Preface

The book is devoted to the spectral theory of commutative C^*-algebras of Toeplitz operators on Bergman spaces, and its applications. For each such commutative algebra we construct a unitary operator which reduces each Toeplitz operator from this algebra to a certain multiplication operator, thus also providing its spectral type representation. This gives us a powerful research tool allowing direct access to the majority of the important properties of the Toeplitz operators studied herein. The presence and exploitation of these spectral type representations forms the basis for an essential part of the results presented in this book.

We give a criterion of when the algebras are commutative on each commonly considered weighted Bergman space. For Toeplitz operators generating such commutative algebras we describe their boundedness, compactness, and spectral properties. Furthermore, the above commutative algebras serve as model or local cases for a number of problems treated in the book, thus making their solutions possible.

We note that in the Bergman space case considered in the book the underlying manifold (the unit disk equipped with the hyperbolic metric) possesses a richer geometric structure in comparison with the Hardy space case (the unit circle). This fact has an important reflection in the presented theory.

We mention as well that from the general operator point of view the Toeplitz operators, both on Hardy space and on Bergman space, are compressions of multiplication operators onto certain subspaces, and thus they represent two interesting different models of operators having similar structure. At the same time the presented results show clearly essential differences between the theories for these two species of operators.

The book is addressed to a wide audience of mathematicians, from graduate students to researchers, whose primary interests lie in complex analysis and operator theory. The prerequisites for reading this book include a basic knowledge in one-dimensional complex analysis, functional analysis, and operator theory. An acquaintance with some facts of the theory of Banach and C^*-algebras will be useful as well. Among various excellent sources which may serve for the preliminary reading we mention, for example, the books by I. Gohberg, S. Goldberg, and M. Kaashoek [81], and R. Douglas [58].

The author is greatly indebted to his colleagues Serguei Grudsky, his coauthor in many papers, and Michael Porter who read the manuscript and made many important suggestions essentially improving the book. The author would like to address special words of gratitude to Olga Grudskaia who tragically died in a car accident in February 2004. She generously assisted in the preparation of this book, beautifully elaborating all the figures of the last three chapters.

The author is sincerely grateful to Israel Gohberg, the editor of the "Operator Theory: Advances and Applications" book series, for his invitation to publish the book in this series and whose friendly remarks and advice greatly assisted in the final stage of its preparation.

The author thankfully acknowledges CONACYT (Consejo Nacional de Ciencia y Tecnologia), Mexico, which has supported him in various projects.

Introduction

The notion of a Toeplitz operator goes back to the work of Otto Toeplitz at the beginning of the 20th century. The Toeplitz operator is normally defined in terms of the so-called Toeplitz matrix

$$A = \begin{pmatrix} a_0 & a_{-1} & a_{-2} & \dots \\ a_1 & a_0 & a_{-1} & \dots \\ a_2 & a_1 & a_0 & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix},$$

where $a_n \in \mathbb{C}$, $n \in \mathbb{Z}$. The result by O. Toeplitz of 1911 says that the matrix A defines a bounded operator on $l_2 = l_2(\mathbb{Z}_+)$, where $\mathbb{Z}_+ = \{0\} \cup \mathbb{N}$, if and only if the numbers $\{a_n\}$ are the Fourier coefficients of a function $a \in L_\infty(S^1)$, where S^1 is the unit circle. The (discrete) inverse Fourier transform is a unitary operator which maps $l_2(\mathbb{Z})$ onto $L_2(S^1)$ and $l_2(\mathbb{Z}_+)$ onto the so-called Hardy space $H^2_+(S^1)$. The last one consists of all $L_2(S^1)$-functions admitting an analytic continuation on the unit disk and whose discrete Fourier transform vanishes for all negative indices. That is, the operator defined by the matrix A is unitary equivalent to the operator T_a which acts on the Hardy space $H^2_+(S^1)$ by the rule

$$T_a : \ f(t) \in H^2_+(S^1) \longmapsto (P_+af)(t) \in H^2_+(S^1),$$

where P_+ is the so-called Szegö orthogonal projection of $L_2(S^1)$ onto $H^2_+(S^1)$, and the Fourier coefficients of the function a are given by the sequence $\{a_n\}$. The operator T_a is also called the Toeplitz operator, and the function $a \in L_\infty(S^1)$ is called the defining symbol[1] of T_a. As the Szegö projection P_+ is given in terms of the one-dimensional singular integral operator with the Cauchy kernel, the theories of the Toeplitz and singular integral operators are very related, especially in questions connected with the algebras generated by such operators. The first fundamental results on this direction for continuous symbols go back to the paper by I. Gohberg of 1952, and to papers by L. Coburn, R. Douglas, I. Gohberg and N. Krupnik of late 1960s – early 1970s for more general classes of defining symbols.

Another concept of a subspace of analytic functions of the L_2-space and of the corresponding orthogonal projection was introduced by S. Bergman in 1950. For a bounded simply connected domain D in $\mathbb{C}$, the Bergman space $\mathcal{A}^2(D)$ consists of all $L_2(D)$-functions which are analytic in D. We denote by B_D the orthogonal (Bergman) projection of $L_2(D)$ onto the Bergman space $\mathcal{A}^2(D)$. Then the Toeplitz operator T_a with defining symbol $a \in L_\infty(D)$ is given on the Bergman space in quite the same way,

$$T_a : \ \varphi(z) \in \mathcal{A}^2(D) \longmapsto (B_D a\varphi)(z) \in \mathcal{A}^2(D).$$

We note that in the Bergman space case the underlying manifold possesses a richer geometric structure in comparison with the Hardy space case. The unit disk

[1] The term "symbol" and its usage in the book is discussed in Section 2.1

equipped with the hyperbolic metric is both symplectic and Kähler. Therefore the Toeplitz operators on the Bergman space on the unit disk, and more generally on the bounded symmetric domains in $\mathbb{C}^n$, appear naturally, for example, under the Berezin quantization procedure.

At the same time, contrary to the Hardy space case, such Toeplitz operators do not admit the above matrix form with respect to the standard monomial basis even for the case of the unit disk.

We note as well that from the general point of view the Toeplitz operator is nothing but the compression of a bounded operator (in our case a multiplication operator) onto a certain subspace of a Hilbert space, representing thus an important model case in operator theory.

Although the basic properties and some results in the theories of Toeplitz operators acting on the Hardy space and on the Bergman space look similar, these two species of operators turn out to be quite different, and many deep results in both theories reflect and make clear their differences.

A quite unexpected and recently discovered phenomenon in the theory of Toeplitz operators on the Bergman space is the existence of a rich family of commutative C^*-algebras generated by Toeplitz operators with non-trivial defining symbols. The main goal of the book is a systematic exploration of this phenomenon.

A reader familiar with the theory of Toeplitz operators on the Hardy space may ask a legitimate question: *How can it be possible? What is this book about?* Indeed, the only possible commutative C^*-algebra of Toeplitz operators on the Hardy space over the unit circle is the algebra generated by Toeplitz operators with *constant* defining symbols. In this case the Toeplitz operators are just scalar multiples of the identity. Another possible option is the C^*-algebra with identity (Toeplitz operator with defining symbol 1) generated by a single self-adjoint element (a Toeplitz operator with real-valued defining symbol). But this is just a general operator theory statement which does not use any specific feature of Toeplitz operators. Furthermore, it is unknown whether such an algebra contains any other Toeplitz operator apart from its initial generator, nor how to single out such operators from the algebra in case of their presence.

In 1995 B. Korenblum and K. Zhu proved that the Toeplitz operators with radial defining symbols acting on the Bergman space over the unit disk can be diagonalized with respect to the standard monomial basis in this Bergman space. The C^*-algebra generated by such Toeplitz operators is therefore obviously commutative. Four years later the author of this book showed that the C^*-algebra generated by Toeplitz operators acting on the Bergman space over the upper half-plane and with defining symbols depending only on $\operatorname{Im} z$ is commutative as well. Further it was understood that there exists a rich family of commutative C^*-algebras of Toeplitz operators. Moreover it turned out that the smoothness properties of the symbols do not play any role in the commutativity: the symbols can be merely measurable. Surprisingly everything is governed by the geometry of the underlying manifold, the unit disk equipped with the hyperbolic metric. The precise description is as follows. *Each pencil of hyperbolic geodesics determines* *the set of symbols which are constant on the corresponding cycles, the orthogonal trajectories to geodesics forming a pencil. The C^*-algebra generated by Toeplitz operators with such defining symbols is commutative.*

An important feature of such algebras is that they remain commutative for Toeplitz operators acting on each of the commonly considered weighted Bergman spaces. Moreover, assuming some natural conditions on "richness" of symbol classes, we have the following complete characterization. *A C^*-algebra generated by Toeplitz operators is commutative on each weighted Bergman space if and only if the corresponding defining symbols are constant on cycles of some pencil of hyperbolic geodesics.* Apart from its inherent beauty this result reveals an extremely deep influence of the geometry of the underlying manifold on the properties of Toeplitz operators over this manifold.

In each case of the above commutative algebra we construct a unitary operator which reduces the corresponding Toeplitz operators to certain multiplication operators, thus also providing their spectral type representations. This gives us a powerful tool in the subject, in particular, yielding direct access to the majority of the important properties (such as boundedness, compactness, spectral properties, invariant subspaces) of the Toeplitz operators under study.

Furthermore, the presence and exploitation of these spectral type representations forms the core for many results presented in the book. Some of these results are a systematic study of Toeplitz operators with unbounded defining symbols; a clarification of the difference between compactness of commutators and semi-commutators of Toeplitz operators; the theory of Toeplitz and related operators with defining symbols having more than two limit values at boundary points; and a kind of semi-classical analysis of spectral properties of Toeplitz operators.

A detailed description of the main results of the book is given in the next section "Highlights of the chapters".

http://www.springer.com/978-3-7643-8725-9

Commutative Algebras of Toeplitz Operators on the
Bergman Space
Vasilevski, N.
2008, XXIX, 418 p., Hardcover
ISBN: 978-3-7643-8725-9
A product of Birkhäuser Basel