if we denote with $b_{i,k}$ the coefficients of $q^{(k)}(t)$, i.e.

$$q^{(k)}(t) = \sum_{i=0}^{n-k} b_{i,k}\,(t-t_0)^i \tag{2.18}$$

the expressions of the position, velocity, acceleration, ... profiles become

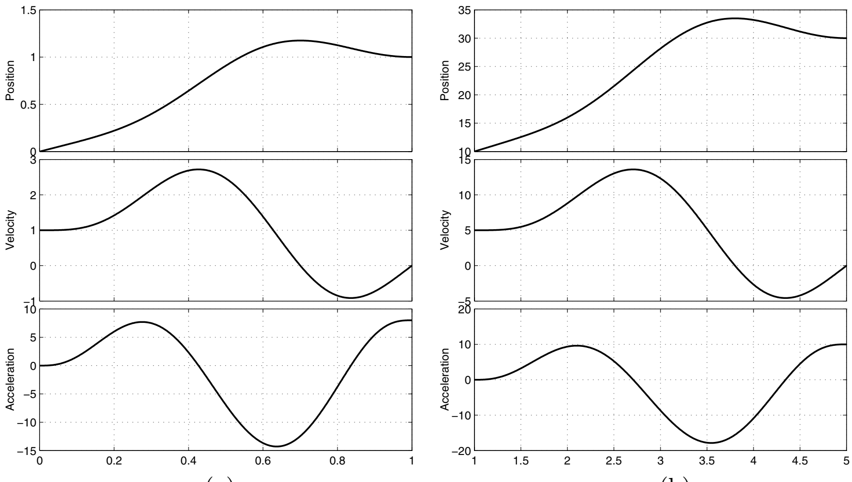


Fig. 2.13. Normalized polynomial trajectory of degree 9 (a) and corresponding trajectory from (t_0, q_0) to (t_1, q_1) (b), Example 2.12.

$$
\begin{array}{rlcl}
\text{position:} & q(t) = \displaystyle\sum_{i=0}^{n} b_{i,0}(t-t_0)^i & \rightarrow & b_{i,0} = \begin{cases} q_0 + h\,a_0, & i = 0 \\ \dfrac{h}{T^i}\,a_i, & i > 0 \end{cases} \\
\text{velocity:} & \dot{q}(t) = \displaystyle\sum_{i=0}^{n-1} b_{i,1}(t-t_0)^i & \rightarrow & b_{i,1} = (i+1)\,\dfrac{h}{T^{i+1}}\,a_{i+1} \\
\text{acceleration:} & \ddot{q}(t) = \displaystyle\sum_{i=0}^{n-2} b_{i,2}(t-t_0)^i & \rightarrow & b_{i,2} = (i+1)(i+2)\,\dfrac{h}{T^{i+2}}\,a_{i+2} \\
 & & \vdots & \\
d\text{-th derivative:} & q^{(d)}(t) = \displaystyle\sum_{i=0}^{n-d} b_{i,d}(t-t_0)^i & \rightarrow & b_{i,d} = \dfrac{(i+d)!}{i!}\,\dfrac{h}{T^{i+d}}\,a_{i+d}.
\end{array}
\tag{2.19}
$$

Of particular interest is the case of null boundary conditions:

$$
\begin{array}{ll}
\mathtt{v}_0 = 0, & \mathtt{v}_1 = 0 \\
\mathtt{a}_0 = 0, & \mathtt{a}_1 = 0 \\
\mathtt{j}_0 = 0, & \mathtt{j}_1 = 0 \\
\vdots & \vdots
\end{array}
$$

Under this hypothesis the control points, which determine (2.9) and are solution of (2.14) and (2.15), are

$$
\boldsymbol{p} = [\underbrace{0,\, 0,\, 0,\, 0, \ldots,\, 0}_{n_{ci}+1},\, \underbrace{1,\, 1,\, 1,\, 1, \ldots,\, 1}_{n_{cf}+1}]^T.
$$

The corresponding expression of the coefficients a_i in (2.7) can be determined from $\boldsymbol{p}$ with (2.10). Their values, for polynomials $q_N(\tau)$ up to degree 21, are reported in Tab. 2.1.

	3	5	7	9	11	13	15	17	19	21
a_0	0	0	0	0	0	0	0	0	0	0
a_1	0	0	0	0	0	0	0	0	0	0
a_2	3	0	0	0	0	0	0	0	0	0
a_3	-2	10	0	0	0	0	0	0	0	0
a_4	-	-15	35	0	0	0	0	0	0	0
a_5	-	6	-84	126	0	0	0	0	0	0
a_6	-	-	70	-420	462	0	0	0	0	0
a_7	-	-	-20	540	-1980	1716	0	0	0	0
a_8	-	-	-	-315	3465	-9009	6435	0	0	0
a_9	-	-	-	70	-3080	20020	-40040	24310	0	0
a_{10}	-	-	-	-	1386	-24024	108108	-175032	92378	0
a_{11}	-	-	-	-	-252	16380	-163800	556920	-755820	352716
a_{12}	-	-	-	-	-	-6006	150150	-1021020	2771340	-3233230
a_{13}	-	-	-	-	-	924	-83160	1178100	-5969040	13430340
a_{14}	-	-	-	-	-	-	25740	-875160	8314020	-33256080
a_{15}	-	-	-	-	-	-	-3432	408408	-7759752	54318264
a_{16}	-	-	-	-	-	-	-	-109395	4849845	-61108047
a_{17}	-	-	-	-	-	-	-	12870	-1956240	47927880
a_{18}	-	-	-	-	-	-	-	-	461890	-25865840
a_{19}	-	-	-	-	-	-	-	-	-48620	9189180
a_{20}	-	-	-	-	-	-	-	-	-	-1939938
a_{21}	-	-	-	-	-	-	-	-	-	184756

Table 2.1. Per column: coefficients a_i of the normalized polynomials $q_N(\tau)$ with degree $n = 3, 5, \ldots, 21$, with null boundary conditions on their derivatives up to order 10. The degree of the polynomials is $n = 2n_c + 1$, being n_c the number of null initial (and final) conditions.

The polynomial functions obtained in this manner, i.e with null boundary conditions and $h = 1,\ T = 1$, have some peculiar properties:

1. $q_N(\tau) = 1 - q_N(1 - \tau)$.

2. $a_0 = a_1 = \ldots = a_{n_{ci}} = 0$.

3. $a_i \in \mathbb{N}$.

4. $\mathrm{sign}(a_{n_{ci}+1}) = 1, \quad \mathrm{sign}(a_{n_{ci}+2}) = -1, \quad \mathrm{sign}(a_{n_{ci}+3}) = 1, \ \ldots$

5. $\sum_{i=0}^{n} a_i = 1$.

From the coefficients of Tab. 2.1 and the above equations (2.19) it is simple to compute the coefficients of the polynomials of the normalized velocity, acceleration, ..., profiles (functions $\dot{q}_N(\tau)$, $\ddot{q}_N(\tau)$, ...) or of the polynomials $q(t)$, $\dot{q}(t)$, $\ddot{q}(t), \ldots$ for a generic displacement. The coefficients of $\dot{q}_N(\tau)$ and $\ddot{q}_N(\tau)$ are reported in Appendix A.1.
The position, velocity, acceleration and jerk profiles for these polynomials are shown in Fig. 2.14. Note the increasing smoothness of the profiles, and the corresponding higher values for the maximum velocity, acceleration and jerk, whose numerical values are reported in Tab. 2.2, denoted with C_v, C_a, and C_j respectively.

Example 2.13 Let us define a polynomial function with the following conditions

$$\begin{array}{lllll} q_0 = 10, & \mathtt{v}_0 = 0, & \mathtt{a}_0 = 0, & \mathtt{j}_0 = 0, & \mathtt{s}_0 = 0 \\ q_1 = 30, & \mathtt{v}_1 = 0, & \mathtt{a}_1 = 0, & \mathtt{j}_1 = 0, & \mathtt{s}_1 = 0 \end{array}$$

and $t_0 = 1$, $t_1 = 5$. There are 10 conditions to be satisfied, and therefore the polynomial must be at least of degree 9. The expression of the normalized polynomial $q_N(\tau)$ in the Bézier/Bernstein form (2.9) with null boundary conditions is:

$$q_N(\tau) = 126(1-\tau)^4\tau^5 + 84(1-\tau)^3\tau^6 + 36(1-\tau)^2\tau^7 + 9(1-\tau)\tau^8 + \tau^9.$$

n	C_v	$\Delta\%$	C_a	$\Delta\%$	C_j	$\Delta\%$
3	1.5	0	6	0	12	0
5	1.875	25	5.7735	-3.78	60	400
7	2.1875	45.83	7.5132	25.22	52.5	337.5
9	2.4609	64.06	9.372	56.2	78.75	556.25
11	2.707	80.47	11.2666	87.78	108.2813	802.34
13	2.9326	95.51	13.1767	119.61	140.7656	1073.05
15	3.1421	109.47	15.0949	151.58	175.957	1366.31
17	3.3385	122.56	17.018	183.63	213.6621	1680.52
19	3.5239	134.93	18.9441	215.73	253.7238	2014.36
21	3.7001	146.68	20.8723	247.87	296.011	2366.76

Table 2.2. Maximum values of velocity (C_v), acceleration (C_a) and jerk (C_j) for normalized polynomials of degree 3 - 21: smoother (higher degree) polynomials present higher velocity and acceleration values. The variations with respect to the 3-rd degree polynomial are also reported.

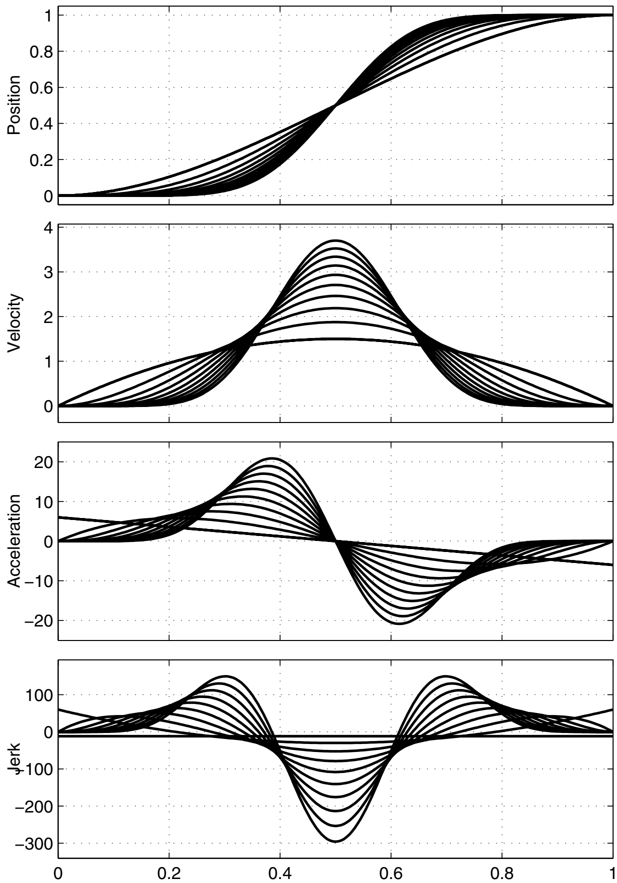


Fig. 2.14. Position, velocity, acceleration and jerk profiles for normalized polynomial functions of degree 3 - 21 with null boundary conditions.

From Tab. 2.1, the coefficients $\boldsymbol{a} = [a_0,\ a_1,\ \ldots,\ a_9]^T$ of the standard polynomial form are:

$$\boldsymbol{a} = [0,\ 0,\ 0,\ 0,\ 0,\ 126,\ -420,\ 540,\ -315,\ 70]^T.$$

By using (2.16), the desired trajectory with displacement $h = 20$ and duration $T = 4$ is computed as

$$q(t) = 10 + 20\left(126\tau^5 - 420\tau^6 + 540\tau^7 - 315\tau^8 + 70\tau^9\right), \quad \text{with } \tau = \left(\frac{t-1}{4}\right).$$

Alternatively, from (2.19), one can directly write the expression of $q(t)$ and of its derivatives:

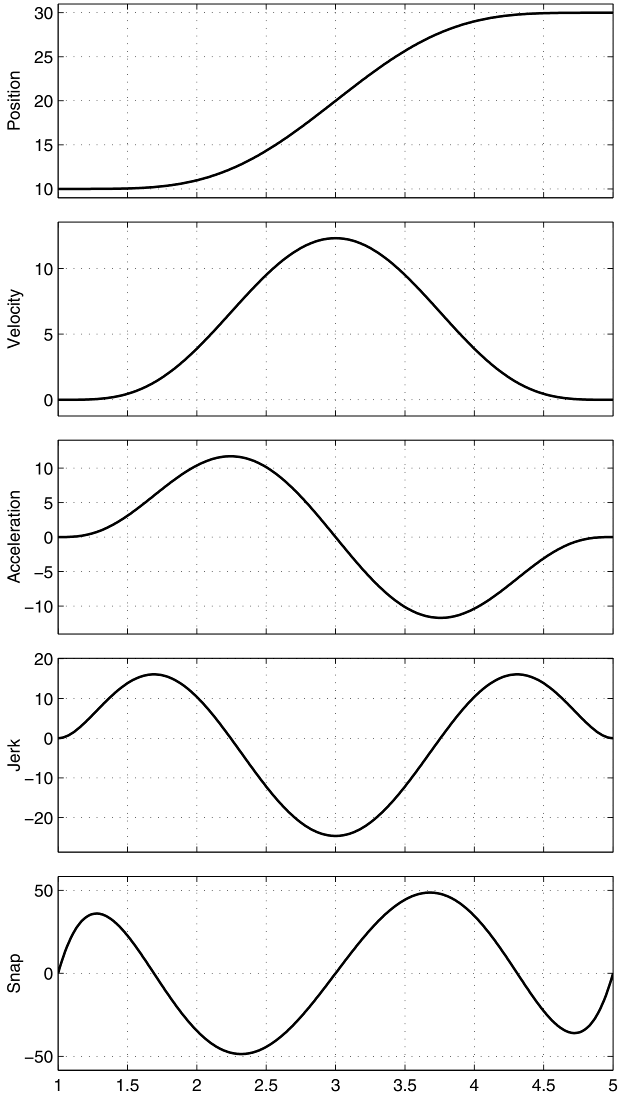


Fig. 2.15. Polynomial function of degree 9 of Example 2.13.

$$
\begin{aligned}
q(t) &= 10 + 20\frac{126}{4^5}(t-1)^5 + 20\frac{-420}{4^6}(t-1)^6 + 20\frac{540}{4^7}(t-1)^7 + \\
&\quad +20\frac{-315}{4^8}(t-1)^8 + 20\frac{70}{4^9}(t-1)^9 \\
&= 10 + 2.4609(t-1)^5 - 2.0508(t-1)^6 + 0.6592(t-1)^7 + \\
&\quad -0.0961(t-1)^8 + 0.0053(t-1)^9
\end{aligned}
$$

$$
\begin{aligned}
\dot{q}(t) &= 5 \cdot 20 \frac{126}{4^5}(t-1)^4 + 6 \cdot 20 \frac{-420}{4^6}(t-1)^5 + 7 \cdot 20 \frac{540}{4^7}(t-1)^6 + \\
&\quad +8 \cdot 20 \frac{-315}{4^8}(t-1)^7 + 9 \cdot 20 \frac{70}{4^9}(t-1)^8 \\
&= 12.3047(t-1)^4 - 12.3047(t-1)^5 + 4.6143(t-1)^6 + \\
&\quad -0.7690(t-1)^7 + 0.0481(t-1)^8
\end{aligned}
$$

$$
\begin{aligned}
\ddot{q}(t) &= 5 \cdot 4 \cdot 20 \frac{126}{4^5}(t-1)^3 + 6 \cdot 5 \cdot 20 \frac{-420}{4^6}(t-1)^4 + 7 \cdot 6 \cdot 20 \frac{540}{4^7}(t-1)^5 + \\
&\quad +8 \cdot 7 \cdot 20 \frac{-315}{4^8}(t-1)^6 + 9 \cdot 8 \cdot 20 \frac{70}{4^9}(t-1)^7 \\
&= 49.2188(t-1)^3 - 61.5234(t-1)^4 + 27.6855(t-1)^5 + \\
&\quad -5.3833(t-1)^6 + 0.3845(t-1)^7.
\end{aligned}
$$

These functions are shown in Fig. 2.15. □

The maximum value of the velocity, acceleration, jerk, ..., of a (normalized) polynomial $q_N(\tau)$ increases with the degree n, as illustrated in Fig. 2.14 and reported in Tab. 2.2. It is interesting to note, as illustrated in Fig. 2.16, that the rates of growth of C_v, C_a and C_j are proportional to $\sqrt{n}$, n, and n^2 respectively.

Although the determination of polynomials in the Bézier/Bernstein form is quite robust from the numerical point of view, for large values of n (eg. 37, 39, ...) the computation of polynomials is in any case affected by relevant numerical errors, and therefore it is advisable to use other functions to define smooth motion profiles, like trigonometric or exponential functions.

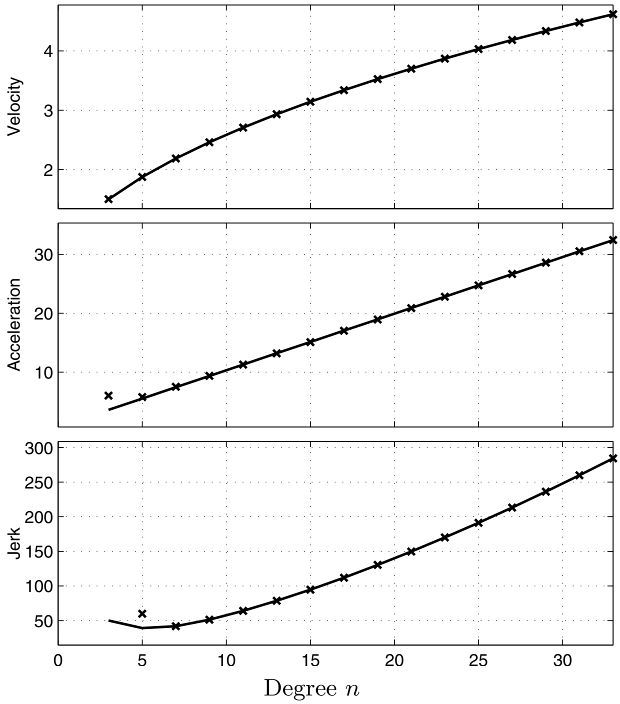


Fig. 2.16. Maximum values of the velocity, acceleration and jerk profiles of normalized polynomials of degree 3 - 33 with null boundary conditions, plotted as function of n (x-marks); interpolation with functions depending respectively on $\sqrt{n}$, n, n^2 (solid lines).

2.2 Trigonometric Trajectories

In this section, the analytical expressions of trajectories based on trigonometric functions are described. These trajectories present non-null continuous derivatives for any order of derivation in the interval $(t_0,\ t_1)$. However, these derivatives may be discontinuous in t_0 and t_1.

2.2.1 Harmonic trajectory

An harmonic motion is characterized by an acceleration profile that is proportional to the position profile, with opposite sign. The mathematical formulation of the harmonic motion can be also deduced graphically, see Fig. 2.17.

Let the point q be the projection on the diameter of point p. If point p moves on the circle with constant velocity, the motion of q, called *harmonic*, is described by

$$s(\theta) = R(1 - \cos\theta) \tag{2.20}$$

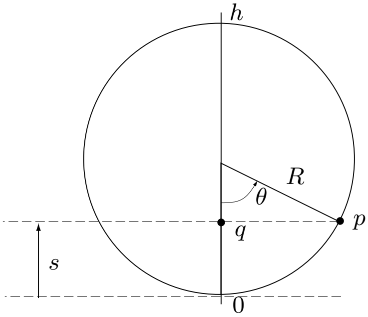


Fig. 2.17. Geometric construction of the harmonic motion.

where R is the radius of the circle. In a more general form, the harmonic trajectory can be defined as

$$q(t) = \frac{h}{2} \left(1 - \cos\frac{\pi(t - t_0)}{T}\right) + q_0 \tag{2.21}$$

with $h = q_1 - q_0$ and $T = t_1 - t_0$, from which

$$\begin{cases} \dot{q}(t) &= \dfrac{\pi h}{2T} \sin\left(\dfrac{\pi(t - t_0)}{T}\right) \\ \ddot{q}(t) &= \dfrac{\pi^2 h}{2T^2} \cos\left(\dfrac{\pi(t - t_0)}{T}\right) \\ q^{(3)}(t) &= -\dfrac{\pi^3 h}{2T^3} \sin\left(\dfrac{\pi(t - t_0)}{T}\right). \end{cases}$$

Example 2.14 Fig. 2.18 reports the position, velocity, acceleration and jerk of an harmonic trajectory with the conditions $t_0 = 0,\ t_1 = 8,\ q_0 = 0,\ q_1 = 10.$ □

2.2.2 Cycloidal trajectory

As shown in Fig. 2.18, the harmonic trajectory presents a discontinuous acceleration and, therefore, infinite instantaneous jerk at t_0, t_1. As already discussed, a discontinuous acceleration profile may generate undesired effects when flexible mechanisms are present. A continuous acceleration profile is obtained with the cycloidal trajectory, described by a circle with circumference h rolling along a line see Fig. 2.19,

$$\begin{aligned} q(t) &= (q_1 - q_0)\left(\frac{t - t_0}{t_1 - t_0} - \frac{1}{2\pi} \sin\frac{2\pi(t - t_0)}{t_1 - t_0}\right) + q_0 \\ &= h\left(\frac{t - t_0}{T} - \frac{1}{2\pi} \sin\frac{2\pi(t - t_0)}{T}\right) + q_0 \end{aligned} \tag{2.22}$$

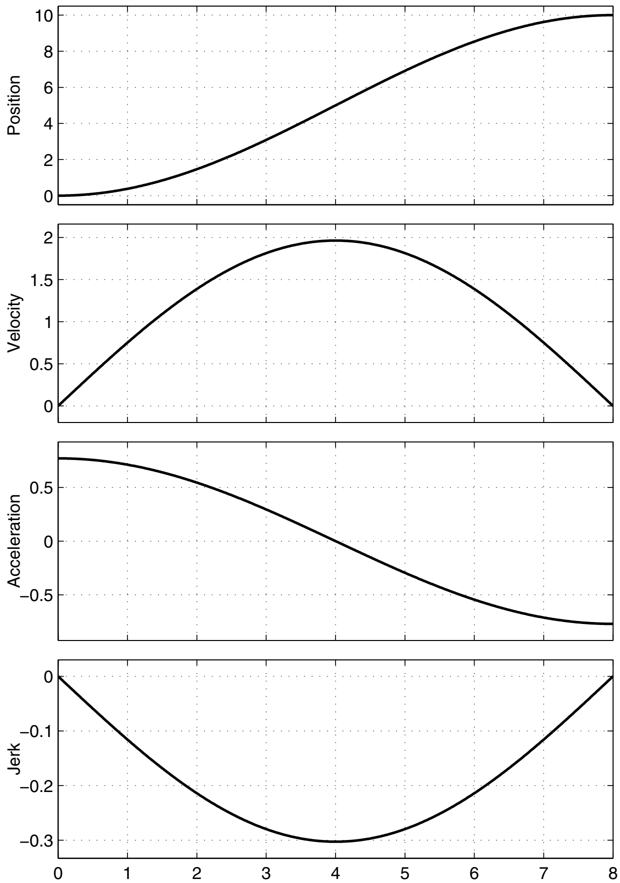


Fig. 2.18. Position, velocity, acceleration and jerk of an harmonic trajectory when $t_0 = 0,\ t_1 = 8,\ q_0 = 0,\ q_1 = 10$.

from which

$$\begin{aligned}\dot{q}(t) &= \frac{h}{T}\left(1 - \cos\frac{2\pi(t-t_0)}{T}\right)\\ \ddot{q}(t) &= \frac{2\pi h}{T^2}\ \sin\frac{2\pi(t-t_0)}{T}\\ q^{(3)}(t) &= \frac{4\pi^2 h}{T^3}\ \cos\frac{2\pi(t-t_0)}{T}.\end{aligned}$$

In this case, the acceleration is null in $t = t_0, t_1$, and therefore it presents a continuous profile.

Example 2.15 Fig. 2.20 shows position, velocity, acceleration and jerk for a cycloidal trajectory with the same conditions as in the previous example. □

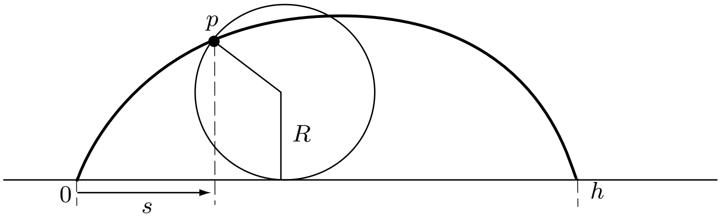


Fig. 2.19. Geometric construction of the cycloidal motion.

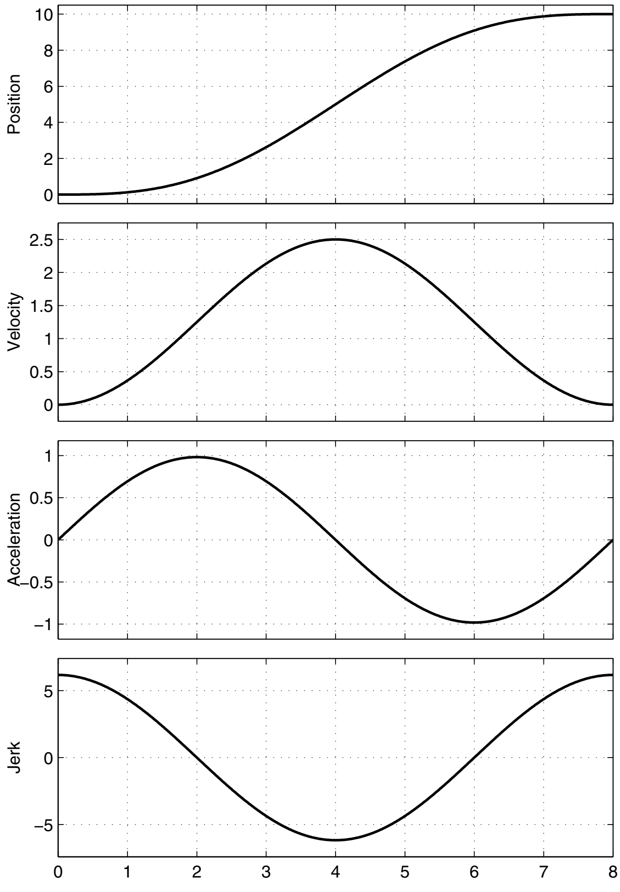


Fig. 2.20. Position, velocity, acceleration and jerk of a cycloidal trajectory with $t_0 = 0,\ t_1 = 8,\ q_0 = 0,\ q_1 = 10$.

2.2.3 Elliptic trajectory

As shown in Fig. 2.17, the harmonic motion can be obtained graphically by projecting on the diameter a point moving on a circle. An elliptic motion is

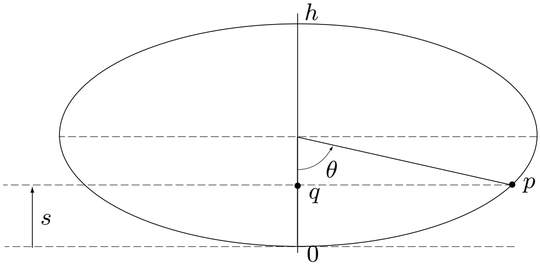


Fig. 2.21. Geometric construction of the elliptic motion.

obtained by projecting the motion of a point moving on an ellipse on the minor axis, of length equal to the desired displacement $h = q_1 - q_0$, see Fig. 2.21. The resulting equation is

$$q(t) = \frac{h}{2}\left(1 - \frac{\cos\frac{\pi(t-t_0)}{T}}{\sqrt{1-\alpha\ \sin^2\frac{\pi(t-t_0)}{T}}}\right) + q_0 \tag{2.23}$$

where $\alpha = \frac{n^2-1}{n^2}$, and n is the ratio between the major and minor ellipse axes. The velocity and the acceleration are

$$\dot{q}(t) = \frac{\pi h}{2T}\ \frac{\sin\frac{\pi(t-t_0)}{T}}{n^2\ \sqrt{\left(1-\alpha\ \sin^2\frac{\pi(t-t_0)}{T}\right)^3}}$$

$$\ddot{q}(t) = \frac{\pi^2 h}{2T^2}\ \cos\left(\frac{\pi(t-t_0)}{T}\right)\ \frac{1+2\ \alpha \sin^2\frac{\pi(t-t_0)}{T}}{n^2\ \sqrt{\left(1-\alpha\ \sin^2\frac{\pi(t-t_0)}{T}\right)^5}}.$$

Obviously, the harmonic trajectory is obtained by setting $n = 1$.

Example 2.16 Fig. 2.22 shows position, velocity, acceleration and jerk of this trajectory. Fig. 2.23 reports the profiles of position, velocity and acceleration with different choices of n. Note that the maximum values of velocity and acceleration increase with n. □

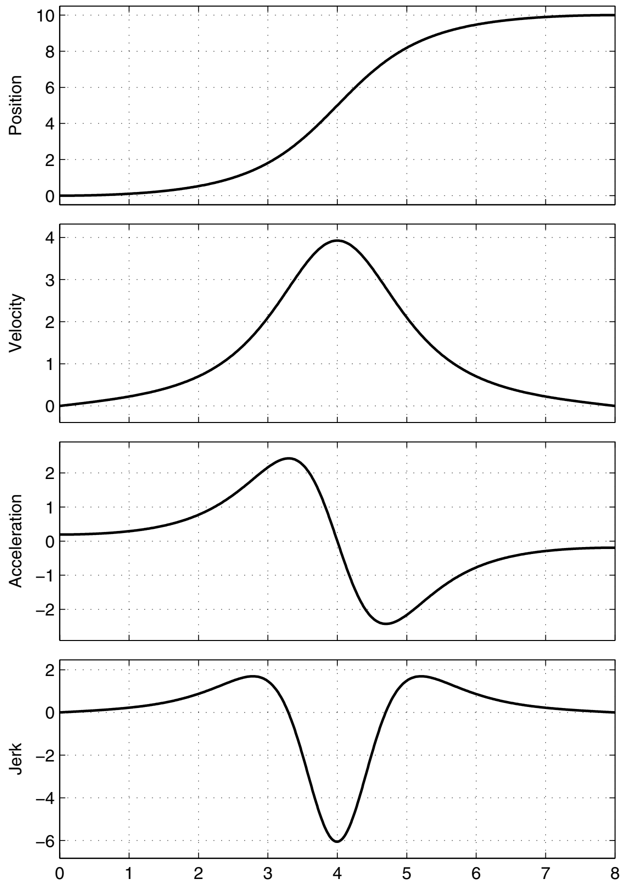


Fig. 2.22. Position, velocity, acceleration and jerk of an elliptic trajectory with $t_0 = 0,\ t_1 = 8,\ q_0 = 0,\ q_1 = 10,\ n = 2$.

2.3 Exponential Trajectories

As discussed in Chapter 7, natural vibrations induced on the machine by the actuation system should always be minimized.

This involves also the choice of proper motion profiles, since discontinuities in the desired trajectory may generate vibrations in the machine due to the induced discontinuities in the applied forces and the elastic effects of the mechanical system itself. Therefore, it may be convenient to introduce trajectories whose smoothness can be adjusted according to the needs, [14].

For this purpose, it is possible to consider an exponential function for the velocity, as

$$\dot{q}(\tau) = \mathrm{v}_c \, e^{-\sigma\, f(\tau,\lambda)}$$

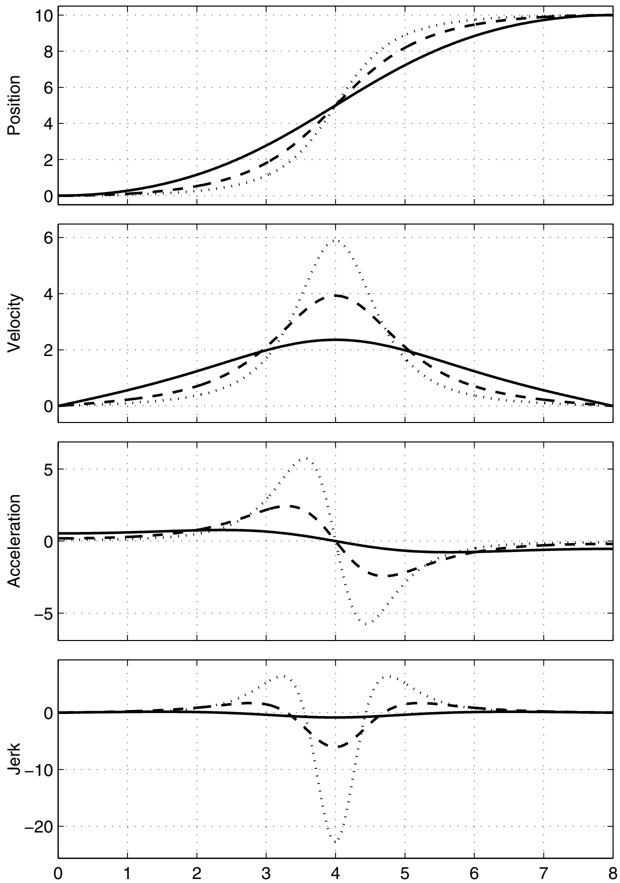


Fig. 2.23. Elliptic trajectories when: $n = 1.2$ (solid), $n = 2$ (dashed), $n = 3$ (dotted).

where σ and λ are free parameters. Possible choices for the function $f(\tau, \lambda)$ are

$$f_a(\tau, \lambda) = \frac{(2\tau)^2}{|1 - (2\tau)^2|^\lambda} \qquad \text{or} \qquad f_b(\tau, \lambda) = \frac{\sin^2 \pi\tau}{|\cos \pi\tau|^\lambda} \ .$$

If a normalized motion profile is considered, i.e. with unit displacement and duration, and in particular with the conditions $q_0 = -0.5$, $q_1 = 0.5$, and $\tau_0 = -0.5$, $\tau_1 = 0.5$, then the constant v_c can be computed as

$$\mathrm{v}_c = \frac{1}{2\displaystyle\int_0^{\frac{1}{2}} -\sigma f(\tau, \lambda) d\tau}.$$

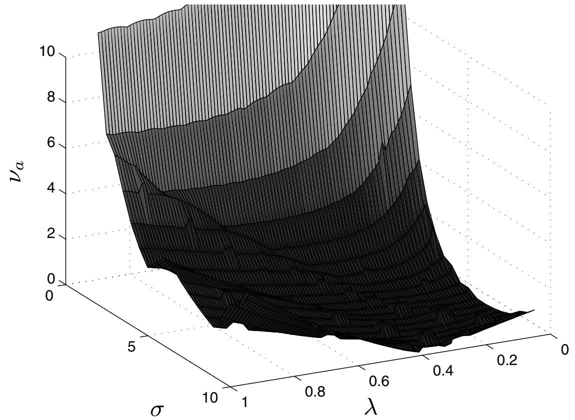


Fig. 2.24. Maximum values of the residual spectrum ν_a of the exponential trajectory for different values of σ and λ.

At this point, the normalized motion $q_N(\tau)$ is defined by the following equations

$$\begin{cases} q_N(\tau) = \mathrm{v}_c \displaystyle\int_0^{\tau} e^{-\sigma\, f(\tau,\lambda)} d\tau \\ \dot{q}_N(\tau) = \mathrm{v}_c\, e^{-\sigma\, f(\tau,\lambda)} \\ \ddot{q}_N(\tau) = -\mathrm{v}_c\, \sigma\, \dfrac{f(\tau,\lambda)}{d\tau} e^{-\sigma\, f(\tau,\lambda)}. \end{cases} \tag{2.24}$$

The choice of the function $f_a(\tau, \lambda)$ or $f_b(\tau, \lambda)$ has only a little influence on the actual motion profile and therefore, being f_a simpler from a computational point of view, it is adopted in the following discussion. More important is the choice of σ and λ, whose values may be assigned in order to minimize the maximum amplitude of the high frequency components of the acceleration profile, responsible of vibrations induced in the machine. The maximum values of the residual spectrum ${\nu_a}^2$ of $\ddot{q}_N$ for frequencies greater than 5 Hz, obtained for several values of the parameters σ, λ, are shown in Fig. 2.24.

In particular, the numerical values of ν_a obtained for some values of σ and with the corresponding λ which minimizes the residual spectrum are reported in Tab. 2.3. It is possible to show that the minimum value $\nu_{a,min} = 0.018$ is obtained for $\lambda = 0.20,\ \sigma = 7.1$, [14].

In case of a trajectory from an initial point q_0 to a final one q_1, with $h = q_1 - q_0$, and time instants t_0 and t_1, with $T = t_1 - t_0$, the actual position $q(t)$, velocity $\dot{q}(t)$ and acceleration $\ddot{q}(t)$ profiles may be obtained from (2.24)

[2] The residual spectrum is defined here as the maximum amplitude of the frequency spectrum of the acceleration profile for frequencies higher than a given threshold.

σ	2.0	2.5	3.0	3.5	4.0	4.5	5.0	5.5	6.0	6.5	7.0	7.5
λ	0.61	0.49	0.41	0.34	0.29	0.25	0.22	0.19	0.18	0.18	0.19	0.28
ν_a	4.364	2.736	1.697	1.034	0.625	0.370	0.217	0.125	0.071	0.039	0.019	0.043

Table 2.3. Parameters σ and λ for exponential trajectories and the related maximum amplitude of the frequency content of the acceleration profiles (> 5 *Hz*).

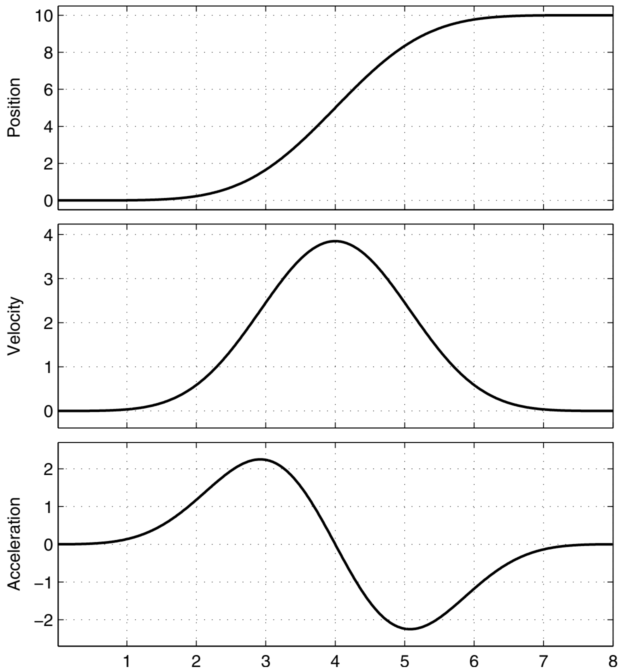


Fig. 2.25. Position, velocity, acceleration profiles of an exponential trajectory with $\sigma = 7.1$ and $\lambda = 0.2$.

as

$$q(t) = q_0 + h\left(\frac{1}{2} + q_N(\tau)\right), \qquad \dot{q}(t) = \frac{h}{T}\dot{q}_N(\tau), \qquad \ddot{q}(t) = \frac{h}{T^2}\ddot{q}_N(\tau)$$

with $\tau = \left(\dfrac{t - t_0}{T} - 0.5\right)$, see also Chapter 5.

Example 2.17 An exponential trajectory with the conditions

$$q_0 = 0, \qquad q_1 = 10, \qquad t_0 = 0, \qquad t_1 = 8, \qquad \lambda = 0.20, \qquad \sigma = 7.1$$

is shown in Fig. 2.25. □

Example 2.18 The exponential trajectories obtained with the conditions

$$q_0 = 0, \qquad q_1 = 10, \qquad t_0 = 0, \qquad t_1 = 8$$

and the parameters σ, λ as in Tab. 2.3 are shown in Fig. 2.26. □

A final comment concerns the computation of eq. (2.24), where an integral function explicitly appears. If the computation of $q_N(\tau)$ by using integrals, with possibly variable upper bounds, is unsuitable for the online generation of the motion profile, it is possible to adopt a series expansion of the function $q_N(\tau)$, truncated at a proper order r, as

$$\begin{cases} q_N(\tau) = a_0\ \tau + \displaystyle\sum_{k=1}^{r} a_{2k} \sin(2k\pi\tau) \\ \dot{q}_N(\tau) = a_0 + 2\pi \displaystyle\sum_{k=1}^{r} k a_{2k} \cos(2k\pi\tau) \\ \ddot{q}_N(\tau) = -4\pi^2 \displaystyle\sum_{k=1}^{r} k^2 a_{2k} \sin(2k\pi\tau) \end{cases}$$

where

$$a_0 = 1, \qquad\qquad a_{2k} = \frac{2}{\pi k} \int_0^{\frac{1}{2}} \dot{q}(\tau) \cos(2k\pi\tau)\ d\tau.$$

2.4 Trajectories Based on the Fourier Series Expansion

Besides quite obvious conditions about continuity of the position profile and its derivatives up to a given order, and the given boundary constraints, it might be of interest to pursue also other goals. Among the different possibilities, it could be desirable to minimize the amplitude of the acceleration profile, in order to avoid efforts on the load due to inertial forces or vibrational effects of the mechanical structure.

The minimization of the amplitude of the acceleration in general is in contrast with the continuity of the profile: a discontinuous acceleration profile minimizes the peak of acceleration but, on the other hand, may generate oscillations and/or vibrations because of the related discontinuity of the inertial forces. For example, the trapezoidal velocity trajectory (discussed in the following Chapter 3) presents, other conditions being equal, smaller values for the acceleration but, at the same time, an higher harmonic content that usually implies possible vibrations in the mechanical structure. On the contrary, the cycloidal trajectory is characterized by a low harmonic content but presents higher acceleration values. It is possible to define trajectories that represent a compromise between these two opposite features. As an example, trajectories

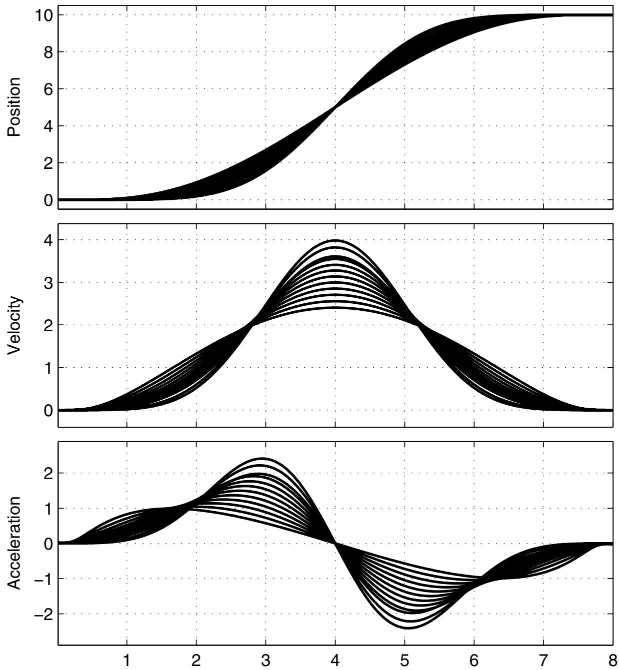


Fig. 2.26. Position, velocity, acceleration profiles of exponential trajectories with σ and λ as in Tab. 2.3.

derived from a Fourier series expansion of the motion profiles illustrated in the previous sections are now considered.

It is well known that a fundamental tool for the analysis in the frequency domain ω of a signal $x(t)$ defined in the time domain is the *Fourier Transform* $X(\omega) = \mathcal{F}\{x(t)\}$, see Appendix D. On the other hand, it is worth noticing that trajectories for high speed automatic machines are often a cyclic repetition of a basic motion: therefore, the trajectory $q(t)$ can be assumed to be periodic. Under this hypothesis, $q(t)$ can be analyzed by means of the *Fourier series expansion*.

The Fourier series is a mathematical tool often used for analyzing periodic functions by decomposing them into a weighted sum of sinusoidal component functions, sometimes referred to as *normal Fourier modes*, or simply *modes*. Given a piecewise continuous function $x(t)$, periodic with period T, and square-integrable over the interval $[-T/2,\ T/2]$, that is

$$\int_{-T/2}^{T/2} |x(t)|^2 \, dt < +\infty,$$

the corresponding Fourier series expansion is

$$x(t) = \frac{1}{2}a_0 + \sum_{k=1}^{\infty}[a_k \cos(k\omega_0 t) + b_k \sin(k\omega_0 t)]$$

where $\omega_0 = 2\pi/T$ is the fundamental frequency (rad/sec) of the function and, for any non-negative integer k,

$$a_k = \frac{2}{T}\int_{-T/2}^{T/2} x(t)\cos(k\omega_0 t)\,dt \qquad \text{are the even Fourier coefficients of } x(t)$$

$$b_k = \frac{2}{T}\int_{-T/2}^{T/2} x(t)\sin(k\omega_0 t)\,dt \qquad \text{are the odd Fourier coefficients of } x(t).$$

An alternative expression of the Fourier series expansion is

$$x(t) = v_0 + \sum_{k=1}^{\infty} v_k \cos(k\omega_0 t - \varphi_k) \tag{2.25}$$

where

$$v_0 = \frac{a_0}{2}, \qquad v_k = \sqrt{a_k^2 + b_k^2}, \qquad \varphi_k = \arctan\left(\frac{b_k}{a_k}\right).$$

Eq. (2.25) defines the signal as a linear combination of a constant term (v_0) and of an infinite number of sinusoidal functions (the *harmonic functions*) at frequencies $k\omega_0$; v_k represents the weight of the k-th harmonic function on $x(t)$, and φ_k its phase. The *maximum frequency* of the signal corresponds to the maximum k for which $v_k \neq 0$ from a practical point of view. On the basis of the Fourier series expansion of a signal, it is then possible to understand its properties in the frequency domain.

The basic idea of the techniques for planning the motion profiles illustrated below is to compute a Fourier series expansion of a function $q(t)$ defined by one of the methods presented in the previous sections and, then, define a new trajectory $q_f(t)$ by considering only the first N terms of the series. In this manner, it is possible to obtain a function that presents specific properties in the frequency domain, see also Sec. 7.3.

2.4.1 Gutman 1-3

This trajectory is obtained as Fourier series expansion of the parabolic profile, Sec. 2.1.2, by taking into consideration only the first two elements, [15]:

$$\begin{cases} q(t) &= q_0 + h\left(\dfrac{(t-t_0)}{T} - \dfrac{15}{32\pi}\sin\dfrac{2\pi(t-t_0)}{T} - \dfrac{1}{96\pi}\sin\dfrac{6\pi(t-t_0)}{T}\right) \\ \dot{q}(t) &= \dfrac{h}{T}\left(1 - \dfrac{15}{16}\cos\dfrac{2\pi(t-t_0)}{T} - \dfrac{1}{16}\cos\dfrac{6\pi(t-t_0)}{T}\right) \\ \ddot{q}(t) &= \dfrac{h\pi}{8T^2}\left(15\sin\dfrac{2\pi(t-t_0)}{T} + 3\sin\dfrac{6\pi(t-t_0)}{T}\right) \\ q^{(3)}(t) &= \dfrac{h\pi^2}{4T^3}\left(15\cos\dfrac{2\pi(t-t_0)}{T} + 9\cos\dfrac{6\pi(t-t_0)}{T}\right) \end{cases}$$

where h is the displacement and T the time duration. The maximum acceleration is $5.15\, h/T^2$, i.e. 28.75% larger than the maximum acceleration of the parabolic trajectory ($4h/T^2$) and, for example, 18.04% smaller than the maximum acceleration of the cycloidal trajectory ($2\pi h/T^2$). On the other hand, the frequency content is lower with respect to the parabolic profile, and higher than the cycloidal one, see Chapter 7.

Example 2.19 Fig. 2.27 reports the position, velocity, acceleration and jerk for the Gutman 1-3 trajectory with $h = 20$ and $T = 10$ ($q_0 = 0$, $t_0 = 0$). □

2.4.2 Freudenstein 1-3

As in the previous case, only the first and the third terms of the Fourier series expansion of the parabolic trajectory are considered, but the trajectory is defined as [16]

$$\begin{cases} q(t) &= q_0 + \dfrac{h(t-t_0)}{T} - \dfrac{h}{2\pi}\left(\dfrac{27}{28}\sin\dfrac{2\pi(t-t_0)}{T} + \dfrac{1}{84}\sin\dfrac{6\pi(t-t_0)}{T}\right) \\ \dot{q}(t) &= \dfrac{h}{T}\left(1 - \dfrac{27}{28}\cos\dfrac{2\pi(t-t_0)}{T} - \dfrac{1}{28}\cos\dfrac{6\pi(t-t_0)}{T}\right) \\ \ddot{q}(t) &= \dfrac{2\pi h}{T^2}\left(\dfrac{27}{28}\sin\dfrac{2\pi(t-t_0)}{T} + \dfrac{3}{28}\sin\dfrac{6\pi(t-t_0)}{T}\right) \\ q^{(3)}(t) &= \dfrac{4\pi^2 h}{T^3}\left(\dfrac{27}{28}\cos\dfrac{2\pi(t-t_0)}{T} + \dfrac{9}{28}\cos\dfrac{6\pi(t-t_0)}{T}\right). \end{cases}$$

This trajectory has a maximum acceleration value equal to $5.39\, h/T^2$, i.e 34.75% larger than the parabolic profile and 14.22% smaller than the cycloidal trajectory.

Example 2.20 Fig. 2.28 shows the position, velocity, acceleration and jerk for the Freudenstein 1-3 trajectory, with $h = 20$ and $T = 10$ ($q_0 = 0$, $t_0 = 0$). □

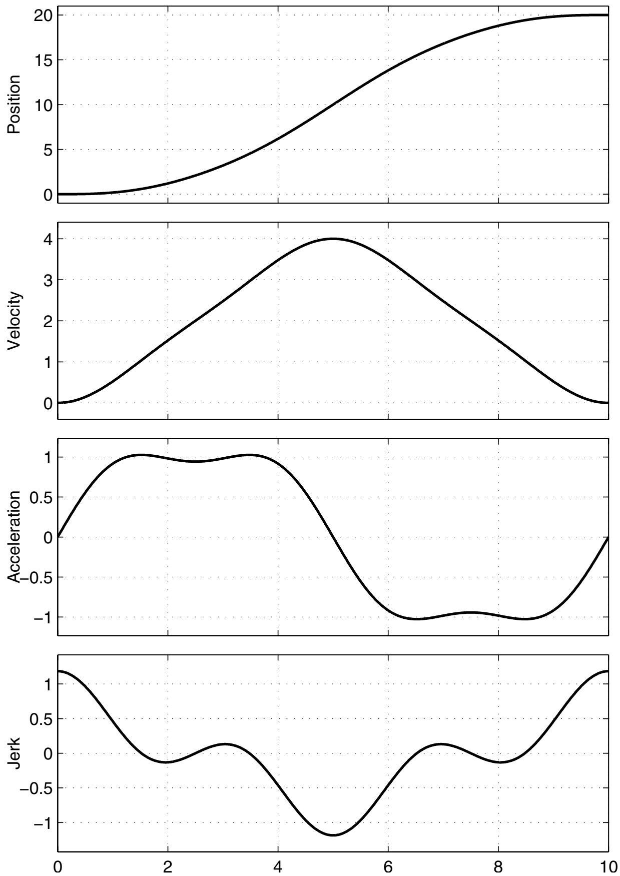


Fig. 2.27. Position, velocity, acceleration and jerk of the Gutman 1-3 trajectory with $h = 20$ and $T = 10$.

2.4.3 Freudenstein 1-3-5

This trajectory is defined as

$$\begin{cases} q(t) = q_0 + \dfrac{h(t-t_0)}{T} - \dfrac{h}{2\pi}\alpha\left(\sin\dfrac{2\pi(t-t_0)}{T} + \dfrac{1}{54}\sin\dfrac{6\pi(t-t_0)}{T} + \dfrac{1}{1250}\sin\dfrac{10\pi(t-t_0)}{T}\right) \\ \dot{q}(t) = \dfrac{h}{T}\left[1 - \alpha\left(\cos\dfrac{2\pi(t-t_0)}{T} + \dfrac{1}{18}\cos\dfrac{6\pi(t-t_0)}{T} + \dfrac{1}{250}\cos\dfrac{10\pi(t-t_0)}{T}\right)\right] \\ \ddot{q}(t) = \dfrac{2\pi h}{T^2}\,\alpha\left(\sin\dfrac{2\pi(t-t_0)}{T} + \dfrac{1}{6}\sin\dfrac{6\pi(t-t_0)}{T} + \dfrac{1}{50}\sin\dfrac{10\pi(t-t_0)}{T}\right) \\ q^{(3)}(t) = \dfrac{4\pi^2 h}{T^3}\,\alpha\left(\cos\dfrac{2\pi(t-t_0)}{T} + \dfrac{1}{2}\cos\dfrac{6\pi(t-t_0)}{T} + \dfrac{1}{10}\cos\dfrac{10\pi(t-t_0)}{T}\right) \end{cases}$$

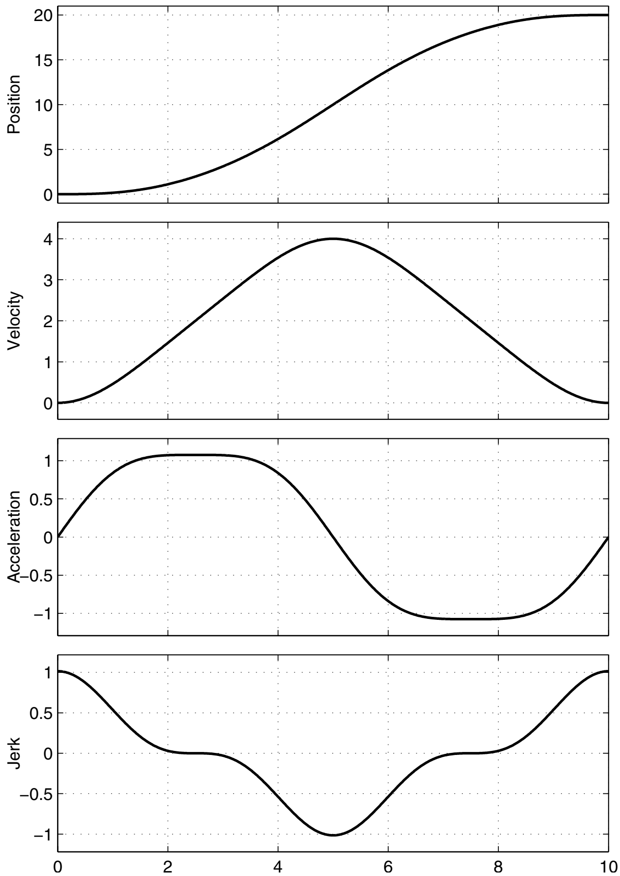


Fig. 2.28. Position, velocity, acceleration and jerk of the Freudenstein 1-3 trajectory with $h = 20$ and $T = 10$.

where $\alpha = \frac{1125}{1192} = 0.9438$. This trajectory has a maximum acceleration value equal to $5.06\, h/T^2$, i.e. 26.5% larger than the parabolic motion, and 19.47% smaller than the cycloidal profile.

Example 2.21 Fig. 2.29 shows the position, velocity, acceleration and jerk for the Freudenstein 1-3-5 trajectory, with $h = 20$ and $T = 10$ ($q_0 = 0$, $t_0 = 0$). □

If a larger number of terms of the Fourier series expansion is considered, profiles with lower acceleration values but higher frequency components are obtained. As discussed in Chapter 7, this could generate undesired vibrations in the mechanical structure. A compromise has then to be obtained between the requirements of low acceleration values and frequency bandwidth of the corresponding signal. An empiric rule could be to limit the maximum frequency

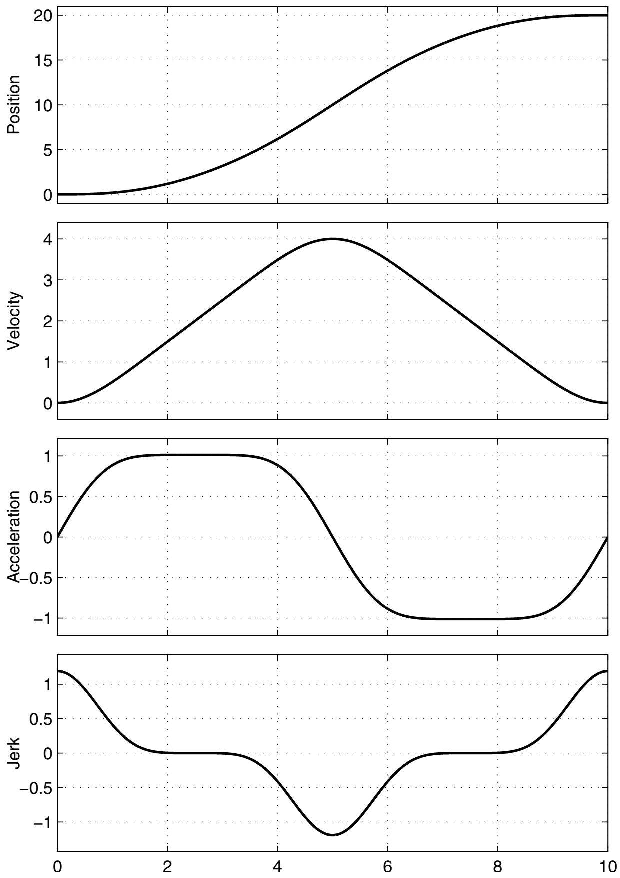


Fig. 2.29. Position, velocity, acceleration and jerk of the Freudenstein 1-3-5 trajectory with $h = 20$ and $T = 10$.

of the trajectory to $\omega_r/10$, being ω_r the lower resonance frequency of the mechanical structure under consideration. This can be obtained by truncating, for example, the Fourier series expansion to N, where $N = \text{floor}\left(\frac{\omega_r/10}{\omega_0}\right)$, where $\omega_0 = 2\pi/T$, T is the period of the trajectory, and $\text{floor}(x)$ is the function which gives the largest integer less than or equal to x.

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