

# Preface

## Hermann Weyl's Lectures on Function Theory

Hermann Weyl gave two courses on complex function theory at the Georg-August-Universität Göttingen in the winter terms of 1910–11 and 1911–12, two years after he finished his thesis on eigenfunctions of differential operators in 1908. This book is a transcription of the first of these two courses. The second one was the basis of Hermann Weyl's book *Die Idee der Riemannschen Fläche* [26], which is often regarded as one of the most important books on mathematics of its era. Nevertheless, it is not at all easy to read because it is the second of two courses. The first part already paves the way for the integration of Riemann's geometric approach with Weierstraß' analytic approach, which is the hallmark of [26]. Several important concepts that are nowadays treated differently, are explained carefully in the first part. An example of this is Weierstraß' version of complex function theory, which has been subsumed in the sheaf-theoretic language of today.

Before going further a remark is called for here. From the period around 1825 onwards—roughly the time Crelle's Journal was established—pure mathematics developed as a separate discipline in Germany as it did nowhere else. The use of the words “reine” and “angewandte” on the title “Journal für die reine und angewandte Mathematik” represents an early use of the words “pure” and “applied” (in translation) in essentially the senses we use today. When Gauß was growing up he was more or less entirely isolated as a mathematician. The Prussian state, which was recovering from the trauma of the Napoleonic period, made mathematics, and for the first time pure mathematics, a central academic concern. Gauß became the model for the first (Dirichlet, Jacobi) and second (Weierstraß, Kummer, Riemann, Eisenstein) generations of German mathematicians and their successors (Dedekind, Kronecker, Klein, Cantor, Hilbert). The development from 1825 on was very dynamic. It exposed the imprecision of much of the foundations of mathematics and forced mathematicians to confront problems of a foundational nature. This process began in the 1870s with the development of ideal theory in algebraic number theory (Dedekind, Kronecker), in the theory of trigonometric series (Riemann, Cantor), in analysis (Weierstraß), the theory of algebraic functions (Riemann, Weierstraß, Klein). In the end, it was necessary to go back to

the notion of a real number (Dedekind) and of sets (Cantor) and to the foundations of mathematics itself (Frege, Hilbert). All of these developments took place in a relatively short time; the beginnings were in the 1870s and 1880s and there was great activity up to 1900–1910. A very large number of them were rooted in the German mathematical community, although they widened to other countries (especially Italy) in the following decades.

Very different points of view were expressed, and more by chance than anything else there emerged two schools, the “Berlin school” and the “Göttingen school”. The first was based on an arithmetical approach to mathematics—typical of this were Weierstraß’  $\varepsilon$ – $\delta$  approach to analysis and Kronecker’s constructive approach to algebraic number theory. The “Göttingen school” combined two sets of ideas that were not immediately or obviously connected with one another. One was a geometric (“anschaulich”) approach to mathematics propagated by Felix Klein. The other was a set-theoretic foundation, which was one of Dedekind’s most revolutionary innovations. He was the first to define or construct mathematical objects as sets. The best known examples of his work are the theory of ideals and his construction of the real numbers using Dedekind cuts. Dedekind had been a student of Gauß in Göttingen and taught later in Braunschweig. Hilbert was profoundly influenced by Dedekind’s ideas and took them much further. These ideas were very suspect to some members of the Berlin mathematical community, especially Kronecker, for they were not rigorous by his standards. See [19] for a discussion of the two schools.

Several protagonists in these debates were very strong personalities, and there were passionate arguments, which collectively went under the name of “Grundlagenkrise.” Besides the mathematical differences, there were more material grounds for an antipathy between Berlin and Göttingen: Klein had the ear of the Ministerialrat F. Althoff in Berlin, and in Berlin this was taken badly, especially by G. Frobenius. These debates continued, in varying forms, for a long time. One of the most ferocious outbreaks was the locking of horns of Hilbert and L. E. J. Brouwer over the editorship of *Mathematische Annalen* (see [24]). Weyl himself was very much involved in this debate, see [27, 28]; see his remarks on the events of that time in the Nachtrag to [28]. The heat went out of most of these arguments after Gödel’s work appeared in the early 1930s.

I. Lakatos ([12]) stressed the role of dialectic in the development of mathematics, in bringing together differing points of view. This is often done by mathematicians of one generation absorbing the achievements of their predecessors and uniting them in their minds. In this fashion, debates of the past lose their force; we, for example, do not really decide between Kronecker’s and Dedekind’s approach to algebraic number theory; they are combined into one theory and we use what is most appropriate to the problem in hand. One can regard the arguments as being essentially productive even if they were not so experienced by the participants.

This is the case with Weyl’s “Idee der Riemannschen Fläche” [26]. At the time Weyl gave his course, Göttingen was at its zenith as a centre of mathematics, not only in Germany but in the world. What Weyl united here were Weierstraß’ rig-

orous and ungeometrical approach to the theory of “analytische Gebilde” (roughly the theory of the structure sheaf) with Riemann’s much more geometrical viewpoint, especially as propagated by Klein. In particular, as early as 1874, Klein had begun to think of Riemann surfaces as abstract entities, see [9, page x]. (Klein ascribes his epiphany to Prym but, as R. Remmert discovered, Prym claimed that he had neither told Klein anything of this nature, nor had he thought it himself, see [17].)

We can see this idea already in Riemann’s “Über die Hypothesen, welche der Geometrie zugrunde liegen” ([18, pages 304–317]; Riemann speaks of “Massverhältnisse” for what we would now call local coordinates); it was with Klein that Riemann’s germ took root and flourished. Klein liked to stress the visual (anschaulich) aspects of mathematics and had little time for foundational questions. His approach was an anathema to Weierstraß and his followers, and this led to a Göttingen school which in many respects was the obverse of the strict Berlin school. Klein’s notion was not rigorous in a sense that the Berlin school could recognise. What was needed was a formal definition of a Riemann surface, and this is the crucial step taken by Weyl in [26]. The definition follows Dedekind and Hilbert in defining a Riemann surface as a set with further properties, namely an atlas and transition functions. In order to do this he used Weierstraß’ ideas. For a very compact argument, in the case of algebraic functions, see [14].

One problem before this was that one could not always talk about a Riemann surface. At the time of Weyl’s lectures, the general uniformisation theorem of P. Koebe and H. Poincaré was still quite new (1909), and function theory was correspondingly very topical. See [1] for a discussion of this. Even the uniformisation theorem was not expressed as a theorem about Riemann surfaces, but rather as a statement about (a connected component of) solutions of systems of analytic equations which were one-dimensional (an “analytisches Gebilde”). This makes both the formulation and the proof of the uniformisation theorem considerably more difficult. After Weyl’s formal definition, the whole theory became much more transparent and flexible. It is worth noting the Koebe ends his paper [10] with the remark, “Damit ist ein Problem, welches, wie man wohl sagen darf, im Sinne der Weierstraß’schen Gedankenbildung liegt, durch die Anwendung von Prinzipien gelöst, welche dem Riemannschen Ideenkreis eigentümlich sind.” See here the discussion in [23].

The present-day formulation of the uniformisation theorem considers a Riemann surface  $R$  and a signature  $\sigma$  on it. The signature is a function on  $R$  taking values in  $\{1, 2, \dots, \infty\}$  and such that  $R \setminus \sigma^{-1}(1)$  is discrete. One calls a point where the weight is  $\infty$  *parabolic* and where it is  $\geq 1$  but finite *elliptic*. Then one has to construct a “covering” surface  $\tilde{R}$  and a map  $p: \tilde{R} \rightarrow R \setminus \sigma^{-1}(\{\infty\})$  which is a covering map on  $\sigma^{-1}(1)$  and so that if  $P$  is such that  $\sigma(P) > 1$  and  $P'$  such that  $p(P') = P$  then for local variables  $z'$  at  $P'$  and  $z$  at  $P$ , one has that  $z'^{\sigma(P)}/z \circ p$  is holomorphic and non-vanishing in a neighbourhood of  $P'$ . The general uniformisation theorem then states that  $\tilde{R}$  is holomorphically equivalent to the unit disc, the plane or the Riemann sphere. Moreover to a parabolic element  $P$  of  $\sigma$  there

exists a parabolic subgroup  $H$  of the group  $G$  of deck transformations for which there exists a horocyclic neighbourhood  $U$  of the fixed point so that  $H \backslash U$  can be identified biholomorphically with a punctured neighbourhood of  $P$ . (A horocyclic neighbourhood of  $\infty$  on the boundary of the upper half-plane  $\mathbb{H}$  is a set of the form  $\{z \mid \operatorname{Imag}(z) > c\}$  for  $c > 0$ .)

The apparent core of the proof is the demonstration of Riemann's mapping theorem, namely that a simply connected Riemann surface is biholomorphically equivalent to the Riemann sphere, the plane or the unit disc. One fixes a point and attempts to construct a harmonic function  $u$  with a logarithmic singularity at that point and which "tends to zero at infinity." This is generally done by an exhaustion argument which is based on the Poincaré–Volterra "Lemma" (proved in 1888); this essentially shows that  $\hat{R}$  is paracompact if  $R$  is, as one always assumes, see [22]. The construction of  $\hat{R}$  is subtle and is rarely given in full generality. For an admirable exception, see [13][Kap.4]. If we assume that  $R$  is compact, so that  $\sigma$  has a finite support, the whole argument is much simpler.

If  $R$  is compact, then a combinatorial argument allows to determine the group of deck transformations. This most remarkable fact goes back to Möbius and Jordan. It is one of the reasons why the theory of Riemann surfaces and Fuchsian groups (that is, the groups of deck transformations) is much more complete than in any other situation.

These results are the goals of [26]. In this first lecture course, we see that he had not yet completed his major step but is thinking towards it. He gave a course with its own distinctive flavour, as Weyl always did, but not revolutionary. Most results covered in this course are contained in the introductory text by Burckhardt [2] and the treatises by Osgood [15] and Hensel and Landsberg [6]. But the tasteful selection of topics already hints at Weyl's later achievements.

As befits a student of the Göttingen school, he emphasised visual aspects and connections to mathematical physics wherever possible. He discussed kinematical aspects of fractional linear transformations in remarkable detail—following Klein's ideas on mathematics education (see [11]). This led him to an excursion on one-parameter subgroups of Lie groups. The care taken in discussing topology was very much cutting-edge at the time. In the final chapter, he introduced Riemann surfaces—still without the rigour of [26]—and Weierstraß' theory of analytic continuation, and argued that both approaches are equivalent.

The interest of this lecture course is, first of all, that it clarifies some of the concepts that for a modern reader are difficult in "Die Idee der Riemannschen Fläche" and, secondly, that it shows the path taken by Weyl to his conception. For these reasons, this course is of considerable interest both to the historian of mathematics and to the historically interested mathematician.

Samuel J. Patterson

# Zur Entstehung dieses Buches

Dieses Buch beruht auf der Vorlesung „*Einleitung in die Funktionentheorie*“, die Hermann Weyl im Wintersemester 1910–11 an der Georg-August-Universität Göttingen gehalten hat. Wie damals üblich, wurde einer der Hörer – hier Fritz Frankfurth – damit beauftragt, eine autorisierte Vorlesungsmitschrift zu erstellen; diese wurde vom Dozenten gegebenenfalls kommentiert und korrigiert und verblieb dann in der Bibliothek des Mathematischen Instituts, die entsprechend heute über eine große Zahl von Vorlesungsmitschriften aus dieser Zeit verfügt.

Dass diese Handschrift schließlich als Buch herausgebracht werden kann, ist eine Nebenwirkung der Studienreform. Bei dieser Gelegenheit wurde in Göttingen nämlich ein  $\text{\LaTeX}$ -Kurs im Bereich Schlüsselqualifikationen eingeführt, in dem die Studierenden als Projektarbeit auch einen etwa zehnsseitigen mathematischen Text in  $\text{\LaTeX}$  erstellen. Als ich diesen Kurs im Oktober 2007 erstmals hielt, habe ich auf Anregung von Yuri Tschinkel und Samuel Patterson den Studierenden angeboten, jeweils zehn Seiten der Vorlesungsmitschrift von Weyls Vorlesung zu transkribieren. Mehrere Studierende griffen diesen Vorschlag bereitwillig auf. Insofern beruht dieses Buch hauptsächlich auf der Arbeit von Kathrin Becker, Malte Böhle, Sandra Bösiger, Steven Gassel, Jan-Niklas Grieb, Alexander Hartmann, Robert Hesse, Carolin Homann, Torsten Klampff, Simon Naarmann, Patrick Neumann, Christian Otto, Mila Runnwerth, Sophia Scholtka und Hannes Vennekate, denen ich hier nochmals meinen besonderen Dank ausspreche. Dieser gilt auch Prof. Dr. Samuel Patterson, der das Vorwort beisteuerte, und Robert Schieweck, der die knapp 200 Skizzen und Illustrationen im Original mit dem Programm Asymptote nachzeichnete.

Der Inhalt dieses Buches ist im Wesentlichen identisch mit dem Original. Ich hielt es aber für angemessen, einige Kleinigkeiten zu verbessern, ohne dies jeweils im Detail anzugeben:

- Das Original kennt keine nummerierten Sätze und kaum nummerierte Gleichungen, und Textverweise beziehen sich immer auf die Seitennummer. Ich habe mir erlaubt, einige wichtige Sätze, entsprechend den heutigen Gepflogenheiten, als nummerierte Sätze aus dem Text herauszuheben, und Verweise darauf entsprechend angepasst.

- Die Abbildungen wurden ebenfalls teilweise nummeriert und mit erklärenden Bildunterschriften versehen, die von Robert Schieweck oder von mir stammen.
- Alle Fußnoten mit Ausnahmen der Fußnoten in Abschnitt 1.10 stammen von mir. Im Original gibt es keine Fußnoten, sondern nur spätere Ergänzungen, die sich fast immer direkt in den Text einfügen.
- Ich habe eine Reihe von Tippfehlern im Original korrigiert und in einigen Punkten die Notation dem modernem Gebrauch angepasst. Letzteres betrifft unter anderem die Schreibung von Grenzwerten, die Bezeichnung von Gebieten, und die Verwendung von Klammern. (Im Original wird ein Gebiet meist mit  $g$  bezeichnet, daraus wurde  $G$ ; geschweifte oder eckige Klammern im Original wurden durch größere runde Klammern ersetzt, weil diese Klammern heute spezielle Bedeutungen haben.)
- Die Rechtschreibung wurde an die heutige Norm angepasst, an manchen Stellen habe ich die Satzstellung verändert.

Außerdem wurden zusammen mit der Vorlesung noch ein korrigiertes Übungsblatt und zwei aus Papier ausgeschnittene Versuche eines Modells der aufgeschnittenen Riemannschen Fläche aus Abbildung 5.11 überliefert. Es ist unklar, inwieweit es noch mehr Übungen gab außer dieser. Das Übungsblatt wurde als Abschnitt 1.10 in das Buch aufgenommen, aber mit den Papiermodellen ließ sich wenig anfangen.

Mein eigener Beitrag zu diesem Buch ist vor allem die Endredaktion und die Zusammenführung der verschiedenen Beiträge.

Ralf Meyer, Mai 2008

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