# Chapter 2
# Norm comparison theorems

In the previous chapter, we have discussed various versions of the Hardy–Littlewood inequality for a pair of solutions $u$ and $v$ of the conjugate $A$-harmonic equation. The purpose of this chapter is to present some norm comparison inequalities for differential forms satisfying the conjugate $A$-harmonic equations, which have been recently established in [104]. Since the proofs display a general method to obtain $L^p$-estimates, we include most of them in this chapter. Also, we always assume that $1 < p < \infty$ and $p^{-1} + q^{-1} = 1$ throughout this chapter.

## 2.1 Introduction

In the first chapter, we have introduced the following conjugate $A$-harmonic equation:

$$A(x, du) = d^{\star} v. \tag{2.1.1}$$

In this chapter, we study a more general type of the conjugate $A$-harmonic equation: the nonhomogeneous conjugate $A$-harmonic equation

$$A(x, g + du) = h + d^{\star} v \tag{2.1.2}$$

for differential forms, where $g, h \in D'(\Omega, \wedge^l)$ and $A : \Omega \times \wedge^l(\mathbf{R}^n) \to \wedge^l(\mathbf{R}^n)$ satisfies the following conditions:

$$|A(x, \xi)| \le a|\xi|^{p-1} \quad \text{and} \quad < A(x, \xi), \xi > \ \ge \ |\xi|^p \tag{2.1.3}$$

for almost every $x \in \Omega$ and all $\xi \in \wedge^l(\mathbf{R}^n)$**.** Here $a > 0$ is a constant and $1 < p < \infty$ is a fixed exponent associated with (2.1.2). Some results related to equation (2.1.2) have been obtained in [105]. In this chapter, we first present some local norm inequalities in Section 2.2. Then, we obtain some $A_r(\Omega)$-weighted estimates and the global norm inequalities for solutions of

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the nonhomogeneous conjugate $A$-harmonic equation in Sections 2.3 and 2.4, respectively. Finally, as applications of the results discussed in Sections 2.2, 2.3, and 2.4, we prove the global Sobolev–Poincaré-type imbedding inequality and derive some global $L^p$-estimates for the gradient operator $\nabla$ and the homotopy operator $T$ from the Banach space $L^s(D, \wedge^l)$ to the Sobolev space $W^{1,s}(D, \wedge^{l-1}), l = 1, 2, \ldots, n$. Some of the results presented in this chapter have nice symmetric properties. These results enrich the $L^p$-theory of differential forms and can be used to estimate the integrals of differential forms and to study the integrability of differential forms, also to explore the properties of the operators $\nabla$ and $T$.

## 2.2 The local unweighted estimates

We develop some unweighted norm inequalities for solutions of conjugate $A$-harmonic equations (2.1.1) and (2.1.2). These inequalities have rich symmetric properties and play important role in this chapter.

### 2.2.1 Basic $L^p$-inequalities

In this section, we present some local norm inequalities which will be extended into the weighted cases and the global cases in next two sections. We first discuss the following norm comparison theorem which describes the relationship between the norm of $du$ and the norm of $d^\star v$.

**Theorem 2.2.1.** *Let $u$ and $v$ be a pair of solutions to the nonhomogeneous conjugate $A$-harmonic equation* (2.1.2) *in a domain $\Omega \subset \mathbf{R}^n$. If $g \in L^p(B, \wedge^l)$ and $h \in L^q(B, \wedge^l)$, then $du \in L^p(B, \wedge^l)$ if and only if $d^\star v \in L^q(B, \wedge^l)$. Moreover, there exist constants $C_1, C_2$, independent of $u$ and $v$, such that*

$$\|d^\star v\|_{q,B}^q \le C_1(\|h\|_{q,B}^q + \|g\|_{p,B}^p + \|du\|_{p,B}^p), \tag{2.2.1}$$

$$\|du\|_{p,B}^p \le C_2(\|h\|_{q,B}^q + \|g\|_{p,B}^p + \|d^\star v\|_{q,B}^q) \tag{2.2.2}$$

*for all balls $B$ with $B \subset \Omega \subset \mathbf{R}^n$. Here $p^{-1} + q^{-1} = 1$.*

**Proof.** We only need to prove (2.2.1) and (2.2.2). From equation (2.1.2) and condition (2.1.3), we have

$$|h + d^\star v| = |A(x, g + du)| \le a|g + du|^{p-1}. \tag{2.2.3}$$

Hence, we obtain

$$|d^\star v| = |d^\star v + h - h| \le |h| + |d^\star v + h| \le |h| + a|g + du|^{p-1}. \tag{2.2.4}$$

Using the elementary inequality

$$|\sum_{i=1}^{N} t_i|^s \le N^{s-1} \sum_{i=1}^{N} |t_i|^s, \tag{2.2.5}$$

we find that

$$\begin{aligned} |d^\star v|^q &\le (|h| + a|g + du|^{p-1})^q \\ &\le 2^{q-1}(|h|^q + a^q|g + du|^{(p-1)q}) \\ &= 2^{q-1}(|h|^q + a^q|g + du|^p) \\ &\le 2^{q-1}(|h|^q + 2^{p-1}a^q(|g|^p + |du|^p)) \\ &\le C_1(|h|^q + |g|^p + |du|^p). \end{aligned} \tag{2.2.6}$$

Integrating the above inequality over $B$, we obtain

$$\|d^\star v\|_{q,B}^q \le C_1(\|h\|_{q,B}^q + \|g\|_{p,B}^p + \|du\|_{p,B}^p).$$

This completes the proof of inequality (2.2.1). From (2.1.3) and Schwartz inequality, we have

$$\begin{aligned} |g + du|^p &\le < A(x, g + du), g + du > \\ &= < h + d^\star v, g + du > \\ &\le |h + d^\star v| \cdot |g + du|. \end{aligned}$$

Therefore, we obtain

$$|g + du|^{p-1} \le |h + d^\star v| \le |h| + |d^\star v|. \tag{2.2.7}$$

Hence,

$$|g + du|^p = |g + du|^{q(p-1)} \le (|h| + |d^\star v|)^q.$$

Using the elementary inequality (2.2.5) again, we find that

$$\begin{aligned} |du|^p &= |du + g - g|^p \\ &\le 2^{p-1}(|du + g|^p + |g|^p) \\ &\le 2^{p-1}(|g|^p + (|h| + |d^\star v|)^q) \\ &\le 2^{p-1}(|g|^p + 2^{q-1}(|h|^q + |d^\star v|^q)) \\ &\le C_2(|g|^p + |h|^q + |d^\star v|^q). \end{aligned} \tag{2.2.8}$$

Integrating inequality (2.2.8) over $B$, we obtain (2.2.2). ∎

Note that (2.2.1) and (2.2.2) can be used to estimate the integrals of differential forms and to study the integrability of differential forms. From the proof of Theorem 2.2.1, the following corollary is immediate.

**Corollary 2.2.2.** *Let $u$ and $v$ be a pair of solutions to the nonhomogeneous conjugate $A$-harmonic equation* (2.1.2) *in a domain $\Omega \subset \mathbf{R}^n$. Then,*

$$\|h + d^\star v\|_{q,B}^q \le C_1 \|g + du\|_{p,B}^p,$$

$$\|g + du\|_{p,B}^p \le C_2 \|h + d^\star v\|_{q,B}^q$$

*for all balls $B$ with $B \subset \Omega \subset \mathbf{R}^n$.*

## *2.2.2 Special cases*

Applying Theorem 2.2.1 with $g = 0$ and $h = 0$, we obtain the following corollary immediately.

**Corollary 2.2.3.** *Let $u$ and $v$ be a pair of solutions to the conjugate $A$-harmonic equation* (2.1.1) *in a domain $\Omega \subset \mathbf{R}^n$. Then, $du \in L^p(B, \wedge^l)$ if and only if $d^\star v \in L^q(B, \wedge^l)$. Moreover, there exist constants $C_1, C_2$, independent of $u$ and $v$, such that*

$$C_1 \|du\|_{p,B}^p \le \|d^\star v\|_{q,B}^q \le C_2 \|du\|_{p,B}^p \tag{2.2.9}$$

*for all balls $B$ with $B \subset \Omega \subset \mathbf{R}^n$.*

**Theorem 2.2.4.** *Let $u$ and $v$ be a pair of solutions to the conjugate $A$-harmonic equation* (2.1.1) *in a domain $\Omega \subset \mathbf{R}^n$. Then, there exists a constant $C$, independent of $u$ and $v$, such that*

$$\|d^\star v\|_{q,B}^q \le C diam(B)^{-p} \|u - c\|_{p,\sigma B}^p \tag{2.2.10}$$

*for all balls $B$ with $\sigma B \subset \Omega \subset \mathbf{R}^n$. Here $c$ is any closed form and $\sigma$ is a constant with $\sigma > 1$.*

Note that (2.2.10) can be written as

$$\|d^\star v\|_{q,B} \le C diam(B)^{-p/q} \|u - c\|_{p,\sigma B}^{p/q}. \tag{2.2.11}$$

**Proof.** Since $u$ and $v$ are a pair of solutions to the conjugate $A$-harmonic equation

$$A(x, du) = d^\star v,$$

it follows that $u$ is a solution to the $A$-harmonic equation

$$d^\star A(x, du) = 0.$$

Hence, the Caccioppoli inequality is applicable to $u$. Using Theorem 2.4.1 with $\alpha = 1$ in [180] and Theorem 2.2.1 with $g = 0$ and $h = 0$, we obtain

$$\|d^\star v\|_{q,B}^q \le C\|du\|_{p,B}^p \le C diam(B)^{-p}\|u - c\|_{p,\sigma B}^p. \qquad \blacksquare$$

**Theorem 2.2.5.** *Let $u$ and $v$ be a pair of solutions to the conjugate $A$-harmonic equation* (2.1.1) *in a domain $\Omega \subset \mathbf{R}^n$. Then, there exists a constant $C$, independent of $u$ and $v$, such that*

$$\|u - u_B\|_{p,B}^p \le C diam(B)^p \|d^\star v\|_{q,\sigma B}^q \tag{2.2.12}$$

*for all balls $B$ with $\sigma B \subset \Omega \subset \mathbf{R}^n$, where $\sigma$ is a constant with $\sigma > 1$.*

**Proof.** Similar to the case in Theorem 2.2.4, using Poincaré inequality (Theorem 2.12 with $w(x) = 1$ in [201]) to $u$, and Theorem 2.2.1 with $g = 0$ and $h = 0$, we obtain

$$\|u - u_B\|_{p,B}^p \le C diam(B)^p \|du\|_{p,\sigma B}^p \le C diam(B)^p \|d^\star v\|_{q,\sigma B}^q. \qquad \blacksquare$$

Combining Theorems 2.2.4 and 2.2.5, we have the following norm comparison inequality which has nice symmetric properties. It can be used to estimate the norm of $d^\star v$ in terms of $u$ or $u - c$.

**Theorem 2.2.6.** *Let $u$ and $v$ be a pair of solutions to the conjugate $A$-harmonic equation* (2.1.1) *in a domain $\Omega \subset \mathbf{R}^n$. Then, there exist constants $C_1, C_2$, independent of $u$ and $v$, such that*

$$C_1 diam(B)^{-p}\|u - u_B\|_{p,B}^p \le \|d^\star v\|_{q,\sigma_1 B}^q \le C_2 diam(B)^{-p}\|u - c\|_{p,\sigma_2 B}^p$$

*for all balls $B$ with $\sigma_2 B \subset \Omega \subset \mathbf{R}^n$. Here $c$ is any closed form and $p, q, \sigma_1, \sigma_2$ are constants with $\sigma_2 > \sigma_1 > 1$.*

## 2.3 The local weighted estimates

Here we generalize the inequalities established in the previous section to the $A_r(\Omega)$-weighted versions. We begin with $L^s$-estimates for $d^\star v$.

### *2.3.1 $L^s$-estimates for $d^\star v$*

We first discuss $L^s$-estimates for $d^\star v$ in terms of $g$, $h$ and $du$ that appear in equation (2.1.2).

**Theorem 2.3.1.** *Let $u$ and $v$ be a pair of solutions to the nonhomogeneous conjugate $A$-harmonic equation* (2.1.2) *in a domain $\Omega \subset \mathbf{R}^n$. Assume that*

*$w \in A_r(\Omega)$ for some $r > 1$. Then, there exists a constant $C$, independent of $u$ and $v$, such that*

$$\|d^\star v\|_{s,B,w^\alpha} \le C|B|^{\alpha r/s}(\|h\|_{t,B,w^{\alpha t/s}} + \||g|^{p/q}\|_{t,B,w^{\alpha t/s}} + \||du|^{p/q}\|_{t,B,w^{\alpha t/s}}) \tag{2.3.1}$$

*for all balls $B$ with $B \subset \Omega \subset \mathbf{R}^n$. Here $\alpha$ is any positive constant with $1 > \alpha r$, $s = (1-\alpha)q$, and $t = s/(1-\alpha r) = qs/(s-\alpha q(r-1))$.*

Note that (2.3.1) can be written in the following symmetric form:

$$|B|^{-1/s}\|d^\star v\|_{s,B,w^\alpha} \le C|B|^{-1/t}(\|h\|_{t,B,w^{\alpha t/s}} + \||g|^{p/q}\|_{t,B,w^{\alpha t/s}} + \||du|^{p/q}\|_{t,B,w^{\alpha t/s}}). \tag{2.3.1}$$

**Proof.** Since $s = (1-\alpha)q < q$, using the Hölder inequality, we have

$$\begin{aligned}\|d^\star v\|_{s,B,w^\alpha} &= \left(\int_B \left(|d^\star v|w^{\alpha/s}\right)^s dx\right)^{1/s}\\ &\le \left(\int_B |d^\star v|^q dx\right)^{1/q}\left(\int_B \left(w^{\alpha/s}\right)^{qs/(q-s)} dx\right)^{(q-s)/qs}\\ &\le \|d^\star v\|_{q,B}\left(\int_B w dx\right)^{\alpha/s}.\end{aligned} \tag{2.3.2}$$

Applying the elementary inequality $|\sum_{i=1}^N t_i|^\tau \le N^{\tau-1}\sum_{i=1}^N |t_i|^\tau$ and Theorem 2.2.1, we obtain

$$\|d^\star v\|_{q,B} \le C_1\left(\|h\|_{q,B} + \|g\|_{p,B}^{p/q} + \|du\|_{p,B}^{p/q}\right). \tag{2.3.3}$$

Since $t = qs/(s-\alpha q(r-1)) > q$, using the Hölder inequality again, we find that

$$\begin{aligned}\|h\|_{q,B} &= \left(\int_B \left(|h|w^{\alpha/s}w^{-\alpha/s}\right)^q dx\right)^{1/q}\\ &\le \left(\int_B |h|^t w^{\alpha t/s} dx\right)^{1/t}\left(\int_B \left(\frac{1}{w}\right)^{\alpha q t/s(t-q)} dx\right)^{(t-q)/qt}\\ &= \|h\|_{t,B,w^{\alpha t/s}}\left(\int_B \left(\frac{1}{w}\right)^{1/(r-1)} dx\right)^{\alpha(r-1)/s}.\end{aligned} \tag{2.3.4}$$

Now, choose

$$k = \frac{sp + \alpha pt(r-1)}{s} = p + \frac{\alpha pt(r-1)}{s},$$

so that $k > p$. Once again using the Hölder inequality, we have

$$
\begin{aligned}
\|g\|_{p,B} &= \left(\int_B |g|^p w^{\alpha t/ks} w^{-\alpha t/ks} dx\right)^{1/p} \\
&\le \left(\int_B |g|^k w^{\alpha t/s} dx\right)^{1/k} \left(\int_B \left(\frac{1}{w}\right)^{\alpha pt/s(k-p)} dx\right)^{(k-p)/kp} \\
&\le \|g\|_{k,B,w^{\alpha t/s}} \left(\int_B \left(\frac{1}{w}\right)^{1/(r-1)} dx\right)^{(k-p)/kp} .
\end{aligned} \tag{2.3.5}
$$

After a simple computation, we find that

$$
\frac{k-p}{kp} = \frac{\alpha(r-1)}{s} \cdot \frac{st}{ps+\alpha pt(r-1)} = \frac{\alpha q(r-1)}{ps},
$$

and hence

$$
\|g\|_{p,B}^{p/q} \le \|g\|_{k,B,w^{\alpha t/s}}^{p/q} \cdot \left(\int_B \left(\frac{1}{w}\right)^{1/(r-1)} dx\right)^{\alpha(r-1)/s} . \tag{2.3.6}
$$

Note that

$$
\begin{aligned}
\|g\|_{k,B,w^{\alpha t/s}}^{p/q} &= \left(\int_B |g|^k w^{\alpha t/s} dx\right)^{p/kq} \\
&= \left(\int_B |g|^{(sp+\alpha pt(r-1))/s} w^{\alpha t/s} dx\right)^{ps/(pqs+\alpha pqt(r-1))} \\
&= \left(\int_B |g|^{pt/q} w^{\alpha t/s} dx\right)^{1/t} \\
&= \||g|^{p/q}\|_{t,B,w^{\alpha t/s}} .
\end{aligned} \tag{2.3.7}
$$

Combination of (2.3.6) and (2.3.7) yields

$$
\|g\|_{p,B}^{p/q} \le \||g|^{p/q}\|_{t,B,w^{\alpha t/s}} \cdot \left(\int_B \left(\frac{1}{w}\right)^{1/(r-1)} dx\right)^{\alpha(r-1)/s} . \tag{2.3.8}
$$

Using a similar method, we have

$$
\|du\|_{p,B}^{p/q} \le \||du|^{p/q}\|_{t,B,w^{\alpha t/s}} \cdot \left(\int_B \left(\frac{1}{w}\right)^{1/(r-1)} dx\right)^{\alpha(r-1)/s} . \tag{2.3.9}
$$

Combination of (2.3.2) and (2.3.3) gives

$$
\|d^\star v\|_{s,B,w^\alpha} \le C_1 \left(\|h\|_{q,B} + \|g\|_{p,B}^{p/q} + \|du\|_{p,B}^{p/q}\right) \left(\int_B w dx\right)^{\alpha/s} . \tag{2.3.10}
$$

Substituting (2.3.4), (2.3.8), and (2.3.9) into (2.3.10), we find that

$$\|d^\star v\|_{s,B,w^\alpha} \le C_1 \left(\|h\|_{t,B,w^{\alpha t/s}} + \||g|^{p/q}\|_{t,B,w^{\alpha t/s}} + \||du|^{p/q}\|_{t,B,w^{\alpha t/s}}\right) \times \left(\int_B w dx\right)^{\alpha/s} \left(\int_B \left(\frac{1}{w}\right)^{1/(r-1)} dx\right)^{\alpha(r-1)/s}. \tag{2.3.11}$$

Since $w(x) \in A_r(\Omega)$, it follows that

$$\begin{aligned} &\left(\int_B w dx\right)^{\alpha/s} \left(\int_B \left(\frac{1}{w}\right)^{1/(r-1)} dx\right)^{\alpha(r-1)/s} \\ &= |B|^{\alpha r/s} \left(\frac{1}{|B|}\int_B w dx\right)^{\alpha/s} \left(\frac{1}{|B|}\int_B \left(\frac{1}{w}\right)^{1/(r-1)} dx\right)^{\alpha(r-1)/s} \\ &\le C_2 |B|^{\alpha r/s}. \end{aligned} \tag{2.3.12}$$

Finally, using (2.3.12) and (2.3.11), we obtain

$$\begin{aligned} &\|d^\star v\|_{s,B,w^\alpha} \\ &\le C_3 |B|^{\alpha r/s}(\|h\|_{t,B,w^{\alpha t/s}} + \||g|^{p/q}\|_{t,B,w^{\alpha t/s}} + \||du|^{p/q}\|_{t,B,w^{\alpha t/s}}). \end{aligned}$$

∎

## 2.3.2 $L^s$-estimates for $du$

Similar to the proof of Theorem 2.3.1, we have the following weighted $L^s$-estimate for $du$.

**Theorem 2.3.2.** *Let $u$ and $v$ be a pair of solutions to the nonhomogeneous conjugate $A$-harmonic equation (2.1.2) in a domain $\Omega \subset \mathbf{R}^n$. Assume that $w \in A_r(\Omega)$ for some $r > 1$. Then, there exists a constant $C$, independent of $u$ and $v$, such that*

$$\begin{aligned} &\|du\|_{s,B,w^\alpha} \\ &\le C|B|^{\alpha r/s}(\|g\|_{t,B,w^{\alpha t/s}} + \||h|^{q/p}\|_{t,B,w^{\alpha t/s}} + \||d^\star v|^{q/p}\|_{t,B,w^{\alpha t/s}}) \end{aligned} \tag{2.3.13}$$

*for all balls $B$ with $B \subset \Omega \subset \mathbf{R}^n$. Here $\alpha$ is any positive constant with $1 > \alpha r$, $s = (1-\alpha)p$, and $t = s/(1-\alpha r) = ps/(s - \alpha p(r-1))$.*

It is easy to see that inequality (2.3.13) is equivalent to

$$\begin{aligned} &|B|^{-1/s}\|du\|_{s,B,w^\alpha} \\ &\le C|B|^{-1/t}(\|g\|_{t,B,w^{\alpha t/s}} + \||h|^{q/p}\|_{t,B,w^{\alpha t/s}} + \||d^\star v|^{q/p}\|_{t,B,w^{\alpha t/s}}). \end{aligned} \tag*{(2.3.13)$'$}$$

### 2.3.3 The norm comparison between $d^\star$ and $d$

**Theorem 2.3.3.** *Let $u$ and $v$ be a pair of solutions to the conjugate $A$-harmonic equation (2.1.1) in a domain $\Omega \subset \mathbf{R}^n$. Assume that $w \in A_r(\Omega)$ for some $r > 1$. Then, for all balls $B$ with $B \subset \Omega \subset \mathbf{R}^n$ and any positive constant $\alpha$ with $1 > \alpha r$, there exist constants $C_1, C_2$, independent of $u$ and $v$, such that*

$$\|d^\star v\|^q_{s,B,w^\alpha} \le C_1 |B|^{\alpha q r/s} \|du\|^p_{pt/q,B,w^{\alpha t/s}} \tag{2.3.14}$$

*for $s = (1-\alpha)q, t = s/(1-\alpha r) = qs/(s-\alpha q(r-1))$; and*

$$\|du\|^p_{s,B,w^\alpha} \le C_2 |B|^{\alpha p r/s} \|d^\star v\|^q_{qt/p,B,w^{\alpha t/s}} \tag{2.3.15}$$

*for $s = (1-\alpha)p, t = s/(1-\alpha r) = ps/(s-\alpha p(r-1))$.*

**Proof.** Applying Theorem 2.3.1 with $g = 0$ and $h = 0$, we obtain

$$\|d^\star v\|_{s,B,w^\alpha} \le C_1 |B|^{\alpha r/s} \||du|^{p/q}\|_{t,B,w^{\alpha t/s}}. \tag{2.3.16}$$

Note that

$$\||du|^{p/q}\|_{t,B,w^{\alpha t/s}} = \|du\|^{p/q}_{pt/q,B,w^{\alpha t/s}}. \tag{2.3.17}$$

Combination of (2.3.16) and (2.3.17) yields

$$\|d^\star v\|^q_{s,B,w^\alpha} \le C_1 |B|^{\alpha q r/s} \|du\|^p_{pt/q,B,w^{\alpha t/s}}.$$

Similarly, using Theorem 2.3.2 with $g = 0$ and $h = 0$, we have

$$\|du\|^p_{s,B,w^\alpha} \le C_2 |B|^{\alpha p r/s} \|d^\star v\|^q_{qt/p,B,w^{\alpha t/s}}. \qquad \blacksquare$$

**Theorem 2.3.4.** *Let $u$ and $v$ be a pair of solutions to the conjugate $A$-harmonic equation (2.1.1) in a domain $\Omega \subset \mathbf{R}^n$. Assume that $w \in A_r(\Omega)$ for some $r > 1$. Then, for all balls $B$ with $B \subset \Omega \subset \mathbf{R}^n$ and any positive constant $\alpha$ with $1 > \alpha r$, there exist constants $C_1, C_2$, independent of $u$ and $v$, such that*

$$\|u - u_B\|_{s,B,w^\alpha} \le C_1 diam(B) |B|^{\alpha r/s} \||d^\star v|^{q/p}\|_{t,\sigma B,w^{\alpha t/s}} \tag{2.3.18}$$

*for $s = (1-\alpha)p, t = s/(1-\alpha r) = ps/(s-\alpha p(r-1))$; and*

$$\|d^\star v\|_{s,B,w^\alpha} \le C_2 diam(B)^{-p/q} |B|^{\alpha r/s} \||u - c|^{p/q}\|_{t,\sigma B,w^{\alpha t/s}} \tag{2.3.19}$$

*for $s = (1-\alpha)q, t = s/(1-\alpha r) = qs/(s-\alpha q(r-1))$. Here $\sigma > 1$ is a constant.*

**Proof.** Applying the Hölder inequality with $p$ and $s = (1-\alpha)p$ and Theorem 2.2.5, we find that

$$
\begin{aligned}
\|u-u_B\|_{s,B,w^{\alpha}} &= \left(\int_B \left(|u-u_B|w^{\alpha/s}\right)^s dx\right)^{1/s} \\
&\le \left(\int_B |u-u_B|^p dx\right)^{1/p} \left(\int_B \left(w^{\alpha/s}\right)^{ps/(p-s)} dx\right)^{(p-s)/ps} \\
&\le \|u-u_B\|_{p,B} \left(\int_B w dx\right)^{\alpha/s} \\
&\le C_1 diam(B) \|d^\star v\|^{q/p}_{q,\sigma B} \left(\int_B w dx\right)^{\alpha/s}.
\end{aligned}
\tag{2.3.20}
$$

Let

$$
k=\frac{sq+\alpha qt(r-1)}{s}=q+\frac{\alpha qt(r-1)}{s},
$$

so that $k>q$. Using the Hölder inequality again, it follows that

$$
\begin{aligned}
&\|d^\star v\|_{q,\sigma B} \\
&\quad = \left(\int_{\sigma B} |d^\star v|^q w^{\alpha t/ks} w^{-\alpha t/ks} dx\right)^{1/q} \\
&\quad \le \left(\int_{\sigma B} |d^\star v|^k w^{\alpha t/s} dx\right)^{1/k} \left(\int_{\sigma B} \left(\tfrac{1}{w}\right)^{\alpha qt/s(k-q)} dx\right)^{(k-q)/kq} \\
&\quad \le \|d^\star v\|_{k,\sigma B, w^{\alpha t/s}} \left(\int_B \left(\tfrac{1}{w}\right)^{1/(r-1)} dx\right)^{(k-q)/kq}.
\end{aligned}
\tag{2.3.21}
$$

Note that

$$
\frac{k-q}{kq}=\frac{\alpha(r-1)}{s}\cdot\frac{st}{q+\alpha qt(r-1)}=\frac{\alpha p(r-1)}{qs}.
$$

Thus,

$$
\|d^\star v\|^{q/p}_{q,\sigma B} \le \|d^\star v\|^{q/p}_{k,\sigma B, w^{\alpha t/s}} \cdot \left(\int_{\sigma B}\left(\frac{1}{w}\right)^{1/(r-1)} dx\right)^{\alpha(r-1)/s}. \tag{2.3.22}
$$

Also, we have

$$
\|d^\star v\|^{q/p}_{k,\sigma B, w^{\alpha t/s}} = \||d^\star v|^{q/p}\|_{t,\sigma B, w^{\alpha t/s}}. \tag{2.3.23}
$$

Combination of (2.3.20), (2.3.22), and (2.3.23) gives

$$
\begin{aligned}
&\|u-u_B\|_{s,B,w^{\alpha}} \\
&\quad \le C_1 diam(B) \||d^\star v|^{q/p}\|_{t,\sigma B, w^{\alpha t/s}} \cdot \left(\int_B w dx\right)^{\alpha/s} \\
&\qquad \times \left(\int_{\sigma B} \left(\tfrac{1}{w}\right)^{1/(r-1)} dx\right)^{\alpha(r-1)/s}.
\end{aligned}
\tag{2.3.24}
$$

Now, using the condition $w(x)\in A_r(\Omega)$, we find that

$$\begin{aligned}&\left(\int_B wdx\right)^{\alpha/s}\left(\int_{\sigma B}\left(\frac{1}{w}\right)^{1/(r-1)}dx\right)^{\alpha(r-1)/s}\\&\le|\sigma B|^{\alpha r/s}\left(\frac{1}{|\sigma B|}\int_{\sigma B} wdx\right)^{\alpha/s}\left(\frac{1}{|\sigma B|}\int_{\sigma B}\left(\frac{1}{w}\right)^{1/(r-1)}dx\right)^{\alpha(r-1)/s}\\&\le C_2|B|^{\alpha r/s}.\end{aligned} \tag{2.3.25}$$

Finally, substituting (2.3.25) into (2.3.24) we obtain

$$\|u-u_B\|_{s,B,w^\alpha}\le C_3 diam(B)|B|^{\alpha r/s}\||d^\star v|^{q/p}\|_{t,\sigma B,w^{\alpha t/s}}.$$

This completes the proof of (2.3.18). The proof of (2.3.19) is similar to that of (2.3.18). ∎

**Note.** In this section, we have only proved the $A_r(\Omega)$-weighted norm comparison theorems. Using a similar technique, we can obtain the local comparison theorems with other kinds of weights discussed in Section 1.4, including two-weight, etc.

## 2.4 The global estimates

We have discussed the weighted local estimates for $\|d^\star v\|_{s,B,w^\alpha}$ and $\|du\|_{s,B,w^\alpha}$ in the previous section. In this section, we prove some global norm comparison theorems using the local results.

### *2.4.1 Global estimates for $d^\star v$*

Using the local weighted estimate developed above, we prove the following global estimate for $d^\star v$.

**Theorem 2.4.1.** *Let $u$ and $v$ be a pair of solutions to the nonhomogeneous conjugate $A$-harmonic equation* (2.1.2) *in a bounded domain $\Omega\subset\mathbf{R}^n$. Assume that $w\in A_r(\Omega)$ for some $r>1$. Then, there exists a constant $C$, independent of $u$ and $v$, such that*

$$\|d^\star v\|_{s,\Omega,w^\alpha}\le C(\|h\|_{t,\Omega,w^{\alpha t/s}}+\||g|^{p/q}\|_{t,\Omega,w^{\alpha t/s}}+\||du|^{p/q}\|_{t,\Omega,w^{\alpha t/s}}), \tag{2.4.1}$$

*where $\alpha$ is any positive constant with $1>\alpha r$, $1\le s=(1-\alpha)q$, and $t=s/(1-\alpha r)=qs/(s-\alpha q(r-1))$.*

**Proof.** Applying Theorem 2.3.1 and the covering lemma (Theorem 1.5.3), we have

$$
\begin{aligned}
&\|d^\star v\|_{s,\Omega,w^\alpha}\\
&\quad= \left(\int_\Omega |d^\star v|^s w^\alpha dx\right)^{1/s}\\
&\quad\le \sum_{B\in\mathcal{V}} \left(\int_B |d^\star v|^s w^\alpha dx\right)^{1/s}\\
&\quad\le C_1 \sum_{B\in\mathcal{V}} |B|^{\frac{\alpha r}{s}} \left(\|h\|_{t,B,w^{\frac{\alpha t}{s}}} + \||g|^{\frac{p}{q}}\|_{t,B,w^{\frac{\alpha t}{s}}} + \||du|^{\frac{p}{q}}\|_{t,B,w^{\frac{\alpha t}{s}}}\right)\\
&\quad\le C_1 \sum_{B\in\mathcal{V}} |\Omega|^{\frac{\alpha r}{s}} \left(\|h\|_{t,\Omega,w^{\frac{\alpha t}{s}}} + \||g|^{\frac{p}{q}}\|_{t,\Omega,w^{\frac{\alpha t}{s}}} + \||du|^{\frac{p}{q}}\|_{t,\Omega,w^{\frac{\alpha t}{s}}}\right)\\
&\quad\le C_2\left(\|h\|_{t,\Omega,w^{\alpha t/s}} + \||g|^{p/q}\|_{t,\Omega,w^{\frac{\alpha t}{s}}} + \||du|^{p/q}\|_{t,\Omega,w^{\frac{\alpha t}{s}}}\right)\cdot N\\
&\quad\le C_3 \left(\|h\|_{t,\Omega,w^{\alpha t/s}} + \||g|^{p/q}\|_{t,\Omega,w^{\alpha t/s}} + \||du|^{p/q}\|_{t,\Omega,w^{\alpha t/s}}\right)
\end{aligned}
$$

since $\Omega$ is bounded. ∎

If we put $g = 0$ and $h = 0$ in Theorem 2.4.1, we obtain the following global norm comparison result for $d^\star v$ and $du$.

**Corollary 2.4.2.** *Let $u$ and $v$ be a pair of solutions to the conjugate $A$-harmonic equation* (2.1.1) *in a bounded domain $\Omega \subset \mathbf{R}^n$. Assume that $w \in A_r(\Omega)$ for some $r > 1$. Then, there exists a constant $C$, independent of $u$ and $v$, such that*

$$\|d^\star v\|_{s,\Omega,w^\alpha} \le C\||du|^{p/q}\|_{t,\Omega,w^{\alpha t/s}} = C\|du\|^{p/q}_{pt/q,\Omega,w^{\alpha t/s}}, \tag{2.4.2}$$

*where $\alpha$ is any positive constant with $1 > \alpha r$, $s = (1-\alpha)q$, and $t = s/(1-\alpha r) = qs/(s-\alpha q(r-1))$.*

### *2.4.2 Global estimates for du*

Using Theorem 2.3.2 and the covering lemma, we obtain the following global $L^s$-estimate for $du$.

**Theorem 2.4.3.** *Let $u$ and $v$ be a pair of solutions to the nonhomogeneous conjugate $A$-harmonic equation* (2.1.2) *in a bounded domain $\Omega \subset \mathbf{R}^n$. Assume that $w \in A_r(\Omega)$ for some $r > 1$. Then, there exists a constant $C$, independent of $u$ and $v$, such that*

$$\|du\|_{s,\Omega,w^\alpha} \le C(\|g\|_{t,\Omega,w^{\alpha t/s}} + \||h|^{q/p}\|_{t,\Omega,w^{\alpha t/s}} + \||d^\star v|^{q/p}\|_{t,\Omega,w^{\alpha t/s}}). \tag{2.4.3}$$

*Here $\alpha$ is any positive constant with $1 > \alpha r$, $s = (1-\alpha)p$, and $t = s/(1-\alpha r) = ps/(s-\alpha p(r-1))$.*

Similarly, if we choose $g = 0$ and $h = 0$ in Theorem 2.4.3, we obtain the following global norm estimate for $du$ in terms of $d^\star v$.

**Corollary 2.4.4.** *Let $u$ and $v$ be a pair of solutions to the conjugate $A$-harmonic equation* (2.1.1) *in a bounded domain $\Omega \subset \mathbf{R}^n$. Assume that $w \in A_r(\Omega)$ for some $r > 1$. Then, there exists a constant $C$, independent of $u$ and $v$, such that*

$$\|du\|_{s,\Omega,w^\alpha} \le C\||d^\star v|^{q/p}\|_{t,\Omega,w^{\alpha t/s}} = \|d^\star v\|^{q/p}_{qt/p,\Omega,w^{\alpha t/s}}. \tag{2.4.4}$$

*Here $\alpha$ is any positive constant with $1 > \alpha r$, $s = (1-\alpha)p$, and $t = s/(1-\alpha r) = ps/(s - \alpha p(r-1))$.*

### *2.4.3 Global $L^p$-estimates*

We denote the space of all $l$-forms that are $L^s$-integrable in $\Omega$ with respect to the measure $\mu$ by $L^s(\Omega, \wedge, \mu)$. Now we prove the following theorem that characterizes the integrability of a pair $du$ and $d^\star v$.

**Theorem 2.4.5.** *Let $u$ and $v$ be a pair of solutions to the nonhomogeneous conjugate $A$-harmonic equation* (2.1.2) *in a bounded domain $\Omega \subset \mathbf{R}^n$. If $g \in L^p(B, \wedge^l, \mu)$ and $h \in L^q(B, \wedge^l, \mu)$, then $du \in L^p(B, \wedge^l, \mu)$ if and only if $d^\star v \in L^q(B, \wedge^l, \mu)$, where the measure $\mu$ is defined by $d\mu = w(x)^\alpha dx$, $w(x) \in A_r(\Omega)$ for some $r > 1$. Moreover, there exist constants $C_1, C_2$, independent of $u$ and $v$, such that*

$$\|d^\star v\|^q_{q,\Omega,w^\alpha} \le C_1(\|h\|^q_{q,\Omega,w^\alpha} + \|g\|^p_{p,\Omega,w^\alpha} + \|du\|^p_{p,\Omega,w^\alpha}), \tag{2.4.5}$$

$$\|du\|^p_{p,\Omega,w^\alpha} \le C_2(\|h\|^q_{q,\Omega,w^\alpha} + \|g\|^p_{p,\Omega,w^\alpha} + \|d^\star v\|^q_{q,\Omega,w^\alpha}) \tag{2.4.6}$$

*for any real number $\alpha$ with $\alpha > 0$.*

**Proof.** We only need to prove (2.4.5) and (2.4.6). Multiplying (2.2.6) by $w^\alpha$, we have

$$|d^\star v|^q w^\alpha \le C_1(|h|^q w^\alpha + |g|^p w^\alpha + |du|^p w^\alpha) \tag{2.4.7}$$

since $w^\alpha > 0$. Integrating (2.4.7) over $\Omega$ we find that (2.4.5) holds. Similarly, from (2.2.8), we obtain

$$|du|^p w^\alpha \le C_2(|g|^p w^\alpha + |h|^q w^\alpha + |d^\star v|^q w^\alpha). \tag{2.4.8}$$

Hence, inequality (2.4.6) follows immediately. ∎

Setting $g = 0$ and $h = 0$ in Theorem 2.4.5, we have the following global norm comparison theorem that provides a powerful tool for the study of the global integrability of differential forms $du$ and $d^\star v$.

**Corollary 2.4.6.** *Let $u$ and $v$ be a pair of solutions to the conjugate $A$-harmonic equation* (2.1.1) *in a bounded domain $\Omega \subset \mathbf{R}^n$. Then, $du \in L^p(\Omega, \wedge^l, \mu)$ if and only if $d^\star v \in L^q(\Omega, \wedge^l, \mu)$, where the measure $\mu$ is defined by $d\mu = w(x)^\alpha dx$, $w \in A_r(\Omega)$ for some $r > 1$. Moreover, there exist constants $C_1, C_2$, independent of $u$ and $v$, such that*

$$C_1 \|du\|^p_{p,\Omega,w^\alpha} \le \|d^\star v\|^q_{q,\Omega,w^\alpha} \le C_2 \|du\|^p_{p,\Omega,w^\alpha} \tag{2.4.9}$$

*for any real number $\alpha$ with $\alpha > 0$.*

Using Theorem 2.3.4 and the covering lemma, we find the following norm comparison theorem for $u - c$ and $d^\star v$.

**Theorem 2.4.7.** *Let $u$ and $v$ be a pair of solutions to the conjugate $A$-harmonic equation* (2.1.1) *in a bounded domain $\Omega \subset \mathbf{R}^n$. Assume that $w(x) \in A_r(\Omega)$ for some $r > 1$. Then, for any positive constant $\alpha$ with $1 > \alpha r$, there exist constants $C_1, C_2$, independent of $u$ and $v$, such that*

$$\|u - u_B\|_{s,\Omega,w^\alpha} \le C_1 \| |d^\star v|^{q/p} \|_{t,\Omega,w^{\alpha t/s}} \tag{2.4.10}$$

*for $s = (1-\alpha)p, t = s/(1-\alpha r) = ps/(s - \alpha p(r-1))$; and*

$$\|d^\star v\|_{s,\Omega,w^\alpha} \le C_2 \| |u - c|^{p/q} \|_{t,\Omega,w^{\alpha t/s}} \tag{2.4.11}$$

*for $s = (1-\alpha)q, t = s/(1-\alpha r) = qs/(s - \alpha q(r-1))$.*

### *2.4.4 Global $L^s$-estimates*

We have obtained the $L^p$-estimate for $du$ and the $L^q$-estimate for $d^\star v$ with $p^{-1} + q^{-1} = 1$ in Theorem 2.4.5. However, in applications, we often need $L^s$-estimates for some $s$ with $1 < s < \infty$. Hence, we now state and prove the following $L^s$-norm comparison theorem.

**Theorem 2.4.8.** *Let $u$ and $v$ be a pair of solutions to the nonhomogeneous conjugate $A$-harmonic equation* (2.1.2) *in a bounded domain $\Omega \subset \mathbf{R}^n$ and $1 < s < \infty$. Assume that $w \in A_r(\Omega)$ for some $r > 1$. Then,*

$$\|d^\star v\|_{s,\Omega,w^\alpha} \le C_1 (\|h\|_{s,\Omega,w^\alpha} + \|g\|^{p-1}_{s(p-1),\Omega,w^\alpha} + \|du\|^{p-1}_{s(p-1),\Omega,w^\alpha}), \tag{2.4.12}$$

$$\|du\|_{s,\Omega,w^\alpha} \le C_2 (\|g\|_{s,\Omega,w^\alpha} + \|h\|^{q-1}_{s(q-1),\Omega,w^\alpha} + \|d^\star v\|^{q-1}_{s(q-1),\Omega,w^\alpha}), \tag{2.4.13}$$

*where $C_1, C_2$ are constants.*

**Proof.** From (2.2.4), we know that

$$|d^\star v|^s w^\alpha \le (|h| + a|g + du|^{p-1})^s w^\alpha \tag{2.4.14}$$

for any weight $w(x) > 0$. Integrating (2.4.14) over $\Omega$ and using the Minkowski inequality, we find that

$$\begin{aligned}\|d^\star v\|_{s,\Omega,w^\alpha} &\le \left(\int_\Omega (|h| + a|g+du|^{p-1})^s w^\alpha dx\right)^{1/s}\\ &\le \|h\|_{s,\Omega,w^\alpha} + a\|g+du\|^{p-1}_{s(p-1),\Omega,w^\alpha}\\ &\le C_1(\|h\|_{s,\Omega,w^\alpha} + \|g\|^{p-1}_{s(p-1),\Omega,w^\alpha} + \|du\|^{p-1}_{s(p-1),\Omega,w^\alpha}).\end{aligned}$$

This completes the proof of (2.4.12). From (2.2.7), we find that

$$|g+du| \le |h + d^\star v|^{1/(p-1)},$$

and hence

$$\begin{aligned}|du|^s w^\alpha &= |du + g - g|^s w^\alpha\\ &\le (|du+g| + |g|)^s w^\alpha\\ &\le (|g| + |h + d^\star v|^{1/(p-1)})^s w^\alpha.\end{aligned} \tag{2.4.15}$$

Integrating (2.4.15) over $\Omega$ and using (2.2.5) and Minkowski inequality again, we obtain

$$\begin{aligned}&\|du\|_{s,\Omega,w^\alpha}\\ &\le \left(\int_\Omega \left(|g| + (|h| + |d^\star v|)^{1/(p-1)}\right)^s w^\alpha dx\right)^{1/s}\\ &\le \|g\|_{s,\Omega,w^\alpha} + \left(\int_\Omega (|h| + |d^\star v|)^{s/(p-1)} w^\alpha dx\right)^{1/s}\\ &\le \|g\|_{s,\Omega,w^\alpha} + \left(C_2 \int_\Omega \left(|h|^{1/(p-1)} + |d^\star v|^{1/(p-1)}\right)^s w^\alpha dx\right)^{1/s}\\ &\le \|g\|_{s,\Omega,w^\alpha} + C_3\left(\left(\int_\Omega |h|^{s/(p-1)} w^\alpha dx\right)^{1/s} + \left(\int_\Omega |d^\star v|^{s/(p-1)} w^\alpha dx\right)^{1/s}\right)\\ &\le C_4\left(\|g\|_{s,\Omega,w^\alpha} + \|h\|^{1/(p-1)}_{s/(p-1),\Omega,w^\alpha} + \|d^\star v\|^{1/(p-1)}_{s/(p-1),\Omega,w^\alpha}\right)\\ &\le C_4(\|g\|_{s,\Omega,w^\alpha} + \|h\|^{q-1}_{s(q-1),\Omega,w^\alpha} + \|d^\star v\|^{q-1}_{s(q-1),\Omega,w^\alpha})\end{aligned}$$

since $1/(p-1) = q-1$. ∎

**Remark.** We have extended only few local norm inequalities into the global ones. Using similar methods, we can generalize other local results into the global ones. We can also obtain the global comparison theorems with other weights, including two-weight. Also, we notice that the global comparison norm inequalities contain a parameter $\alpha$. Hence, by choosing different value of $\alpha$, we have different versions of the global norm inequalities. For example, choosing $\alpha = p > 1$ in (2.4.5) and (2.4.6), we obtain

$$\|d^\star v\|^q_{q,\Omega,w^p} \le C_1(\|h\|^q_{q,\Omega,w^p} + \|g\|^p_{p,\Omega,w^p} + \|du\|^p_{p,\Omega,w^p})$$

and

$$\|du\|^p_{p,\Omega,w^p} \le C_2(\|h\|^q_{q,\Omega,w^p} + \|g\|^p_{p,\Omega,w^p} + \|d^\star v\|^q_{q,\Omega,w^p}),$$

respectively.

## 2.5 Applications

In this section, we discuss some applications of the global norm comparison theorems. We prove the global Sobolev–Poincaré-type inequality, and obtain some versions of imbedding theorems for the homotopy operator $T$ and the gradient operator $\nabla$.

### *2.5.1 Imbedding theorems for differential forms*

From Corollary 4.1 in [99], for any $u \in D'(B, \wedge^l)$ with $du \in L^p(B, \wedge^{l+1})$, we have

$$\|u - u_B\|_{W^{1,p}(B,\wedge^l)} \le C|B|\|du\|_{p,B}. \tag{2.5.1}$$

Combining (2.2.9) and (2.5.1), we obtain the following analog of Sobolev–Poincaré-type imbedding theorem for differential forms satisfying the conjugate $A$-harmonic equation.

**Theorem 2.5.1.** *Let $u$ and $v$ be a pair of solutions to the conjugate $A$-harmonic equation* (2.1.1) *in a domain $\Omega \subset \mathbf{R}^n$. Then,*

$$\|u - u_B\|^p_{W^{1,p}(B,\wedge^l)} \le C\|d^\star v\|^q_{q,B} \tag{2.5.2}$$

*for all balls $B$ with $B \subset \Omega$, where $C$ is a constant.*

Using (2.5.2) and the covering lemma, we prove the following global Sobolev–Poincaré-type imbedding theorem.

**Theorem 2.5.2.** *Let $u \in D'(\Omega, \wedge^0)$ and $v \in D'(\Omega, \wedge^2)$ be a pair of solutions to the conjugate $A$-harmonic equation* (2.1.1) *in a $\delta$-John domain $\Omega \subset \mathbf{R}^n$. Then,*

$$\|u - u_{B_0}\|^p_{W^{1,p}(\Omega,\wedge^l)} \le C\|d^\star v\|^q_{q,\Omega},$$

*where $C$ is a constant and $B_0 \subset \Omega$ is a fixed ball.*

**Notes to Chapter 2.** (i) We should note that all the results established in this chapter are about $l$-forms, $l = 0, 1, \ldots, n$, and that the real functions in $\mathbf{R}^n$ are 0-forms.
(ii) It is known that if $f(x) = (f^1, f^2, \ldots, f^n)$ is $K$-quasiregular in $\mathbf{R}^n$, then

$$u = f^l df^1 \wedge df^2 \wedge \cdots \wedge df^{l-1}, \quad l = 1, 2, \ldots, n-1,$$

and

$$v = \star f^{l+1} df^{l+2} \wedge \cdots \wedge df^n, \quad l = 1, 2, \ldots, n-1,$$

are solutions of the conjugate $A$-harmonic equation (2.1.1). Thus, our results, such as Theorems 2.5.1 and 2.5.2, can be used to study the $K$-quasiregular mapping $f$.

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