Preface

Various aspects of numerical analysis for equations arising in boundary integral equation methods have been the subject of several books published in the last 15 years [95, 102, 183, 196, 198]. Prominent examples include various classes of one-dimensional singular integral equations or equations related to single and double layer potentials. Usually, a mathematically rigorous foundation and error analysis for the approximate solution of such equations is by no means an easy task. One reason is the fact that boundary integral operators generally are neither integral operators of the form identity plus compact operator nor identity plus an operator with a small norm. Consequently, existing standard theories for the numerical analysis of Fredholm integral equations of the second kind are not applicable. In the last 15 years it became clear that the Banach algebra technique is a powerful tool to analyze the stability problem for relevant approximation methods [102, 103, 183, 189]. The starting point for this approach is the observation that the stability problem is an invertibility problem in a certain Banach or C^*-algebra. As a rule, this algebra is very complicated – and one has to find relevant subalgebras to use such tools as local principles and representation theory.

However, in various applications there often arise continuous operators acting on complex Banach spaces that are not linear but only additive – i.e.,

$$A(x+y) = Ax + Ay$$

for all x, y from a given Banach space. It is easily seen that additive operators are $\mathbb{R}$-linear provided they are continuous[1]. As an example, let us mention the one-dimensional singular integral operators with conjugation often arising in mechanics. It is known that the study of such operators can be reduced to $\mathbb{C}$-linear operators, but with matrix-valued coefficients. In passing note that this observation is one of a number of motivations to study singular integral operators with matrix-valued coefficients.

The present book is devoted to numerical analysis for certain classes of additive operators and related equations, including singular integral operators with conjugation, the Riemann-Hilbert problem, Mellin operators with conjugation and the famous Muskhelishvili equation. Until now, most relevant material is only

[1]Here and subsequently, $\mathbb{R}$ and $\mathbb{C}$ denote the fields of real and complex numbers, respectively.

found in journal papers, and there is no book offering a systematic study of this topic. Banach algebras play an important role in this book. However, the algebras that arise are not complex but real, and are not as familiar as complex algebras. Therefore, here we present certain results on real algebras and demonstrate their use in stability problems. In particular, we obtain stability conditions for various approximation methods, including spline Galerkin, collocation, qualocation and quadrature methods for equations with additive operators for both smooth and non-smooth data. Error analysis and convergence rates are present only occasionally, since rather more standard.

This book is addressed to a wide audience. We hope that it can be useful for both mathematicians working in theoretical fields of numerical analysis and engineers wishing to have practically realizable concepts for computations. Let us give a short overview of the content of this book.

Chapter 1 contains theoretical background. Here we have collected facts of functional analysis, necessary for understanding the approach proposed. Since real C^*-algebras play an important role in our investigations, elementary properties of such algebras are discussed. Moreover, a method to obtain real C^*-algebras by extending complex C^*-algebras (by adding a special element m) is described. Features of this procedure have previously been used in the study of one-dimensional singular integral equations with conjugation. As already mentioned, the stability problem for operator sequences can be interpreted as an invertibility problem in suitable real or complex Banach algebras. Thus we are accustomed to studying invertibility in Banach algebras or, more specifically, in C^*-algebras. Over the last 40 years certain concepts known as local principles were worked out. We present related results with special attention paid to the case of real algebras. The concluding part of Chapter 1 is devoted to the theory of singular integral operators and to Mellin operators. It is notable that all operators in this book have the property that *locally* they are Mellin operators.

Chapter 2 deals with polynomial and spline approximation methods for the Cauchy singular integral equation

$$(A\varphi)(t) = a(t)\varphi(t) + \frac{b(t)}{\pi i}\int\limits_{\Gamma_0} \frac{\varphi(\tau)}{\tau - t}d\tau + \overline{c(t)\varphi(t)} + \overline{\frac{d(t)}{\pi i}\int\limits_{\Gamma_0} \frac{\varphi(\tau)}{\tau - t}d\tau} = f(t)\,,$$

in the space $L^2(\Gamma_0)$, where Γ_0 is the unit circle with center at the origin and the functions a, b, c, d are continuous or piecewise continuous. For operators A without conjugation (i.e., $c = d \equiv 0$), there is a vast literature concerning the approximation methods under consideration (see e.g., [102, 183] and comments and remarks for the related chapters of the present book). Thus various complex C^*-algebras generated by approximation sequences for singular integral operators are completely described. Such algebras can be extended to real C^*-algebras that contain operator sequences associated with approximation methods for singular integral equations with conjugation. In particular, a real C^*-algebra generated by paired circulants and by the operator of complex conjugation is studied. This algebra contains a variety of approximation sequences, including spline Galerkin and spline collocation methods sequences, and others arising in quadrature and qualocation methods. The stability result is that a sequence from this algebra is stable if and only if a family of associated operators consists of invertible elements only. In the case of a simple closed Lyapunov contour Γ, the study of approximation methods for the operators mentioned can be reduced to the case of Γ_0.

Chapter 3 presents approximation methods for the following Riemann-Hilbert problem: Given an $(m \times m)$-matrix function G and a real vector-function f on Γ_0, find a vector-function φ that is analytic in the unit disc $\mathbf{D} := \{z \in \mathbb{C}, |z| < 1\}$, and such that $\operatorname{Im} \varphi(0) = 0$, and

$$\frac{1}{2}\left(G\varphi + \overline{G\varphi}\right) = f$$

on Γ_0. The new aspect is that the operator corresponding to this problem acts in a pair of spaces, so the algebraic methods used to study the stability of related approximation sequences have to be modified. This is done by using para-algebras. The same concept is employed to study approximation sequences associated with the generalized Riemann-Hilbert-Poincaré problem.

Chapter 4 is again concerned with approximation methods for the Cauchy singular integral equations with conjugation, but more general conditions are imposed on the curve Γ. Thus we now assume that Γ is a simple open or closed piecewise smooth curve in the complex plane $\mathbb{C}$. It is notable that the double layer potential operator is contained in the aforementioned class of operators. Given smooth boundaries, the stability of the corresponding projection methods for the double layer potential operator can be studied without great effort, since this operator is compact. However, if Γ is piecewise smooth, the algebras of Mellin operators with conjugation have to be invoked. Using this approach, we study various approximation sequences. As before, the stability of these sequences relies on the invertibility of the members of families of associated operators. The invertibility of the occurring operators is extremely difficult to check, especially if they are connected with corner points, so approximation methods based on cut-off techniques are also studied. This approach allows us to simplify conditions of the applicability of the corresponding methods.

Chapter 5 is devoted to the famous Muskelishvilli equation

$$(R\varphi)(t) = -k\overline{\varphi(t)} - \frac{k}{2\pi i}\int\limits_{\Gamma} \overline{\varphi(\tau)}\, d\log\frac{\overline{\tau}-\overline{t}}{\tau - t} - \frac{1}{2\pi}\int\limits_{\Gamma} \varphi(\tau)\, d\frac{\overline{\tau}-\overline{t}}{\tau - t} = f_0(t)\,, \tag{1}$$

and its approximate solution. The Muskhelishvili equation arose in investigation and solution of various biharmonic problems, especially in elasticity theory and hydrodynamics. Notwithstanding its exceptional importance, approximation methods for this equation have not been developed, mainly due to the fact that the operator R is not invertible in the functional spaces under interest. Fortunately,

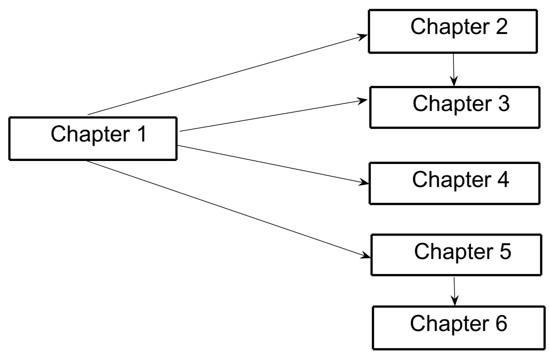

Figure 1: Book structure

this can be corrected by introducing a new operator that possesses all the necessary properties in order to find approximate solutions of the Muskhelishvili equation.

The idea of such correction is due to D.I. Sherman, although he only studied the solvability of the equation – not the invertibility of the associated operator, which is very important for the stability of approximation methods. Here we present all the results needed to construct and study projection methods for equation (1) in spaces L_p with weight. Let us note that we again use the fact that *locally* the operator R is a Mellin convolution operator.

Finally, Chapter 6 presents a few numerical results showing that the proposed approximation methods behave fairly well.

How to read this book? Probably, the best way is to single out a topic of interest and immediately read the related chapter. (If necessary, the reader can consult Chapter 1 for some background.) Of course, the later chapters of the book contain some material from chapters other, then the first but all connections can be easily traced. In particular, Chapter 3 also uses results from Chapter 2. The connection between different parts of the book is shown in Figure 1.

Although the attitudes and approaches of this book are solely the responsibility of the authors, we are indebted to our colleagues, friends, and collaborators for useful suggestions and ideas. It is a pleasure to mention here Roger Hosking and Steffen Roch, who read the early drafts of the manuscript and saved us from a number of embarrassing solecisms and ambiguities with detailed criticisms and generous advises. Ezio Venturino provided numerical examples and graphs presented in Chapter 6. We are grateful to Wolfgang Sprössig who has significantly facilitated our work. Finally our gratitude is due to an anonymous reviewer who contributed substantial improvement of this book.

Springer

http://www.springer.com/978-3-7643-8750-1

Approximation of Additive Convolution-Like Operators
Real C*-Algebra Approach
Didenko, V.; Silbermann, B.
2008, XII, 306 p., Softcover
ISBN: 978-3-7643-8750-1
A product of Birkhäuser Basel