# CHAPTER 2

# THE FACTORIAL FUNCTION *n*!

The factorial function occurs widely in function theory; especially in the denominators of power series expansions of many transcendental functions. It also plays an important role in combinatorics [Section 2:14]. Because they too arise in the context of combinatorics, Stirling numbers of the second kind are discussed in Section 2:14 [those of the first kind find a home in Chapter 18].

Double and triple factorial functions are described in Section 2:14.

## 2:1 NOTATION

The factorial function of $n$, also spoken of as "$n$ factorial", is generally given the symbol $n!$. It is represented by $\lfloor\underline{n}$ in older literature. The symbol $\Pi(n)$ is occasionally encountered.

## 2:2 BEHAVIOR

The factorial function is defined only for nonnegative integer argument and is itself a positive integer. Because of its explosive increase, a plot of $n!$ versus $n$ is not very informative. Figure 2-1 is a graph of the logarithm (to base 10) of $n!$ versus $n$. Note that $70! \approx 10^{100}$.

## 2:3 DEFINITIONS

The factorial function of the positive integer $n$ equals the product of all positive integers up to and including $n$:

2:3:1 $$n! = 1\times 2\times 3\times\cdots\times(n-1)\times n = \prod_{j=1}^{n} j \qquad n = 1,2,3,\cdots$$

This definition is supplemented by the value

2:3:2 $$0! = 1$$

conventionally accorded to zero factorial.

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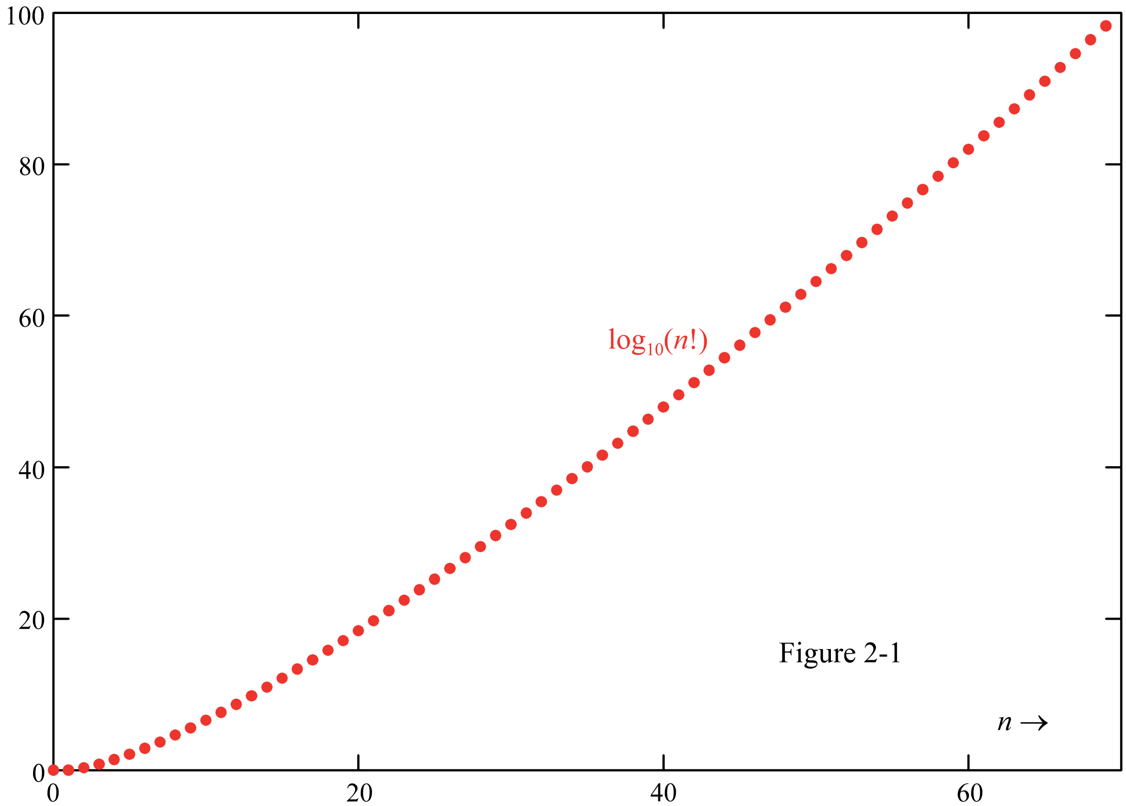



Figure 2-1

The exponential function [Chapter 26] is a generating function [Section 0:3] for the reciprocal of the factorial function

2:3:3 $$\exp(t) = \sum_{n=0}^{\infty} \frac{1}{n!} t^n$$

## 2:4 SPECIAL CASES

There are none.

## 2:5 INTRARELATIONSHIPS

The most important property of the factorial function is its recurrence

2:5:1 $$(n+1)! = n!(n+1) \qquad n = 0,1,2,\cdots$$

which may be iterated to produce the argument-addition formula

2:5:2 $$(n+m)! = n!(n+1)(n+2)\cdots(n+m) = n!(n+1)_m \qquad n,m = 0,1,2,\cdots$$

where $(n+1)_m$ is a Pochhammer polynomial [Chapter 18]. Formula 2:5:2 leads to an expression for the ratio of two factorials. An alternative expression is

2:5:3 $$\frac{n!}{(n-m)!} = n(n-1)(n-2)\cdots(n-m+1) = (-)^m(-n)_m \qquad n > m$$

Setting $m = n$ in equation 2:5:2 provides a *duplication formula* for the factorial function, enabling $(2n)!$ to be expressed with the help of a Pochhammer polynomial. Alternative duplication formulas are available from equations 2:12:3 and 2:12:4, which may be rewritten as

2:5:4
$$n! = \begin{cases} 2^n \left(\tfrac{1}{2}\right)_{n/2} \left(\tfrac{1}{2}n\right)! & n = 2,4,6,\cdots \\ 2^n \left(\tfrac{1}{2}\right)_{(n+1)/2} \left(\tfrac{1}{2}n - \tfrac{1}{2}\right)! & n = 1,3,5,\cdots \end{cases}$$

There are analogous triplication formulas that can be developed from the equations in 2:12:5.

The frequent occurrence of factorials as coefficients of power series permits the summation of such series as

2:5:5
$$\frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \cdots = \sum_{j=0}^{\infty} \frac{1}{j!} = \exp(1) = 2.7182\ 81828\ 45905$$

and

2:5:6
$$\frac{1}{(0!)^2} + \frac{1}{(1!)^2} + \frac{1}{(2!)^2} + \cdots = \sum_{j=0}^{\infty} \frac{1}{(j!)^2} = \mathrm{I}_0(2) = 2.2795\ 85302\ 33607$$

where $\mathrm{I}_0$ is the modified Bessel function [Chapter 49]. The corresponding series with alternating signs sum similarly to $\exp(-1)$ and to the particular value $\mathrm{J}_0(2)$ of the zero-order Bessel function [Chapter 52]. There is even the intriguing asymptotic result [see equation 37:13:4]

2:5:7
$$0! - 1! + 2! - \cdots = \sum_{j=0}^{\infty} (-)^j j! \sim \int_0^{\infty} \frac{\exp(-t)}{1+t} \mathrm{d}t = 0.59634\ 73623\ 23194$$

Moreover, the series $\Sigma(-1)^n/(2n)!$ sums to $\cos(1)$ and there are several analogous summations.

## 2:6 EXPANSIONS

*Stirling's formula* [see also Section 43:6]

2:6:1
$$n! \sim \sqrt{2\pi n}\, \exp(-n) n^n \left[1 + \frac{1}{12n} + \frac{1}{288n^2} - \frac{139}{51840n^3} - \cdots\right] \qquad n \to \infty$$

provides an expansion for the factorial function. Though they are technically asymptotic [Section 0:6], this expansion and a similar one for the logarithm of the factorial function

2:6:2
$$\ln(n!) \sim \ln\sqrt{2n\pi} + n\left[\ln(n) - 1 + \sum_{j=1} \frac{\mathrm{B}_j}{j(j-1)} \left(\frac{1}{n}\right)^j\right]$$
$$= n\ln(n) - n + \frac{\ln(2n\pi)}{2} + \frac{1}{12n} - \frac{1}{360n^3} + \frac{1}{1260n^5} - \cdots \qquad n \to \infty$$

are remarkably accurate, even for small $n$. $\mathrm{B}_j$ is the $j^{\text{th}}$ Bernoulli number [Chapter 4].

## 2:7 PARTICULAR VALUES

| 0! | 1! | 2! | 3! | 4! | 5! | 6! | 7! | 8! | 9! | 10! | 11! | 12! |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 2 | 6 | 24 | 120 | 720 | 5040 | 40320 | 362880 | 3628800 | 39916800 | 479001600 |

## 2:8 NUMERICAL VALUES

The decimal integer representing $n!$ has exactly $\mathrm{Int}(n/5)+\mathrm{Int}(n/25)+\mathrm{Int}(n/125)+\cdots$ terminal zeros; for example 31! ends with seven zeros. This rule is useful in calculating exact numerical values of large factorials. Int is the integer-value function describe in Chapter 8.

*Equator*'s factorial function routine (keyword **!**) provides values of $n!$. For integer input in the range $0 < n \le 170$, a simple algorithm based on recursion 2:5:1, followed by rounding, is used to compute $n!$. Exact output is reported up to $20! = 2.4329\ 02008\ 17664\ \mathrm{E}{+}18$. For $21 \le n \le 170$, *Equator* provides a floating point approximation of $n!$ precise to 15 digits.

Separately, values of the natural and decadic logarithms, $\ln(n!)$ and $\log_{10}(n!)$, are provided by the logarithmic factorial function and logarithm to base 10 of the factorial function routines (**ln!** and **log10!**). Such logarithmic values are useful when $n$ is large because $n!$ itself is then prohibitively huge. For input up to $n = 170$, $\ln(n!)$ is computed by simply taking the logarithm of the output from the routine described above. For integer input in the range $171 \le n \le 1\mathrm{E}305$, *Equator* uses Stirling's formula in the truncated and concatenated form

2:8:1 $$\ln(n!) = \frac{\ln(2n\pi)}{2} - n\left(1-\ln(n)-\frac{1}{12n^2}\left(1-\frac{1}{30n^2}\left(1-\frac{2}{7n^2}\right)\right)\right)$$

Division by 2.3025 85092 99405 generates $\log_{10}(n!)$.

## 2:9 LIMITS AND APPROXIMATIONS

For large argument, the limiting formula

2:9:1 $$n! \to \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \qquad n\to\infty$$

applies, $e$ being the base of natural logarithms [Section 1:7]. The related approximation

2:9:2 $$n! \approx \mathrm{Round}\left\{(1+12n)\sqrt{\frac{\pi}{72n}}\left(\frac{n}{e}\right)^n\right\}$$

is surprisingly good, and even exact, for small positive integers. Round is the rounding function, described in Section 8:13.

## 2:10 OPERATIONS OF THE CALCULUS

No operations of the calculus are possible on a function such as $n!$ that is defined only for discrete arguments.

## 2:11 COMPLEX ARGUMENT

In view of relation 2:12:1, the gamma function formulas given in Section 43:11 may be used to ascribe meaning to $(n+im)!$.

## 2:12 GENERALIZATIONS

The factorial function is a special case of the gamma function [Chapter 43]

2:12:1 $$n! = \Gamma(n+1) \qquad n = 0,1,2,\cdots$$

and of the Pochhammer polynomial [Chapter 18]

2:12:2 $$n! = (1)_n \qquad n = 0,1,2,\cdots$$

The latter identity permits us to write

2:12:3 $$(2n)! = 4^n (\tfrac{1}{2})_n (1)_n$$

and

2:12:4 $$(2n+1)! = 4^n (1)_n (\tfrac{3}{2})_n$$

Similarly

2:12:5 $$(3n)! = 3^{3n}\left(\tfrac{1}{3}\right)_n\left(\tfrac{2}{3}\right)_n(1)_n, \qquad (3n+1)! = 3^{3n}\left(\tfrac{2}{3}\right)_n(1)_n\left(\tfrac{4}{3}\right)_n, \qquad (3n+2)! = 3^{3n}(1)_n\left(\tfrac{4}{3}\right)_n\left(\tfrac{5}{3}\right)_n$$

and so on.

## 2:13 COGNATE FUNCTIONS: multiple factorials

See 6:3:4 for the close relationship between the factorial function and binomial coefficients.

The *double factorial* or *semifactorial function* is defined by

2:13:1 $$n!! = \begin{cases} 1 & n = -1,0 \\ n\times(n-2)\times(n-4)\times\cdots\times 5\times 3\times 1 & n = 1,3,5,\cdots \\ n\times(n-2)\times(n-4)\times\cdots\times 6\times 4\times 2 & n = 2,4,6,\cdots \end{cases}$$

For even argument it reduces to

2:13:2 $$n!! = 2^{n/2}\left(n/2\right)! \qquad n = 0,2,4,\cdots$$

while for odd $n$ it may be expressed in terms of factorials, or as a gamma function [Chapter 43] or as a Pochhammer polynomial [Chapter 18]

2:13:3 $$n!! = \frac{n!}{2^{(n-1)/2}\left(\dfrac{n-1}{2}\right)!} = 2^{n/2}\sqrt{\frac{2}{\pi}}\,\Gamma\left(1+\frac{n}{2}\right) = 2^{(n+1)/2}\left(\frac{1}{2}\right)_{(n+1)/2} \qquad n = 1,3,5,\cdots$$

Equivalent to the last equation is

2:13:4 $$(2n+1)!! = \frac{(2n+1)!}{2^n n!}$$

These formulas are used by *Equator*'s double factorial function routine (keyword **!!**) to compute values of $n!!$ for integers in the range $-1 \le n \le 300$. Of course

2:13:5 $$n!!(n-1)!! = n! \qquad n = 0,1,2,\cdots$$

Some early members of the double-factorial family are listed below.

| $(-1)!!$ | $0!!$ | $1!!$ | $2!!$ | $3!!$ | $4!!$ | $5!!$ | $6!!$ | $7!!$ | $8!!$ | $9!!$ | $10!!$ | $11!!$ | $12!!$ | $13!!$ | $14!!$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 2 | 3 | 8 | 15 | 48 | 105 | 384 | 945 | 3840 | 10395 | 46080 | 135135 | 645120 |

Note that, apart from 0!! = 1, the double factorial $n!!$ shares the parity of $n$. Also notice that, to accord with the $n=-1$ instance of the general recursion formula

2:13:6 $$(n+2)!!=(n+2)n!!$$

$(-1)!!$ is assigned the value of unity. With a similar rationale, one sometimes encounters the values $(-3)!! = -1$, $(-5)!! = \tfrac{1}{3}$, etc.

Of frequent occurrence [for example in Sections 6:4, 32:5, 61:6 and 62:12] is the ratio $(n-1)!!/n!!$ of the double factorials of consecutive integers. For odd $n$, the ratio is expressible by the integral

2:13:7 $$\frac{(n-1)!!}{n!!}=\frac{2^{n-1}}{n!}\left[\left(\frac{n-1}{2}\right)!\right]^2 = \int_0^{\pi/2}\sin^n(t)\,\mathrm{d}t \qquad n=1,3,5,\cdots$$

while for even $n$ it is given by *Wallis's formula* (John Wallis, English mathematician and cryptographer, 1616–1703)

2:13:8 $$\frac{(n-1)!!}{n!!}=\frac{n!}{2^n\left[(n/2)!\right]^2}=\frac{2}{\pi}\int_0^{\pi/2}\sin^n(t)\,\mathrm{d}t \qquad n=0,2,4,\cdots$$

This important ratio has the asymptotic expansion

2:13:9 $$\frac{(n-1)!!}{n!!}\sim\begin{cases}\sqrt{\dfrac{2}{\pi n}}\left[1-\dfrac{1}{4n}+\dfrac{1}{32n^2}-\cdots\right] & \text{even } n\to\infty\\[2ex] \sqrt{\dfrac{\pi}{2n}}\left[1-\dfrac{1}{4n}+\dfrac{1}{32n^2}-\cdots\right] & \text{odd } n\to\infty\end{cases}$$

Finite sums of some such ratios obey the simple rule

2:13:10 $$1+\frac{1}{2}+\frac{3}{8}+\frac{15}{48}+\cdots+\frac{(2n-1)!!}{(2n)!!}=\sum_{j=0}^{n}\frac{(2j-1)!!}{(2j)!!}=\frac{(2n+1)!!}{(2n)!!} \qquad n=0,1,2,\cdots$$

and there is the related infinite summation due to Ross:

2:13:11 $$\frac{1}{2}+\frac{3}{16}+\frac{15}{144}+\frac{105}{1536}+\cdots=\sum_{j=1}^{\infty}\frac{(2j-1)!!}{j(2j)!!}=\ln(4)$$

The *triple factorial* is defined analogously

2:13:12 $$n!!!=\begin{cases}1 & n=-2,-1,0\\ n\times(n-3)\times(n-6)\times\cdots\times7\times4\times1 & n=1,4,7,\cdots\\ n\times(n-3)\times(n-6)\times\cdots\times8\times5\times2 & n=2,5,8,\cdots\\ n\times(n-3)\times(n-6)\times\cdots\times9\times6\times3 & n=3,6,9,\cdots\end{cases}$$

and finds application in connections with Airy functions [Chapter 56]. Some early values are:

| (−2)!!! | (−1)!!! | 0!!! | 1!!! | 2!!! | 3!!! | 4!!! | 5!!! | 6!!! | 7!!! | 8!!! | 9!!! | 10!!! | 11!!! | 12!!! |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 2 | 3 | 4 | 10 | 18 | 28 | 80 | 162 | 280 | 880 | 1944 |

The extension to a *quadruple factorial* $n!!!!$ is obvious; it is useful in Sections 43:4 and 59:7.

## 2:14 RELATED TOPIC: combinatorics and Stirling numbers of the second kind

The factorial function appears very often in applications involving *combinatorics*. For example, the number of *permutations* (arrangements) of $n$ objects, all different, is $n!$. If not all of the $n$ objects are different, the number of permutations is reduced to

2:14:1 $$\frac{n!}{(n_1)!(n_2)!\cdots(n_J)!} \qquad n = n_1 + n_2 + \cdots + n_J$$

where there are $n_1$ samples of object 1, $n_2$ samples of object 2, $\cdots$, $n_J$ samples of object $J$, so that $\Sigma n_j = n$.

If from a group of $n$ objects, all different, one withdraws $m$ objects, one at a time, the number of *variations* (possible withdrawal sequences) is

2:14:2 $$\frac{n!}{(n-m)!} \qquad m \le n$$

If one ignores the order of withdrawal, 2:14:2 is reduced to

2:14:3 $$\frac{n!}{(n-m)!m!} \qquad m \le n$$

which then represents the number of ways in which $m$ objects can be chosen from among $n$, all different, and is known as the number of *combinations*. Expression 2:14:3 is, in fact, the binomial coefficient addressed in Chapter 6.

The number of *partitions* (different ways in which $n$ distinct objects may be placed in $m$ identical boxes so that each box contains at least one object) is given by a *Stirling number of the second kind*. There is no standardized notation for such functions; this *Atlas* uses the symbol $\sigma_n^{(m)}$. Clearly, no partitioning is possible if $m = 0$ or if $n < m$ and accordingly the second Stirling number is zero in such circumstances. Otherwise, a general formula is

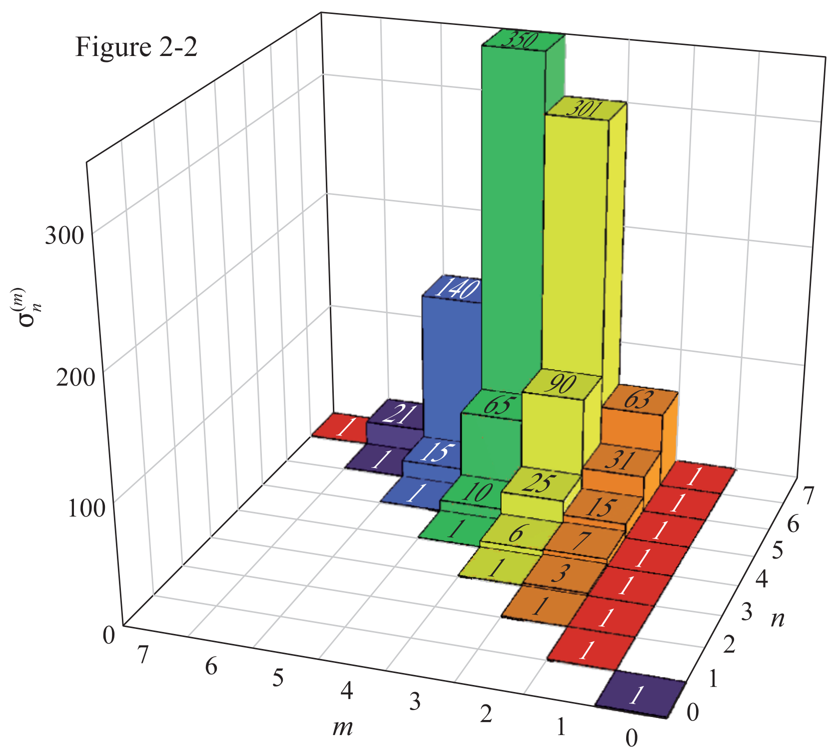
2:14:4 $$\sigma_n^{(m)} = \sum_{j=0}^{m} \frac{(-)^{m-j}\, j^n}{(m-j)!\, j!}$$

Non-zero values of Stirling numbers of the second kind are, of course, positive integers and some are shown in Figure 2-2. Others may be calculated via the recursion formula

2:14:5 $$\sigma_n^{(m)} = m\sigma_{n-1}^{(m)} + \sigma_{n-1}^{(m-1)}$$

for $m = 1,2,3,\cdots$ and $n = 1,2,3,\cdots$. This recursion forms the basis of *Equator*'s Stirling number of the second kind routine (keyword **sigmanum**). First, $\sigma_0^{(1\to m)}$ and $\sigma_{1\to n}^{(0)}$ are initialized to zero while $\sigma_0^{(0)}$ is set equal to 1. Then $\sigma_1^{(1\to m)}$, $\sigma_2^{(1\to m)}$, $\cdots$, $\sigma_n^{(1\to m)}$ are calculated via recursion 2:14:5. Because Stirling numbers are integers, rounding ensures that the 15 digits that *Equator* generates are exact.

Figure 2-2



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