

Contents

Preface to the Second Edition	ix
Introduction	1
Notation	18
 I Gradient Flow in Metric Spaces	 21
1 Curves and Gradients in Metric Spaces	23
1.1 Absolutely continuous curves and metric derivative	23
1.2 Upper gradients	26
1.3 Curves of maximal slope	30
1.4 Curves of maximal slope in Hilbert and Banach spaces	32
 2 Existence of Curves of Maximal Slope	 39
2.1 Main topological assumptions	42
2.2 Solvability of the discrete problem and compactness of discrete tra- jectories	44
2.3 Generalized minimizing movements and curves of maximal slope .	45
2.4 The (geodesically) convex case	49
 3 Proofs of the Convergence Theorems	 59
3.1 Moreau-Yosida approximation	59
3.2 A priori estimates for the discrete solutions	66
3.3 A compactness argument	69
3.4 Conclusion of the proofs of the convergence theorems	71
 4 Generation of Contraction Semigroups	 75
4.1 Cauchy-type estimates for discrete solutions	82
4.1.1 Discrete variational inequalities	82
4.1.2 Piecewise affine interpolation and comparison results	84

4.2	Convergence of discrete solutions	89
4.2.1	Convergence when the initial datum $u_0 \in \underline{D(\phi)}$	89
4.2.2	Convergence when the initial datum $u_0 \in \overline{D(\phi)}$	92
4.3	Regularizing effect, uniqueness and the semigroup property	93
4.4	Optimal error estimates	97
4.4.1	The case $\lambda = 0$	97
4.4.2	The case $\lambda \neq 0$	99
II	Gradient Flow in the Space of Probability Measures	103
5	Preliminary Results on Measure Theory	105
5.1	Narrow convergence, tightness, and uniform integrability	106
5.1.1	Unbounded and l.s.c. integrands	109
5.1.2	Hilbert spaces and weak topologies	113
5.2	Transport of measures	118
5.3	Measure-valued maps and disintegration theorem	121
5.4	Convergence of plans and convergence of maps	124
5.5	Approximate differentiability and area formula in Euclidean spaces	128
6	The Optimal Transportation Problem	133
6.1	Optimality conditions	135
6.2	Optimal transport maps and their regularity	139
6.2.1	Approximate differentiability of the optimal transport map	142
6.2.2	The infinite dimensional case	147
6.2.3	The quadratic case $p = 2$	149
7	The Wasserstein Distance and its Behaviour along Geodesics	151
7.1	The Wasserstein distance	151
7.2	Interpolation and geodesics	158
7.3	The curvature properties of $\mathcal{P}_2(X)$	160
8	A.C. Curves in $\mathcal{P}_p(X)$ and the Continuity Equation	167
8.1	The continuity equation in \mathbb{R}^d	169
8.2	A probabilistic representation of solutions of the continuity equation	178
8.3	Absolutely continuous curves in $\mathcal{P}_p(X)$	182
8.4	The tangent bundle to $\mathcal{P}_p(X)$	189
8.5	Tangent space and optimal maps	194
9	Convex Functionals in $\mathcal{P}_p(X)$	201
9.1	λ -geodesically convex functionals in $\mathcal{P}_p(X)$	202
9.2	Convexity along generalized geodesics	205
9.3	Examples of convex functionals in $\mathcal{P}_p(X)$	209

9.4	Relative entropy and convex functionals of measures	215
9.4.1	Log-concavity and displacement convexity	220
10	Metric Slope and Subdifferential Calculus in $\mathcal{P}_p(X)$	227
10.1	Subdifferential calculus in $\mathcal{P}_2^r(X)$: the regular case	229
10.1.1	The case of λ -convex functionals along geodesics	231
10.1.2	Regular functionals	232
10.2	Differentiability properties of the p -Wasserstein distance	234
10.3	Subdifferential calculus in $\mathcal{P}_p(X)$: the general case	240
10.3.1	The case of λ -convex functionals along geodesics	244
10.3.2	Regular functionals	246
10.4	Example of subdifferentials	254
10.4.1	Variational integrals: the smooth case	254
10.4.2	The potential energy	255
10.4.3	The internal energy	257
10.4.4	The relative internal energy	265
10.4.5	The interaction energy	267
10.4.6	The opposite Wasserstein distance	269
10.4.7	The sum of internal, potential and interaction energy	272
10.4.8	Relative entropy and Fisher information in infinite dimensions	276
11	Gradient Flows and Curves of Maximal Slope in $\mathcal{P}_p(X)$	279
11.1	The gradient flow equation and its metric formulations	280
11.1.1	Gradient flows and curves of maximal slope	283
11.1.2	Gradient flows for λ -convex functionals	284
11.1.3	The convergence of the “Minimizing Movement” scheme	286
11.2	Gradient flows for λ -convex functionals along generalized geodesics	295
11.2.1	Applications to Evolution PDE’s	298
11.3	Gradient flows in $\mathcal{P}_p(X)$ for regular functionals	304
12	Appendix	307
12.1	Carathéodory and normal integrands	307
12.2	Weak convergence of plans and disintegrations	308
12.3	PC metric spaces and their geometric tangent cone	310
12.4	The geometric tangent spaces in $\mathcal{P}_2(X)$	314
	Bibliography	331
	Index	333

Gradient Flows

In Metric Spaces and in the Space of Probability
Measures

Ambrosio, L.; Gigli, N.; Savare, G.

2008, IX, 334 p., Softcover

ISBN: 978-3-7643-8721-1

A product of Birkhäuser Basel