Preface

Integer programming (IP) is a fascinating topic. Indeed, while linear programming (LP), its continuous analogue, is well understood and extremely efficient LP software packages exist, solving an integer program can remain a formidable challenge, even for some small size problems. For instance, the following small (5-variable) IP problem (called the *unbounded knapsack problem*)

$$\begin{aligned}
&\min\{213x_1 - 1928x_2 - 11111x_3 - 2345x_4 + 9123x_5\}\\
&\text{s.t. } \ 12223x_1 + 12224x_2 + 36674x_3 + 61119x_4 + 85569x_5 = 89643482,\\
&\qquad x_1, x_2, x_3, x_4, x_5 \in \mathbb{N},
\end{aligned}$$

taken from a list of difficult knapsack problems in Aardal and Lenstra [2], is not solved even by hours of computing, using for instance the last version of the efficient software package CPLEX.

However, this is *not* a book on integer programming, as very good ones on this topic already exist. For standard references on the theory and practice of integer programming, the interested reader is referred to, e.g., Nemhauser and Wolsey [113], Schrijver [121], Wolsey [136], and the more recent Bertsimas and Weismantel [21]. On the other hand, this book could provide a complement to the above books as it develops a rather unusual viewpoint.

Indeed, one first goal of this book is to analyze and develop some striking analogies between four problems, all related to an integer program $\mathbf{P}_d$ (the subscript "*d*" being for *discrete*); namely its associated *linear* programming problem $\mathbf{P}$, its associated *linear integration* problem $\mathbf{I}$, and its associated *linear summation* (or *linear counting*) problem $\mathbf{I}_d$. In fact, while $\mathbf{P}$ and $\mathbf{P}_d$ are the respective $(\max,+)$-algebra analogues of $\mathbf{I}$ and $\mathbf{I}_d$, $\mathbf{P}_d$, and $\mathbf{I}_d$ are the respective discrete analogues of $\mathbf{P}$ and $\mathbf{I}$. In addition, the same simple relationship links the value of $\mathbf{P}$ with that of $\mathbf{I}$ on the one hand, and the value of $\mathbf{P}_d$ with that of $\mathbf{I}_d$ on the other hand.

If LP duality is of course well understood (as a special case of Legendre–Fenchel duality in convex analysis), IP duality is much less developed although there is a well-known *superadditive* dual associated with $\mathbf{P}_d$. On the other hand, the linear integration and linear counting problems $\mathbf{I}$ and $\mathbf{I}_d$ have well-defined respective dual problems $\mathbf{I}^*$ and $\mathbf{I}_d^*$, although they are not qualified as such in the literature. Indeed, $\mathbf{I}^*$ (resp., $\mathbf{I}_d^*$) is obtained from the inverse Laplace transform (resp., inverse $\mathbb{Z}$-transform) applied to the Laplace transform (resp., $\mathbb{Z}$-transform) of the value function, exactly in the same way the LP dual $\mathbf{P}^*$ is obtained from the Legendre–Fenchel transform applied to the Legendre–Fenchel transform of the value function. Moreover, recent results by people like

Barvinok, Brion, and Vergne, and Pukhlikov and Khovanskii have provided nice and elegant exact formulas for $\mathbf{I}$ and $\mathbf{I}_d$. One purpose of this book is to show that a careful analysis of these formulas permit us to shed some interesting light on the links and analogies between the (continuous) integration and (discrete) counting programs $\mathbf{I}$ and $\mathbf{I}_d$.

In addition, in view of connections and analogies among $\mathbf{P}, \mathbf{P}_d, \mathbf{I}$, and $\mathbf{I}_d$ on the one hand, and duality results already available for $\mathbf{P}, \mathbf{I}$, and $\mathbf{I}_d$ on the other, another goal is to provide new insights and results on duality for integer programming, and to reinterpret some already existing results in light of these new results and analogies.

This book is an attempt to reach this goal, and throughout all chapters our investigation is guided by these analogies, which are not just formal but rest on a rigorous mathematical analysis. We hope to convince the reader that they are also useful to better understand in particular the difference between the discrete and continuous cases and reasons why the former is significantly more difficult. We also hope that some of the material presented here could be introduced in graduate courses in optimization and operations research, as this new viewpoint makes a link between problems that after all are not so different when looked at through a distinct lens. Indeed, very often the discrete and continuous cases are treated separately (as are integration and optimization) and taught in different courses. The associated research communities are also distinct. On a more practical side, some duality results presented in the last chapters of the book may also provide new ideas for generating efficient *cuts* for integer programs, a crucial issue for improving the powerful software packages already available, like CPLEX and XPRESS-MP codes.

Finally, let us mention that IP is also studied in the algebraic geometry research community, and standard algebraic tools like Gröbner basis, Gröbner fan, and toric ideals permit us to better understand IP as an arithmetic refinement of LP and to re-interpret some known (algebraic) concepts, like Gomory relaxations, for example.

The plan of the book is as follows: We first introduce the four related problems $\mathbf{P}, \mathbf{P}_d, \mathbf{I}$, and $\mathbf{I}_d$ in Chapter 1. Next, in Part I we analyze problem $\mathbf{I}$ and compare with $\mathbf{P}$ in Chapters 2 and 3. In Part II, we do the same with $\mathbf{I}_d$ and $\mathbf{P}_d$ in Chapters 4 and 5. Part III is then devoted to various duality results, including (i) the link between the old concept of Gomory relaxations and exact formula obtained by Brion, and Brion and Vergne, in Chapters 6 and 7, (ii) a discrete Farkas lemma and a characterization of the integer hull in Chapters 8 and 9, and (iii) the link with the superadditive dual in Chapter 10.

Some elementary background on Legendre–Fenchel, Laplace, and $\mathbb{Z}$-transforms, as well as Cauchy residue theorem in complex analysis, can be found in the Appendix.

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