Chapter 2 Statics

2.0 Introduction

A *Statics* analysis is generally the first step in determining how an engineering system deforms under load, and how the system supports the load internally. For ***equilibrium***, the vector sum of all the forces $\mathbf{F}_i$ that act on a system must equal zero. Likewise, the vector sum of all the moments $\mathbf{M}_i$ that act on a system must equal zero. However complex the system, it must support the applied loads without accelerating.

In mathematical terms, *equilibrium of forces* and *equilibrium of moments* are represented by the vector equations:

$$\sum \mathbf{F}_i = 0 \qquad \text{[Eq. 2.1]}$$

$$\sum \mathbf{M}_i = 0 \qquad \text{[Eq. 2.2]}$$

Free body diagrams (FBDs) of the entire system, of individual components, and of parts of individual components, are vital to the solution of any problem. In a FBD, the body of interest is first isolated from its surroundings. All of the forces and moments that act on the body from its surroundings are then represented with force and moment vectors. The coordinate system is also indicated. The examples in this chapter, and throughout the text, illustrate the importance of FBDs.

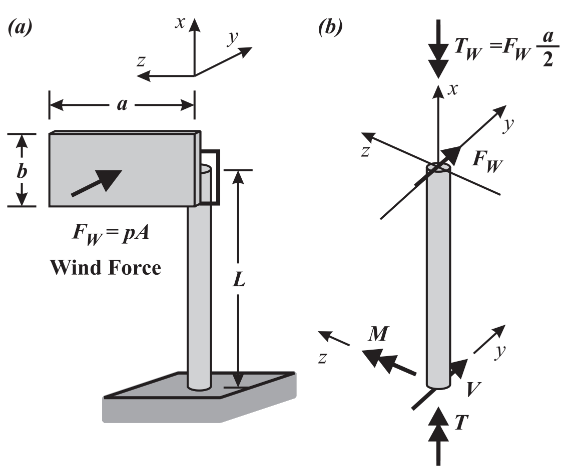

Figure 2.1. A free body diagram isolates a system or part of a system, and shows the forces acting on it. **(a)** A highway sign and support mast under a wind load. **(b)** FBD of the support mast.

An example of a FBD is given in *Figure 2.1*. A highway sign is acted on by wind load F_W; the weight of the sign and support mast are presently ignored. A FBD of the support mast is shown in *Figure 2.1b*. The mast has

F.A. Leckie, D.J. Dal Bello, *Strength and Stiffness of Engineering Systems*, Mechanical Engineering Series, DOI 10.1007/978-0-387-49474-6_2, © Springer Science+Business Media, LLC 2009

been removed from its surroundings, and all the forces and moments that act on it are represented. Here, the wind load causes a force F_W at the top of the mast, requiring a reaction shear force at the base V. The wind causes torque T_W, requiring reaction torque T. In engineering systems, a torque is a moment that causes twisting of an axial member or shaft. Finally, reaction moment M at the base keeps the mast from falling over due to the wind load. All the reactions have been drawn in the positive directions of the axes, not necessarily in the directions that they physically act. This problem is studied in more detail in *Example 2.9*.

Carefully drawn FBDs make the loads acting on a structure easier to visualize, and help to communicate to others how the system operates. The FBDs are the basis for the equations of equilibrium. Every problem and solution should include an FBD.

2.1 Axial Members

An ***axial member*** is a straight component that only supports a force P parallel to its axis (*Figure 2.2*). This ***axial force*** must pass through the ***centroid*** (center of area) of the component's cross-section so that the response at any cross-section is uniform (the same at every point). Loads that stretch the component are ***tensile loads***, and those that shorten the component are ***compressive loads***. *Figure 2.2a* shows a tensile force P and *Figure 2.2b* shows a compressive force.

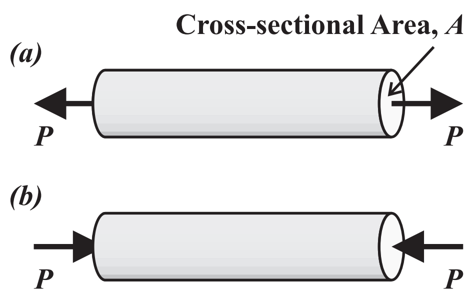

Figure 2.2. An axial member under **(a)** a tensile load and **(b)** a compressive load.

If axial force P is consistently drawn assuming that it is a tensile force, as in *Figure 2.2a*, then a calculated positive value for P ($P > 0$) means that it is tensile, while a negative value for P ($P < 0$) means that it physically acts opposite drawn, i.e., the force is compressive.

An axial member is typically a ***two-force member***. The forces that act at each end of a two-force member are *equal*, *opposite,* and *co-linear.* A straight two-force member supports a constant force P normal to any cross-section along its entire length. An assembly of straight two-force members appropriately pinned together is a *truss*.

The internal force that an axial member supports may vary along its length. These changes may occur at discrete locations due to point loads, or continuously due to distributed loads. The cross-sectional area may also change abruptly or continuously.

A few statics examples with axial members follow.

Example 2.1 Flower Pots on Hanging Shelves

Given: Three shelves are hung from the ceiling by means of four rope cords. Plants are evenly distributed on each shelf as shown in *Figure 2.3a*. The total weight of the plants, pots, and earth on each shelf is $W = 100$ lb, which is assumed to be uniformly distributed over the shelf area (*Figure 2.3b*).

Required: Determine the force in each segment of each cord.

Solution: Assuming that weight W is evenly distributed on each shelf, that the cords are symmetrically placed, and that the shelves are level, then the four cords share equally in supporting each shelf.

Step 1. A FBD of the entire system (*Figure 2.3b*) is made by taking a cut at A, replacing the physical ceiling supports with reaction forces T; the plants are replaced with total weight W on each shelf. Equilibrium in the vertical (y-) direction gives the reaction at each ceiling support:

$$\sum F_y = 0: \; 4T - 3W = 0$$

Answer: $T = \dfrac{3}{4}W = 75$ lb

Step 2. By taking a cut between levels A and B, the same equation is used to show that each cord segment AB carries tensile load $T_{AB} = 75$ lb. The FBD in 2D is shown in *Figure 2.3c*.

Step 3. By taking a cut through the cords below the top shelf (*Figure 2.3d*), equilibrium gives the tension in each cord between B and C:

$$\sum F_y = 0: \; 4T_{BC} - 2W = 0$$

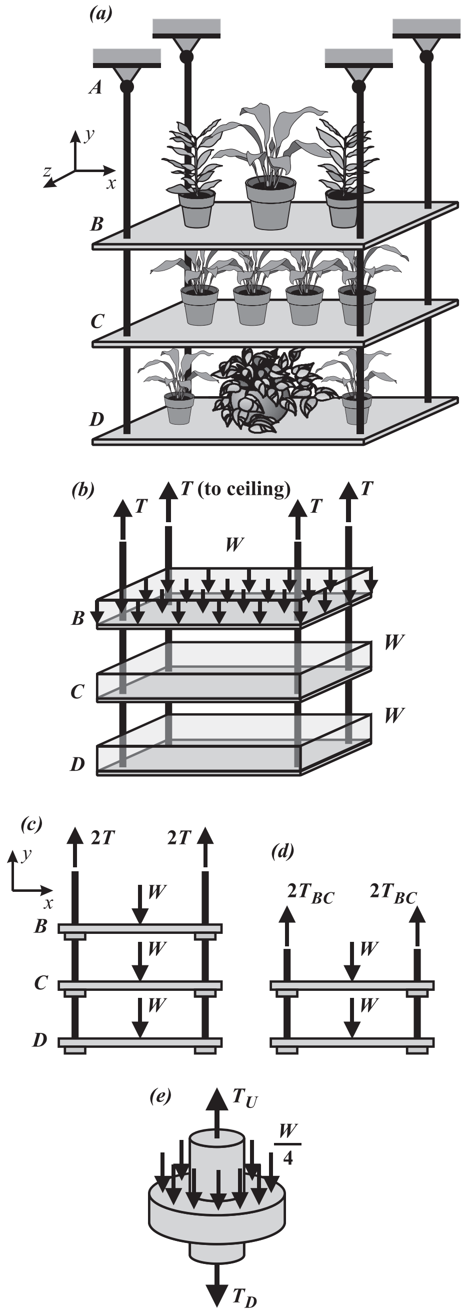

Figure 2.3. (a) Three hanging shelves. **(b)** 3D FBD of entire system. **(c)** 2D FBD of entire system. **(d)** 2D FBD cut at middle cords. **(e)** FBD of connector.

Answer: $T_{BC} = \frac{2}{4}W = 50$ lb

Step 4. The reader should verify by cutting the cords between C and D that the tension in each of the lowest cords is $T_{CD} = 25$ lb.

A connector is necessary to transfer the load from each shelf to the cord (*Figure 2.3e*). Each connector supports a load $W/4$ applied by the shelf, and the connector system must be strong enough to transfer that load to the cord. The tension in the cord above the connector T_U equals its share of the weight of the shelf $W/4$ and the tension in the cord below the connector T_D.

Example 2.2 Tower Crane – Method of Joints

Given: The tower crane shown in *Figure 2.4* consists of tower DCE fixed at the ground, and two jibs AC and CB. The jibs are supported by tie bars AD and DB, and are assumed to be attached to the tower by pinned connections. The counterweight W_C weighs 390 kips and the crane has a lifting capacity of $W_{max} = 250$ kips. Neglect the weight of the crane itself.

Required: Determine (a) the reactions at the base of the tower when the crane is lifting its capacity, (b) the axial forces in tie bars AD and DB, and jibs AC and CB, and (c) the internal forces and moment in the tower at point F, 40 ft below joint D.

Solution: *Step 1. Reactions*. The FBD of the entire crane lifting load W is shown in *Figure 2.4b*. Equilibrium requires:

$$\sum F_y = 0: \; R - W_C - W = 0$$

$$\sum M_z = 0:$$

$$M + (W_C \times a) - (W \times b) = 0$$

In the moment calculation, moments that cause counterclockwise rotation are taken as positive.

Substituting the values of a, b, W_C, and $W = W_{max}$ into the above equations gives:

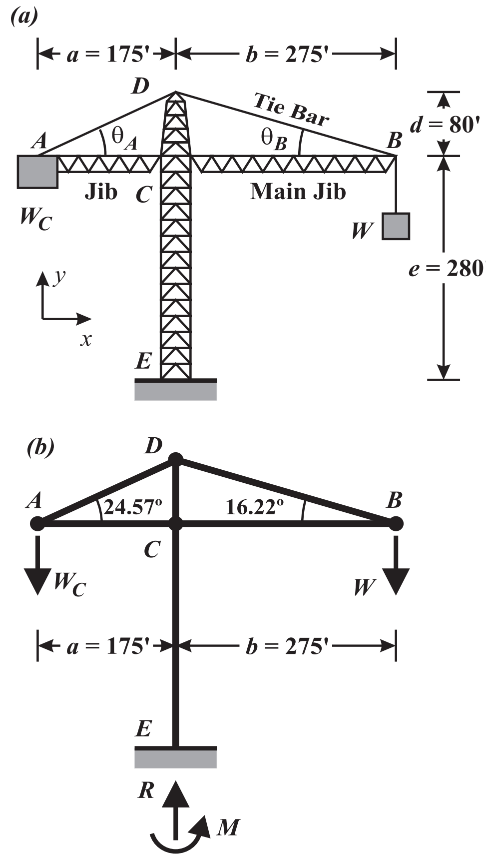

Figure 2.4. **(a)** Tower crane lifting load W with the counterweight W_C at its maximum distance, a. **(b)** FBD of entire crane.

Answer: $\underline{R = 640 \text{ kips}}$

Answer: $\underline{M = 500 \text{ kip-ft}}$

Note that the calculated moment is relatively small. If the base of the crane is 25 ft wide, then the forces of the equivalent couple are 20.0 kips (25×20=500), which is small when compared with $R = 640$ kips. For any general load W, the counterweight W_C is moved along AC to balance the moment caused by the load. This action minimizes the moment at the base, reducing the tendency for the tower to overturn. Ideally, the moment at the base is $M = 0$.

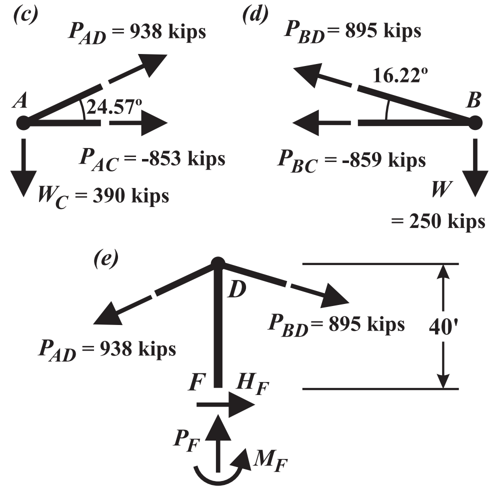

Figure 2.4. (c) FBD of joint *A*. **(d)** FBD of joint *B*. **(e)** FBD of *DF*; point *F* is 40 ft below point *D*.

Step 2. Forces in members AC, *AD*, *BD* and *BC* are solved using the ***Method of Joints*** by isolating joints *A* and *B*, and considering their FBDs (*Figures 2.4c* and *d*). The forces in the tie bars and jibs are assumed to be in *tension*, so are drawn *acting away* from the joint. If a calculation results in a negative force, then the force actually acts opposite that drawn; i.e., the force is *compressive*.

From the geometry, angles θ_A and θ_B are found:

$$\tan\theta_A = \frac{80 \text{ ft}}{175 \text{ ft}} = 0.4571 \Rightarrow \theta_A = \tan^{-1}(0.4571) = 24.57°$$

$$\tan\theta_B = \frac{80 \text{ ft}}{275 \text{ ft}} = 0.2909 \Rightarrow \theta_B = \tan^{-1}(0.2909) = 16.22°$$

Applying force equilibrium at joint *A*:

$$\sum F_y = 0: \; -W_C + P_{AD}\sin\theta_A = 0$$

Answer: $P_{AD} = \dfrac{390 \text{ kips}}{\sin(24.57°)} = 937{,}940 \text{ lb} \Rightarrow \underline{P_{AD} = 938 \text{ kips}}$

$$\sum F_x = 0: \; P_{AC} + P_{AD}\cos\theta_A = 0$$

Answer: $P_{AC} = -(937.9 \text{ kips})\cos(24.57°) \Rightarrow \underline{P_{AC} = -853 \text{ kips}}$

The value of the force in jib *AC*, P_{AC}, is *negative*, meaning that the force is *compressive* (the jib pushes against joint *A*; if the jib was not there, the counterweight would swing downward). Force P_{AD} is *positive*, meaning the force acts in the direction drawn; the tie bar force is *tensile*.

Step 3. Applying equilibrium at joint *B*:

$$\sum F_y = 0: \ -W + P_{BD}\sin\theta_B = 0$$

Answer: $P_{BD} = \dfrac{250 \text{ kips}}{\sin(16.22°)} = 895.0 \text{ kips} \Rightarrow \underline{P_{BD} = 895 \text{ kips}}$

$$\sum F_x = 0: \ -P_{BC} - P_{BD}\cos\theta_B = 0$$

Answer: $P_{BC} = -(895.0 \text{ kips})\cos(16.22°) \text{ kips} \Rightarrow \underline{P_{BC} = -859 \text{ kips}}$

At point *F*, 40 ft below joint *D* on member *DC* (*Figure 2.4e*), equilibrium requires that:

$$\sum F_y = 0: \ -P_{AD}\sin\theta_A - P_{BD}\sin\theta_B + P_F = 0$$

Answer: $P_F = (938)\sin(24.57°) + (895)\sin(16.22°) \Rightarrow \underline{P_F = 640 \text{ kips}}$

$$\sum F_x = 0: \ -P_{AD}\cos\theta_A + P_{BD}\cos\theta_B + H_F = 0$$

Answer: $H_F = (938)\cos(24.57°) - (895)\cos(16.22°) \Rightarrow \underline{H_F = -6.3 \text{ kips}}$

$$\sum M_{z,F} = 0: \ (P_{AD}\cos\theta_A)(DF) - P_{BD}(\cos\theta_B)(DF) + M_F = 0$$

Answer: $\underline{M_F = 252 \text{ kip-ft}}$

Note that $P_F = 640$ kips is the vertical reaction force at the base. The downward forces of weights W and W_C are carried up through the tie bars to joint *D*, and the tower carries the vertical load to ground. The negative result for H_F indicates that it acts in the opposite direction drawn.

Example based on the K10000 tower crane by Kroll Giant Towercranes, as cited at: http://www.towercrane.com/ Accessed May 2008. Values are approximate.

Example 2.3 Truss System – Method of Joints and Method of Sections

Background: *Trusses* are used in such applications as cranes, railway bridges, supermarket roofs, ships, aircraft, and space structures. Axial members are pinned together to form a beam-like structure. Trusses are a very effective means of spanning large distances.

Given: A simple truss bridge is shown in *Figure 2.5a*. Two plane trusses are constructed of 15 ft long axial members assembled with pins into equilateral triangles. Crossbeams connected to the trusses at the pin (node) locations maintain the spacing between the trusses, while diagonal bracing keeps them from moving laterally with respect to each other. The lower crossbeams support a roadway (deck) 20 ft wide. For the design, the roadway is to carry a uniformly distributed load of 80 lb per square foot (psf). Neglect the weight of the bridge itself.

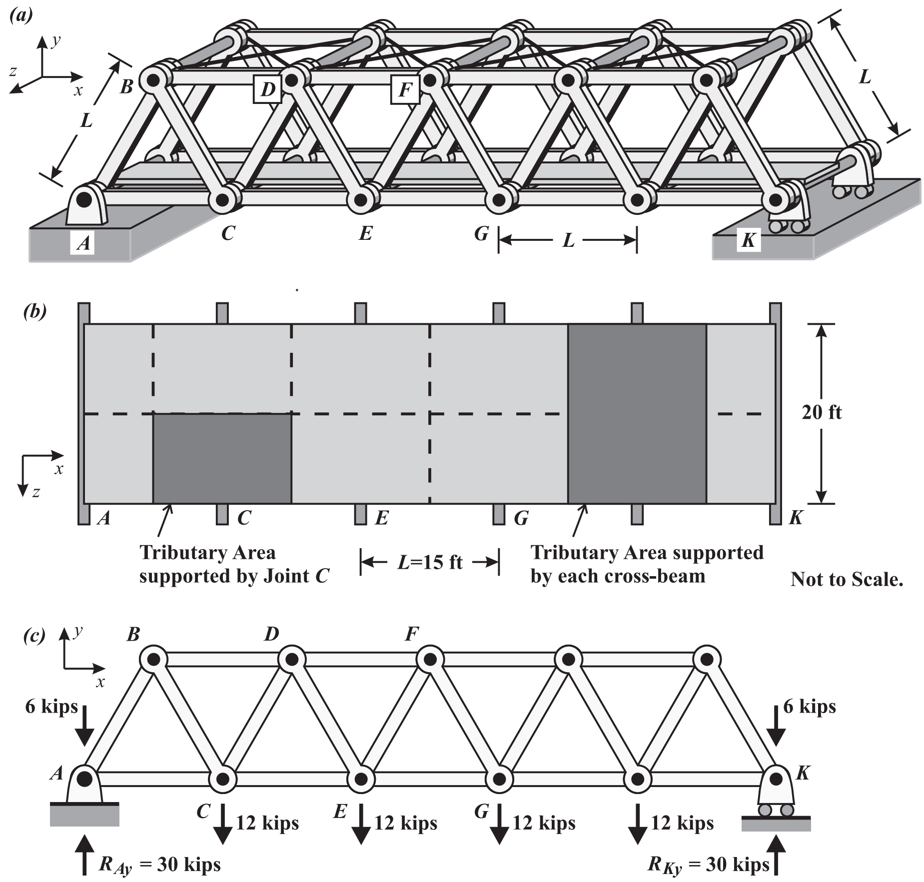

Figure 2.5. (a) A truss-bridge. The roadway rests on crossbeams that are connected to the trusses at the lower joints (e.g., joints *A, C, E, G, K*, etc.). **(b)** Top view of roadway supported by the lower crossbeams, showing *tributary area* of each crossbeam and of each pin. **(c)** 2D view of front truss.

Required: Considering only the roadway load, determine the forces (a) in members *AB* and *AC* using the ***Method of Joints*** and (b) in *DF*, *EF* and *EG* using the ***Method of Sections***.

Solution: *Step 1. Load path.* Each lower crossbeam supports a deck area of 15×20 ft – a ***tributary area*** of 300 ft^2 (*Figure 2.5b*). The force on each crossbeam is

$$(300\ \text{ft}^2)(80\ \text{psf}) = 24{,}000\ \text{lb} = 24.0\ \text{kips}$$

Each end of the crossbeam is supported by a truss. Since the load is uniformly distributed on the roadway, and the geometry is symmetric about the center of the roadway, both trusses carry the same load; only one truss needs to be analyzed.

The force supported at any lower pin is half the value of the load on each crossbeam, or 12,000 lb = 12.0 kips. At the supports, the tributary area on the crossbeams is half the standard tributary area, so the downward force at each support is 6.0 kips (*Figure 2.5c*).

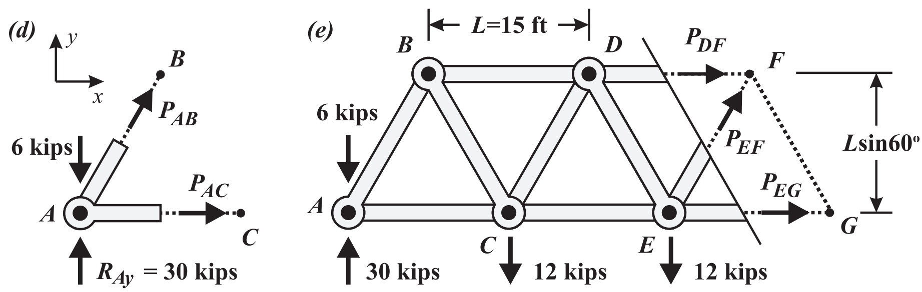

Figure 2.5. (d) FBD of joint *A*. **(e)** FBD of truss cut through members *DF*, *EF*, and *EG* .

Step 2. Reactions. From equilibrium and symmetry considerations, the vertical reaction loads are:

$$R_{Ay} = R_{Ky} = 30{,}000 \text{ lb} = 30.0 \text{ kips}$$

Because one end of the truss is supported by a roller and no horizontal loads are applied to the truss, the horizontal reactions are $R_{Ax} = R_{Kx} = 0$.

Step 3. Forces in members AB and AC (*Figure 2.5d*). Applying the *method of joints* at joint *A*, the forces in members *AB* and *AC* are determined:

$$\sum F_y = 0: \quad 30 \text{ kips} + P_{AB}\sin(60°) - 6 \text{ kips} = 0$$

$$\textit{Answer}: \; P_{AB} = \frac{(6-30 \text{ kips})}{\sin(60°)} \Rightarrow \underline{P_{AB} = -27.7 \text{ kips}}$$

$$\sum F_x = 0: \quad P_{AC} + P_{AB}\cos(60°) = 0$$

$$\textit{Answer}: \; P_{AC} = -\frac{1}{2}P_{AB} \Rightarrow \underline{P_{AC} = 13.86 \text{ kips}}$$

Since P_{AB} is negative, member *AB* is in compression.

Step 4. Forces in DF, *EF*, and *EG*, (*Figure 2.5e*). The *method of joints* can be used to solve for the forces in all of the members, one joint at a time. However, using the *method of sections*, the force in any inner member can be determined directly. For example, take a cut through members *DF*, *EF* and *EG*, as shown in *Figure 2.5e*.

Considering moment equilibrium about joint *E* to eliminate forces P_{EF} and P_{EG} :

$$\sum M_{z,E} = 0: \quad (6-30 \text{ kips})(30 \text{ ft}) + (12 \text{ kips})(15 \text{ ft}) - P_{DF}(15 \text{ ft})\sin 60° = 0$$

$$\textit{Answer}: \; P_{DF} = \frac{-540 \text{ kips-ft}}{(15 \text{ ft})\sin 60°} \Rightarrow \underline{P_{DF} = -41.6 \text{ kips (compression)}}$$

To solve for P_{EG}, take moments about joint *F*, eliminating P_{EF} and P_{DF} from the calculation. Note that the horizontal distance from point *A* to point *F* is $2.5L = 37.5$ ft.

$$\sum M_{z,\,F} = 0\text{: } (6-30)(37.5) + (12)(22.5) + (12)(7.5) + P_{EG}(15)\sin 60° = 0$$

$$\textit{Answer}\text{: } P_{EG} = -\frac{-540 \text{ kip-ft}}{(15 \text{ ft})\sin 60°} \Rightarrow \underline{P_{EG} = 41.6 \text{ kips (tension)}}$$

Vertical equilibrium requires:

$$\sum F_y = 0\text{: } \quad (30-6-12-12) + (P_{EF}\ \sin 60°) = 0$$

$$\textit{Answer}\text{: } \underline{P_{EF} = 0 \text{ kips}}$$

In general, the force in diagonal member *EF* is not zero; it is zero here because the shear force goes to zero at the center of a symmetrically loaded simply-supported truss (beam).

As a check, consider horizontal equilibrium in *Figure 2.5e*:

$$\sum F_x = 0\text{: } \quad P_{EG} + P_{DF} + P_{EF}\cos 60° = -41.6 + 41.6 + 0 = 0 \quad \textbf{OK}$$

2.2 Torsion Members

A ***torsion member*** is a component that transmits torque T (*Figure 2.6*). The torque twists the member about its axis, which passes through the *centroid* (center of area) of its cross-section.

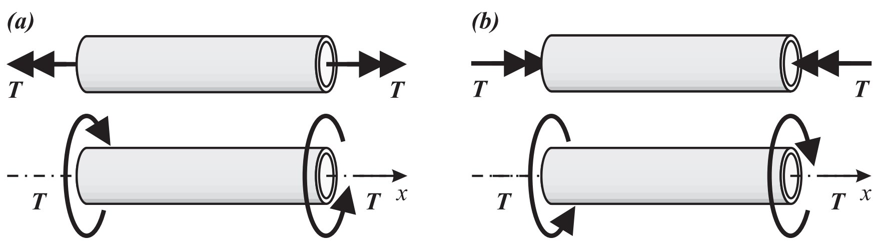

Figure 2.6. Torsion members. **(a)** A positive torque represented alternatively by a double-headed vector or a curved arrow. A torque within a torsion member is termed *positive* if it points in the same direction as the outward-pointing normal vector of the cross-section on which it acts (e.g., on a positive face in a positive direction or on a negative face in a negative direction). If torque is consistently drawn positive, a negative value indicates that it physically acts as shown in **(b)**.

Example 2.4 Drive Shaft in a Machine Shop

Given: The individual machines of classical machine shops were powered by belts driven by drive shafts. An example is shown in *Figure 2.7*, in which three machines, *B*, *C*, and *D*, draw torque from the main shaft according to *Table 2.1*. Bearing *E* is assumed to be frictionless, and therefore draws no torque.

Required: (a) Determine the torque anywhere along the drive shaft and (b) draw the torque diagram, $T(x)$ vs. x.

Table 2.1. Torque drawn by each machine.

Machine	Torque (lb-ft)
B	15
C	30
D	20

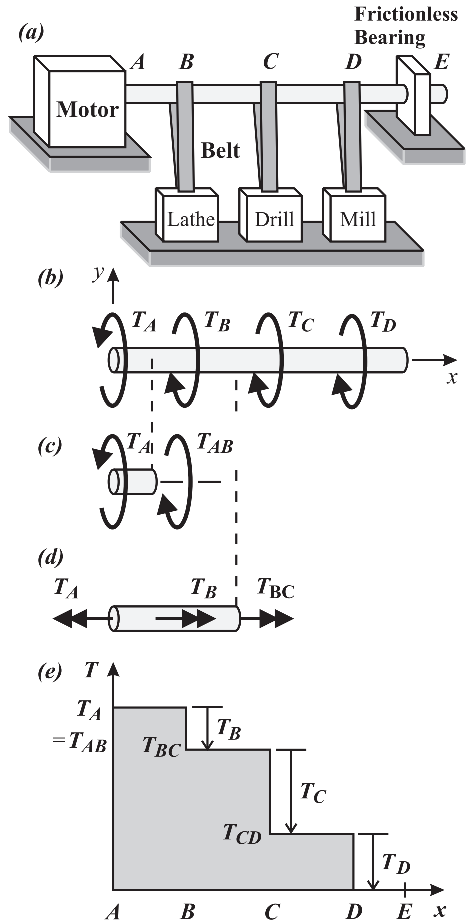

Figure 2.7. (a) Rotating shaft powering three machines. **(b)** FBD of the entire shaft. **(c)** FBD of shaft cut between points *A* and *B*. Internal torque T_{AB} is drawn in its positive sense – counterclockwise about the positive *x*-axis. **(d)** FBD of shaft cut between points *B* and *C*; torques represented with double-headed vectors. **(e)** Torque diagram.

Solution: *Step 1.* The FBD of the entire drive shaft is shown in *Figure 2.7b*. Torque T_A is the input torque. From equilibrium, the sum of the torques about the *x*-axis must be zero:

$$\sum T_x = 0: \quad -T_A + T_B + T_C + T_D = 0$$

$$\Rightarrow T_A = (15 + 30 + 20)\text{lb-ft} = 65 \text{ lb-ft}$$

Step 2. The internal torque supported at any cross-section is found by taking a cut at that section, and a FBD of the remaining structure is considered. The torque carried inside the shaft between *A* and *B*, T_{AB}, is found by taking a cut between *A* and *B* (*Figure 2.7c*) and applying equilibrium to the external and internal torques.

$$\sum T_x = 0: \quad -T_A + T_{AB} = 0$$

Answer: $\underline{T_{AB} = T_A = 65 \text{ lb-ft}}$

Step 3. Likewise, the internal torque between *B* and *C* (*Figure 2.7d*) is:

$$\sum T_x = 0: \quad -T_A + T_B + T_{BC} = 0$$

$$\Rightarrow T_{BC} = (65 - 15) \text{ lb-ft}$$

Answer: $\underline{T_{BC} = 50 \text{ lb-ft}}$

Step 4. Verify for yourself that the torque in segment *CD* is $T_{CD} = 20$ lb-ft.

Note that cuts to determine the torque in a torsion member should never be taken at the point of application of a point torque; always take cuts between the point loads.

Step 5. The torque diagram in *Figure 2.7e* is used to display the internal torque carried by the shaft. The torque diagram is analogous to the shear force and moment diagrams for beams.

Example 2.5 Classic Lug Wrench

Given: A lug nut is tightened by applying a downward force F on the lug wrench's right arm and an upward force F on the wrench's left arm (*Figure 2.8a*). Linear motion is converted into angular motion; force is converted into torque. The forces are assumed to be of equal magnitude.

Required: Determine the magnitude of the torque applied to the lug nut.

Solution: The torque, or *couple*, applied to the wrench stem at point A is:

$$T_A = 2(F)\left(\frac{d}{2}\right) = Fd$$

and is clockwise with respect to the $+x$-axis. The reaction torque T_B applied by the lug nut against the wrench's stem is shown in *Figure 2.8b*. The torque applied to the lug nut by the stem is equal and opposite to the reaction torque T_B.

The torque applied by the wrench to the nut is the same as that applied by the user to the wrench. The magnitude of this torque is:

Answer: $\underline{|T_{nut}| = Fd}$

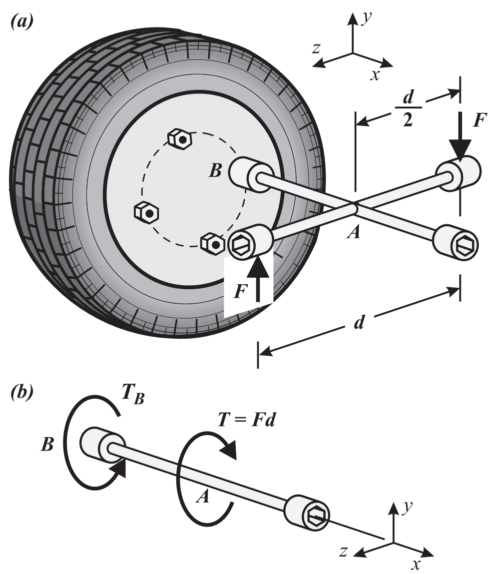

Figure 2.8. (a) Tightening a lug nut with a classic lug wrench. **(b)** Torque $T = Fd$ is applied at point A. The curved arrow at point A represents the direction that the applied torque physically acts, so its value is written as positive.

Note that since the applied forces are equal and opposite, there is no shear force acting in the lug wrench stem; the two forces form a *couple*.

2.3 Beams

Beams are components that support loads transverse to their main structural axis. Examples include aircraft wings, floor, and ceiling joists in buildings, bridges, atomic force microscopes, robotic arms in space structures, tree branches, etc. (*Figure 2.9*). The internal loads in beams are ***bending moments*** and ***shear forces*** (*Figure 2.10*).

In general, the internal bending moment *M* and shear force *V* vary with distance *x* along the beam. In *Figure 2.10b*, they are drawn in their ***positive senses*** as defined by the convention of this text, and described in the following paragraphs.

The internal bending moment is *positive* if it causes *compression* on the top of the beam; the moment is *negative* if it causes compression at the bottom of the beam.

The shear force is *positive* if it acts on a *positive face* in a *positive direction*, or on a *negative face* in a *negative direction*. Otherwise, the shear force is negative.

Figure 2.10b is a FBD of a length of the beam exposing a cross-section that faces in the +*x*-direction – a *positive face*. At the cut, a *positive moment M* is drawn acting about the +*z*-axis, out of the paper (check this with the right-hand rule), and a *positive shear force V* is drawn acting in the +*y*-direction. Drawn in their *positive senses*, moment and shear force both act on a *positive face* in a *positive direction*.

Figure 2.10c is the complementary FBD of *Figure 2.10b*. The FBD of *L–x* exposes a cross-section that faces in the *negative x*-direction. Drawn in their *positive senses*, moment and shear force both act on a *negative face* in a *negative direction*: the moment about the –*z*-axis and the shear force in the –*y*-direction.

Calculations that determine the sign of the internal moment and shear force thus determine the directions in which they act. Internal axial forces and torques follow the same convention.

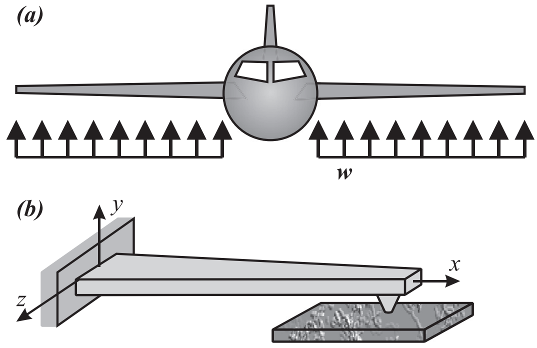

Figure 2.9. (a) Airplane wings act as beams loaded by air pressure to keep the plane aloft. **(b)** An atomic force microscope is a cantilever beam (built-in at one end, free at the other) loaded at its tip.

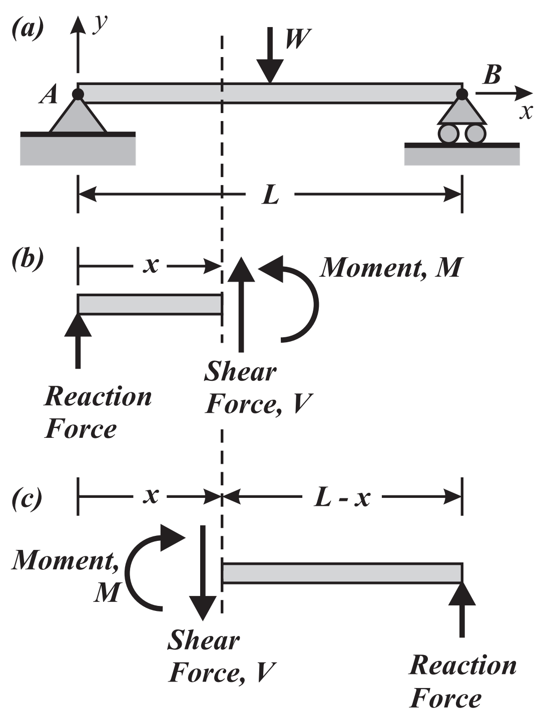

Figure 2.10. (a) A simply-supported beam (supported by a pin and a roller) loaded by central force *W*. **(b)** A FBD of length *x* of the beam, and **(c)** a FBD of complementary length *L–x*. Internal moment *M* and shear force *V* are drawn in their *positive senses* per the convention of this text: **(b)** positive face–positive direction and **(c)** negative face–negative direction.

Example 2.6 Park Bench: Modeled as a Simply-Supported Beam under a Point Load

Given: A person sits in the middle of a park bench (*Figure 2.11*). The slats of the bench are supported at each end by a set of legs. The bench is modeled as a beam supported by a *pin* at the left end and a *roller* at the right end, with a point load applied at the center (*Figure 2.11b*). Pinned supports allow rotation and cannot support or resist a moment. A beam with pinned supports (e.g., a pin and a roller) is called a *simply-supported beam*.

Required: (a) Determine the *shear force* and the *bending moment* as functions of x along the length of the beam. (b) Draw the *shear force* and *bending moment diagrams*.

Solution: *Step 1.* The FBD of the entire beam is shown in *Figure 2.11c*. Since the loading and geometry are both *symmetric*, then the vertical reactions are equal:

$$R_{Ay} = R_{By} = R = \frac{W}{2}$$

Since there is no load applied horizontally, the horizontal reaction at point A is zero.

Step 2. Shear force and *bending moment* for $0 < x < L/2$.

The shear force and bending moment at any section D to the left of the load are found from the FBD in *Figure 2.11d*. From equilibrium of segment AD (taking moments about point D):

$$\sum F_y = 0: \ \frac{W}{2} + V_D = 0$$

$$\Rightarrow V_D = -\frac{W}{2}$$

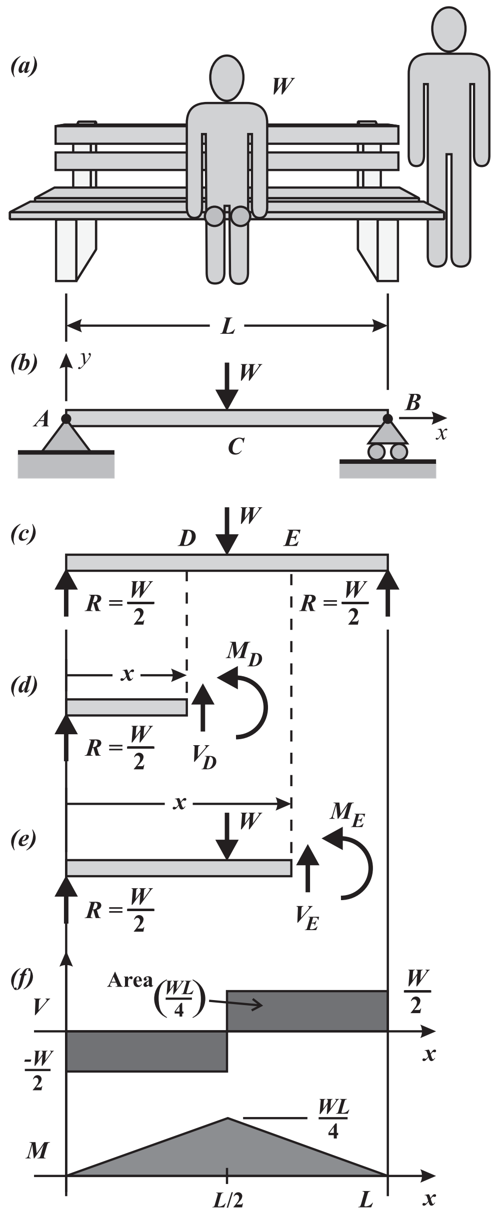

Figure 2.11. (a) A person sitting on a park bench. **(b)** The system modeled as a simply-supported beam under a central point load *W*. **(c)** FBD of entire beam. **(d)** FBD for $x < L/2$. **(e)** FBD for $x > L/2$. **(f)** Shear force diagram $V(x)$ and bending moment diagram $M(x)$.

$$\sum M_{z,\,D} = 0: \quad -\frac{W}{2}x + M_D = 0$$

$$\Rightarrow M_D = \frac{W}{2}x$$

If equilibrium of the right-hand FBD (a FBD from point D to point B) is considered, then the same results are obtained. Check this statement.

Step 3. Shear force and *bending moment* for $L/2 < x < L$.

The *shear force* and *bending moment* on any cross-section E to the right of the load are found from the FBD in *Figure 2.11e*. From equilibrium of segment AE:

$$\sum F_y = 0: \quad \frac{W}{2} - W + V_E = 0$$

$$\Rightarrow V_E = \frac{W}{2}$$

$$\sum M_{z,\,E} = 0: \quad -\frac{W}{2}x + W\left(x - \frac{L}{2}\right) + M_E = 0$$

$$\Rightarrow M_E = \frac{W}{2}(L - x)$$

In summary:

Answer:

- For $0 < x < \frac{L}{2}$: $\quad V(x) = -\frac{W}{2}; \quad M(x) = \frac{W}{2}x$
- For $\frac{L}{2} < x < L$: $\quad V(x) = \frac{W}{2}; \quad M(x) = \frac{W}{2}(L - x)$

Note that cuts to determine the shear and moment in a beam should never be taken at the point of application of a point load (force or moment); always take cuts between point loads.

Step 4. The variations of *shear force* and *bending moment* along the beam are shown in *Figure 2.11f*. These plots are the *shear force* and *bending moment diagrams*.

A simply-supported beam under a central point load is referred to as *three-point bending*. This form of loading is often used in experiments to determine the strength of a material.

Example 2.7 Park Bench: Modeled as a Simply-Supported Beam; Uniformly Distributed Load

Given: The park bench in the previous example is now completely full (*Figure 2.12*). The beam is assumed to have the same geometric boundary conditions (simple supports). Since the beam is full, the load is modeled as a *uniformly distributed load* (force per unit length). The distributed load is:

$$w = \frac{nW}{L}$$

where n is the number of people on the bench, W is the weight of each person (assumed to be the same), and L is the distance between supports. The beam model is shown in *Figure 2.12b.*

Required: (a) Determine the *shear force* and the *bending moment* along the length of the beam. (b) Draw the shear force and bending moment diagrams.

Solution: *Step 1.* The FBD of the entire beam is shown in *Figure 2.12c*. Since the loading and geometry are both symmetric, $R_{Ay} = R_{By} = R$. From vertical equilibrium:

$$\sum F_y = 0: \ -wL + R_{Ay} + R_{By} = 0$$

$$\Rightarrow R_{Ay} = R_{By} = R = \frac{wL}{2}$$

Step 2. The internal shear force and bending moment at any section D distance x from the origin may be found from the FBD shown in *Figure 2.12d*. From vertical equilibrium:

$$\sum F_y = 0: \ \frac{wL}{2} - wx + V_D = 0$$

$$\textit{Answer}: \ \underline{V_D = V(x) = w\left(x - \frac{L}{2}\right)}$$

Moment equilibrium about point D gives:

$$\sum M_{z,D} = 0$$

$$\left(-\frac{wL}{2}\right)x + (wx)\left(\frac{x}{2}\right) + M_D = 0$$

$$\textit{Answer}: \ \underline{M_D = M(x) = \frac{w}{2}(Lx - x^2)}$$

Note that the second term in the moment equilibrium equation is the product of the equivalent force wx due to the distributed

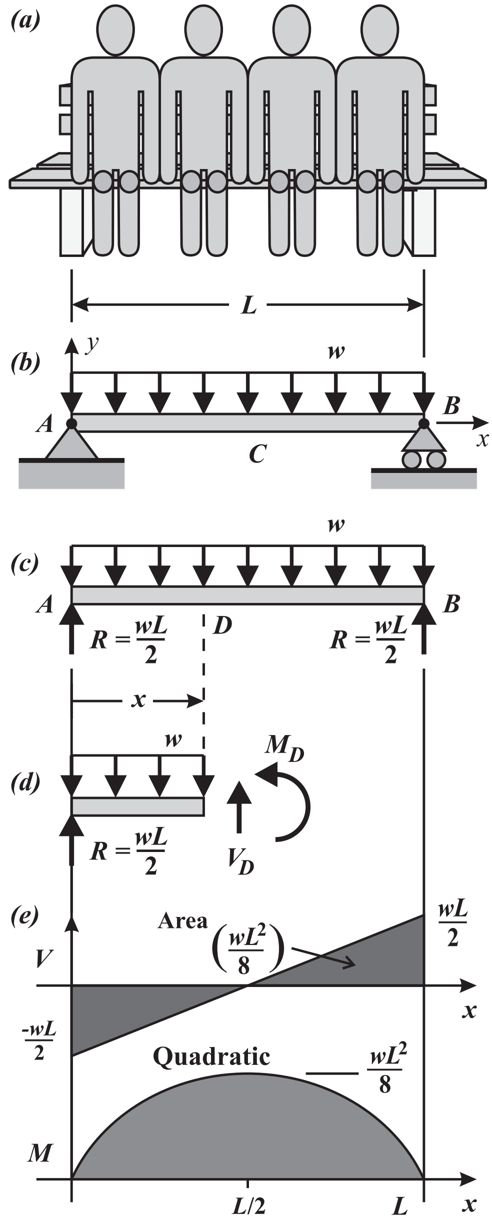

Figure 2.12. (a) A fully-loaded park bench. **(b)** The bench modeled as a simply-supported beam under uniformly distributed load w (force per unit length). **(c)** FBD of entire beam. **(d)** FBD at any distance x from the left end. **(e)** Shear force diagram $V(x)$ and bending moment diagram $M(x)$.

load and its lever arm ($x/2$) with respect to the cut at point D.

Step 3. The *shear force* and *bending moment diagrams* are shown in *Figure 2.12e*.

For this problem, the shear force is *linear*, with a maximum magnitude of $wL/2$ that occurs at each support. The bending moment is *parabolic*, with a maximum value of $wL^2/8$ at the center of the beam. The maximum bending moment occurs when the shear force is equal to zero. Because of the symmetry of the geometry and applied load, the response (shear force and moment) are symmetric.

Note that only one equation each for the shear and moment was required. In the previous example (*Example 2.6*), two equations for each load type were required. The difference is that the loading in this example is constant over the entire length of the beam. Whenever there is a sudden change in the beam's loading (e.g., at the point load in *Example 2.6*), an additional set of shear and moment equations must be considered.

Example 2.8 Atomic Force Microscope: A Cantilever with a Point Load

Background: The principal component of the atomic force microscope (AFM), used to measure the micro-geometry of surfaces and the forces in biological systems, consists of a *cantilever beam*. A cantilever beam is *built-in* (*fixed* against displacement and rotation) at one end and free at the other end (*Figure 2.13*).

Given: Force P is applied at the free end of the AFM cantilever. A representative load at this scale is $P = 20$ nN (20×10^{-9} N) and the beam length is $L = 60$ μm (60×10^{-6} m).

Required: **(a)** Determine the *shear force* and the *bending moment* along the length of the beam. **(b)** Draw the shear force and moment diagrams.

Solution: *Step 1.* The FBD of the entire beam is shown in *Figure 2.13c*.

From force equilibrium in the y-direction and moment equilibrium about the z-axis, the reaction force and reaction moment are:

$$\sum F_y = 0: \quad R + P = 0$$

$$\Rightarrow R = -P$$

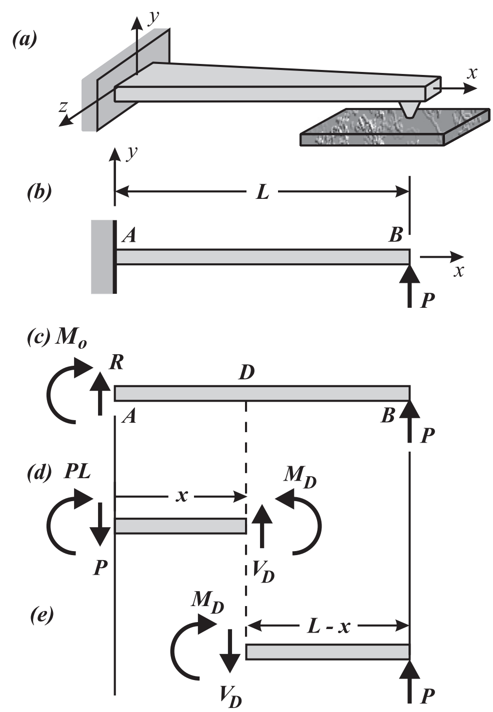

Figure 2.13. **(a)** An atomic force microscope scans a material surface. **(b)** The AFM modeled as a cantilever beam under tip load P. **(c)** FBD of entire beam. **(d)** Left-hand FBD at any distance x from the left end. **(e)** Right-hand FBD at any distance x from the left end.

$$\sum M_{z,A} = 0: \quad -M_o + PL = 0$$

$$\Rightarrow M_o = PL$$

Step 2. To investigate how the shear force and moment vary with distance x along the beam, a cut is taken at an arbitrary cross-section D. Since the load on the beam does not change over its length, only one cut needs to be taken.

Taking equilibrium of the *right-hand* FBD, segment DB (*Figure 2.13e*):

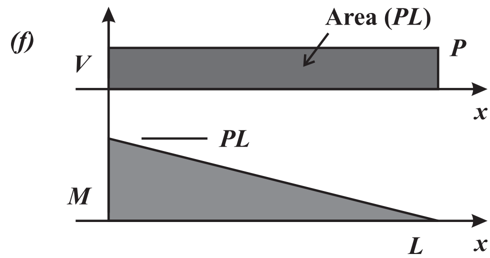

Figure 2.13. (f) shear force diagram $V(x)$, and bending moment diagram $M(x)$.

$$\sum F_y = -V_D + P = 0$$

Answer: $\underline{V_D = V(x) = P}$

The shear force is constant throughout the beam.

The moment along the beam is given by:

$$\sum M_{z,D} = -M_D + P(L-x) = 0$$

Answer: $\underline{M_D = M(x) = P(L-x)}$

Note that the moment equation checks with the expected values at each end of the beam: at the clamped end $M(x=0) = PL$ and at the free end $M(x=L) = 0$. The general FBD of *Figure 2.13e* reduces to *Figure 2.13c* for $x = 0$.

If equilibrium of the left-hand side of the beam AD was considered (*Figure 2.13d*), then the same results for the shear force and moment would be obtained.

Step 3. The shear force and moment diagrams are shown in *Figure 2.13f*. Using the given representative values, the maximum bending moment is $M_{max} = PL = (20\text{ nN})(60\ \mu\text{m}) = 1.2\times10^{-12}$ N·m. Shear force V_D is plotted as positive since it acts upward on a $+x$-face (or downward on a $-x$-face). Moment M_D is plotted positive as it causes compression on the top of the beam (it is a $+z$-moment on the $+x$-face or a $-z$-moment on a $-x$-face).

2.4 Combined Loading

Components are frequently subjected to several types of loading at the same time. Two examples of combined loading follow.

Example 2.9 Highway Sign with Wind Load

Given: Signs overhanging highways are often supported by steel masts as shown in *Figure 2.14*. The sign is $b = 4.0$ ft high and $a = 12$ ft wide, and its center is $L = 16$ ft above the road. Wind blows against the sign at $V = 100$ mph. The sign weighs $W_S = 400$ lb and the mast weighs $W_M = 1000$ lb.

Required: Determine the reactions at the base of the mast.

Solution: *Step 1. Loading*. For wind, the equivalent static pressure is (*Equation 1.1*):

$$p = 0.00256V^2$$

where p is in pounds per square foot (psf) and wind velocity V is in miles per hour (mph). For a wind speed of 100 mph, the static pressure is:

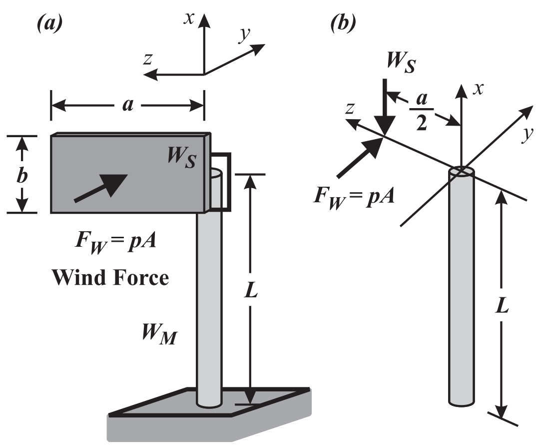

Figure 2.14. (a) Highway sign under wind load F_W. **(b)** Sketch of the wind load and weight of the sign acting at distance $a/2$ from the axis of the mast. Note that this is not a FBD.

$$p = 0.00256(100)^2 = 25.6 \text{ psf}$$

Assuming the entire static pressure acts against the sign, the wind load is:

$$F_W = pA = (25.6 \text{ psf})(48 \text{ ft}^2) = 1229 \text{ lb} = 1.23 \text{ kips}$$

and acts at the centroid of the sign. The x-axis is taken to coincide with the axis of the mast and the wind force is taken to act in the positive y-direction.

The wind force F_W and weight of the sign W_S, both act distance $a/2$ from the x-axis of the mast (*Figure 2.14b*). The wind force causes a shear force F_W and a torque $T_W = F_W[a/2]$ (clockwise about the $+x$-axis) on the mast. The weight of the sign causes an axial force W_S in the mast and a moment $M_S = W_S[a/2]$ (clockwise about the $+y$-axis) applied at the top of the mast.

Step 2. Reactions. The FBD of the entire mast is shown in *Figure 2.14c*. The reactions at the ground due to the wind load are: shear force V, bending moment M_2, and torque T. The reactions due to the weight of the sign are: part of the axial reaction force R, and bending moment M_1. The reaction due to the weight of the mast also makes up part of the axial reaction R.

Applying equilibrium to the FBD of the entire structure (*Figure 2.14c*):

$$\sum F_x = 0: \; -W_S - W_M + R = 0$$

Answer: $R = 400 + 1000 \Rightarrow \underline{R = 1.40 \text{ kips}}$

$$\sum F_y = 0: \quad F_W + V = 0$$

Answer: $V = -F_W \Rightarrow \underline{V = -1.23 \text{ kips}}$

A negative sign indicates that force V acts opposite drawn.

Taking moments about the x-axis:

$$\sum M_x = 0: \quad -F_W\left(\frac{a}{2}\right) + T = 0$$

$$\Rightarrow T = (1.23 \text{ kips})(6 \text{ ft})$$

Answer: $\underline{T = 7.38 \text{ kip-ft}}$

Taking moments about the base, first about the y-axis, and then about the z-axis:

$$\sum M_y = 0: \quad -W_S\left(\frac{a}{2}\right) + M_1 = 0$$

$$\Rightarrow M_1 = (400 \text{ lb})(6 \text{ ft})$$

Answer: $\underline{M_1 = 2.40 \text{ kip-ft}}$

$$\sum M_z = 0: \quad F_W L + M_2 = 0$$

$$\Rightarrow M_2 = -(1.23\text{kips})(16 \text{ ft})$$

Answer: $\underline{M_2 = -19.7 \text{ kip-ft}}$

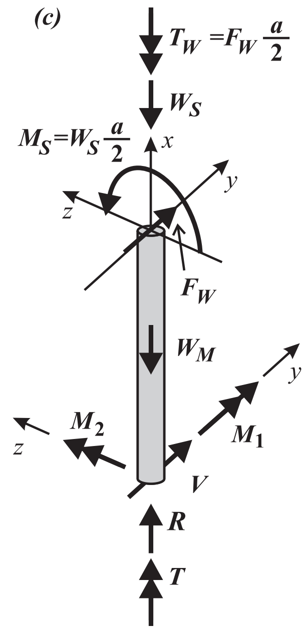

Figure 2.14. (c) FBD of the mast. Although the weight of the mast W_M is distributed along its length, its equivalent force acts at the center of gravity. At the ground, all the reactions are drawn acting in the positive direction of the appropriate axis, not necessarily in the directions that they physically act.

Moment M_2 acts opposite drawn. Note that the shear force, torque, and bending moment about the y-axis are constant along the length of the mast. The bending moment about the z-axis increases from zero at the top of the mast to its maximum at ground level.

Example 2.10 Single-Arm Lug Wrench

Given: The loading on the compact lug wrenches that come in modern automobiles is similar to the wind loading on the sign of the previous example (*Figure 2.15*). In these lug wrenches, the lug nut is tightened by applying a downward force F at point C when the wrench arm is on the right side of the stem AB.

Required: Determine the reactions at the lug nut.

Solution: To support force F, the nut–wrench interface must have the following shear force, torque, and moment reactions (*Figure 2.15c*):

Answer: $\underline{R_B = F}$

Answer: $\underline{T_B = -\dfrac{Fd}{2}}$;

Answer: $\underline{M_B = -Fl}$

A negative sign indicates that the reaction load acts opposite drawn.

In classic lug wrenches (*Example 2.5*), essentially a pure torque is applied to the lug nut; the bending moment and shear force at the nut are zero. The new lug wrenches cause extra loads on the nut–wrench interface. The bending moment tends to cause the single-arm lug wrench to slip off the nut. Additionally, the force applied to the single-arm lug wrench must be about twice that applied to the classic lug wrench to obtain the same torque. This leads too often to the unsafe practice of standing on the lug wrench to get a large enough torque to loosen or tighten the nut.

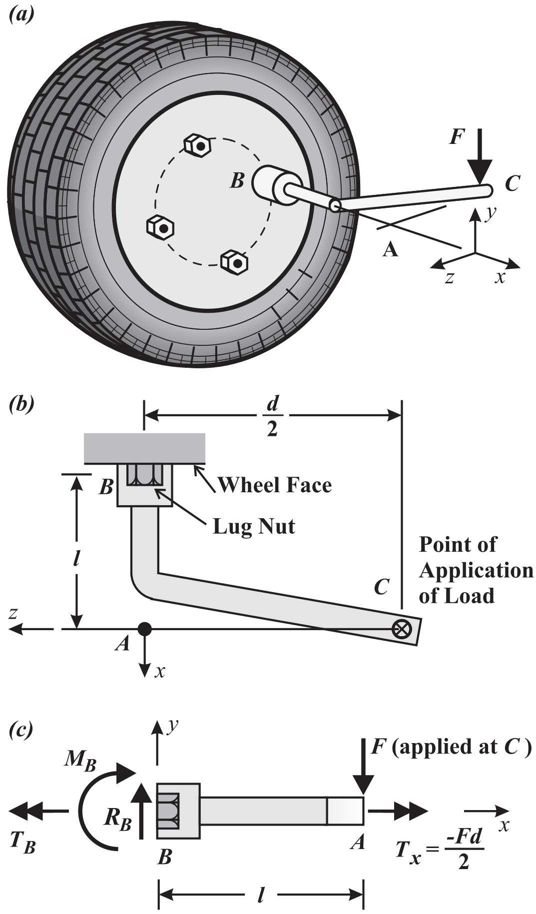

Figure 2.15. (a) The single-arm lug wrench as supplied in modern automobiles. **(b)** Top view of lug wrench. **(c)** Side view of single-arm wrench with reaction loads at lug nut. Unlike the classic lug wrench of *Example 2.5*, the nut–wrench interface must now transfer a shear force and a moment.