Preface

Kvasz's book is a contribution to the history and philosophy of mathematics, or, as one might say, the historical approach to the philosophy of mathematics. This approach is for mathematics what the history and philosophy of science is for science. Yet the historical approach to the philosophy of science appeared much earlier than the historical approach to the philosophy of mathematics. The first significant work in the history and philosophy of science is perhaps William Whewell's *Philosophy of the Inductive Sciences, founded upon their History.* This was originally published in 1840, a second, enlarged edition appeared in 1847, and the third edition appeared as three separate works published between 1858 and 1860. Ernst Mach's *The Science of Mechanics: A Critical and Historical Account of Its Development* is certainly a work of history and philosophy of science. It first appeared in 1883, and had six further editions in Mach's lifetime (1888, 1897, 1901, 1904, 1908, and 1912). Duhem's *Aim and Structure of Physical Theory* appeared in 1906 and had a second enlarged edition in 1914. So we can say that history and philosophy of science was a well-established field by the end of the 19^{th} and the beginning of the 20^{th} century.

By contrast the first significant work in the history and philosophy of mathematics is Lakatos's *Proofs and Refutations*, which was published as a series of papers in the years 1963 and 1964. Given this late appearance of history and philosophy of mathematics relative to history and philosophy of science, we would expect the early development of history and philosophy of mathematics to be strongly influenced by ideas which had been formulated and discussed by those working in the history and philosophy of science. This proves to be the case. Lakatos's own pioneering work was, as the title indicated, developed from Popper's model of conjectures and refutations which Popper had devised to explain the growth of science.

In 1992, I edited a collection of papers with the general title: *Revolutions in Mathematics*. Once again the title showed clearly that ideas drawn from the history and philosophy of science were being applied to mathematics. The publication of Kuhn's 1962 *The Structure of Scientific Revolutions* led to debates about whether revolutions occurred in science and, if so, what was their nature. The 1992 collection carried over this debate to mathematics.

Kvasz's book on history and philosophy of mathematics breaks to some extent with this tradition of importing ideas from the history and philosophy of science into mathematics. This is because he adopts a *linguistic approach*. Kvasz's idea (p. 6)[1] is to interpret the development of mathematics as a sequence of linguistic innovations. Kvasz points out (p. 7) several advantages of this approach. These include the fact that languages have many objective aspects that can easily be studied, and so are more accessible to analysis than, for example, heuristics, or psychological acts of discovery.

As a result of his linguistic approach, Kvasz draws more on ideas from general analytic philosophy than from philosophy of science. More specifically he makes use of the classic works of Frege and the early Wittgenstein. However, Kvasz develops the ideas of Frege and the early Wittgenstein in a number of novel ways. Perhaps most importantly he introduces a historical dimension to the study of language. Both Frege and Wittgenstein treat language as timeless, but Kvasz, by contrast, focusses on the historical changes by which an older language can develop into a stronger, richer new language which has greater expressive power.

Perhaps, however, Kvasz has not broken away completely from philosophy of science, because it is worth noting that Kuhn too adopted a linguistic approach in his later period. This is shown in the 2000 book *The Road Since Structure* which contains essays by Kuhn from the period 1970–1993. On p. 57 of this book, Kuhn goes as far as to say:

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[1] Page references on their own are to Kvasz's book of which this is the preface.

> "If I were now rewriting *The Structure of Scientific Revolutions*, I would emphasize language change more and the normal/revolutionary distinction less." (Kuhn)

This passage comes from a talk which Kuhn gave in 1982, twenty years after the publication of *Structure*.

Yet although Kuhn adopted a linguistic approach, he did not manage to produce results in his new research programme at all comparable in significance to those of his earlier period. By contrast, as we shall see, Kvasz does indeed produce a number of novel and exciting results. I will argue later that one reason for Kvasz's greater success using the linguistic approach is that it is more suitable for the analysis of mathematics than for the analysis of science, though the linguistic approach does still have some value for science.

Kvasz, using his linguistic approach, formulates three patterns of change in the development of mathematics. These are: (1) *re-codings*, (2) *relativizations*, and (3) *re-formulations*. I will now briefly describe these in turn.

A re-coding occurs (p. 8) when there are changes in the formation rules of terms and formulae, or in the rules for construction of geometrical figures. Kvasz takes a quotation from Frege's 1891 paper *Funktion und Begriff* as the starting point for his analysis of re-coding. In this quotation (given on p. 15), Frege traces a development which starts with simple arithmetical assertions, such as $2 + 3 = 5$, goes on to general algebraic laws, such as $(a + b).c = a.c + b.c$, and then to the coinage of the technical term "function", and the statement of general laws about functions. Each step in this development is, according to Frege, the transition to a higher level.

Kvasz develops this idea in a number of significant ways. First of all he points out that Frege concentrates on symbolic languages, which are those dealing with arithmetic, algebra and analysis. Kvasz suggests that we consider, on a par with such symbolic languages, what he calls *iconic languages*, in a similar fashion, with geometrical figures. In this approach (pp. 12–13) geometrical diagrams are not just heuristic aids, but an integral part of the geometrical theory.

The introduction of iconic languages leads to one of Kvasz's most interesting claims which is that the path of development does not go directly from one symbolic language to another as the Frege quotations seem to suggest, but rather via an iconic intermediate level. So, according to Kvasz, there is not a direct transition from elementary arithmetic to algebra. Rather the transition is first from elementary arithmetic to synthetic geometry. This was carried out by the Greeks. Next there is a transition from synthetic geometry to algebra which was carried out later by the Arabs. Indeed according to Kvasz, this oscillation continues, since algebra leads to the iconic language of analytic geometry which in turn leads to the symbolic language of differential and integral calculus. A diagram showing all these transitions is to be found on p. 86.

Here it can be remarked that, although Kvasz's main focus is on linguistic change, he does often mention wider cultural factors which may have influenced mathematical development. Thus he says (pp. 29–30):

> "It seems that there must have been some obstacle that prevented the Greeks from entering the sphere of algebraic thought. The first who entered this new land were the Arabs. There is no doubt that they learned from the Greeks what is a proof, what is a definition, what is an axiom. But the Arabic culture was very different from the Greek one. Its center was Islam, a religion which denied that transcendence could be grounded in the metaphor of sight. Therefore the close connection between knowledge and sight which formed the core of the Greek *epistéme*, was lost." (Kvasz)

Kvasz's next pattern of change is relativization (pp. 8–9). This differs from re-coding (pp. 7–8) in that the ways of generating descriptions remain unchanged, but there are changes in the relation between the linguistic expressions and the objects that they stand for. Kvasz takes some ideas from Wittgenstein's *Tractatus* as the starting point of his analysis of relativization. In the *Tractatus*, Wittgenstein presents the picture theory of language. Regarding pictures, he makes the following important observation:

> "2.172 A picture cannot, however, depict its pictorial form:
> it displays it." (Wittgenstein)

On the picture theory of language, then, language must have a form which is displayed in the language, but cannot be expressed in the language. To Kvasz, this suggests a way in which an initial language L_1 say can be transformed historically into a more powerful language L_2. We consider the form of the language of L_1 which cannot be expressed within L_1 itself. However by adding this form to L_1 we create a new language L_2 which has more expressive power than L_1. The creation of more powerful languages in this manner does indeed, according to Kvasz, occur frequently in the development of mathematics. It is what he calls relativization.

Kvasz first example of relativization gives another example of his cultural leanings. He considers the system of perspective used by Renaissance painters. This was of course mathematical in character and its language had a form which Kvasz calls the perspectivist form. Perspective paintings are designed to be seen from a particular viewpoint, but this viewpoint lies outside the painting. It is something which cannot be expressed in the perspectivist form of language. Suppose, however, that we incorporate this point (renamed the centre of projection) into our system. We then get projective geometry whose language goes beyond that of perpectivism.

This is a striking and convincing example of what Kvasz calls 'relativization'. However, it is far from the only one. He goes on to analyse, using this concept, the emergence of ever more complicated forms of synthetic geometry – including most notably non-Euclidean geometry. This approach casts new light on some of the puzzling features of the discovery of non-Euclidean geometry.

The pattern of relativization seems to be naturally suited to developments in geometry, but Kvasz's next surprising claim is that it applies to algebra as well. The sequence of successive relativizations in algebra is not quite the same as the sequence in geometry, as can be seen by comparing the diagrams on pp. 160 and 200. However there is still a great deal of similarity. Thus while with recodings we have an oscillation between the symbolic and the iconic, with relativizations the symbolic and the iconic develop independently but in parallel, passing through most of the same stages.

Kvasz's third pattern of change, re-formulation, introduces, as the name suggests, less dramatic changes than the other two. However, reformulations can still be of great importance in bringing about advances in mathematics. Kvasz first examples of re-formulations are taken from Lakatos's *Proofs and Refutations*, and this leads him in Chapter 4 to give an interesting analysis of Lakatos's work. Kvasz claims that the changes which Lakatos considers, both in mathematics and science, are all re-formulations, and so relatively small in character. Lakatos never deals with the bigger changes which occur in re-codings or relativizations. This seems to me quite plausible. Lakatos has the unique honour of having been the first to introduce the historical approach into philosophy of mathematics. However, being a pioneer in this respect, it was likely that he would only succeed in dealing with one type of change – leaving others to be discovered later.

One impressive feature of Kvasz's book is the wealth of detailed historical case studies from the history of mathematics with which he illustrates his three patterns of change. Now there might be some arguments as to whether one or other of the examples given are really examples of the pattern which they are supposed to exemplify. However, these are likely to be disagreements on the margin, and there seems to me little doubt that the three patterns described by Kvasz are indeed patterns which have characteristically recurred in the development of mathematics. The concepts of recoding and relativization are quite novel and original, and so must be considered as constituting a real and substantial contribution to history and philosophy of mathematics. Both concepts go beyond anything to be found in Lakatos. They show that Kvasz has made genuine progress with his linguistic approach, and in this respect has done better than Kuhn whose linguistic approach did not lead to any such striking results. One reason for this situation, in my view, is that mathematics is more suited than science to the linguistic approach. I will now explain why I think this to be the case.

Let us compare a big innovation in science such as the development of special relativity with a big innovation in mathematics such as the development of calculus. The development of calculus brought a lot of linguistic changes. There were the introduction of signs for differentials such as $\mathrm{d}y/\mathrm{d}x$, and for integrals, such as $\int$. The results of the new calculus were expressed in formulae which were quite different from those of previous mathematics and would have been as incomprehensible to earlier mathematicians as hieroglyphics were to Egyptians living in the 18th Century. The introduction of the calculus was a recoding in Kvasz's sense. When we turn now to the introduction of special relativity, this brought about enormous conceptual changes in science. Einsteinian mass is quite different from Newtonian mass. Yet the language used to express the new results of special relativity differed hardly at all from the immediately preceding language of Newtonian mechanics. This example, and others like it, indicated that mathematics is the place to look for those who want to study how significant changes in language occur. The language of every day life does indeed change, but at a slow rate which can take centuries. In a few decades mathematicians can develop strikingly new languages. Thus mathematics is a kind of artifical laboratory in which large linguistic changes can be observed and studied with ease.

The importance of mathematics for the study of linguistic change shows that Kvasz's results and the various schemas which he introduces such as recoding and relativization should be of interest not just to philosophers of mathematics, but also to those with a general interest in the philosophy of language. Perhaps some of the results obtained by Kvasz will have application in other fields. Indeed Kvasz himself considers some examples from the history of art, and notions such as 'iconic language' could well have a fruitful application in the field of aesthetics.

There is however a problem because the book presupposes a considerable knowledge of mathematics and its history which could well render its results unintelligible to someone without this background. The way round this problem for philosophers of language who are not mathematicians would perhaps be to read a selection from the book which gives the main ideas without too many technical details. A suitable such selection might be the following: Introduction (pp. 1–10), Ch. 1 to the end of 1.1.3 (pp. 11–36) + first section of 1.2 (pp. 85–89), Ch. 2 to the end of of 2.1.3 (pp. 107–124) + 2.2 to the end of 2.2.2 (pp. 160–173), Ch. 3 to the end of 3.1 (pp. 225–232), Ch. 4 (pp. 239–251). Even for the more technically minded reader, it might be worth reading these sections first to get an overview of Kvasz's system, before plunging into the details of the examples from the history of mathematics which Kvasz uses to support his ideas.

So to sum up. Kvasz has made a considerable advance in the field of history and philosophy of mathematics by adopting a linguistic approach. This has enabled him to formulate three patterns of change which are at the same time patterns of linguistic innovation. At least two of these (recoding and relativization) are quite novel and nothing like them has so far been discussed. His work therefore deserves careful study by anyone interested in history and philosophy of mathemetics. Moreover, because the patterns of linguistic innovation may perhaps also be found in other areas where new linguistic forms develop, Kvasz's work also deserves the attention of those with more general interests outside mathematics in the philosophy of language.

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