Preface

Operator theory and functional analysis have a long tradition, initially being guided by problems from mathematical physics and applied mathematics. Much of the work in Banach spaces from the 1930s onwards resulted from investigating how much real (and complex) variable function theory might be extended to functions taking values in (function) spaces or operators acting in them. Many of the first ideas in geometry, basis theory and the isomorphic theory of Banach spaces have vector measure-theoretic origins and can be credited (amongst others) to N. Dunford, I.M. Gelfand, B.J. Pettis and R.S. Phillips. Somewhat later came the penetrating contributions of A. Grothendieck, which have pervaded and influenced the shape of functional analysis and the theory of vector measures/integration ever since.

Today, each of the areas of functional analysis/operator theory, Banach spaces, and vector measures/integration is a strong discipline in its own right. However, it is not always made clear that these areas grew up together as cousins and that they had, and still have, enormous influences on one another. One of the aims of this monograph is to reinforce and make transparent precisely this important point.

The monograph itself contains mostly new material which has not appeared elsewhere and is directed at advanced research students, experienced researchers in the area, and researchers interested in the interdisciplinary nature of the mathematics involved. We point out that the monograph is self-contained with complete proofs, detailed references and (hopefully) clear explanations. There is an emphasis on many and varied examples, with most of them having all the explicit details included. They form an integral part of the text and are designed and chosen to enrich and illuminate the theory that is developed. Due to the relatively new nature of the material and its research orientation, many open questions are posed and there is ample opportunity for the diligent reader to enter the topic with the aim of “pushing it further”. Indeed, the theory which is presented, even though rather complete, is still in its infancy and its potential for further development and applications to concrete problems is, in our opinion, quite large. As can be gleaned from the above comments, the reader is assumed to have some proficiency in certain aspects of general functional analysis, linear operator theory, measure theory and the theory of Banach spaces, particularly function spaces. More specialized material is introduced and developed as it is needed.

Chapter 1 should definitely be read first and in its entirety. It explains completely and in detail what the monograph is all about: its aims, contents, philosophy and the whole structure and motivation that it encompasses. The actual details and the remainder of the text then follow in a natural and understandable way.

Chapter 2 treats certain aspects of the class of spaces on which the linear operators in later chapters will be defined. These are the quasi-Banach *function* spaces (over a finite measure space), with the emphasis on $\mathbb{C}$-valued functions. This is in contrast to much of the existing literature which is devoted to spaces over $\mathbb{R}$. Of particular importance is the notion of the *p-th power* $X_{[p]}$, $0 < p < \infty$, of a given quasi-Banach function space X. This associated family of quasi-Banach function spaces $X_{[p]}$, which is intimately connected to the base space X, is produced via a procedure akin to that which produces the Lebesgue L^p-spaces from L^1 and plays a crucial role in the sequel.

Let X be a quasi-Banach function space and $T : X \to E$ be a continuous linear operator, with E a Banach space. Then $m_T : A \mapsto T(\chi_A)$, for A a measurable set, defines a finitely additive E-valued vector measure with the property that $T(s) = \int s \, dm_T$ for each $\mathbb{C}$-valued simple function $s \in X$. Under appropriate conditions on X and/or T, it follows that m_T is actually σ-additive and so $\int f \, dm_T \in E$ is defined for each m_T-integrable function $f \in L^1(m_T)$. The crucial point is that the linear map $I_{m_T} : f \mapsto \int f \, dm_T$ is then typically defined for many more functions $f \in L^1(m_T)$ than just those coming from the domain X of T (which automatically satisfies $X \subseteq L^1(m_T)$). That is, I_{m_T} is an E-valued *extension* of T. This is one of the crucial points that pervades the entire monograph. In order to fully develop this idea, it is necessary to make a detailed study of the Banach lattice $L^1(m_T)$ and the related spaces $L^p(m_T)$, for $1 \le p \le \infty$, together with various properties of the integration operator $I_{m_T} : L^1(m_T) \to E$. This is systematically carried out in Chapter 3, not just for m_T but, for general E-valued vector measures ν. Of course, there is already available a vast literature on the theory of vector measures and integration, of which we surely make good use. However, for applications in later chapters we need to further develop many new aspects concerning the spaces $L^p(\nu)$ and the integration operators $I_\nu^{(p)} : L^p(\nu) \to E$, for $1 < p < \infty$, in particular, their ideal properties in relation to compactness, weak compactness and complete continuity.

Chapter 4 presents a detailed and careful analysis of the particular extension $I_{m_T} : L^1(m_T) \to E$ of an operator $T : X \to E$. In addition to always existing, the remarkable feature is that $L^1(m_T)$ and I_{m_T} turn out to be *optimal*, in the sense that if Y is any σ-order continuous quasi-Banach function space (over the same measure space as for X) for which $X \subseteq Y$ continuously and such that there exists a continuous linear operator $T_Y : Y \to E$ which coincides with T on X, then necessarily Y is continuously embedded in the *Banach* function space $L^1(m_T)$

and T_Y coincides with I_{m_T} restricted to Y. The remainder of the monograph then develops and applies the many consequences that follow from the existence of this optimal extension.

For instance, a continuous linear operator $T : X \to E$ is called *p-th power factorable*, for $1 \le p < \infty$, if there exists a continuous linear operator $T_{[p]} : X_{[p]} \to E$ which coincides with T on $X \subseteq X_{[p]}$. There is no a priori reason to suspect any connection between the p-th power factorability of T and its associated E-valued vector measure m_T. The purpose of Chapter 5 is to show that such a connection does indeed exist and that it has some far-reaching consequences. The spaces $L^p(m_T)$, for $1 \le p < \infty$, which are treated in Chapter 3, also exhibit certain optimality properties and play a vital role in Chapter 5. Many operators of interest coming from various branches of analysis are p-th power factorable; quite some effort is devoted to studying these in detail.

According to the results of Chapter 5, a p-th power factorable operator $T : X \to E$ factorizes through $L^p(m_T)$. However, whenever possible, the factorization of T through a *classical* L^p-space (i.e., of a scalar measure) is preferable. The Maurey–Rosenthal theory provides such a factorization, albeit under various assumptions. In Chapter 6 we investigate norm inequalities in the following sense. Let $0 < q < \infty$ and $1 \le p < \infty$ be given and X be a q-convex quasi-Banach function space. Then T is required to satisfy

$$\Big(\sum_{j=1}^{n} \|T(f_j)\|_E^{q/p}\Big)^{1/q} \le C\Big\|\Big(\sum_{j=1}^{n} |f_j|^{q/p}\Big)^{1/q}\Big\|_X \tag{0.1}$$

for some constant $C > 0$ and all finite collections $f_1, \dots, f_n$ of elements from X. This inequality already encompasses the main geometrical condition of p-th power factorability. In fact, we will treat a more general inequality which can be applied even when X is not q-convex, namely

$$\Big(\sum_{j=1}^{n} \|T(f_j)\|_E^{q/p}\Big)^{1/q} \le C\Big\|\sum_{j=1}^{n} |f_j|^{q/p}\Big\|_{\mathrm{b},X_{[q]}}^{1/q}, \tag{0.2}$$

where $\|\cdot\|_{b,X_{[q]}}$ is a suitable *seminorm* which is continuous and defined on the quasi-Banach function space $X_{[q]}$. If X is q-convex, then (0.1) and (0.2) are equivalent; if, in addition, $p = 1$, then (0.2) reduces to q-concavity of T. In this case, we are in the (abstract) setting of the Maurey–Rosenthal Theorem, a version due to A. Defant. The main aim of Chapter 6 is to establish various equivalences with (0.2), thereby providing factorizations for a *subclass* of the p-th power factorable operators for which the required hypotheses of the Maurey–Rosenthal Theorem are not necessarily fulfilled. In view of the fact that the spaces $L^p(m_T)$ are always p-convex, this has some interesting consequences for the associated integration operator $I_{m_T}^{(p)} : L^p(m_T) \to E$.

Chapter 7 is the culmination of all of the ideas in this monograph applied to two important classes of operators arising in classical harmonic analysis (over a compact abelian group G). One class of operators consists of the *Fourier transform* F from $X = L^p(G)$, for $1 \leq p < \infty$, into either $E = c_0(\Gamma)$ or $E = \ell^{p'}(\Gamma)$, where Γ is the dual group of G and $\frac{1}{p} + \frac{1}{p'} = 1$. Concerning the optimal domain $L^1(m_F)$ and an analysis of the extension $I_{m_F} : L^1(m_F) \to E$, the situation depends rather dramatically on whether $p = 1$ or $1 < p < \infty$ and whether $E = c_0(\Gamma)$ or $E = \ell^{p'}(\Gamma)$. This is particularly true of the ideal properties of I_{m_F} and whether or not the inclusion $L^p(G) \subseteq L^1(m_F)$ is proper. The second class of operators consists of the *convolution operators* $C_\lambda : L^p(G) \to L^p(G)$ defined via $f \mapsto \lambda * f$, where λ is any $\mathbb{C}$-valued, regular Borel measure on G. Again there are significant differences depending on whether $p = 1$ or $1 < p < \infty$ and whether λ is absolutely continuous or not with respect to Haar measure on G. Also relevant are those measures λ whose Fourier–Stieltjes transform $\widehat{\lambda}$ belongs to $c_0(\Gamma)$. There is also a difference for certain features exhibited between the two classes of operators. For instance, with $E = \ell^{p'}(\Gamma)$ the Fourier transform map $F : L^p(G) \to E$ is (for certain groups G) never q-th power factorable for any $1 \leq q < \infty$. On the other hand, the q-th power factorability of the convolution operator $C_\lambda : L^p(G) \to L^p(G)$ is completely determined as to whether or not λ is a so-called L^r*-improving measure* (a large class of measures introduced by E.M. Stein). Many other questions are also treated, e.g., when is $L^1(m_{C_\lambda}) = L^1(G)$ as large as possible, when is $L^1(m_{C_\lambda}) = L^p(G)$ as small as possible, what happens if the vector measure m_F or m_{C_λ} has finite variation, and so on?

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