

Chapter 2

Distributions Expressed as Copulas

2.1 Introduction

A feature common to all the distributions in this chapter is that $H(x, y)$ is a simple function of the uniform marginals $F(x)$ and $G(y)$. These types of joint distributions are known as copulas, as mentioned in the last chapter, and will be denoted by $C(u, v)$; the corresponding random variables will be denoted by U and V , respectively.

When the marginals are uniform, independence of U and V implies a flat p.d.f., and any deviation from this will indicate some form of dependence.

Most of the copulas presented in this chapter are of simple forms although in some cases [e.g., the distribution of Kimeldorf and Sampson (1975a) discussed in Section 2.12] they have a rather complicated expression. Some are obtained through marginal transformations, while several others already have uniform marginals and need no transformations to bring them to that form.

The great majority of the copulas described in this chapter have a single parameter that reflects the strength of mutual dependence between U and V . To emphasize its role, we could have chosen to use the same symbol in all these cases. We have not done this, however, since for some distributions it is customary to find α used, others θ , and yet others c .

Throughout this chapter, we assume that U and V are uniform with $C(u, v)$ as their joint distribution function and $c(u, v)$ as the corresponding density function. Thus, the supports of the bivariate distributions are unit squares. For each case, we state some simple properties such as the correlation coefficient and conditional properties. Also, we should note that for bivariate copulas, Pearson's product moment correlation coefficient is the same as the grade coefficient (Spearman's coefficient), as mentioned in Section 1.7.

Unless otherwise specified, the supports of all the distributions are over the unit square. Also, the distribution functions are in fact the cumulative distribution functions. Following this introduction, we discuss the Farlie–Gumbel–Morgenstern (F-G-M) copula and its generalization in Section 2.2.

Next, in Sections 2.3 and 2.4, we discuss the Ali–Mikhail–Haq and Frank distributions. The distribution of Cuadras and Augé and its generalization are presented in Section 2.5. In Section 2.6, the Gumbel–Hougaard copula and its properties are detailed. Next, the Plackett and bivariate Lomax distributions are described in Sections 2.7 and 2.8, respectively. The Lomax copula is presented in Section 2.9. In Sections 2.10 and 2.12, the Gumbel type I bivariate exponential and Kimeldorf and Sampson’s distributions are discussed, respectively. The Gumbel–Barnett copula and some other copulas of interest are described in Sections 2.11 and 2.14, respectively. In Section 2.13, the Rodríguez-Lallena and Úbeda-Flores families of bivariate copulas are discussed. Finally, in Section 2.15, some references to illustrations are presented for the benefit of readers.

2.2 Farlie–Gumbel–Morgenstern (F-G-M) Copula and Its Generalization

Formula for Distribution Function

$$C(u, v) = uv[1 + \alpha(1 - u)(1 - v)], \quad -1 \leq \alpha \leq 1. \quad (2.1)$$

Formula for Density Function

$$c(u, v) = 1 + \alpha(1 - 2u)(1 - 2v). \quad (2.2)$$

Correlation Coefficient

The correlation coefficient is $\rho = \frac{\alpha}{3}$, which clearly ranges from $-\frac{1}{3}$ to $\frac{1}{3}$. After the marginals have been transformed to distributions other than uniform, Gumbel (1960a) and Schucany et al. (1978) showed that (i) ρ cannot exceed $\frac{1}{3}$ and (ii) determined it for some well-known distributions—for example, $\frac{\alpha}{\pi}$ for normal marginals and $\frac{\alpha}{4}$ for exponential ones.

Conditional Properties

The regression $E(V|U = u)$ is linear in u .

Dependence Properties

- Lai (1978) has shown that, for $0 \leq \alpha \leq 1$, U and V are positively quadrant dependent (PQD) and positively regression dependent (PRD).
- For $0 \leq \alpha \leq 1$, U and V are likelihood ratio dependent (LRD) (TP₂) [Drouet-Mari and Kotz (2001)].
- For $-1 \leq \alpha \leq 0$, its density is RR₂; see Drouet-Mari and Kotz (2001).

Remarks

- This copula is not Archimedean [Genest and MacKay (1986)].
- The p.d.f. is symmetric about the point $(\frac{1}{2}, \frac{1}{2})$, i.e., it is the same as at $(1-u, 1-v)$ as it is at (u, v) , and so the survival (complementary) copula is the same as the original copula.
- Among the results established by Mikhail, Chasnov, and Wooldridge (1987) are the regression curves when the marginals are exponential. Drouet-Mari and Kotz (2001, pp. 115–116) have also provided expressions for the conditional mean and conditional variance when the marginal distributions are F and G .
- Mukherjee and Sasmal (1977) have worked out some properties of a two-component system whose components' lifetimes have the F-G-M distribution, with standard exponential marginals, such as the densities, m.g.f.'s, and tail probabilities of $\min(X, Y)$, $\max(X, Y)$, and $X + Y$, these being of relevance to series, parallel, and standby systems, respectively. Mukherjee and Sasmal (1977) have compared the densities and means of $\min(X, Y)$ and $\max(X, Y)$ with those of Downton (1970) and Marshall and Olkin (1967) distributions.
- Tolley and Norman (1979) obtained some results relevant to epidemiological applications with the marginals being exponential.
- Lingappaiah (1984) was also concerned with properties of the F-G-M distribution with gamma marginals in the context of reliability.
- Building a paper by Phillips (1981), Kotz and Johnson (1984) considered a model in which components 1 and 2 were subjected to “revealed” and “unrevealed” faults, respectively, with (Y, Z) having an F-G-M distribution, where Y is the time between unrevealed faults and Z is the time from an unrevealed fault to a revealed fault.
- In the context of sample selection, Ray et al. (1980) have presented results for the distributions having logistic marginals, with the copula being the F-G-M or the Pareto.

2.2.1 Applications

- Cook and Johnson (1986) used this distribution (with lognormal marginals) for fitting data on the joint occurrence of certain trace elements in water.
- Halperin et al. (1979) used this distribution, with exponential marginals, as a starting point when considering how a population p.d.f. $h(x, y)$ is altered in the surviving and nonsurviving groups by a risk function $a(x, y)$. (X and Y were blood pressure and cigarette smoking, respectively, in this study.).
- Durling (1974) utilized this distribution with logistic marginals for y , re-analyzing seven previously published datasets on the effects of mixtures of poisons.
- Chinchilli and Breen (1985) used a six-variate version of this distribution with logistic marginals to analyze multivariate binary response data arising in toxicological experiments—specifically, tumor incidence at six different organ sites of mice exposed to one of five dosages of a possible carcinogen [data from Brown and Fears (1981)].
- Thinking now of “lifetimes” in the context of component reliability, Teichmann (1986) used this distribution for (U_1, U_2) , with U_i being a measure of association between an external factor and the failure of the i th unit—specifically, it was the ratio of how much the external factor increases the probability of failure compared with how much an always fatal factor would increase the probability of failure.
- With exponential marginals, Lai (1978) used the F-G-M distribution to model the joint distribution of two adjacent intervals in a Markov-dependent point process.
- In the context of hydrology, Long and Krzysztofowicz (1992) also noted that the F-G-M model is limited to describing weak dependence since $|\rho| \leq 1/3$.

2.2.2 Univariate Transformations

The following cases have been considered in the literature: the case of exponential marginals by Gumbel (1960a,b); of normal marginals by Gumbel (1958, 1960b); of logistic marginals by Gumbel (1961, Section 6); of Weibull marginals by Johnson and Kotz (1977) and Lee (1979); of Burr type III marginals by Rodriguez (1980); of gamma marginals by D’Este (1981); of Pareto marginals by Arnold (1983, Section 6.2.5), who cites Conway (1979); of “inverse Rayleigh” marginals (i.e., $F = \exp(-\theta/x^2)$) by Mukherjee and Saran (1984); and of Burr type XII marginals by Bagchi and Samanta (1985).

Drouet-Mari and Kotz (2001, pp. 122–124) have presented a detailed discussion on the bivariate F-G-M distribution with Weibull marginals. Kotz and Van Dorp (2002) have studied the F-G-M family with marginals as a two-sided power distribution.

2.2.3 A Switch-Source Model

For general marginals, the density is $f(x)g(y)\{1 + \alpha[1 - 2F(x)][1 - 2G(y)]\}$. The density

$$a(x)a(y)[1 + \alpha b(x)b(y)] \quad (2.3)$$

arises from a mixture model governed by a Markov process. Imagine a source producing observations from a density f_1 , another source producing observations from a density f_2 , a switch connecting one or the other of these sources to the output, a Markov process governing the operation of the switch, and X and Y being observations at two points in time; see Willett and Thomas (1985, 1987).

2.2.4 Ordinal Contingency Tables

The nonidentical marginal case of (2.3) is $a(x)b(y)[1 + \alpha b(x)d(y)]$. This looks very much like the “rank-2 canonical correlation model” used to describe structure in ordinary contingency tables; see Gilula (1984), Gilula et al. (1988), and Goodman (1986).

Now, instead of generalizing (2.1) and comparing it with contingency table models, we shall explicitly write (2.1) in the contingency form and see what sort of restrictions are effectively being imposed on the parameters of a contingency table model. The probability within a rectangle $\{x_0 < X < x_1, y_0 < Y < y_1\}$ is $H_{11} - H_{01} - H_{10} + H_{00}$ (in an obvious notation), which equals

$$\begin{aligned} & (x_1 - x_0)(y_1 - y_0) + \alpha[x_1(1 - x_1) - x_0(1 - x_0)][y_1(1 - y_1) - y_0(1 - y_0)] \\ &= (x_1 - x_0)(y_1 - y_0)[1 + \alpha(1 - x_1 - x_0)(1 - y_1 - y_0)]. \end{aligned}$$

Comparing this with equation (2.2) of Goodman (1986), we see that $(1 - x_1 - x_0)$ and $(1 - y_1 - y_0)$ play the role of row scores and column scores—in effect, Goodman’s model U .

2.2.5 Iterated F-G-M Distributions

For the singly iterated case, the distribution function C and p.d.f. c are, respectively, given by

$$C(u, v) = uv[1 + \alpha(1 - u)(1 - v) + \beta uv(1 - u)(1 - v)], \quad (2.4)$$

$$c(u, v) = [1 + \alpha(1 - 2u)(1 - 2v) + \beta uv(2 - 3u)(2 - 3v)], \quad (2.5)$$

where the valid combinations of α and β are $-1 \leq \alpha \leq 1$ and $-1 - \alpha \leq \beta \leq (3 - \alpha + \sqrt{9 - 6\alpha - 3\alpha^2})/2$. This distribution is obtained [Johnson and Kotz (1977) and Kotz and Johnson (1977)] by realizing that (2.1) may alternatively be written in terms of the survival function \bar{C} as

$$\bar{C} = (1 - u)(1 - v)(1 + \alpha uv). \quad (2.6)$$

Now replacing the independent survival function $(1 - u)(1 - v)$ in (2.1) by this survival function of an F-G-M distribution, having a possibly different associated parameter, β/α (say) instead of α , we obtain the result in (2.4). This process can be repeated, of course. The correlation coefficient is $\text{corr}(U, V) = \frac{\alpha}{3} + \frac{\beta}{12}$.

Note

For normal marginals, $\text{corr}(X, Y) = \frac{\alpha}{\pi} + \frac{\beta}{4\pi}$. The first iteration increases the maximum attainable correlation to over 0.4. However, very little increase of the maximum correlation is achievable with further iterations, as noted by Kotz and Johnson (1977).

Lin (1987) suggested another way of iterating the F-G-M distribution: Start with (2.6), and replace uv by (2.1). After substituting for \bar{C} in terms of C , we obtain

$$C(u, v) = uv[1 + \alpha(1 - u)(1 - v) + \beta(1 - u)^2(1 - v)^2]$$

at the first step.

Zheng and Klein (1994) studied an iterated F-G-M distribution of the form

$$C(u, v) = uv + \sum_j \alpha_j (uv)^{1/2} [(1 - u)(1 - v)]^{(j+1)/2}, \quad -1 \leq \alpha_j \leq 1.$$

2.2.6 Extensions of the F-G-M Distribution

We shall discuss here a number of extensions of F-G-M copulas developed primarily to increase the maximal value of the correlation coefficient. Most of these are polynomial-type copulas (copulas that are expressed in terms of polynomials in u and v).

Huang and Kotz Extension

Huang and Kotz (1999) considered

$$C(u, v) = uv[1 + \alpha(1 - u^p)(1 - v^p)]. \quad (2.7)$$

The corresponding p.d.f. is

$$c(u, v) = 1 + \alpha \left(1 - (1 + p)u^p\right) \left(1 - (1 + p)v^p\right). \quad (2.8)$$

The admissible range for α is given by

$$-(\max\{1, p^2\})^{-2} \leq \alpha \leq p^{-1}.$$

The range for $\rho = \text{corr}(U, V) = 3\alpha(\frac{p}{p+2})^2$ is

$$-3(p+2)^{-2} \min\{1, p^2\} \leq \rho \leq \frac{3p}{(p+2)^2}.$$

Thus, for $p = 2$, $\rho_{\max} = \frac{3}{8}$, and for $p = 1$, $\rho_{\min} = \frac{-3}{16}$.

It is clear that the introduction of the parameter p has enabled us to increase the maximal correlation for the F-G-M copula.

Another extension of the bivariate F-G-M copula is given by

$$C(u, v) = uv[1 + \alpha(1 - u)^p(1 - v)^p], \quad p > 0, \quad (2.9)$$

with p.d.f.

$$c(u, v) = 1 + \alpha(1 - u)^{p-1}(1 - v)^{p-1} \left(1 - (1 + p)u\right) \left(1 - (1 + p)v\right). \quad (2.10)$$

The admissible range of α is (for $p > 1$)

$$-1 \leq \alpha \leq \left(\frac{p+1}{p-1}\right)^{p-1}.$$

The range is empty for $p < 1$. The correlation

$$\rho = \text{corr}(U, V) = 12\alpha \left(\frac{1}{(p+1)(p+2)}\right)^2$$

in this case has the range

$$-12 \left(\frac{1}{(p+1)(p+2)}\right)^2 \leq \rho \leq 12 \frac{(p-1)^{1-p}(p+1)^{p-3}}{(p+2)^2}.$$

Thus, for $p = 1.877$, $\rho_{\max} = 0.3912$ and $\rho_{\min} = \frac{-1}{3}$, showing that the maximal correlation is even higher than the one attained by the first extension in (2.7).

Sarmanov's Extension

Sarmanov (1974) considered the following copula:

$$C(u, v) = uv \left\{ 1 + 3\alpha(1-u)(1-v) + 5\alpha^2(1-u)(1-2u)(1-v)(1-2v) \right\}. \quad (2.11)$$

The corresponding density function is

$$c(u, v) = 1 + 3\alpha(2u-1)(2v-1) + \frac{5}{4}\alpha^2[3(2u-1)^2-1][3(2v-1)^2-1].$$

Equation (2.11) is a probability distribution when $|\alpha| \leq \frac{\sqrt{7}}{5} \simeq 0.55$.

Bairamov-Kotz Extension

Bairamov and Kotz (2000a) considered a two-parameter extension of the F-G-M copula given by

$$C(u, v) = uv[1 + \alpha(1-u^a)^b(1-v^a)^b], \quad a > 0, b > 0, \quad (2.12)$$

with the corresponding p.d.f.

$$c(u, v) = 1 + \alpha(1-u^a)^{b-1}(1-v^a)^{b-1}[1-u^a(1+ab)][1-v^a(1+ab)]. \quad (2.13)$$

The admissible range of α is as follows: For $b > 1$,

$$-\min \left\{ 1, \left[\frac{1}{a^b} \left(\frac{ab+1}{b-1} \right)^{b-1} \right]^2 \right\} \leq \alpha \leq \left[\frac{1}{a^b} \left(\frac{ab+1}{b-1} \right)^{b-1} \right],$$

and for $b = 1$, the quantity inside the square bracket is taken to be 1. It can be shown in this case that $\text{corr}(U, V) = 12\alpha \left[\frac{b}{ab+2} \frac{\Gamma(b)\Gamma(\alpha/2)}{\Gamma(b+\frac{\alpha}{2})} \right]^2$. For $a = 2.8968$ and $b = 1.4908$, we have $\rho_{\max} = 0.5015$. For $a = 2$ and $b = 1.5$, $\rho_{\min} = -0.48$.

Another extension that does not give rise to a copula is

$$C(u, v) = u^p v^p [1 + \alpha(1-u^q)^n(1-v^q)^n], \quad p, q \geq 0, \quad n > 1, \quad (2.14)$$

with marginals u^p and v^p , respectively.

Lai and Xie Extension

Lai and Xie (2000) considered the copula

$$C(u, v) = uv + \alpha u^b v^b (1-u)^a (1-v)^a, \quad a, b \geq 1, \quad (2.15)$$

and showed that it is PQD for $0 \leq \alpha \leq 1$. The corresponding p.d.f. is

$$c(u, v) = 1 + \alpha(uv)^{b-1}[(1-u)(1-v)]^{a-1}[b - (a+b)u][b - (a+b)v]. \quad (2.16)$$

The correlation coefficient is given by $\text{corr}(U, V) = 12\alpha[B(b+1, a+1)]^2$. Bairamov and Kotz (2000b) observed that (2.15) is a bivariate copula for α over a wider range satisfying

$$\min \left\{ \frac{1}{[B^+(a, b)]^2}, \frac{1}{[B^-(a, b)]^2} \right\} \leq \alpha \leq \frac{1}{B^+(a, b)B^-(a, b)},$$

where B^+ and B^- are functions of a and b .

Bairamov–Kotz–Bekci Generalization

Bairamov et al. (2001) presented a four-parameter extension of the F-G-M copula as

$$C(u, v) = uv \left\{ 1 + \alpha(1 - u^{p_1})^{q_1}(1 - v^{p_2})^{q_2} \right\}, \quad p_1, p_2 \geq 1, \quad q_1, q_2 \geq 1. \quad (2.17)$$

2.2.7 Other Related Distributions

- Farlie (1960) introduced the more general expression

$$H(x, y) = F(x)G(y)\{1 + \alpha A[F(x)]B[G(y)]\}.$$

- Rodriguez (1980, p. 48), in the context of Burr type III marginals, made passing references to $H = FG[1 + \alpha(1 - F^a)(1 - G^b)]$.
- Cook and Johnson (1986) discussed a compound F-G-M distribution.
- Regarding a distribution obtained by a Khintchine mixture using the F-G-M distribution as the bivariate F-G-M copula, see Johnson (1987, pp. 157–159).
- Cambanis (1977) has mentioned $C(u, v) = uv[1 + \beta(1 - u) + \beta(1 - v) + \alpha(1 - u)(1 - v)]$, which arises as the conditional distribution in a multivariate F-G-M distribution.
- The following distribution was denoted u_8 in Kimeldorf and Sampson (1975b):

$$C(u, v) = uv[1 + \alpha(1 - u)(1 - v) + \beta(1 - u^2)(1 - v^2)], \quad (2.18)$$

$$c(u, v) = 1 + \alpha(1 - 2u)(1 - 2v) + \beta(1 - 3u^2)(1 - 3v^2), \quad (2.19)$$

with correlations $\tau = \frac{2\alpha}{9} + \frac{\beta}{2} + \frac{\alpha\beta}{450}$ and $\rho_S = \frac{\alpha}{3} + \frac{3\beta}{4}$.

2.3 Ali–Mikhail–Haq Distribution

$$C(u, v) = \frac{uv}{1 - \alpha(1-u)(1-v)} \quad (2.20)$$

and

$$c(u, v) = \frac{1 - \alpha + 2\alpha \frac{uv}{1 - \alpha(1-u)(1-v)}}{[1 - \alpha(1-u)(1-v)]^2}. \quad (2.21)$$

Correlation Coefficients

The range of product-moment correlation is $(-0.271, 0.478)$ for uniform marginals, $(-0.227, 0.290)$ for exponential marginals, and approximately $(-0.300, 0.600)$ for normal marginals; see Johnson (1987, pp. 202–203), crediting these results to Conway (1979).

Derivation

This distribution was introduced by Ali et al. (1978). They proposed searching for copulas for which the survival odds ratio satisfies

$$\frac{1 - C_\alpha(u, v)}{C_\alpha(u, v)} = \frac{1 - u}{u} + \frac{1 - v}{v} + (1 - \alpha) \frac{1 - u}{u} \times \frac{1 - v}{v}.$$

Solving $C_\alpha(u, v)$ yields the Ali–Mikhail–Haq family given in (2.20).

Remarks

- This distribution is an example of an Archimedean copula:

$$\log \left[\frac{1 + \alpha(C - 1)}{C} \right] = \log \left[\frac{1 + \alpha(u - 1)}{u} \right] + \log \left[\frac{1 + \alpha(v - 1)}{v} \right];$$

i.e., the generator is $\varphi = \log \frac{1 + \alpha(u - 1)}{u}$.

- The distribution may be written as

$$C(u, v) = uv[1 + \alpha(1 - u)(1 - v)] + \sum_{i=2}^{\infty} \alpha^i (1 - u)^i (1 - v)^i,$$

with the first term being the F-G-M copula.

- Ali et al. (1978) showed that the copula is PQD, LTD, and PRD.
- Mikhail et al. (1987a) presented some further results, including the (mean) regression curves when the marginals are logistic. They also corrected errors in the calculations of the median regression by Ali et al. (1978).

Genest and MacKay (1986) showed that

$$\tau = \frac{3\alpha - 2}{3\alpha} - \frac{2(1 - \alpha)^2}{3\alpha^2} \log(1 - \alpha).$$

To obtain ρ_S , the second integration requires finding $\int_0^1 (1 - u)^{-1} \log(1 - \alpha + \alpha u) du$. By substituting $x = \alpha(1 - u)$, it becomes $\int_0^\alpha x^{-1} \log(1 - x) dx$, which is $\text{dilin}(1 - \alpha)$, dilin being the dilogarithm function.

The final expression for ρ_S is then

$$\rho_S = -\frac{12(1 + \alpha)}{\alpha^2} \text{dilin}(1 - \alpha) - \frac{3(12 + \alpha)}{\alpha} - \frac{24(1 - \alpha)}{\alpha^2} \log(1 - \alpha).$$

2.3.1 Bivariate Logistic Distributions

A bivariate distribution that corresponds to (2.20),

$$C(u, v) = \frac{uv}{1 - \alpha(1 - u)(1 - v)}$$

is

$$H(x, y) = [1 + e^{-x} + e^{-y} + (1 - \alpha)e^{-x-y}]^{-1}, \quad -1 \leq \alpha \leq 1, \quad (2.22)$$

[Ali et al. (1978)].

Properties

- The marginals are standard logistic distributions.
- When $\alpha = 0$, X and Y are independent.
- When $\alpha = 1$, we have Gumbel's bivariate logistic distribution discussed in Section 11.17:

$$H(x, y) = (1 + e^{-x} + e^{-y})^{-1}.$$

- Gumbel's logistic lacks a parameter which limits its usefulness in applications. The generalized bivariate logistic (2.22) makes up for this lack.

2.3.2 Bivariate Exponential Distribution

The copula in (2.20) with $\alpha = 1$ also corresponds to the survival copula of a bivariate exponential distribution whose survival function is given by

$$\bar{H}(x, y) = (e^x + e^y - 1)^{-1}.$$

Clearly, X and Y are standard exponential random variables.

2.4 Frank's Distribution

$$C(u, v) = \log_{\alpha} \left[1 + \frac{(\alpha^u - 1)(\alpha^v - 1)}{\alpha - 1} \right] \quad (2.23)$$

and

$$c(u, v) = \frac{(\alpha - 1) \log_{\alpha} \alpha^{u+v}}{[\alpha - 1 + (\alpha^u - 1)(\alpha^v - 1)]^2}. \quad (2.24)$$

Correlation and Dependence

- (i) For $0 < \alpha < 1$, we have (positive) association.
- (ii) As $\alpha \rightarrow 1$, we have independence.
- (iii) For $\alpha > 1$, we have negative association.

Nelsen (1986) has given an expression for Blomqvist's medial correlation coefficient. Nelsen (1986) and Genest (1987) have shown that

$$\tau = 1 + 4[D_1(\alpha^*) - 1]/\alpha,$$

$$\rho_S = 1 + 12[D_2(\alpha^*) - D_1(\alpha^*)]/\alpha^*,$$

where $\alpha^* = -\log(\alpha)$ and D_1 and D_2 are Debye functions defined by

$$D_k(\beta) = \frac{k}{\beta^k} \int_0^{\beta} \frac{t^k}{e^t - 1} dt.$$

Derivation

This is the distribution such that both C and $\hat{C} = u + v - C$ are associative, meaning $C[C(u, v), w] = C[u, C(v, w)]$ and similarly for \hat{C} [Frank (1979)].

There does not seem to be a probabilistic interpretation of this associative property.

Remarks

- This distribution is an example of an Archimedean copula [Genest and MacKay (1986)],

$$\log \left(\frac{1 - \alpha^C}{1 - \alpha} \right) = \log \left(\frac{1 - \alpha^u}{1 - \alpha} \right) + \log \left(\frac{1 - \alpha^v}{1 - \alpha} \right),$$

so that $\varphi(t) = \log\left(\frac{1-\alpha^t}{1-\alpha}\right)$.

- The p.d.f. is symmetric about $(\frac{1}{2}, \frac{1}{2})$, and consequently the copula and the survival (complementary) copula are the same. In fact, this family is the only copula that satisfies the functional equation $\hat{C}(u, v) = C(u, v)$.
- When $0 < \alpha < 1$, this distribution is positive likelihood ratio dependent [Genest (1987)].
- This distribution has the “monotone regression dependence” property [Bilodeau (1989)].

2.5 Distribution of Cuadras and Augé and Its Generalization

This distribution, put forward by Cuadras and Augé (1981), is given by

$$C(u, v) = uv[\max(u, v)]^{-c} = uv[\min(u^{-c}, v^{-c})], \quad (2.25)$$

with c being between 0 and 1. It is usually met with identical exponential marginals in the form of Marshall and Olkin given by

$$\bar{H}(x, y) = \exp(-\lambda x - \lambda y - \lambda_{12} \max(x, y)).$$

2.5.1 Generalized Cuadras and Augé Family (Marshall and Olkin's Family)

The Marshall and Olkin bivariate exponential distribution in the original form is

$$\bar{H}(x, y) = \exp(-\lambda_1 x - \lambda_2 y - \lambda_{12} \max(x, y)).$$

Nelsen (2006, p. 53) considered the uniform representation of the survival function above. In order to obtain it, we rewrite the preceding equation in the form

$$\begin{aligned}\bar{H}(x, y) &= \exp(-(\lambda_1 + \lambda_{12})x - (\lambda_2 + \lambda_{12})y + \lambda_{12} \min(x, y)) \\ &= \bar{F}(x)\bar{G}(y) \min\{\exp(\lambda_{12}x), \exp(\lambda_{12}y)\}.\end{aligned}\quad (2.26)$$

Set $u = \bar{F}(x)$ and $v = \bar{G}(y)$, and let $\alpha = \frac{\lambda_{12}}{(\lambda_1 + \lambda_{12})}$, and $\beta = \frac{\lambda_{12}}{(\lambda_2 + \lambda_{12})}$. Then, $\exp(\lambda_{12}x) = u^{-\alpha}$ and $\exp(\lambda_{12}y) = v^{-\beta}$, with the survival copula (complementary copula) \hat{C} given by

$$\hat{C}(u, v) = uv \min(u^{-\alpha}, v^{-\beta}) = \min(uv^{1-\beta}, u^{1-\alpha}v). \quad (2.27)$$

Since the λ 's are all positive, it follows that α and β satisfy $0 < \alpha, \beta < 1$. Hence, the survival copula for the Marshall and Olkin bivariate exponential distribution yields a two-parameter family of copulas given by

$$C_{\alpha, \beta}(u, v) = \min(u^{1-\alpha}, v^{1-\beta}) = \begin{cases} u^{1-\alpha}v, & u^\alpha \geq v^\beta \\ uv^{1-\beta}, & u^\alpha \leq v^\beta \end{cases}. \quad (2.28)$$

This family is known as the *Marshall and Olkin family* and the *generalized Cuadras and Augé family*. When $\alpha = \beta = c$, (2.28) reduces to the Cuadras and Augé family in (2.25). Hanagal (1996) studied the distribution above with Pareto distributions of the first kind as marginals.

A slight complicating factor with this is that the p.d.f. has a singularity along $y = x$. For $\alpha = \beta = c$, Cuadras and Augé determined Pearson's correlation to be $3c/(4 - c)$. Since the marginals are uniform, ρ_S is the same value. It may also be shown that $\tau = c/(2 - c)$, and so $\rho_S = 3\tau/(2 + \tau)$.

Nelsen (2006, Chapter 5) showed that $\rho_S = 3\tau/(2 + \tau)$ also holds for the asymmetric case $H = \min(xy^{1-\beta}, x^{1-\alpha}y)$, but $\tau = \frac{\alpha\beta}{\alpha - \alpha\beta + \beta}$.

2.6 Gumbel–Hougaard Copula

The copula satisfies the equation

$$[-\log C(u, v)]^\alpha = (-\log u)^\alpha + (-\log v)^\alpha. \quad (2.29)$$

Rewriting it in a different form gives

$$C(u, v) = \exp\left(-\left[(-\log u)^\alpha + (-\log v)^\alpha\right]^{1/\alpha}\right). \quad (2.30)$$

Letting $-\log u = e^{-x}$, $-\log v = e^{-y}$ in (2.30), we can verify that the joint distribution of X and Y is

$$H(x, y) = \exp \left[- \left(e^{-\alpha x} + e^{-\alpha y} \right)^{1/\alpha} \right], \quad (2.31)$$

which is the type B bivariate extreme-value distribution with type 1 extreme-value marginals, see Kotz et al. (2000, p. 628) and Nelsen (2006, p. 28).

Correlation Coefficient

Kendall's τ is $(\alpha-1)/\alpha$ [Genest and MacKay (1986)]. The correlation between $\log U$ and $\log V$ is $1 - \alpha^2$.

Derivation

Perhaps surprisingly, the survival copula corresponding to (2.30) can be derived by compounding [Hougaard (1986)].

Suppose there are two independent components having failure rate functions given by $\theta\lambda(x)$ and $\theta\lambda(y)$. Then the joint survival probability is $e^{-\theta[\Lambda(x)+\Lambda(y)]}$. Now assuming θ has a stable distribution with the Laplace transform $E(e^{-\theta s}) = e^{-s^\gamma}$, then $E(e^{-\theta[\Lambda(x)+\Lambda(y)]}) = e^{-[\Lambda(x)+\Lambda(y)]^\gamma}$. Finally, we might suppose that $\lambda(u)$ is of the Weibull form $\varepsilon\alpha u^{\alpha-1}$, in which case $\Lambda(t) = \varepsilon t^\alpha$, so that

$$\bar{H}(x, y) = \exp[-(\varepsilon x^\alpha + \varepsilon y^\alpha)^\gamma], \quad x, y > 0. \quad (2.32)$$

Set $\gamma = 1/\alpha$, and it follows that

$$\bar{H}(x, y) = \exp[-(\varepsilon x^\alpha + \varepsilon y^\alpha)^{1/\alpha}], \quad x, y > 0.$$

Clearly, $\bar{H}(x, y) = C(\bar{F}(x), \bar{G}(y))$ where C is the Gumbel–Hougaard copula and $\bar{F}(x) = e^{-\varepsilon^{1/\alpha}x}$ and $\bar{G}(y) = e^{-\varepsilon^{1/\alpha}y}$.

It now follows from (1.4) that the Gumbel–Hougaard copula is the survival copula of the bivariate exponential distribution given by (2.31).

The Pareto distribution is obtained in a similar manner, but with θ having a gamma distribution. Hougaard (1986, p. 676) has mentioned the possibility of using a distribution that subsumes both gamma and positive stable distributions in order to arrive at a bivariate distribution that subsumes both the Gumbel–Hougaard and Pareto copulas.

Independently, Crowder (1989) had the same idea but added a new wrinkle to it. His distribution, in the bivariate form, is

$$\bar{H}(x, y) = \exp[\kappa^\alpha - (\kappa + \varepsilon x^\gamma + \varepsilon y^\gamma)^\alpha], \quad (2.33)$$

where we see an extra parameter κ ; also, note that ε 's and γ 's are allowed to be different for X and Y . An interpretation of κ is in terms of selection

based on $Z > z_0$ from a population having trivariate survival distribution $\exp[-(\varepsilon x^\gamma + \varepsilon y^\gamma + \varepsilon z^\gamma)^\alpha]$. Crowder has discussed further the dependence and association properties, hazard functions and failure rates, the marginal distributions, the density functions, the distribution of minima, and the fitting of the model to data.

Remarks

- We have called this a Gumbel–Hougaard copula since it appeared in the works of Gumbel (1960a, 1961) and a derivation of it has been given by Hougaard (1986).
- Clearly, from the form of (2.29), it is an Archimedean copula [Genest and MacKay (1986)].

Fields of Applications

- Gumbel and Mustafi (1967) fitted this distribution, in the extreme value form, to data on the sizes of annual floods (1918–1950) of the Fox River (Wisconsin) at two points.
- Hougaard (1986) used a trivariate version of this distribution to analyze data on tumor appearance in rats with 50 liters of a drug treated and two control animals.
- Hougaard (1986) analyzed insulation failure data using a trivariate form of the Weibull version of this distribution.
- Crowder (1989) fitted (2.33) to data on the sensitivity of rats to tactile stimulation of rats that did or did not receive an analgesic drug.

2.7 Plackett's Distribution

The distribution function is derived from the functional equation

$$\frac{C(1 - u - v + C)}{(u - C)(v - C)} = \psi. \quad (2.34)$$

The equation above can be interpreted as (having the support divided into four regions by dichotomizing U and V)

$$\frac{\text{Probability in lower-left region} \times \text{Probability in upper-right region}}{\text{Probability in upper-left region} \times \text{Probability in lower-right region}} = \text{a constant}$$

independent of where the variates are dichotomized. Expressed alternatively,

$$C = \frac{[1 + (\psi - 1)(u + v)] - \sqrt{[1 + (\psi - 1)(u + v)]^2 - 4\psi(\psi - 1)uv}}{2(\psi - 1)}. \quad (2.35)$$

It needs to be noted that the other root is not a proper distribution function, not falling within the Fréchet bounds.

The probability density function is

$$c = \frac{\psi[(\psi - 1)(u + -2uv) + 1]}{\{[1 + (\psi - 1)(u + v)]^2 - 4\psi(\psi - 1)uv\}^{3/2}}.$$

Correlation Coefficient

Spearman's correlation is $\rho_S = \frac{\psi+1}{\psi-1} - \frac{2\psi}{(\psi-1)^2} \log \psi$. Kendall's τ does not seem to be known as a function of ψ . For the product-moment correlation when the marginals are normal, see Mardia (1967).

Conditional Properties

The regression of V on U is linear. After the marginals have been transformed to be normal, the conditional densities are skew and the regression of Y on X is nonlinear [Pearson (1913)].

Remarks

- Interest in this distribution was stimulated by the papers of Plackett (1965) and Mardia (1967), but in fact it can be traced in the contingency table literature back to the days of Yule and Karl Pearson [see Goodman (1981)].
- As compared with the bivariate normal distribution, the outer contours of the p.d.f. of this distribution with normal marginals are more nearly circular—their ellipticity is less than that of the inner ones [Pearson (1913) and Anscombe (1981, pp. 306–310)].
- For low correlation, this distribution is equivalent to the F-G-M in the sense that, if we set $\psi = 1 - \alpha$ in (2.33), expand in terms of α , and then let α be small so that we can neglect α^2 and higher terms, we arrive at (2.1).
- Arnold (1983, Section 6.2.5) has made brief mention of the Pareto-marginals version of this distribution, citing Conway (1979).
- Another account of this distribution is by Conway (1986).

Fields of Application

- This distribution has received considerable attention in the contingency table literature, where it is known as the *constant global cross ratio model*. Suppose one has a square table of frequencies, the categories of the dimensions being ordinals. Then, if the model of independence fails and a degree of positive (or negative) association is evident, one model that has a single degree of freedom to describe the association is the bivariate normal. But this is inconvenient to handle computationally with most of the present-day packages for modeling tables of frequencies. Another model consisting of a single association model is Plackett's distribution, which is much easier computationally. Work in this direction has been carried out by Mardia (1970a, Example 8.1), Wahrendorf (1980), Anscombe (1981, Chapter 12), Goodman (1981), and Dale (1983, 1984, 1985, 1986).
- In the context of bivariate probit models, Amemiya (1985, p. 319) has mentioned that Lee (1982) applied Plackett's distribution with logistic marginals to the data of Ashford and Sowden (1970) and Morimune (1979).
- Mardia (1970b) fitted the S_U -marginals version of this distribution to Johansen's bean data.

2.8 Bivariate Lomax Distribution

The joint survival function of the bivariate Lomax distribution (Durling–Pareto distribution) is given by

$$\bar{H}(x, y) = (1 + ax + by + \theta xy)^{-c}, \quad 0 \leq \theta \leq (c + 1)ab, \quad a, b, c > 0, \quad (2.36)$$

with probability density function

$$h(x, y) = \frac{c[c(b + \theta x)(a + \theta y) + ab - \theta]}{(1 + ax + by + \theta xy)^{c+2}}. \quad (2.37)$$

Marginal Properties

It has Lomax (Pareto of the second kind) marginals with

$$E[X] = \frac{1}{a(c-1)}, \quad E[Y] = \frac{1}{b(c-1)}, \quad c > 1$$

(the mean exists only if $c > 1$) and

$$\text{var}(X) = \frac{c}{(c-1)^2(c-2)a^2}, \quad \text{var}(Y) = \frac{c}{(c-1)^2(c-2)b^2}, \quad c > 2$$

(the variance exists only if $c > 2$).

Derivations

- Begin with two exponential random variables X and Y with parameters θ_1 and θ_2 , respectively. Conditional on (θ_1, θ_2) , X and Y are independent. We now assume that (θ_1, θ_2) has Kibble's bivariate gamma distribution with density $h(\theta_1, \theta_2)$ (see Section 8.2). Then

$$\Pr(X > x, Y > y) = \int_0^\infty \int_0^\infty \exp(-\theta_1 x, \theta_2 y) h(\theta_1, \theta_2) d\theta_1 d\theta_2$$

will have the same form as (2.36).

- Begin with Gumbel's bivariate distribution of the type

$$\bar{F}(x, y) = \exp\left(-\eta(\alpha x + \beta y + \lambda xy)\right).$$

Assuming that η has a gamma distribution with scale parameter m and shape parameter c , then (2.36) will be obtained by letting $a = \alpha/m$, $b = \beta/m$, and $\theta = \lambda/m$; see Sankaran and Nair (1993).

Properties of Bivariate Dependence

Lai et al. (2001) established the following properties:

- For the bivariate Lomax survival function, X and Y are positively (negatively) quadrant dependent if $0 \leq \theta \leq ab$ ($ab < \theta \leq (c+1)ab$).
- The Lomax distribution is RTI if $\theta \leq ab$ and RTD if $\theta \geq ab$.
- X and Y are associated if $\theta \leq ab$.

Correlation Coefficients

- Lai et al. (2001) have shown that

$$\rho = \frac{(1-\theta)(c-2)}{c^2} F[1, 2; c+1; (1-\theta)], \quad 0 \leq \theta \leq (c+1), \quad a = b = 1,$$

where $F(a, b; c; z)$ is Gauss' hypergeometric function; see, for example, Chapter 15 of Abramowitz and Stegun (1964).

- For $a \neq 1, b \neq 1$, the correlation is

$$\rho = \frac{(ab - \theta)(c - 2)}{abc^2} F[1, 2; c + 1; (1 - \theta/ab)], \quad 0 \leq \theta \leq (c + 1)ab.$$

- For $c = n$ an integer and $ab = 1$,

$$\begin{aligned} \text{corr}(X, Y) &= \frac{\frac{\theta^{n-2}}{(n-1)(\theta-1)^{n-1}} \log \theta - \sum_{i=2}^{n-1} \frac{\theta^{n-1-i}}{n(i-1)(\theta-1)^{n-i}} - \frac{1}{(n-1)^2}}{\frac{n}{(n-1)^2(n-2)}} \\ &= \frac{\frac{\theta^{n-2}}{(\theta-1)^{n-1}} \log \theta - \sum_{i=2}^{n-1} \frac{\theta^{n-1-i}}{(i-1)(\theta-1)^{n-i}} - \frac{1}{n-1}}{\frac{n}{(n-1)(n-2)}}, \quad n \geq 3. \end{aligned}$$

- (i) For $c = n = 2$, and $ab = 1$, in particular,

$$\text{cov}(X, Y) = \frac{\log \theta}{\theta - 1} - 1.$$

Thus, the covariance exists for $c = 2$ even though the correlation does not exist since the marginal variance does not exist for $c = 2$.

- (ii) For $c = n = 3$, and $ab = 1$, in particular,

$$\rho = \text{corr}(X, Y) = \left[\frac{2}{3(\theta - 1)^2} \theta \log \theta - \frac{2}{3(\theta - 1)} - \frac{1}{3} \right].$$

- For a given c and $ab = 1$, the correlation ρ decreases as θ increases. However, it does not decrease uniformly over c .
- For a given c and $ab = 1$, the value of ρ lies in the interval

$$-\frac{(c-2)}{c} F(1, 2; c+1; -c) \leq \rho \leq 1/c.$$

Thus, the admissible range for ρ is $(-0.403, 0.5)$.

- This reasonably wide admissible range compares well with the well-known Farlie–Gumbel–Morgenstern bivariate distribution having the ranges of correlation (i) $-\frac{1}{3}$ to $\frac{1}{3}$ for uniform marginals, (ii) $-\frac{1}{4}$ to $\frac{1}{4}$ for exponential marginals, and (iii) $-\frac{1}{\pi}$ to $\frac{1}{\pi}$ for normal marginals, as mentioned earlier.

Remarks

- In order to have a well-defined bivariate Lomax distribution, we need to restrict ourselves to the case $c > 2$ so that the second moments exist.
- The bivariate Lomax distribution is also known as the Durling–Pareto distribution.
- Durling (1975) actually proposed an extra term in the Takahasi–Burr distribution rather than in the simpler Pareto form. Some properties of Durling’s distribution were established by Bagchi and Samanta (1985).

- Durling has given the (product-moment) correlation coefficient for the general case in which x and y are each raised to some power.
- An application of this distribution in the special case where $c = 1$, considered in the literature, is in modeling the severity of injuries to vehicle drivers in head-on collisions between two vehicles of equal mass.
- Several reliability properties have been discussed by Sankaran and Nair (1993). Lai et al. (2001) have discussed some additional properties pertaining to reliability analysis.
- Rodriguez (1980) introduced a similar term into the bivariate Burr type III distribution, resulting in $H = (1 + x^{-a} + y^{-b} + kx^{-a}y^{-b})^{-c}$. He included a number of plots of probability density surfaces of this distribution in the report. This distribution (with location and scale parameters present) was used by Rodriguez and Taniguchi (1980) to describe the joint distribution of customers' and expert raters' assessments of octane requirements of cars.
- The special case

$$\bar{H}(x, y) = \frac{1}{(1 + ax + by)^c}, \quad c > 0, \quad (2.38)$$

is also known as the bivariate Pareto and has been studied in detail by several authors, including Lindley and Singpurwalla (1986).

- Sums, products, and ratios for the special case given in (2.38) are derived in Nadarajah (2005).
- Shoukri et al. (2005) studied inference procedures for $\gamma = \Pr(Y < X)$ of the special case above; in particular, the properties of the maximum likelihood estimate $\hat{\gamma}$ are derived.

2.8.1 The Special Case of $c = 1$

Suppose now that we have a number of 2×2 contingency tables, each of which corresponds to some particular x and some particular y , and we want to fit the distribution $\bar{H} = (1 + ax + by + kabxy)^{-1}$ to them. Notice that the parameter θ depends on a and b . This special case with $c = 1$ is very convenient in these circumstances because we have $p_{11} = (1 + ax + by + kabxy)^{-1}$, $p_{10} + p_{11} = (1 + ax)^{-1}$, and $p_{01} + p_{11} = (1 + by)^{-1}$. We can then estimate a and b from the marginals by

$$\frac{1 - (p_{10} + p_{11})}{p_{10} + p_{11}} = ax \quad \text{and} \quad \frac{1 - (p_{01} + p_{11})}{p_{01} + p_{11}} = by,$$

and k can be estimated by

$$\frac{\frac{1-p_{11}}{p_{11}} - \frac{1-(p_{10}+p_{11})}{p_{10}+p_{11}} - \frac{1-(p_{01}+p_{11})}{p_{01}+p_{11}}}{\left[\frac{1-(p_{10}+p_{11})}{p_{10}+p_{11}} \right] + \left[\frac{1-(p_{01}+p_{11})}{p_{01}+p_{11}} \right]}.$$

Applications of this distribution in transformed form have been discussed by Morimune (1979) and Amemiya (1975).

2.8.2 Bivariate Pareto Distribution

In this case, we have

$$\bar{H}(x, y) = (1 + x + y)^{-c}. \quad (2.39)$$

The marginal is known as the *Pareto distribution of the second kind* (sometimes the Lomax distribution). The p.d.f. is

$$h(x, y) = c(c+1)(1+x+y)^{-(c+2)}. \quad (2.40)$$

Correlation Coefficients and Conditional Properties

Pearson's product-moment correlation is $1/c$ for $c > 2$. The regression of Y on X is linear, $E(Y|X = x) = (x+1)/c$, and the conditional variance is quadratic, $\text{var}(Y|X = x) = \frac{c+1}{(c-1)c^2}(x+1)^2$ for $c > 1$. In fact, $Y|X = x$ is also Pareto.

Derivation

Starting with X and Y having independent exponential distributions with the same scale parameter and then taking the scale parameter to have a gamma distribution, this distribution is obtained by compounding. More generally, starting with $\Pr(X > x) = [1 - A(x)]^\theta$ and $\Pr(Y > y) = [1 - B(y)]^\theta$, where A and B are distribution functions, and then taking θ to have a gamma distribution, the distribution (2.39) is obtained by compounding, with the only difference being that monotone transformations have been applied to X and Y .

If compounding of the scale parameter is applied to an F-G-M distribution that has exponential marginals instead of an independent distribution with exponential marginals, a distribution proposed and used by Cook and Johnson (1986) results.

Remarks

- Barnett (1979, 1983b) has considered testing for the presence of an outlier in a dataset assumed to come from this distribution; see also Barnett and Lewis (1984, Section 9.3.3). An alternative proposal given by Barnett (1983a) involves transformations to independent normal variates.
- The bivariate failure rate is decreasing [Nayak (1987)].
- The product moment is $E(X^r Y^s) = \Gamma(c - r - s)\Gamma(r + 1)\Gamma(s + 1)/\Gamma(c)$ if $r + s < c$ and ∞ otherwise.
- Mardia (1962) wrote this distribution in the form $h \propto (bx + ay - ab)^{-(c+2)}$, with $x > a > 0$, $y > b > 0$. In this case, Malik and Trudel (1985) have derived the distributions of XY and X/Y .

Univariate Transformation

In the bivariate Burr type XII (Takahasi–Burr) distribution, x and y in the distribution function are replaced by their powers; see Takahasi (1965). Further results, oriented toward the repeated measurements experimental paradigm, for this case have been given by Crowder (1985). For generation of random variates following the method of the distribution’s derivation (scale mixture), see Devroye (1986, pp. 557–558). Arnold (1983, p. 249) has referred to this as a type IV Pareto distribution.

Rodriguez (1980) has discussed the bivariate Burr distribution, $H(x, y) = (1 + x^{-a} + y^{-b})^{-c}$. In that report, there is a derivation (by compounding an extreme-value distribution with a gamma), algebraic expressions for the conditional density, conditional distributions, conditional moments, and correlation, and a number of illustrations of probability density surfaces. Satterthwaite and Hutchinson (1978) replaced x and y in the distribution function by e^{-x} and e^{-y} . Gumbel (1961) had previously done this in the special case $c = 1$, thus getting a distribution whose marginals are logistic; however, it lacks an association parameter.

Cook and Johnson (1981) and Johnson (1987, Chapter 9) have treated this copula [whether in Takahasi (1965) form, or Satterthwaite–Hutchinson (1978) form] systematically and have also provided several plots of densities. Cook and Johnson (1986) and Johnson (1987, Section 9.2) have generalized the distribution further.

2.9 Lomax Copula

Consider the bivariate Lomax distribution with the survival function given by (2.36). As $\bar{H}(x, y) = \bar{C}(\bar{F}(x), \bar{G}(y))$, we observe that (2.36) can be obtained from the survival copula

$$\hat{C}(u, v) = uv \left\{ 1 - \alpha(1 - u^{\frac{1}{c}})(1 - v^{\frac{1}{c}}) \right\}^{-c}, \quad -c \leq \alpha \leq 1, \quad (2.41)$$

by taking $\alpha = 1 - \frac{\theta}{ab}$. Recall that the survival function of C is related to the survival copula through $\bar{C}(u, v) = 1 - u - v + C(u, v) = \hat{C}(1 - u, 1 - v)$, and so the copula that corresponds to (2.41) is

$$C(u, v) = \frac{(1 - u)(1 - v)}{\left\{ 1 - \alpha[1 - (1 - u)^{\frac{1}{c}}][1 - (1 - v)^{\frac{1}{c}}] \right\}^{\frac{1}{c}}} + u + v - 1. \quad (2.42)$$

- Case $\theta = 0$ ($\alpha = 1$), so $\hat{C}(u, v) = (u^{-1/c} + v^{-1/c} - 1)^c$ is known as Clayton's copula.
- The case $\alpha = 0$ (i.e., $\theta = ab$) corresponds to the case of independence. Fang et al. (2000) have also shown that U and V are also independent as $c \rightarrow \infty$.
- When $c = 1$, the survival copula (2.41) becomes

$$\hat{C}(u, v) = \frac{uv}{1 - \alpha(1 - u)(1 - v)}, \quad -1 < \alpha < 1,$$

which is nothing but the Ali–Mikhail–Haq family of an Archimedean copula with generator $\log \frac{1 - \alpha(1 - t)}{t}$. Thus, the survival copula in (2.41) can be considered to be a generalization of the Ali–Mikhail–Haq family.

- Fang et al. (2000) have shown that the correlation coefficient of the copula is

$$\rho = 3\{ {}_3F_2(1, 1, c : 1 + 2c, 1 + 2c; \alpha) - 1 \}, \quad 0 \leq \alpha \leq 1, \quad c > 0,$$

where

$${}_3F_2(a, b, c; d, e; x) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k (c)_k}{(d)_k (e)_k} \frac{x^k}{k!}.$$

- It is noted in Fang et al. (2000) that the copula is LRD if $\alpha > 0$.
- For $\alpha = 1$, the survival copula is also known as the Pareto copula, which is discussed next.

2.9.1 Pareto Copula (Clayton Copula)

$$\hat{C}(u, v) = (u^{-1/c} + v^{-1/c} - 1)^{-c}, \quad c > 0. \quad (2.43)$$

This is the survival copula that corresponds to the bivariate Pareto distribution in (2.39). This is not symmetric about $(\frac{1}{2}, \frac{1}{2})$. Equation (2.43) is

also called the Clayton copula by Genest and Rivest (1993). Clearly, this is a special case of the Lomax copula in (2.41).

The survival function of the copula that corresponds to the bivariate Pareto distribution in (2.43) is given by

$$\bar{C}(u, v) = [(1 - u)^{-1/c} + (1 - v)^{-1/c} - 1]^{-c}, \quad c > 0,$$

which has been discussed by many authors, including Oakes (1982, 1986) and Cox and Oakes (1984, Section 10.3). Note that the copula that corresponds to the bivariate Pareto distribution is given by

$$C(u, v) = [(1 - u)^{-1/c} + (1 - v)^{-1/c} - 1]^{-c} + u + v - 1.$$

Remarks

- Johnson (1987, Section 9.1) has given a detailed account of this distribution and has paid more attention to the marginals than we have done here. Johnson has referred to this as the Burr–Pareto–logistic family.
- This distribution is an example of an Archimedean copula [Genest and MacKay (1986)] with generator $\varphi(t) = t^{-1/c} - 1$.
- Ray et al. (1980) have presented results relevant in the context of sample selection.
- This distribution has the “monotone regression dependence” property [Bilodeau (1989)].
- It is possible [see Drouet-Mari and Kotz (2001, p. 86)] to extend the Pareto copula in (2.43) to have negative dependence by allowing $c < 0$. In that case, $\hat{C}(u, v) = \max(u^{-1/c} + v^{-1/c} - 1, 0)^{-c}$, $c < 0$. As $c \rightarrow -1$, this distribution then tends to the lower Fréchet bound.

Fields of Applications

- Cook and Johnson (1981, 1986) fitted this distribution, among others, with lognormal marginals to data on the joint distribution of certain trace elements (e.g., cesium and scandium) in water.
- Concerning association in bivariate life tables, Clayton (1978) deduced that a bivariate survival function must be of the form $\bar{H}(x, y) = [1 + a(x) + b(y)]^{-c}$ if

$$h\bar{H}(x, y) = c \int_x^\infty h(u, y) du \int_y^\infty h(x, v) dv.$$

Clayton’s context is in terms of deaths of fathers and sons from some chronic disease, with association stemming from common environmental or genetic influences. The equation above arises as follows:

- Consider the ratio of the age-specific death rate for sons given that the father died at age y to the age-specific death rate for sons given that the father survived beyond age y . This ratio is assumed to be independent of the son's age.
- As a symmetric form of association is being considered, an analogous assumption holds for the ratio of fathers' age-specific death rates.
- The proportionality property $\frac{h}{H} = c \frac{\partial}{\partial x}(-\log \bar{H}) \frac{\partial}{\partial y}(-\log \bar{H})$ then holds. (The left-hand side of this equation is the bivariate failure rate, and the right-hand side is c times the product of the hazard function for sons of fathers who survive until y and the hazard function for sons of fathers who survive until x .) See also Oakes (1982, 1986) and Clayton and Cuzick (1985a,b).
- Klein and Moeschberger (1988) have used this form of association in the “competing risks” context.
- The bivariate Burr distribution, both with and without the extra association term introduced by Durling (1975), was used by Durling (1974) in reanalyzing seven previously published datasets on the effects of mixtures of poisons.
- The Takahasi–Burr distribution, in its quadrivariate form, was applied by Crowder (1985) in a repeated measurements context—specifically, in analyzing response times of rats to pain stimuli at four intervals after receiving a dose of an analgesic drug.

2.9.2 Summary of the Relationship Between Various Copulas

For ease of reference, we summarize the relationship between the Lomax copula and its special cases.

The Lomax copula (α, c) is given in (2.41):

- (i) $\alpha = 1 \Rightarrow$ Pareto copula (Clayton copula) as given in (2.43).
- (ii) $c = 1 \Rightarrow$ Ali–Mikhail–Haq copula as given in (2.20).

2.10 Gumbel's Type I Bivariate Exponential Distribution

Again, we depart from our usual pattern by describing this distribution, with exponential marginals, before the copula.

Formula for Cumulative Distribution Function

$$H(x, y) = 1 - e^{-x} - e^{-y} + e^{-(x+y+\theta xy)}, \quad 0 \leq \theta \leq 1. \quad (2.44)$$

Formula for Density Function

$$h(x, y) = e^{-(x+y+\theta xy)}[(1 + \theta x)(1 + \theta y) - \theta]. \quad (2.45)$$

Univariate Properties

Both marginals are exponential.

Correlation Coefficients and Conditional Properties

$$\rho = -1 + \int_0^\infty \frac{e^{-y}}{1 + \theta y} dy.$$

Gumbel (1960a) has plotted ρ as a function of θ . A compact expression may be obtained in terms of the exponential integral (but care is always necessary with this function, as the nomenclature and notation are not standardized). For $\theta = 0$, X and Y are independent and $\rho = 0$. As θ increases, ρ increases, reaching -0.404 at $\theta = 1$. Thus, this distribution is unusual in being oriented towards negative correlation. (Of course, positive correlation can be obtained by changing X to $-X$ or Y to $-Y$.)

Barnett (1983a) has discussed the maximum likelihood method for estimating θ as well as a method based on the product-moment correlation.

Gumbel (1960a) has further given the following expressions:

$$\begin{aligned} g(y|x) &= e^{-y(1+\theta x)}[(1 + \theta x)(1 + \theta y) - \theta], \\ E(Y|X = x) &= (1 + \theta + \theta x)(1 + \theta x)^{-2}, \\ \text{var}(Y|X = x) &= \frac{(1 + \theta + \theta x)^2 - 2\theta^2}{(1 + \theta x)^4}. \end{aligned}$$

Remarks

- This distribution is characterized by

$$\begin{aligned} E(X - x|X > x \text{ and } Y > y) &= E(X|Y > y), \\ E(Y - y|X > x \text{ and } Y > y) &= E(Y|X > x), \end{aligned} \quad (2.46)$$

which is a form of the lack-of-memory property; see K.R.M. Nair and N.U. Nair (1988) and N.U. Nair and V.K.R. Nair (1988).

- Barnett (1979, 1983b) and Barnett and Lewis (1984, Section 9.3.2) have discussed testing for the presence of an outlier in a dataset assumed to come from this distribution. An alternative proposal by Barnett (1983a) involves transformation to independent normal variates.
- In the context of structural reliability, Der Kiureghian and Liu (1986) utilized this distribution (with $\theta = 1$) in the course of demonstrating a procedure to approximate multivariate integrals by transforming the marginals to normality and assuming multivariate normality; see also Grigoriu (1983, Example 2).

An Application

In describing this, let us quote the opening words of the paper by Moore and Clarke (1981): “The rainfall runoff models referred to in the title of this paper are (1) those that attempt to describe explicitly both the storage of precipitated water within a river basin and the translation or routing of water that is in temporary storage to the basin outfall, and (2) those in which the parameters of the model are estimated from existing records of mean areal rainfall, Penman potential evaporation E_T , or some similar measure of evaporation demand, and stream flow.” On pp. 1373–1374 of the paper is a section entitled “A Bivariate Exponential Storage-Translation Model.” This introduces distribution (2.44), the justification being that it has exponential marginals and that the correlation is negative (“a basin with thin soils in the higher altitude areas that are furthest from the basin outfall is likely to have s and t negatively correlated”). The variables s and t are, respectively, the depth of a (hypothesized) storage element and the time taken for runoff to reach the catchment outfall.

Moore and Clarke did not present in detail the results using (2.44), saying, “Application of storage-translation models using more complex distribution functions ... did not lead to any appreciable improvement in model performance ... One exception ... gives a correlation of -0.37 between s and t .”

2.11 Gumbel–Barnett Copula

Gumbel (1960a,b) suggested the exponential-marginals form of this copula; many authors refer to this copula as another Gumbel family. We call it the *Gumbel–Barnett copula* since Barnett (1980) first discussed it in terms of

the uniform marginals among the distributions he considered. The survival function of the copula $C(u, v)$ that corresponds to Gumbel's type 1 bivariate exponential distribution (2.44) is given by

$$\bar{C}(u, v) = (1 - u)(1 - v)e^{-\theta \log(1-u) \log(1-v)},$$

so that

$$C(u, v) = u + v - 1 + (1 - u)(1 - v)e^{-\theta \log(1-u) \log(1-v)} \quad (2.47)$$

because of the relationship $C(u, v) = \bar{C}(u, v) + u + v - 1$. The density of the copula is

$$c(u, v) = \{-\theta + [1 - \theta \log(1 - u)][1 - \theta \log(1 - v)]\} e^{-\theta \log(1-u) \log(1-v)}. \quad (2.48)$$

The survival copula that corresponds to (2.47) is

$$\hat{C}(u, v) = \bar{C}(1 - u, 1 - v) = uve^{-\theta \log u \log v}. \quad (2.49)$$

2.12 Kimeldorf and Sampson's Distribution

Kimeldorf and Sampson (1975a) studied a bivariate distribution on the unit square, with uniform marginals and p.d.f. as follows:

- β on each of $[\beta]$ squares of side $1/\beta$ arranged corner to corner up to the diagonal from $(0, 0)$ towards $(1, 1)$, $[\beta]$ being the largest integer not exceeding β ;
- $\frac{\beta}{\beta - [\beta]}$ on one smaller square side of $1 - [\beta]/\beta$ in the top-right corner of the unit square (unless β is an integer);
- and 0 elsewhere.

For this distribution, Johnson and Tenenbein (1979) showed that

$$\rho_S = \frac{[\beta]}{\beta} \frac{3\beta^2 - 3\beta[\beta] + [\beta^2] - 1}{\beta^2},$$

and Nelsen (in a private communication) showed that

$$\tau = \frac{[\beta]}{\beta} \frac{2\beta - [\beta] - 1}{\beta}.$$

Hence, if $1 \leq \beta < 2$, $\rho_S = 3\tau/2$; and if β is an integer, $\rho_S = 2\tau - \tau^2$.

Remarks

- Clearly (2.49) is an Archimedean copula.
- If C_α and C_β are both Gumbel–Barnett copulas given by (2.49), then their geometric mean is again a Gumbel–Barnett copula given by $C_{(\alpha+\beta)/2}$; see Nelsen (2006, p. 133).

2.13 Rodríguez-Lallena and Úbeda-Flores' Family of Bivariate Copulas

Rodríguez-Lallena and Úbeda-Flores (2004) defined a new class of copulas of the form

$$C(u, v) = uv + f(u)g(v), \quad (2.50)$$

where f and g are two real functions defined on $[0, 1]$ such that

- (i) $f(0) = f(1) = g(0) = g(1)$;
- (ii) f and g are absolutely continuous;
- (iii) $\min\{\alpha\delta, \beta\gamma\} \geq -1$, where $\alpha = \inf\{f'(u), u \in A\} < 0$, $\beta = \sup\{f'(u), u \in A\} > 0$, $\gamma = \inf\{g'(v), v \in B\} < 0$, and $\delta = \sup\{g'(v), v \in B\} > 0$, with $A = \{u \in [0, 1] : f'(u) \text{ exists}\}$ and $B = \{v \in [0, 1] : g'(v) \text{ exists}\}$.

Example 2.1. The family studied by Lai and Xie (2000), $C(u, v) = uv + \lambda u^a v^b (1-u)^c (1-v)^d$, $u, v \in [0, 1]$, $0 \leq \lambda \leq 1$, $a, b, c, d \geq 1$, is a special case of Rodríguez-Lallena and Úbeda-Flores' family.

Properties

- $\tau = 8 \int_0^1 f(t) dt \int_0^1 g(r) dr$, $\rho_S = 3\tau/2$.
- Let (X, Y) be a continuous random pair whose associated copula is a member of Rodríguez-Lallena and Úbeda-Flores' family. Then X and Y are positively quadrant dependent if and only if either $f \geq 0$ and $g \geq 0$ or $f \leq 0$ and $g \leq 0$.

2.14 Other Copulas

Table 4.1 of Nelsen (2006) presents some important one-parameter families of Archimedean copulas, along with their generators, the range of the parameter, and some special cases and limiting cases. We have discussed some of these here, and for the rest we refer the reader to this reference. Many other copulas are discussed throughout the book of Nelsen (2006), wherein we can find a comprehensive treatment of copulas.

2.15 References to Illustrations

We will now outline five important references that contain illustrations of distributions discussed in this chapter as well as some others to follow.

Conway (1981). Conway's graphs are contours of bivariate distributions; that is, for uniform marginals $F(x) = x$ and $G(y) = y$, y as a function of x has been plotted such that a contour of the (cumulative) distribution is the result (i.e., $H(x, y) = c$) a constant. The paper (i) presents such contours for various c for three reference distributions (upper and lower Fréchet bounds, and the independence), (ii) gives the $c = 0.2$ contour for distributions having various strengths of correlations drawn from the Farlie–Gumbel–Morgenstern, Ali–Mikhail–Haq, Plackett, Marshall–Olkin, and Gumbel–Hougaard families, and (iii) presents some geometric interpretations of properties of bivariate distributions.

Barnett (1980). The contours in this paper are of probability density functions. The distributions are again transformed to have uniform marginals; the bivariate normal, Farlie–Gumbel–Morgenstern, Plackett, Cauchy, and Gumbel–Barnett are the ones included.

Johnson et al. (1981). This contains both contours and three-dimensional plots of the p.d.f.'s of a number of distributions after their marginals have been transformed to be either normal or exponential. The well-known distributions included are the Farlie–Gumbel–Morgenstern, Plackett, Cauchy, and Gumbel's type I exponentials, plus a bivariate normal transformed to exponential marginals. However, the main purpose of this work is to give similar plots for distributions obtained by a trivariate reduction technique and by the Khintchin mixture.

Johnson et al. (1984). In this, there are 18 small contour plots of the p.d.f.'s of distributions after their marginals have been transformed to be normal. The well-known distributions included are the bivariate normal, Farlie–Gumbel–Morgenstern, Ali–Mikhail–Haq, Plackett, Gumbel's type I exponential, and the bivariate Pareto.

Johnson (1987). Chapters 9 and 10 of this book presents contour and three-dimensional plots of the p.d.f.'s of the following distributions: Farlie–Gumbel–Morgenstern (uniform, normal, and exponential marginals), Ali–Mikhail–Haq (normal marginals), Plackett (contour plots only; uniform, normal, and exponential marginals), Gumbel's type I exponential (uniform, normal, and exponential marginals), bivariate Pareto (uniform and normal marginals; contour plots only for exponential marginals), and Cook and Johnson's generalized Pareto (contour plots only; uniform, normal, and exponential marginals; and one three-dimensional plot of normal marginals).

When thinking of contours of p.d.f.'s, the subject of unimodality (or otherwise) of multivariate distributions comes to mind. An excellent reference

for this topic is the book by Dharmadhikari and Joag-Dev (1985), and we refer readers to this book for all pertinent details.

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