2

Gelfand Theory

Our main objective in this chapter is to develop Gelfand's theory for commutative Banach algebras. Associated with any such algebra A is a locally compact Hausdorff space $\Delta(A)$, the structure space of A, and a norm-decreasing homomorphism Γ_A from A into $C_0(\Delta(A))$ (Section 2.2). If A has an identity, $\Delta(A)$ is compact. The converse is true whenever Γ_A is injective (A is semisimple), a fact that can be shown only later (Chapter 3). This representation of A as an algebra of functions on a locally compact Hausdorff space is fundamental to any thorough study of commutative Banach algebras. Thus basic questions are when Γ_A is injective or surjective. It turns out that Γ_A is an isometric isomorphism onto $C_0(\Delta(A))$ precisely when A is a C^*-algebra (Section 2.4).

If A is unital and (topologically) finitely generated by n elements, say, then $\Delta(A)$ can be canonically identified with a compact subset of $\mathbb{C}^n$ (Section 2.3). There is a complete characterisation of subsets of $\mathbb{C}^n$ arising in this way as structure spaces of finitely generated commutative Banach algebras. This leads to the study of uniform algebras, closed unital subalgebras of $C(X)$, for a compact subset X of $\mathbb{C}^n$, which separate the points of X. In Section 2.5 we investigate the algebras $P(X)$ and $R(X)$ of polynomial and rational functions on X, respectively. Comparison of such algebras is interesting from the approximation theory point of view. Considerably more complicated is the algebra $A(X)$ of continuous functions on X which are holomorphic on the interior of X (Section 2.6).

Following our intention to emphasize the connections with commutative harmonic analysis, we extensively study the convolution algebra $L^1(G)$ of integrable functions on a locally compact Abelian group G. The structure space $\Delta(L^1(G))$ turns out to be homeomorphic with the dual group $\widehat{G}$ of G, and the Gelfand homomorphism is injective, but surjective only when G is finite (Section 2.7). Much more subtle are weighted group algebras $L^1(G, \omega)$. We confine ourselves to showing that $L^1(G, \omega)$ is always semisimple and to determining $\Delta(L^1(G, \omega))$ in some special cases (Section 2.8).

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Proceeding further with algebras of functions associated with locally compact groups, in Section 2.9 we elaborate the Gelfand representation of the Fourier algebra $A(G)$ for an arbitrary locally compact group G. Next, applying the Gelfand theory to the C^*-algebra of almost periodic functions, we establish the existence of the Bohr compactification of a locally compact Abelian group (Section 2.10).

Finally, we investigate tensor products of two commutative Banach algebras A and B, especially the projective tensor product $A\,\widehat{\otimes}_\pi B$. Although $\Delta(A\,\widehat{\otimes}_\pi B)$ is in the obvious manner homeomorphic to $\Delta(A)\times\Delta(B)$, semisimplicity of $A\,\widehat{\otimes}_\pi B$ is a very delicate question (Section 2.11). In fact, by using failure of the approximation property for Banach spaces one can construct semisimple commutative Banach algebras A and B such $A\,\widehat{\otimes}_\pi B$ is not semisimple.

2.1 Multiplicative linear functionals

A linear functional φ on an algebra A is called *multiplicative* if $\varphi(xy) = \varphi(x)\varphi(y)$ for all $x, y \in A$. We start with identifying the multiplicative ones among all linear functionals on A in terms of spectra (Theorem 2.1.2). We do not assume here that A is commutative.

Lemma 2.1.1. *Let A be a real or complex algebra with identity e, and let φ be a linear functional on A satisfying*

$$\varphi(e) = 1 \text{ and } \varphi(x^2) = \varphi(x)^2$$

for all $x \in A$. Then φ is multiplicative.

Proof. By assumption we have

$$\begin{aligned}\varphi(x^2) + \varphi(xy + yx) + \varphi(y^2) &= \varphi(x^2 + xy + yx + y^2)\\ &= \varphi((x+y)^2) = (\varphi(x) + \varphi(y))^2\\ &= \varphi(x^2) + 2\varphi(x)\varphi(y) + \varphi(y^2),\end{aligned}$$

and therefore

$$\varphi(xy + yx) = 2\varphi(x)\varphi(y)$$

for all $x, y \in A$. Thus it remains to verify that $\varphi(yx) = \varphi(xy)$. Now, for $a, b \in A$, the identity

$$(ab - ba)^2 + (ab + ba)^2 = 2[a(bab) + (bab)a]$$

implies

$$\begin{aligned}\varphi(ab - ab)^2 + 4\varphi(a)^2\varphi(b)^2 &= \varphi((ab - ba)^2) + \varphi(ab + ba)^2\\ &= \varphi((ab - ba)^2 + (ab + ba)^2)\\ &= 2\varphi(a(bab) + (bab)a)\\ &= 4\varphi(a)\varphi(bab).\end{aligned}$$

Taking $a = x - \varphi(x)e$, so that $\varphi(a) = 0$, and $b = y$ we obtain $\varphi(ay) = \varphi(ya)$ and hence $\varphi(xy) = \varphi(yx)$. □

The following theorem is often called the *Gleason–Kahane–Zelazko theorem.*

Theorem 2.1.2. *Let A be a unital Banach algebra. For a linear functional φ on A the following conditions are equivalent.*

(i) *φ is nonzero and multiplicative.*
(ii) *$\varphi(e) = 1$ and $\varphi(x) \neq 0$ for every invertible element x of A.*
(iii) *$\varphi(x) \in \sigma_A(x)$ for every $x \in A$.*

Proof. If φ is nonzero and multiplicative, then $\varphi(e) = 1$ and $1 = \varphi(x)\varphi(x^{-1})$ whenever x is invertible. Thus (i) $\Rightarrow$ (ii). Also, (ii) $\Rightarrow$ (iii) is obvious since if $\lambda \in \rho_A(x)$, then $0 \neq \varphi(x - \lambda e) = \varphi(x) - \lambda$.

Now assume (iii) and note first that $\varphi(e) = 1$. We are going to show that $\varphi(x^2) = \varphi(x)^2$ for all $x \in A$. To that end, let $n \geq 2$ and consider the polynomial

$$p(\lambda) = \varphi((\lambda e - x)^n)$$

of degree n. Denoting its roots by $\lambda_1, \dots, \lambda_n$, we have for each i,

$$0 = p(\lambda_i) = \varphi((\lambda_i e - x)^n) \in \sigma_A((\lambda_i e - x)^n).$$

This implies that $\lambda_i \in \sigma_A(x)$ and hence $|\lambda_i| \leq r_A(x)$. Now

$$\prod_{i=1}^n (\lambda - \lambda_i) = p(\lambda) = \lambda^n - n\varphi(x)\lambda^{n-1} + \binom{n}{2}\varphi(x^2)\lambda^{n-2} + \ldots + (-1)^n\varphi(x^n).$$

Comparing coefficients we see that

$$\sum_{i=1}^n \lambda_i = n\varphi(x) \ \text{ and } \sum_{1\leq i<j\leq n} \lambda_i\lambda_j = \binom{n}{2}\varphi(x^2).$$

On the other hand, by the second equation,

$$\left(\sum_{i=1}^n \lambda_i\right)^2 = \sum_{i=1}^n \lambda_i^2 + 2\sum_{1\leq i<j\leq n} \lambda_i\lambda_j = \sum_{i=1}^n \lambda_i^2 + n(n-1)\varphi(x^2).$$

Combining these formulae yields

$$n^2|\varphi(x)^2 - \varphi(x^2)| = \left|-n\varphi(x^2) + \sum_{i=1}^n \lambda_i^2\right| \leq n|\varphi(x^2)| + nr_A(x)^2.$$

This being true for all n, we conclude that $\varphi(x^2) = \varphi(x)^2$ for all $x \in A$. It follows now from Lemma 2.1.1 that φ is multiplicative. □

Throughout the book, for any Banach algebra A, $\Delta(A)$ denotes the set of all nonzero multiplicative linear functionals on A. It is very important to know how $\Delta(A)$ and $\Delta(A_e)$ are related.

Remark 2.1.3. Because $\psi(e) = 1$ for every $\psi \in \Delta(A_e)$, each $\varphi \in \Delta(A)$ has a unique extension $\widetilde{\varphi} \in \Delta(A_e)$ given by

$$\widetilde{\varphi}(x + \lambda e) = \varphi(x) + \lambda, \quad x \in A, \quad \lambda \in \mathbb{C}.$$

Let $\widetilde{\Delta}(A) = \{\widetilde{\varphi} : \varphi \in \Delta(A)\}$. Moreover, let φ_∞ denote the homomorphism from A_e to $\mathbb{C}$ with kernel A, that is, $\varphi_\infty(x + \lambda e) = \lambda$. Then

$$\Delta(A_e) = \widetilde{\Delta}(A) \cup \{\varphi_\infty\}.$$

In fact, if $\psi \in \Delta(A_e)$ and $\psi \neq \varphi_\infty$, then $\psi|_A \in \Delta(A)$ and hence $\psi = \widetilde{\psi|_A}$. Identifying $\Delta(A)$ with $\widetilde{\Delta}(A) \subseteq \Delta(A_e)$ we always regard $\Delta(A)$ as a subset of $\Delta(A_e)$. In this sense, $\Delta(A_e) = \Delta(A) \cup \{\varphi_\infty\}$.

Remark 2.1.4. A simple unitisation argument shows that for a linear functional φ on a Banach algebra A, condition (iii) in Theorem 2.1.2 is equivalent to multiplicativity of φ without assuming that A has an identity.

Lemma 2.1.5. *Let A be a Banach algebra. Every $\varphi \in \Delta(A)$ is a bounded linear functional on A and $|\varphi(x)| \leq r_A(x)$ holds for all $x \in A$. In particular, $\|\varphi\| \leq 1$ and $\|\varphi\| = 1$ if A is unital.*

Proof. We can assume that A has an identity e. If $x \in A$ and $\lambda \in \mathbb{C}$ are such that $|\lambda| > r_A(x)$, then $r_A((1/\lambda)x) < 1$ and hence $\lambda e - x = \lambda\,(e - (1/\lambda)x)$ is invertible in A by Lemma 1.2.6. This implies $\varphi(x) \neq \lambda$ for all such λ, so that $|\varphi(x)| \leq r_A(x)$. This implies that $\|\varphi\| \leq 1$ and actually $\|\varphi\| = 1$ since $\varphi(e) = 1$. $\square$

The obvious problem which we have to encounter is the existence of nonzero multiplicative linear functionals on commutative Banach algebras A. To start with let A be a complex Banach space and define a product on A by setting $xy = 0$ for all $x, y \in A$. If φ is a multiplicative linear functional on A, then

$$\varphi(x)^2 = \varphi(xx) = \varphi(0) = 0.$$

Less trivial examples showing that nonzero multiplicative linear functionals need not exist are the following two. Note that, for any commutative Banach algebra A, $\Delta(A) = \emptyset$ whenever $r_A(x) = 0$ for every $x \in A$.

Example 2.1.6. Let $A = P(\mathbb{D})$ as a Banach space. For $f, g \in A$ define a function $f \circ g$ on $\mathbb{D}$ by

$$f \circ g(z) = z \int_0^1 f(z - tz) g(tz) dt,$$

$z \in \mathbb{D}$. We claim that $f \circ g \in A$. For that notice first that if polynomials

$$p(z) = \sum_{j=0}^{n} a_j z^j \text{ and } q(z) = \sum_{k=0}^{m} b_k z^k,$$

$a_j, b_k \in \mathbb{C}$, are given then, for any $z \in \mathbb{D}$,

$$(p|_{\mathbb{D}} \circ q|_{\mathbb{D}})(z) = \sum_{j=0}^{n} \sum_{k=0}^{m} a_j b_k z^{j+k+1} \int_0^1 t^k (1-t)^j \, dt,$$

so that $p|_{\mathbb{D}} \circ q|_{\mathbb{D}}$ equals the restriction to $\mathbb{D}$ of a polynomial. Now, given arbitrary f and g in A and $\varepsilon > 0$, let p and q be polynomials such that $\|f - p|_{\mathbb{D}}\|_\infty \leq \epsilon$ and $\|g - q|_{\mathbb{D}}\|_\infty \leq \epsilon$. Then, for any $z \in \mathbb{D}$,

$$\begin{aligned}
|f \circ g(z) - p|_{\mathbb{D}} \circ q|_{\mathbb{D}}(z)| &\leq \int_0^1 |f(z-tz)g(tz) - p(z-tz)q(tz)| dt \\
&\leq \int_0^1 |g(tz)| \cdot |f(z-tz) - p(z-tz)| dt \\
&\quad + \int_0^1 |p(z-tz)| \cdot |g(tz) - q(tz)| dt \\
&\leq \|g\|_\infty \|f - p|_{\mathbb{D}}\|_\infty + \|p|_{\mathbb{D}}\|_\infty \|g - q|_{\mathbb{D}}\|_\infty \\
&\leq \epsilon(\|f\|_\infty + \|g\|_\infty + \epsilon).
\end{aligned}$$

Hence $f \circ g$ is the uniform limit of polynomial functions on $\mathbb{D}$, whence $f \circ g \in A$. Clearly, $\|f \circ g\|_\infty \leq \|f\|_\infty \|g\|_\infty$. Moreover, the multiplication $(f, g) \to f \circ g$ is commutative, associative, and distributive. In fact, this is straightforward from the definition of $f \circ g$. Thus A with product $\circ$ is a commutative Banach algebra.

We proceed to show by induction that

$$|f^n(z)| \leq \frac{1}{(n-1)!} \|f\|_\infty^n |z|^{n-1}$$

for every $f \in A$ and all $z \in \mathbb{D}$ and $n \in \mathbb{N}$. The case $n = 1$ being obvious, assume the estimate to hold for n. Let $z \in \mathbb{D}$ and write $z = re^{i\varphi}, 0 \leq r \leq 1, \varphi \in \mathbb{R}$.

Then

$$\begin{aligned}
f^{n+1}(z) &= re^{i\varphi} \int_0^1 f(z - tre^{i\varphi}) f^n(tre^{i\varphi}) dt \\
&= e^{i\varphi} \int_0^r f(z - se^{i\varphi}) f^n(se^{i\varphi}) ds,
\end{aligned}$$

and hence by the inductive hypothesis,

$$\begin{aligned}
|f^{n+1}(z)| &\leq \|f\|_\infty \int_0^r |f^n(se^{i\varphi})|ds \\
&\leq \frac{1}{(n-1)!}\|f\|_\infty^{n+1} \int_0^r s^{n-1}ds \\
&= \frac{1}{n!}\|f\|_\infty^{n+1} r^n \\
&= \frac{1}{n!}\|f\|_\infty^{n+1} |z|^n,
\end{aligned}$$

as required. Thus, for every $f \in A$ and $n \in \mathbb{N}$,

$$\|f^n\|_\infty \leq \frac{\|f\|_\infty^n}{(n-1)!}$$

and hence, by the spectral radius formula,

$$r_A(f) = \lim_{n\to\infty} \|f^n\|_\infty^{1/n} \leq \|f\|_\infty \lim_{n\to\infty} \left(\frac{1}{(n-1)!}\right)^{1/n} = 0.$$

This shows that $\sigma_A(f) = \{0\}$ for all $f \in A$, and therefore $\Delta(A) = \emptyset$.

Example 2.1.7. Define a bounded linear operator T on $C[0,1]$ by

$$Tf(t) = \int_0^t f(s)ds, \quad f \in C[0,1], \quad t \in [0,1].$$

Let A be the norm closure in $\mathcal{B}(C[0,1])$ of the set of all polynomials in T of the form

$$\sum_{i=1}^n a_i T^i,\ a_1, \ldots, a_n \in \mathbb{C}, n \in \mathbb{N}.$$

A is a commutative Banach algebra which does not have an identity. A straightforward induction argument shows

$$|T^n f(t)| \leq \|f\|_\infty \frac{t^n}{n!}$$

for all $t \in [0,1]$ and $n \in \mathbb{N}$. Hence

$$\|T^n f\|_\infty \leq \frac{1}{n!}\|f\|_\infty,$$

and this inequality gives

$$\|T^n\|^{1/n} \leq \left(\frac{1}{n!}\right)^{1/n}$$

for all $n \in \mathbb{N}$. Since $(n!)^{1/n} \to \infty$ as $n \to \infty$, we get $r_A(T) = 0$. The spectral radius r_A is subadditive and submultiplicative (Lemma 1.2.13) and continuous. Therefore it follows that $r_A(S) = 0$ for all $S \in A$.

Of course, the algebra A is reminiscent of the Volterra algebra which was introduced in Exercise 1.6.12. In fact, restricting the convolution of Example 1.6.12 to $C[0,1]$ and denoting by u the constant one function on $[0,1]$, we have

$$Tf(t) = \int_0^t u(t-s)f(s)ds = u * f(t)$$

for all $f \in C[0,1]$ and $t \in \mathbb{R}$. It is now easily verified that u^n, the n-fold convolution product of u, is given by $u^n(t) = t^{n-1}/(n-1)!$, $n \in \mathbb{N}$. Since the polynomials are uniformly dense in $C[0,1]$, it follows that A equals the closure of in $\mathcal{B}(C[0,1])$ of the algebra of all convolution operators

$$T_g : C[0,1] \to C[0,1],\ f \to g * f,$$

$g \in C[0,1]$. Because of this similarity, A is also sometimes termed *Volterra algebra.*

If A is a unital commutative Banach algebra, the anomaly just discussed cannot occur. This is an immediate consequence of the next theorem which forms the basic link between $\Delta(A)$ and ideals in A.

Theorem 2.1.8. *For a commutative Banach algebra A, the mapping*

$$\varphi \to \ker \varphi = \{x \in A : \varphi(x) = 0\}$$

is a bijection between $\Delta(A)$ and $\mathrm{Max}(A)$*, the set of all maximal modular ideals in A.*

Proof. For $\varphi \in \Delta(A)$, $\ker \varphi$ is an ideal and a closed linear subspace of codimension one in A. To verify that $\ker \varphi$ is modular simply choose $u \in A$ such that $\varphi(u) = 1$. Then, for any $x \in A$,

$$\varphi(ux - x) = \varphi(u)\varphi(x) - \varphi(x) = 0,$$

whence $ux - x \in \ker \varphi$. Thus u is an identity modulo $\ker \varphi$, and hence $\ker \varphi$ is a maximal modular ideal.

Let now that $\varphi_1, \varphi_2 \in \Delta(A)$ be such that $\ker \varphi_1 = \ker \varphi_2$ and denote this ideal by I. Let u be an identity modulo I. Then, since the codimension of I is one, each $x \in A$ can be uniquely expressed as

$$x = \lambda u + y, \quad y \in I, \quad \lambda \in \mathbb{C}.$$

As $\varphi(u) = 1$ for every homomorphism φ with $\ker \varphi = I$, we get

$$\varphi_1(x) = \lambda\varphi_1(u) + \varphi_1(y) = \lambda = \lambda\varphi_2(u) + \varphi_2(y) = \varphi_2(x).$$

Finally, let $M \in \mathrm{Max}(A)$ and let u be an identity modulo M. We already know that M is closed in A, so A/M is a Banach algebra. Suppose there exists

$x \in A \setminus M$ such that $x + M$ is not invertible in A/M. Then $A/M(x+M)$ is a proper nonzero ideal in A/M since

$$x + M = (u + M)(x + M) \in A/M(x + M)$$

is nonzero. This contradicts the maximality of M. Thus A/M is a Banach division algebra and hence, by the Gelfand–Mazur theorem (Theorem 1.2.9), isomorphic to the field of complex numbers. Clearly, this isomorphism defines a homomorphism $\varphi : A \to \mathbb{C}$ with $\ker \varphi = M$. □

Definition 2.1.9. Let A be a commutative Banach algebra. The *radical* of A, $\mathrm{rad}(A)$, is defined by

$$\mathrm{rad}(A) = \bigcap \{M : M \in \mathrm{Max}(A)\} = \bigcap \{\ker \varphi : \varphi \in \Delta(A)\},$$

where $\mathrm{rad}(A)$ is understood to be A if $\Delta(A) = \emptyset$. Clearly, $\mathrm{rad}(A)$ is a closed ideal of A. The algebra A is called *semisimple* if $\mathrm{rad}(A) = \{0\}$ and *radical* if $\mathrm{rad}(A) = A$.

In Examples 2.1.6 and 2.1.7 we have already seen examples of radical Banach algebras with nontrivial multiplication. On the other hand, it will follow from Theorem 2.2.5 in the next section that A is semisimple if and only if for every $x \in A$, $r_A(x) = 0$ implies that $x = 0$. Because the spectral radius is subadditive and submultiplicative, this means that A is semisimple if and only if r_A is an algebra norm on A. Thus $\Delta(A) \neq \emptyset$.

Returning to the existence of nonzero multiplicative linear functionals, assume that A is a commutative Banach algebra with identity. Then the proper ideal $\{0\}$ is contained in some maximal ideal which, by Theorem 2.1.8, is the kernel of a homomorphism from A onto $\mathbb{C}$.

We continue with a number of interesting applications of Lemma 2.1.5.

Corollary 2.1.10. *Let ϕ be a homomorphism from a commutative Banach algebra A into a semisimple commutative Banach algebra B. Then ϕ is continuous.*

Proof. By the closed graph theorem it suffices to show that if $x_n \in A, n \in \mathbb{N}$, are such that $x_n \to 0$ and $\phi(x_n) \to b$ for some $b \in B$, then $b = 0$. Let $\varphi \in \Delta(B)$. Then $\varphi \circ \phi \in \Delta(A) \cup \{0\}$ and hence both, φ and $\varphi \circ \phi$, are continuous by Lemma 2.1.5. It follows that

$$\varphi(b) = \lim_{n\to\infty} \varphi(\phi(x_n)) = \lim_{n\to\infty} (\varphi \circ \phi)(x_n) = 0.$$

Since this holds for all $\varphi \in \Delta(B)$ and B is semisimple we get $b = 0$. □

Corollary 2.1.11. *On a semisimple commutative Banach algebra all Banach algebra norms are equivalent.*

Proof. Suppose A is a semisimple commutative Banach algebra, and let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two Banach algebra norms on A. The statement follows by applying Corollary 2.1.10 with ϕ the identity mappings $(A, \|\cdot\|_1) \to (A, \|\cdot\|_2)$ and $(A, \|\cdot\|_2) \to (A, \|\cdot\|_1)$. □

Corollary 2.1.12. *Every involution on a semisimple commutative Banach algebra A is continuous.*

Proof. Let $\|\cdot\|$ be the given norm an A. We define a new norm $|\cdot|$ on A by $|x| = \|x^*\|$. It is clear that $|\cdot|$ is submultiplicative. If $x_n \in A, n \in \mathbb{N}$, form a Cauchy sequence for $|\cdot|$, then $(x_n^*)_n$ is a Cauchy sequence for $\|\cdot\|$. Consequently, $\|x_n^* - x\| \to 0$ for some $x \in A$, and hence $|x_n - x^*| \to 0$. This shows that $(A, |\cdot|)$ is complete. By Corollary 2.1.11 there exists $c > 0$ such that

$$\|x^*\| = |x| \leq c\|x\|$$

for all $x \in A$, as was to be shown. □

Let $C^\infty[0,1]$ denote the algebra of all infinitely many times differentiable functions on $[0,1]$.

Corollary 2.1.13. *The algebra $C^\infty[0,1]$ admits no Banach algebra norm.*

Proof. Suppose there is a Banach algebra norm $\|\cdot\|$ on $C^\infty[0,1]$. Applying Corollary 2.1.10 to the identity mapping from $C^\infty[0,1]$ into $C[0,1]$ we see that there exists $c > 0$ such that

$$\|f\|_\infty \leq c\|f\|$$

for all $f \in C^\infty[0,1]$. Using this inequality, we prove that the differentiation mapping $D : f \to f'$ from $C^\infty[0,1]$ into itself is continuous. Thus, let $f_n \in C^\infty[0,1], n \in \mathbb{N}$, be such that

$$\lim_{n\to\infty} \|f_n\| = 0 \text{ and } \lim_{n\to\infty} \|f_n' - g\| = 0$$

for some $g \in C^\infty[0,1]$. Then

$$\lim_{n\to\infty} \|f_n\|_\infty = 0 \text{ and } \lim_{n\to\infty} \|f_n' - g\|_\infty = 0.$$

Since for each $x, y \in [0,1]$,

$$\begin{aligned} \left|\int_x^y g(t)dt\right| &\leq |f_n(y) - f_n(x)| + \left|\int_x^y (f_n'(t) - g(t))dt\right| \\ &\leq 2\,\|f_n\|_\infty + |y - x| \cdot \|f_n' - g\|_\infty, \end{aligned}$$

it follows that $\int_x^y g(t)dt = 0$. Hence $g = 0$ because x and y are arbitrary. By the closed graph theorem, D is continuous. Thus there exists $d > 0$ such that

$$\|f'\| \leq d\|f\|$$

for all $f \in C^\infty[0,1]$. Now, let $f(t) = e^{2dt}$, $t \in [0,1]$. Then

$$2d\|f\| = \|f'\| \leq d\|f\|.$$

This contradiction shows that there cannot exist a Banach algebra norm on $C^\infty[0,1]$. □

2.2 The Gelfand representation

In this section we develop the basic elements of Gelfand's theory which represents a (semisimple) commutative Banach algebra as an algebra of continuous functions on a locally compact Hausdorff space.

Definition 2.2.1. Let A be a commutative Banach algebra and, as before, $\Delta(A)$ the set of all nonzero (hence surjective) algebra homomorphisms from A to $\mathbb{C}$. We endow $\Delta(A)$ with the weakest topology with respect to which all the functions

$$\Delta(A) \to \mathbb{C}, \quad \varphi \to \varphi(x), \quad x \in A,$$

are continuous. A neighbourhood basis at $\varphi_0 \in \Delta(A)$ is then given by the collection of sets

$$U(\varphi_0, x_1, \ldots, x_n, \epsilon) = \{\varphi \in \Delta(A) : |\varphi(x_i) - \varphi_0(x_i)| < \epsilon, 1 \leq i \leq n\},$$

where $\epsilon > 0, n \in \mathbb{N}$, and $x_1, \ldots, x_n$ are arbitrary elements of A. This topology on $\Delta(A)$ is called the *Gelfand topology*. There are several names in use for the space $\Delta(A)$, equipped with the Gelfand topology: The *structure space*, the *spectrum* or *Gelfand space* of A, and the *maximal ideal space*, the latter notion being justified through the bijective correspondence between $\Delta(A)$ and $\mathrm{Max}(A)$ (Theorem 2.1.8).

Remark 2.2.2. We have seen in Lemma 2.1.5 that $\Delta(A)$ is contained in the unit ball of A^*. The Gelfand topology obviously coincides with the relative w^*-topology of A^* on $\Delta(A)$. When adjoining an identity e to A, $\Delta(A_e) = \Delta(A) \cup \{\varphi_\infty\}$ (Remark 2.1.3) and according to the following theorem the topology on $\Delta(A)$ is the one induced from $\Delta(A_e)$.

Theorem 2.2.3. *Let A be a commutative Banach algebra. Then*

(i) $\Delta(A)$ *is a locally compact Hausdorff space.*
(ii) $\Delta(A_e) = \Delta(A) \cup \{\varphi_\infty\}$ *is the one-point compactification of* $\Delta(A)$.
(iii) $\Delta(A)$ *is compact if A has an identity.*

Proof. It is easy to see that $\Delta(A)$ is a Hausdorff space. Indeed, if φ_1 and φ_2 are distinct elements of $\Delta(A)$, then for some $x \in A, \delta = \frac{1}{2}|\varphi_1(x) - \varphi_2(x)| > 0$ and hence

$$U(\varphi_1, x, \delta) \cap U(\varphi_2, x, \delta) = \emptyset.$$

To prove that $\Delta(A)$ is compact if A has an identity e, let

$$C = \prod_{x \in A} \{z \in \mathbb{C} : |z| \leq \|x\|\}.$$

Equipped with the product topology, C is a compact space by Tychonoff's theorem. Since $|\varphi(x)| \leq \|x\|$ for all $\varphi \in \Delta(A)$ and $x \in A$, we can define a mapping ϕ from $\Delta(A)$ into C by

$$\phi(\varphi) = (\varphi(x))_{x \in A}.$$

Then ϕ is injective and, by definition of the Gelfand topology, a homeomorphism from $\Delta(A)$ onto $\phi(\Delta(A))$. Thus, in order to establish that $\Delta(A)$ is compact it remains to show that $\phi(\Delta(A))$ is closed in C. To this end, let $\lambda = (\lambda_x)_{x \in A} \in C$ lie in the closure of $\phi(\Delta(A))$ and let $x, y \in A, \alpha, \beta \in \mathbb{C}$ and $\varepsilon > 0$ be given. If $\varphi \in \Delta(A)$ is such that $|\varphi(a) - \lambda_a| \leq \varepsilon$ for $a \in \{x, y, xy, \alpha x + \beta y\}$, then

$$\begin{aligned} |\alpha\lambda_x + \beta\lambda_y - \lambda_{\alpha x + \beta y}| &\leq |\alpha|\,|\lambda_x - \varphi(x)| + |\beta|\,|\lambda_y - \varphi(y)| \\ &\quad + |\varphi(\alpha x + \beta y) - \lambda_{\alpha x + \beta_y}| \\ &\leq \epsilon(|\alpha| + |\beta| + 1) \end{aligned}$$

and

$$\begin{aligned} |\lambda_{xy} - \lambda_x\lambda_y| &\leq |\lambda_{xy} - \varphi(xy)| + |\varphi(y)|\,|\varphi(x) - \lambda_x| \\ &\quad + |\lambda_x|\,|\varphi(y) - \lambda_y| \\ &\leq \epsilon(1 + \|y\| + \|x\|). \end{aligned}$$

Since $\epsilon > 0$ was arbitrary, it follows that $\psi : x \to \lambda_x, A \to \mathbb{C}$ is a homomorphism. Moreover, $\psi \in \Delta(A)$ because $\psi(e) = \lambda_e = 1$. This completes the proof of statement (iii).

Now we drop the hypothesis that A be unital and consider $\Delta(A_e)$ and $\Delta(A) \subseteq \Delta(A_e)$. We denote the basic neighbourhoods in $\Delta(A)$ and $\Delta(A_e)$ by U and U_e, respectively. Then, for $\varphi \in \Delta(A), \epsilon > 0$ and a finite subset F of A,

$$U_e(\varphi, F, \epsilon) = \begin{cases} U(\varphi, F, \epsilon) \cup \{\varphi_\infty\} & \text{if } |\varphi(x)| < \epsilon \text{ for all } x \in F, \\ U(\varphi, F, \epsilon) & \text{otherwise.} \end{cases}$$

It follows that the Gelfand topology on $\Delta(A)$ coincides with the relative Gelfand topology of $\Delta(A_e)$. However, the singleton $\{\varphi_\infty\}$ is closed in $\Delta(A_e)$, so that $\Delta(A)$ is open in $\Delta(A_e)$ and hence is locally compact. This proves (i).

Finally, for $x \in A$ and $\epsilon > 0$,

$$\begin{aligned} U_\varepsilon(\varphi_\infty, x, \epsilon) &= \{\varphi_\infty\} \cup \{\varphi \in \Delta(A) : |\varphi(x)| < \epsilon\} \\ &= \Delta(A_e) \setminus \{\psi \in \Delta(A_e) : |\psi(x)| \geq \epsilon\}. \end{aligned}$$

Now, the sets $\{\psi \in \Delta(A_e), |\psi(x)| \geq \epsilon\}$, $x \in A$, are closed in $\Delta(A_e)$ and hence compact. The complement of a basic neighbourhood of φ_∞ is a finite union of such compact sets. Therefore it follows that $\Delta(A_e)$ is the one-point compactification of $\Delta(A)$. □

A natural question arising in view of the preceding theorem is whether a semisimple commutative Banach algebra A has to possess an identity if $\Delta(A)$ is compact. Actually, this is true. This turns out to be a consequence of Shilov's idempotent theorem, the proof of which utilises the several-variable functional calculus. A considerably simpler proof is available when A is regular (Corollary 4.2.11).

Definition 2.2.4. For $x \in A$, we define $\widehat{x} : \Delta(A) \to \mathbb{C}$ by $\widehat{x}(\varphi) = \varphi(x)$. Then $\widehat{x}$ is a continuous function, which is called the *Gelfand transform* of x. It is easily checked that the mapping

$$\Gamma_A : A \to C(\Delta(A)), \quad x \to \widehat{x}$$

is a homomorphism, the *Gelfand homomorphism* or *Gelfand representation* of A. We quite often denote $\Gamma_A(A)$ by $\widehat{A}$.

Fundamental properties of the Gelfand transform and the Gelfand representation are given in the next theorems.

Theorem 2.2.5. *Let A be a commutative Banach algebra. For each $x \in A$,*

$$\sigma_A(x) \setminus \{0\} \subseteq \widehat{x}(\Delta(A)) = \{\varphi(x) : \varphi \in \Delta(A)\} \subseteq \sigma_A(x).$$

If A is unital, then $\widehat{x}(\Delta(A)) = \sigma_A(x)$.

Proof. Suppose first that A has an identity e. Then $\varphi(x) \in \sigma_A(x)$ for every $\varphi \in \Delta(A)$ (see Theorem 2.1.2). Conversely, if $\lambda \in \sigma_A(x)$, then

$$I = (\lambda e - x)A$$

is a proper ideal in A and hence contained in $\ker \varphi$ for some $\varphi \in \Delta(A)$ (Lemma 1.3.2 and Theorem 2.1.8). It follows that $\varphi(\lambda e - x) = 0$, so that $\lambda \in \widehat{x}(\Delta(A))$.

If A fails to be unital, then by the preceding paragraph and the definition of the spectrum,

$$\begin{aligned} \sigma_A(x) \setminus \{0\} &= \sigma_{A_e}(x) \setminus \{0\} = \widehat{x}(\Delta(A_e)) \setminus \{0\} \\ &\subseteq \widehat{x}(\Delta(A)) = \widehat{x}(\Delta(A_e)) = \sigma_{A_e}(x) \\ &= \sigma_A(x), \end{aligned}$$

as was to be shown. □

The following corollary is an immediate consequence of Theorem 2.2.5 and the spectral radius formula.

Corollary 2.2.6. *For $x \in A, \widehat{x} = 0$ if and only if*

$$r_A(x) = \lim_{n\to\infty} \|x^n\|^{1/n} = 0.$$

Theorem 2.2.7. *Let A be a commutative Banach algebra and Γ the Gelfand representation of A.*

(i) *Γ maps A into $C_0(\Delta(A))$ and is norm decreasing.*
(ii) *$\Gamma(A)$ strongly separates the points of $\Delta(A)$.*
(iii) *Γ is isometric if and only if $\|x\|^2 = \|x^2\|$ for all $x \in A$.*

Proof. (i) Since, by Theorem 2.2.3, $\Delta(A_e)$ is the one-point compactification of $\Delta(A)$ and $\widehat{x}(\varphi_\infty) = 0$ for $x \in A$, we have $\widehat{x} \in C_0(\Delta(A))$. Moreover, by Theorem 2.2.5,

$$\|\widehat{x}\|_\infty = r_A(x) \leq \|x\|.$$

(ii) It is clear that $\Gamma(A)$ strongly separates the points of $\Delta(A)$, that is, $\Gamma(A)(\varphi) \neq \{0\}$ for each $\varphi \in \Delta(A)$, and if $\varphi_1 \neq \varphi_2$, then $\widehat{x}(\varphi_1) \neq \widehat{x}(\varphi_1)$ for some $x \in A$.

(iii) If $\|y\|^2 = \|y^2\|$ for all $y \in A$, then $\|x^{2^n}\| = \|x\|^{2^n}$ for every $x \in A$ and $n \in \mathbb{N}$. Hence

$$\|\widehat{x}\|_\infty = r_A(x) = \lim_{n\to\infty} \|x^{2^n}\|^{1/2^n} = \|x\|.$$

Conversely, $\|x^2\| = \|\widehat{x}^2\|_\infty = \|\widehat{x}\|_\infty^2 = \|x\|^2$ when Γ is an isometry. □

We now present three simple examples. More difficult and challenging ones are discussed in subsequent sections.

Example 2.2.8. Let X be a locally compact Hausdorff space. The closed ideals in $C_0(X)$ have been completely determined in Theorem 1.3.6. In particular,

$$x \to M_x = \{f \in C_0(X) : f(x) = 0\}$$

sets up a one-to-one correspondence between the points of X and the maximal modular ideals of $C_0(X)$. On the other hand, by Theorem 2.1.8, we have a bijection

$$\Delta(C_0(X)) \to \mathrm{Max}(C_0(X)), \quad \varphi \to \ker \varphi.$$

This yields a bijection $X \to \Delta(C_0(X)), x \to \varphi_x$ where $\varphi_x(f) = f(x)$ for $f \in C_0(X)$. The map $x \to \varphi_x$ is a homeomorphism. Indeed, given $x \in X$ and an open neighbourhood V of x, by Urysohn's lemma there exists $f \in C_0(X)$ such that $f(x) \neq 0$ and $f|_{X\setminus V} = 0$, and hence V contains the Gelfand neighbourhood $\{y : |\varphi_y(f) - \varphi_x(f)| < |f(x)|\}$ of x. After identifying X with $\Delta(C_0(X))$, the Gelfand homomorphism of $C_0(X)$ is the identity mapping.

Example 2.2.9. Let $A = C^n[a,b]$, and for each $t \in [0,1]$ define $\varphi_t \in \Delta(A)$ by $\varphi_t(f) = f(t)$. We claim that

$$\phi : [a,b] \to \Delta(A), \quad t \to \varphi_t$$

is a homeomorphism. Obviously, ϕ is injective and continuous. Let M be any maximal ideal in A. Then, by the same reasoning as in the proof of Theorem 1.3.6, we find $s \in [a,b]$ such that $M = \{f \in A : f(s) = 0\}$. It follows that $M = \ker \varphi_s$. Hence ϕ is a homeomorphism since $[a,b]$ is compact and $\Delta(A)$ is Hausdorff. As in the previous example, after identifying $[a,b]$ with $\Delta(A)$, the Gelfand homomorphism of A is the identity mapping.

Example 2.2.10. We determine the structure space of $l^1(\mathbb{Z})$. For $z \in \mathbb{T}$, define $\varphi_z : l^1(\mathbb{Z}) \to \mathbb{C}$ by

$$\varphi_z(f) = \sum_{n\in\mathbb{Z}} f(n) z^{-n}.$$

Then, for $f, g \in l^1(\mathbb{Z})$,

$$\begin{aligned} \varphi_z(f * g) &= \sum_{n\in\mathbb{Z}} \left(\sum_{m\in\mathbb{Z}} f(n-m) g(m) \right) z^{-n} \\ &= \sum_{n,m\in\mathbb{Z}} f(n) g(m) z^{-(n+m)} \\ &= \varphi_z(f)\varphi_z(g). \end{aligned}$$

Thus $\varphi_z \in \Delta(l^1(\mathbb{Z}))$ and the map $z \to \varphi_z$ is clearly injective. Conversely, every $\varphi \in \Delta(l^1(\mathbb{Z}))$ is of this form. Indeed, let $z = \varphi(\delta_{-1})$. Then

$$\varphi(\delta_{-n}) = \varphi(\delta_{-1} * \ldots * \delta_{-1}) = \varphi(\delta_{-1})^n = z^n$$

and hence also $\varphi(\delta_n) = 1/\varphi(\delta_{-n}) = z^{-n}$ for all $n \in \mathbb{N}$. Since the finite linear combinations of Dirac functions δ_n, $n \in \mathbb{N}$, are dense in $l^1(\mathbb{Z})$, it follows that $\varphi = \varphi_z$. By routine arguments it is shown that the map $z \to \varphi_z$ is a homeomorphism.

We have seen earlier (Example 1.1.5) that the commutative Banach algebra $AC(\mathbb{T})$ is isomorphic to $l^1(\mathbb{Z})$, the isomorphism being given by $f \to (c_n(f))_n$, where

$$c_n(f) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{it}) e^{-int} dt$$

for $n \in \mathbb{Z}$. Thus, by the preceding example, $\Delta(AC(\mathbb{T}))$ can be identified with $\mathbb{T}$ as follows. For $z \in \mathbb{T}$, let

$$\varphi_z(f) = \sum_{n\in\mathbb{Z}} c_n(f) z^n, \quad f \in AC(\mathbb{T}).$$

Then $z \to \varphi_z$ is a homeomorphism between $\mathbb{T}$ and $\Delta(AC(\mathbb{T}))$. On making this identification,

$$\widehat{f}(z) = \sum_{n\in\mathbb{Z}} c_n(f) z^n = f(z)$$

for all $f \in AC(\mathbb{T})$, so that the Gelfand representation of $AC(\mathbb{T})$ is the identity. As a simple consequence we obtain the following classical result due to Wiener.

Theorem 2.2.11. *If $f \in AC(\mathbb{T})$ is such that $f(z) \neq 0$ for all $z \in \mathbb{T}$, then $1/f \in AC(\mathbb{T})$; that is, $1/f$ has an absolutely convergent Fourier series.*

Proof. With the previous identification of $\Delta(AC(\mathbb{T}))$ with $\mathbb{T}$, the assumption on f means that f belongs to no maximal ideal of $AC(\mathbb{T})$. Thus f is invertible in $AC(\mathbb{T})$ and so $1/f \in AC(\mathbb{T})$. □

Lemma 2.2.12. *Let A and B be commutative Banach algebras. If A and B are algebraically isomorphic, then $\Delta(A)$ and $\Delta(B)$ are homeomorphic.*

Proof. Suppose $\phi : A \to B$ is an algebra isomorphism. Let $\phi^* : \Delta(B) \to \Delta(A)$ be the dual mapping; that is,

$$\phi^*(\varphi)(a) = \varphi(\phi(a)), \quad a \in A, \quad \varphi \in \Delta(B).$$

It is easily checked that ϕ^* is a bijection. ϕ^* is continuous provided that all functions

$$\Delta(B) \to \mathbb{C}, \quad \varphi \to \phi^*(\varphi)(a), \quad a \in A,$$

are continuous. However, that such functions are continuous follows immediately from the definition of ϕ^* and the definition of the topology on $\Delta(B)$. $(\phi^*)^{-1}$ is continuous on the same grounds. □

Corollary 2.2.13. *For locally compact Hausdorff spaces X and Y the following conditions are equivalent.*

(i) *$C_0(X)$ and $C_0(Y)$ are isometrically isomorphic.*
(ii) *$C_0(X)$ and $C_0(Y)$ are algebraically isomorphic.*
(iii) *X and Y are homeomorphic.*

Proof. (i) ⇒ (ii) is trivial, and (ii) ⇒ (iii) is a consequence of the preceding lemma and Example 2.2.8. Finally, if $\phi : X \to Y$ is a homeomorphism, then $f \to f \circ \phi$ is an isometric algebra isomorphism from $C_0(Y)$ to $C_0(X)$. □

We continue with a proposition which often can efficiently be used to identify the Gelfand topology.

Proposition 2.2.14. *Let X be a locally compact Hausdorff space and let A be a family of functions in $C_0(X)$ which strongly separates the points of X. Then the topology of X equals the weak topology with respect to the functions $x \to f(x), f \in A$.*

Proof. The given topology on X is stronger than the weak topology. Thus it suffices to show that given $x \in X$ and an open neighbourhood U of x in X, there exists a set V such that $x \in V \subseteq U$ and V is open in the weak topology. Let $\widetilde{X}$ be X if X is compact, and let $\widetilde{X} = X \cup \{\infty\}$ be the one-point compactification of X if X is noncompact. Every $f \in C_0(X)$ extends continuously to $\widetilde{X}$ by setting $f(\infty) = 0$. Since A strongly separates the points of X, for every $y \in \widetilde{X} \setminus U$ there exists $f_y \in A$ such that

$$\epsilon_y = |f_y(y) - f_y(x)| > 0.$$

Then, for every $y \in \widetilde{X} \setminus U$,

$$V_y = \{z \in \widetilde{X} : |f_y(z) - f_y(y)| < \epsilon_y/2\}$$

is an open neighbourhood of y in $\widetilde{X}$, and because $\widetilde{X} \setminus U$ is compact there are finitely many $y_1, \ldots, y_n \in \widetilde{X} \setminus U$ such that $\widetilde{X} \setminus U \subseteq \bigcup_{j=1}^n V_{y_j}$. Let

$$V = \{z \in X : |f_{y_j}(z) - f_{y_j}(x)| < \epsilon_{y_j}/2 \text{ for all } 1 \leq j \leq n\}.$$

Then $x \in V$ and V is contained in U. Indeed, if $z \in V$ and $z \notin U$, then $z \in V_{y_j}$ for some j, and hence

$$|f_{y_j}(x) - f_{y_j}(y_j)| \leq |f_{y_j}(x) - f_{y_j}(z)| + |f_{y_j}(z) - f_{y_j}(y_j)| < \epsilon_{y_j}.$$

This contradicts the definition of ϵ_{y_j}. □

For a closed ideal I of a commutative Banach algebra A, we now relate the Gelfand topologies on $\Delta(I)$ and on $\Delta(A/I)$ to the Gelfand topology on $\Delta(A)$. For a subset M of A, the *hull* $h(M)$ of M is defined to be

$$h(M) = \{\varphi \in \Delta(A) : \varphi(M) = \{0\}\}.$$

Lemma 2.2.15. *Let I be a closed ideal of A and $q : A \to A/I$ the quotient homomorphism.*

(i) *The map $\varphi \to \varphi \circ q$ is a homeomorphism from $\Delta(A/I)$ onto $h(I)$.*
(ii) *The map $\varphi \to \varphi|_I$ is a homeomorphism from $\Delta(A) \setminus h(I)$ onto $\Delta(I)$.*

Proof. (i) It is obvious that the map is a bijection. It is a homeomorphism since

$$\begin{aligned} U(\varphi, x + I, \epsilon) \circ q &= \{\psi \circ q : \psi \in \Delta(A/I), |\psi(x + I) - \varphi(x + I)| < \epsilon\} \\ &= \{\rho \in h(I) : |\rho(x) - \varphi \circ q(x)| < \epsilon\} \\ &= U(\varphi \circ q, x, \epsilon) \end{aligned}$$

for all $\varphi \in \Delta(A/I), x \in A$ and $\epsilon > 0$.

(ii) If $\varphi_1, \varphi_2 \in \Delta(A) \setminus h(I)$ are such that $\varphi_1|_I = \varphi_2|_I$, then choosing $x \in I$ such that $\varphi_1(x) = 1$, it follows that

$$\varphi_1(y) = \varphi_1(yx) = \varphi_2(yx) = \varphi_2(y)$$

for all $y \in A$. So the map $\varphi \to \varphi|_I$ is injective, and it is clearly continuous. Given $\psi \in \Delta(I)$, again choose $x \in I$ with $\psi(x) = 1$ and define φ on A by $\varphi(y) = \psi(yx)$, $y \in A$. Then φ extends ψ, and it is easily verified that $\varphi \in \Delta(A) \setminus h(I)$. Finally, let $\varphi \in \Delta(A) \setminus h(I)$, $y \in A$, $y \neq 0$, and $\epsilon > 0$ be given and let $\delta = \min\{\epsilon/2, \epsilon/2\|y\|\}$. Then, if $\rho \in \Delta(A)$ is such that $\rho|_I \in U(\varphi|_I, x, yx, \delta)$, it follows that

$$\begin{aligned} |\rho(y) - \varphi(y)| &\leq |\rho(y)| \cdot |\varphi(x) - \rho(x)| + |\rho(yx) - \varphi(yx)| \\ &< \delta\|y\| + \delta \leq \epsilon, \end{aligned}$$

whence $\rho \in U(\varphi, y, \epsilon)$. Thus the map $\varphi \to \varphi|_I$ is also open, hence a homeomorphism. □

By Lemma 2.2.15, for each $y \in A$ there is a unique continuous function f_y on $\Delta(I)$ such that $\widehat{yx}(\varphi) = f_y(\varphi)\widehat{x}(\varphi)$ for all $\varphi \in \Delta(I)$ and $x \in A$. This in particular applies when a commutative Banach algebra A has a bounded approximate identity and hence can be considered as a closed ideal of its multiplier algebra $M(A)$ (Theorem 1.4.12). The following proposition, however, shows that this same conclusion holds if A is merely assumed to be faithful (see Proposition 1.4.11).

Proposition 2.2.16. *Let A be a commutative Banach algebra and let $T \in M(A)$. Then there exists a unique continuous function f on $\Delta(A)$ such that $\widehat{Tx}(\varphi) = f(\varphi)\widehat{x}(\varphi)$ for all $\varphi \in \Delta(A)$ and $x \in A$. Furthermore, f is bounded and $\|f\|_\infty \leq \|T\|$.*

Proof. If $\varphi \in \Delta(A)$ and $x, y \in A$ are such that $\widehat{x}(\varphi) \neq 0$ and $\widehat{y}(\varphi) \neq 0$, then it follows from $(Tx)y = x(Ty)$ that

$$\frac{\widehat{Tx}(\varphi)}{\widehat{x}(\varphi)} = \frac{\widehat{Ty}(\varphi)}{\widehat{y}(\varphi)}.$$

For each $\varphi \in \Delta(A)$ choose $x \in A$ with $\widehat{x}(\varphi) \neq 0$, and define

$$f(\varphi) = \frac{\widehat{Tx}(\varphi)}{\widehat{x}(\varphi)}.$$

The above equation shows that this definition is independent of the choice of x, and hence f is a well-defined continuous function on $\Delta(A)$. Moreover, if $\widehat{x}(\varphi) = 0$ then $\widehat{Tx}(\varphi) = 0$. Indeed, this follows from

$$\widehat{Tx}(\varphi)\widehat{y}(\varphi) = \widehat{x}(\varphi)\widehat{Ty}(\varphi)$$

by choosing y such that $\widehat{y}(\varphi) \neq 0$. Thus the equation $\widehat{Tx}(\varphi) = f(\varphi)\widehat{x}(\varphi)$ holds for all $x \in A$ and $\varphi \in \Delta(A)$.

If g is a second continuous function on $\Delta(A)$ satisfying $\widehat{Tx} = g\widehat{x}$ for all $x \in A$, then $(f(\varphi) - g(\varphi))\widehat{x}(\varphi) = 0$ for all $x \in A$ and $\varphi \in \Delta(A)$, and this implies $f(\varphi) = g(\varphi)$. So f is unique.

To show that f is bounded, observe that

$$|f(\varphi)\widehat{x}(\varphi)| = |\widehat{Tx}(\varphi)| \leq \|\varphi\| \cdot \|Tx\| \leq \|\varphi\| \cdot \|T\| \cdot \|x\|$$

for all $x \in A$ and $\varphi \in \Delta(A)$. Taking $x \in A$ with $\|x\| = 1$, we obtain

$$|f(\varphi)| \cdot \sup\{|\widehat{x}(\varphi)| : \|x\| = 1\} \leq \|\varphi\| \cdot \|T\|,$$

for all $\varphi \in \Delta(A)$ and hence $\|f\|_\infty \leq \|T\|$. □

2.3 Finitely generated commutative Banach algebras

Many naturally occuring Banach algebras are generated (in the sense of the following definition) by finitely many elements. Such algebras admit a particularly satisfying description of their structure spaces and this is the theme of the present section.

Definition 2.3.1. Let A be a commutative Banach algebra with identity e. A subset E of A is said to *generate* A if every closed subalgebra of A containing E and e coincides with A. Equivalently, the set of all finite linear combinations of elements of the form

$$x_1^{n_1} x_2^{n_2} \cdots x_r^{n_r}, \quad x_i \in E, \quad n_i \in \mathbb{N} \cup \{0\}, \quad r \in \mathbb{N},$$

is dense in A. A is called *finitely generated* if there exists a finite subset of A that generates A.

As a very simple example, recall that $l^1(\mathbb{Z})$ is generated by the two Dirac functions δ_1 and δ_{-1}.

Definition 2.3.2. Let A be a commutative Banach algebra with identity and let $x_1, \ldots, x_n \in A$. Then the *joint spectrum* of $x_1, \ldots, x_n$ is the subset $\sigma_A(x_1, \ldots, x_n)$ of $\mathbb{C}^n$ defined by

$$\sigma_A(x_1, \ldots, x_n) = \{(\varphi(x_1), \ldots, \varphi(x_n)) : \varphi \in \Delta(A)\}.$$

Since $\Delta(A)$ is compact and the mapping

$$\Delta(A) \to \mathbb{C}^n, \quad \varphi \to (\varphi(x_1), \ldots, \varphi(x_n))$$

is continuous, $\sigma_A(x_1, \ldots, x_n)$ is a compact subset of $\mathbb{C}^n$. It is also evident from Theorem 2.2.5 that the joint spectrum of a single element x reduces to the spectrum $\sigma_A(x)$ of x.

Lemma 2.3.3. *Let A be a unital commutative Banach algebra, and suppose that $E \subseteq A$ generates A. Then the mapping*

$$\phi : \Delta(A) \to \prod_{x \in E} \sigma_A(x), \quad \varphi \to (\varphi(x))_{x \in E}$$

is a homeomorphism between $\Delta(A)$ and $\phi(\Delta(A)) \subseteq \prod_{x \in E} \sigma_A(x)$. In particular, if E is finite, say $E = \{x_1, \ldots, x_n\}$, then we have a homeomorphism

$$\Delta(A) \to \sigma_A(x_1, \ldots, x_n), \quad \varphi \to (\varphi(x_1), \ldots, \varphi(x_n)).$$

Proof. Assume first that $\varphi_1, \varphi_2 \in \Delta(A)$ are such that $\varphi_1(x) = \varphi_2(x)$ for all $x \in E$. Let B denote the smallest subalgebra of A containing E and the identity. Then B is dense in A, and $\varphi_1(y) = \varphi_2(y)$ for all $y \in B$. Since elements in $\Delta(A)$ are continuous it follows that $\varphi_1 = \varphi_2$. Hence ϕ is injective.

Now $\prod_{x \in E} \sigma_A(x)$ carries the weak topology with respect to the projections

$$p_y : \prod_{x \in E} \sigma_A(x) \to \sigma_A(y), \quad y \in E.$$

Therefore ϕ is continuous provided that all the functions $p_y \circ \phi, y \in E$, are continuous. However, this is clear from $p_y \circ \phi(\varphi) = \varphi(y)$. Thus

$$\phi : \Delta(A) \to \phi(\Delta(A)), \quad \varphi \to (\varphi(x))_{x \in E}$$

is a continuous bijection between a compact space and a Hausdorff space, and hence is a homeomorphism. □

We now aim at characterizing those compact subsets of $\mathbb{C}^n$ which arise in this way as structure spaces of commutative Banach algebras generated by n elements, $n \in \mathbb{N}$ (Theorem 2.3.6). The relevant geometrical notion is that of polynomial convexity.

Definition 2.3.4. A compact subset K of $\mathbb{C}^n, n \in \mathbb{N}$, is said to be *polynomially convex* if for every $z \in \mathbb{C}^n \setminus K$ there exists a polynomial p such that $p(z) = 1$ and $|p(w)| < 1$ for all $w \in K$.

Lemma 2.3.5. *Every compact convex subset K of $\mathbb{C}^n$ is polynomially convex.*

Proof. We view $\mathbb{C}^n$ as a $2n$-dimensional real vector space. Then, given $w \in \mathbb{C}^n \setminus K$, there exist a real linear functional ψ on $\mathbb{C}^n = \mathbb{R}^{2n}$ and $\alpha \in \mathbb{R}$ such that

$$\psi(w) > \alpha \text{ and } \psi(z) < \alpha \text{ for all } z \in K.$$

Let $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$, with $z_j = x_j + iy_j, x_j, y_j \in \mathbb{R}$. Then ψ has the form

$$\psi(z) = \sum_{j=1}^{n} (a_j x_j + b_j y_j),$$

where $a_j, b_j \in \mathbb{R}, 1 \leq j \leq n$. Let $c_j = a_j - ib_j, 1 \leq j \leq n$, and consider the function

$$f(z) = \exp\left(\sum_{j=1}^{n} c_j z_j\right)$$

on $\mathbb{C}^n$. Then

$$|f(z)| = \exp\left(\operatorname{Re}\left(\sum_{j=1}^{n} c_j z_j\right)\right) = \exp\left(\sum_{j=1}^{n}(a_j x_j + b_j y_j)\right) = \exp\psi(z)$$

and hence $|f(w)| > e^\alpha$ and $|f(z)| < e^\alpha$ for all $z \in K$. It follows that, for a suitable $N \in \mathbb{N}$, the polynomial q defined by

$$q(z) = \prod_{j=1}^{n}\left(\sum_{k=0}^{N} \frac{1}{k!} c_j^k\, z_j^k\right)$$

satisfies $|q(w)| > e^\alpha$ and $|q(z)| < e^\alpha$ for all $z \in K$. Finally, the polynomial $p = |q(w)|^{-1} q$ has the properties required in Definition 2.3.4. □

Theorem 2.3.6. *For a compact subset K of $\mathbb{C}^n$ the following conditions are equivalent.*

(i) *There exists a unital commutative Banach algebra A which is generated by n elements $x_1, \ldots, x_n$ such that $K = \sigma_A(x_1, \ldots, x_n)$.*
(ii) *K is polynomially convex.*

Proof. To prove (i) ⇒ (ii), let e denote the identity of A and let

$$\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n \setminus \sigma_A(x_1, \ldots, x_n).$$

Then, given any $\varphi \in \Delta(A), \varphi(x_j) \neq \lambda_j$ for some $1 \leq j \leq n$. Equivalently, for each $M \in \operatorname{Max}(A)$ there exists j such that $x_j - \lambda_j e \notin M$. Consider the ideal

$$I = \left\{\sum_{j=1}^{n}(x_j - \lambda_j e) y_j : y_j \in A\right\}$$

of A. If I were a proper ideal, then $I \subseteq M$ for some $M \in \operatorname{Max}(A)$, but $x_j - \lambda_j e \in I$ and $x_j - \lambda_j e \notin M$ for some j. Thus $I = A$, and hence there exist $y_1, \ldots, y_n \in A$ such that

$$\sum_{j=1}^{n}(x_j - \lambda_j e) y_j = e.$$

Choose $\delta > 0$ such that $\delta \sum_{j=1}^{n} \|x_j - \lambda_j e\| < 1$. Since A is generated by $x_1, \ldots, x_n$, there exist polynomials $p_1, \ldots, p_n$ in n variables such that

$$\|p_j(x_1, \ldots, x_n) - y_j\| \leq \delta$$

for $1 \leq j \leq n$. It follows that

$$\left\| e - \sum_{j=1}^{n} (x_j - \lambda_j e) p_j(x_1, \ldots, x_n) \right\| \leq \sum_{j=1}^{n} \|x_j - \lambda_j e\| \cdot \|y_j - p_j(x_1, \ldots, x_n)\| < 1.$$

Now, define a polynomial p on $\mathbb{C}^n$ by

$$p(z_1, \ldots, z_n) = 1 - \sum_{j=1}^{n} (z_j - \lambda_j) p_j(z_1, \ldots, z_n).$$

Then $p(\lambda_1, \ldots, \lambda_n) = 1$, and for every $\varphi \in \Delta(A)$

$$\begin{aligned}
|p(\varphi(x_1), \ldots, \varphi(x_n))| &= \left| 1 - \sum_{j=1}^{n} (\varphi(x_j) - \lambda_j) p_j(\varphi(x_1), \ldots, \varphi(x_n)) \right| \\
&= \left| \varphi(e) - \sum_{j=1}^{n} \varphi(x_j - \lambda_i e) \varphi(p_j(x_1, \ldots, x_n)) \right| \\
&\leq \left\| e - \sum_{j=1}^{n} (x_j - \lambda_j e) p_j(x_1, \ldots, x_n) \right\| \\
&< 1.
\end{aligned}$$

This proves that $\sigma_A(x_1, \ldots, x_n)$ is polynomially convex.

Conversely, suppose that $K \subseteq \mathbb{C}^n$ is polynomially convex. Let $A = P(K)$, the algebra of all functions $f : K \to \mathbb{C}$ that are uniform limits of polynomial functions on K. Then A is generated by the functions

$$f_j(z) = z_j, \quad z = (z_1, \ldots, z_n) \in K, \quad 1 \leq j \leq n.$$

We are going to show that $K = \sigma_A(f_1, \ldots, f_n)$. For $z \in K$, define $\varphi_z \in \Delta(A)$ by $\varphi_z(f) = f(z)$. As distinct points can be separated by the functions f_j, the mapping

$$\phi : K \to \Delta(A), \quad z \to \varphi_z$$

is injective. ϕ is also continuous since $\Delta(A)$ carries the weak topology with respect to the functions $\varphi \to \varphi(f)$, $f \in A$, and $z \to \varphi_z(f) = f(z)$ is continuous on K. Thus ϕ is a homeomorphism from K onto $\phi(K) \subseteq \Delta(A)$. We claim that $\phi(K) = \Delta(A)$. Towards a contradiction, suppose there exists $\varphi \in \Delta(A) \setminus \phi(K)$ and put

$$\lambda_j = \varphi(f_j), \quad 1 \leq j \leq n, \text{ and } \lambda = (\lambda_1, \ldots, \lambda_n).$$

Then $\lambda \notin K$ since otherwise $\varphi_\lambda(f_j) = f_j(\lambda) = \lambda_j = \varphi(f_j), 1 \leq j \leq n$, and hence $\varphi = \varphi_\lambda$ as A is generated by $f_1, \ldots, f_n$. Because K is polynomially convex, we can choose a polynomial p in n variables such that $|p(z_1, \ldots, z_n)| < 1$ for all $z = (z_1, \ldots, z_n) \in K$ and $p(\lambda) = 1$. Then, as K is compact,

$$\|p|_K\|_\infty = \sup_{z\in K} |p(z)| < 1,$$

and hence $|\psi(p|_K)| < 1$ for all $\psi \in \Delta(A)$. Now, $p|_K$ is a finite linear combination of functions of the form

$$z \to z_1^{m_1} z_2^{m_2} \cdot \ldots \cdot z_n^{m_n} = f_1(z)^{m_1} f_2(z)^{m_2} \cdot \ldots \cdot f_n(z)^{m_n}.$$

As $\varphi(f_j) = \lambda_j, 1 \le j \le n$, we obtain $\varphi(p|_K) = p(\lambda) = 1$, which is a contradiction. It follows that $\phi(K) = \Delta(A)$, and hence

$$\begin{aligned}\sigma_A(f_1, \ldots, f_n) &= \{(\varphi_z(f_1), \ldots, \varphi_z(f_n)) : z \in K\} \\ &= \{(z_1, \ldots, z_n) : z \in K\} = K.\end{aligned}$$

This shows (ii) $\Rightarrow$ (i). □

It is worth emphasising that the proof of (ii) $\Rightarrow$ (i) in Theorem 2.3.6 shows that $\Delta(P(K)) = K$ when K is polynomially convex.

The following theorem provides a topological description of polynomially convex subsets of $\mathbb{C}$.

Theorem 2.3.7. *A compact subset K of $\mathbb{C}$ is polynomially convex if and only if $\mathbb{C} \setminus K$ connected.*

Proof. We first assume that K is polynomially convex and that nevertheless $\mathbb{C} \setminus K$ is not connected. Then $\mathbb{C} \setminus K$ has a bounded connected component $S \neq \emptyset$. Then S is closed in $\mathbb{C} \setminus K$ and also open $\mathbb{C} \setminus K$, since $\mathbb{C} \setminus K$ is locally connected. Hence S is also open in $\mathbb{C}$, and therefore its boundary $\partial(S)$ is contained in K.

By Theorem 2.3.6 there exists a commutative Banach algebra A with identity that is generated by some element $a \in A$ such that $K = \sigma_A(a)$. For every $x \in A$ there is a sequence $p_n, n \in \mathbb{N}$, of polynomials such that $\|p_n(a) - x\| \to 0$. Because

$$|p_n(\varphi(a)) - \varphi(x)| = |\varphi(p_n(a)) - \varphi(x)| \le \|p_n(a) - x\|$$

for all $\varphi \in \Delta(A), (p_n)_{n\in\mathbb{N}}$ converges uniformly on $K = \sigma_A(a) = \widehat{a}(\Delta(A))$ with limit $\widehat{x}$. Since $\partial(S) \subseteq K$, $(p_n)_{n\in\mathbb{N}}$ converges uniformly on all of S by the maximum modulus principle. We now fix some $\lambda \in S$. Note that $\lim_{n\to\infty} p_n(\lambda)$ does not depend on the particular choice of polynomials p_n with $p_n(a) \to x$. Indeed, if $(q_n)_n$ is a second sequence of polynomials such that $q_n(a) \to x$, then for each $\varphi \in \Delta(A)$

$$|p_n(\varphi(a)) - q_n(\varphi(a))| \le |p_n(\varphi(a)) - \varphi(x)| + |q_n(\varphi(a)) - \varphi(x)| \to 0,$$

so that $p_n - q_n$ converges uniformly to zero on K, and hence on S. It follows that

$$\lim_{n\to\infty} p_n(\lambda) = \lim_{n\to\infty} q_n(\lambda).$$

This allows us to define $\psi : A \to \mathbb{C}$ by setting

$$\psi(x) = \lim_{n\to\infty} p_n(\lambda),$$

where $(p_n)_n$ is any sequence of polynomials with $p_n(a) \to x$. It is now easily verified that ψ is a homomorphism. For example, if $p_n(a) \to x$ and $q_n(a) \to y$, then $(p_n q_n)(a) \to xy$ and therefore

$$\psi(xy) = \lim_{n\to\infty} (p_n q_n)(\lambda) = \lim_{n\to\infty} p_n(\lambda) \cdot \lim_{n\to\infty} q_n(\lambda) = \psi(x)\psi(y).$$

With $p_n \equiv 1, n \in \mathbb{N}$, we get $\psi(e) = 1$, so that $\psi \in \Delta(A)$. Finally, choosing $p_n(z) = z$ for all $z \in \mathbb{C}, n \in \mathbb{N}$, we obtain $p_n(a) = a$ and hence

$$\psi(a) = \lim_{n\to\infty} p_n(\lambda) = \lambda.$$

Thus $\lambda \in \widehat{a}(\Delta(A)) = K$, contradicting the fact that $\lambda \in S \subseteq \mathbb{C} \setminus K$.

Conversely, suppose that $\mathbb{C} \setminus K$ is connected, and consider $A = P(K)$ as in the proof of Theorem 2.3.6, (ii) $\Rightarrow$ (i). Then A is generated by the function $f(z) = z$. Moreover, $\sigma_{C(K)}(f)$, the spectrum of f in $C(K)$, equals K since $z \to \lambda - f(z)$ is invertible in $C(K)$ if and only if $\lambda \notin K$. As $\mathbb{C} \setminus K$ is connected, Theorem 1.2.12 implies $K = \sigma_A(f)$, and hence K is polynomially convex by Theorem 2.3.6, (i) $\Rightarrow$ (ii). □

Remark 2.3.8. More generally, it is true for arbitrary $n \in \mathbb{N}$, that if $K \subseteq \mathbb{C}^n$ is polynomially convex, then $\mathbb{C}^n \setminus K$ is connected. This is proved analogously by employing the maximum modulus principle for polynomials of several complex variables. However, the following example shows that for $n \geq 2$ there exist compact subsets of $\mathbb{C}^n$ which fail to be polynomially convex, even though $\mathbb{C}^n \setminus K$ is connected.

Example 2.3.9. Let $n \geq 2$ and

$$K = \{z = (z_1, \ldots, z_n) \in \mathbb{C}^n : |z_j| = 1, 1 \leq j \leq n\}.$$

Assuming that K is polynomially convex we find a polynomial p in n variables such that $|p(z)| < 1$ for all $z \in K$ and $p(0, 1, \ldots, 1) = 1$. Define a polynomial q in one variable by

$$q(w) = p(w, 1, \ldots, 1), \quad w \in \mathbb{C}.$$

Then $|q(w)| < 1$ for all $w \in \mathbb{C}$ with $|w| = 1$ and $q(0) = 1$. This contradicts the maximum modulus principle. Nevertheless, $\mathbb{C}^n \setminus K$ is connected. To see this, let

$$A_j = \{z = (z_1, \ldots, z_n) \in \mathbb{C}^n : |z_j| > 1\}$$

and

$$B_j = \{z = (z_1, \ldots, z_n) \in \mathbb{C}^n : |z_j| < 1\},$$

$1 \leq j \leq n$, we see that $\mathbb{C}^n \setminus K = \bigcup_{j=1}^n (A_j \cup B_j)$. The sets A_j and B_j are arcwise connected, $A_j \cap A_k \neq \emptyset$, $B_j \cap B_k \neq \emptyset$, and, for $j \neq k$, $A_j \cap B_k \neq \emptyset$. It follows that $\mathbb{C}^n \setminus K$ is connected.

2.4 Commutative C^*-algebras

In this section we investigate the question of when the Gelfand homomorphism of a commutative Banach algebra A is an isometric isomorphism onto $C_0(\Delta(A))$. We start with the relevant definition.

Definition 2.4.1. Let A be a Banach algebra with involution $x \to x^*$. Then A is called a C^*-*algebra*, if its norm satisfies the equation $\|x^*x\| = \|x\|^2$ for all $x \in A$. The definition of a C^*-subalgebra is evident.

Note that a C^*-algebra is a Banach $*$-algebra since the equation $\|x\|^2 = \|x^*x\|$ implies $\|x\| \leq \|x^*\|$ and hence $\|x\| = \|x^*\|$ for all $x \in A$.

Now let A be a commutative Banach algebra for which the Gelfand homomorphism is an isometric isomorphism onto $C_0(\Delta(A))$. Notice first that in this case for every $x \in A$ there is a unique element $x^* \in A$ such that $\widehat{x^*} = \overline{\widehat{x}}$. Obviously, the mapping $x \to x^*$ is an involution. Moreover,

$$\|x^*\| = \|\widehat{x^*}\|_\infty = \|\widehat{x}\|_\infty = \|x\|,$$

and hence

$$\|x^*x\| = \|\widehat{x^*x}\|_\infty = \|\overline{\widehat{x}}\,\widehat{x}\|_\infty = \|\widehat{x}\|_\infty^2 = \|x\|^2.$$

Thus A is a C^*-algebra. The main purpose of what follows is to show that conversely for each commutative C^*-algebra A the Gelfand homomorphism is an isometric $*$-isomorphism onto $C_0(\Delta(A))$. This is one of the most striking results in Gelfand's theory.

Example 2.4.2. (1) Let X be an arbitrary topological space. With the involution given by $f^*(x) = \overline{f(x)}$ and the supremum norm $\|\cdot\|_\infty$, $C^b(X)$ is a commutative C^*-algebra. If X is a locally compact Hausdorff space, then $C_0(X)$ is a C^*-subalgebra of $C^b(X)$.

(2) Let H be a complex Hilbert space, and recall that for $T \in \mathcal{B}(H)$, T^* denotes the adjoint operator of T. Then $\mathcal{B}(H)$ is a C^*-algebra since $\|T^*T\| = \|T\|^2$ holds for all $T \in \mathcal{B}(H)$. However, $\mathcal{B}(H)$ is not commutative whenever $\dim H \geq 2$. $\mathcal{K}(H)$, the closed ideal consisting of all compact operators in H, is a C^*-subalgebra of $\mathcal{B}(H)$ because T^* is compact whenever T is.

(3) Suppose $T \in \mathcal{B}(H)$ is normal, that is, $T^*T = TT^*$, and let $A(T)$ denote the smallest closed subalgebra of $\mathcal{B}(H)$ containing T, T^* and the identity operator of H. Then $A(T)$ is a commutative C^*-algebra with identity.

(4) The Gelfand–Naimark theorem [39] states that for every C^*-algebra A there exists a Hilbert space H such that A is isometrically $*$-isomorphic to some C^*-subalgebra of $\mathcal{B}(H)$.

(5) Let G be a locally compact Abelian group. Then $L^1(G)$ is a commutative Banach $*$-algebra. However, whenever $G \neq \{e\}$, the L^1-norm fails to be a C^*-norm. In fact, it is not difficult to construct $f \in L^1(G)$ such that

$$\|f^* * f\|_1 \neq \|f\|_1^2$$

(Exercise 2.12.25).

(6) The assignment $f \to f^*$, where $f^*(z) = \overline{f(\overline{z})}$, defines an involution on the disc algebra $A(\mathbb{D})$ (Example 1.1.7(2)). However, $A(\mathbb{D})$ fails to be a C^*-algebra (Exercise 1.6.15).

If A is a $*$-algebra, then so is A_e once we define

$$(a + \lambda e)^* = a^* + \bar{\lambda} e, \quad a \in A, \quad \lambda \in \mathbb{C}.$$

Then A_e is a normed $*$-algebra with $\|a + \lambda e\| = \|a\| + |\lambda|$, yet in general not a C^*-algebra if A is. The following lemma, where we do not assume A to be commutative, shows that nevertheless a different norm can be introduced on A_e which extends the norm on A and turns A_e into a C^*-algebra.

Lemma 2.4.3. *Let A be a C^*-algebra without identity. There exists a norm $\|\cdot\|_0$ on A_e such that $\|a\|_0 = \|a\|$ for all $a \in A$ and $(A_e, \|\cdot\|_0)$ becomes a C^*-algebra.*

Proof. Let $\|\cdot\|$ denote the above norm on A_e; that is,

$$\|a + \lambda e\| = \|a\| + |\lambda|, \quad a \in A, \quad \lambda \in \mathbb{C}.$$

For $x \in A_e$, let $L_x : A \to A$ be defined by $L_x(a) = xa$, $a \in A$. Then

$$\|L_x a\| \leq \|x\| \cdot \|a\|,$$

so that L_x is bounded and $\|L_x\| \leq \|x\|$.

We claim that $\|x\|_0 = \|L_x\|$ defines a C^*-norm on A_e extending the given norm on A. Note first that, for $a \in A$,

$$\|L_a(a^*)\| = \|aa^*\| = \|a\|^2 = \|a\| \cdot \|a^*\|$$

and hence $\|L_a\| \geq \|a\|$ and therefore $\|L_a\| = \|a\|$. Now, $x \to \|x\|_0$ is a norm on A_e as soon as we have seen that $L_x = 0$ implies $x = 0$. To this end let

$$x = b + \lambda e, \quad b \in A, \quad \lambda \in \mathbb{C},$$

be such that $xa = 0$ for all $a \in A$. If $\lambda \neq 0$, then $a = (-(1/\lambda)b)a$ for all $a \in A$, that is, $u = -(1/\lambda)b$ is a left identity for A. Since

$$u^* = uu^* = (uu^*)^* = (u^*)^* = u,$$

and hence, for all $a \in A$,

$$au = au^* = (ua^*)^* = (a^*)^* = a,$$

u is also a right identity for A. This contradiction yields $x = b \in A$, and therefore $x = 0$ as $\|b\| = \|L_b\|$. Moreover, $\|\cdot\|_0$ is an algebra norm since

$$\|xy\|_0 = \|L_{xy}\| = \|L_x \circ L_y\| \leq \|L_x\|\|L_y\| = \|x\|_0\|y\|_0,$$

and A_e is complete because A is complete and A_e/A is one-dimensional.

Finally, $\|\cdot\|_0$ is a C^*-norm on A_e. Indeed, from

$$\begin{aligned}\|L_x(a)\|^2 &= \|xa\|^2 = \|(xa)^*(xa)\| \\ &= \|a^*(x^*x)a\| \leq \|a^*\| \cdot \|L_{x^*x}a\| \\ &\leq \|a\|^2\|L_{x^*x}\|\end{aligned}$$

it follows that

$$\|x\|_0^2 = \|L_x\|^2 \leq \|L_{x^*x}\| = \|x^*x\|_0 \leq \|L_{x^*}\|\|L_x\| = \|x^*\|_0\|x\|_0,$$

and this in turn gives

$$\|x\|_0 \leq \|x^*\|_0 \text{ and } \|x^*\|_0 \leq \|x^{**}\|_0 = \|x\|_0.$$

Thus $\|x^*\|_0 = \|x\|_0$, and $\|x^*x\|_0 \leq \|x^*\|_0\|x\|_0 = \|x\|_0^2$. □

Lemma 2.4.4. *Let A be a commutative C^*-algebra. Then the Gelfand homomorphism is a $*$-homomorphism; that is, $\widehat{x^*} = \overline{\widehat{x}}$ for all $x \in A$.*

Proof. We have to show that $\varphi(x^*) = \overline{\varphi(x)}$ for $\varphi \in \Delta(A)$ and $x \in A$. Of course, we can assume that A has an identity e. Let

$$\varphi(x) = \alpha + i\beta \text{ and } \varphi(x^*) = \gamma + i\delta,$$

$\alpha, \beta, \gamma, \delta \in \mathbb{R}$. Towards a contradiction, assume that $\beta + \delta \neq 0$ and let

$$y = (\beta + \delta)^{-1}(x + x^* - (\alpha + \gamma)e) \in A.$$

Then $y = y^*$ and

$$\varphi(y) = (\beta + \delta)^{-1}(\alpha + i\beta + \gamma + i\delta - (\alpha + \gamma)) = i.$$

This implies that, for all $t \in \mathbb{R}$,

$$\varphi(y + tie) = \varphi(y) + ti = (t + 1)i,$$

and hence $|t + 1| \leq \|y + tie\|$. Since $y = y^*$, the C^*-norm property gives

$$\begin{aligned}(t+1)^2 &\leq \|y + tie\|^2 = \|(y + tie)(y + tie)^*\| \\ &= \|(y + tie)(y - tie)\| = \|y^2 + t^2e\| \\ &\leq \|y^2\| + t^2.\end{aligned}$$

However, this inequality cannot hold for large t. This shows that $\delta = -\beta$ and therefore

$$\varphi((ix)^*) = \varphi(-ix^*) = -i\varphi(x^*) = -i(\gamma + i\delta) = -\beta - i\gamma.$$

On the other hand $\varphi(ix) = i(\alpha + i\beta) = -\beta + i\alpha$. Applying what we have seen so far with ix in place of x, we obtain $\gamma = \alpha$ and hence $\varphi(x^*) = \overline{\varphi(x)}$. □

We are now ready to prove the first main result of this section.

Theorem 2.4.5. *For a commutative C^*-algebra A the Gelfand homomorphism is an isometric $*$-isomorphism from A onto $C_0(\Delta(A))$.*

Proof. To prove that $x \to \widehat{x}$ is isometric, note first that if $y = y^* \in A$, then $\|y\|^2 = \|y^*y\| = \|y^2\|$ and hence by induction $\|y\|^{2^n} = \|y^{2^n}\|$ for all $n \in \mathbb{N}$, so that

$$r_A(y) = \lim_{n\to\infty} \|y^{2^n}\|^{1/2^n} = \|y\|.$$

If now $x \in A$ is arbitrary, then by what we have just seen,

$$r_A(x^*x) = \|x^*x\| = \|x\|^2.$$

Recalling that $\widehat{x^*} = \overline{\widehat{x}}$ (Lemma 2.4.4) and $\sigma_A(x) \setminus \{0\} \subseteq \widehat{x}(\Delta(A)) \subseteq \sigma_A(x)$ (Theorem 2.2.5) we conclude that

$$\|\widehat{x}\|_\infty^2 = \|\overline{\widehat{x}}\,\widehat{x}\|_\infty = \|(x^*x)^\wedge\|_\infty = r_A(x^*x) = \|x\|^2.$$

Thus $x \to \widehat{x}$ is isometric and, in particular, the image $\widehat{A}$ of A is complete with respect to the supremum norm and hence closed in $C_0(\Delta(A))$. On the other hand, $\widehat{A}$ is a $*$-subalgebra of $C_0(\Delta(A))$ which strongly separates the points of $\Delta(A)$ (Theorem 2.2.7). Thus $\widehat{A}$ is dense in $C_0(\Delta(A))$ by the Stone–Weierstrass theorem. This proves that $\widehat{A} = C_0(\Delta(A))$. □

The preceding theorem, together with the following corollary, sets up a bijection between the homeomorphism classes of locally compact Hausdorff spaces and the isomorphism classes of commutative C^*-algebras.

Corollary 2.4.6. *For two commutative C^*- algebras A and B the following are equivalent.*

(i) *$\Delta(A)$ and $\Delta(B)$ are homeomorphic.*
(ii) *There exists an isometric $*$-isomorphism between A and B.*
(iii) *There exists an algebra isomorphism between A and B.*

Proof. The implication (ii) $\Rightarrow$ (iii) is trivial and, as we have seen earlier (Lemma 2.2.10), the implication (iii) $\Rightarrow$ (i) holds even for general commutative Banach algebras A and B. To prove (i) $\Rightarrow$ (ii), note first that if $\phi : \Delta(A) \to \Delta(B)$ is a homeomorphism, then $f \to f \circ \phi$ is an isometric isomorphism from $C_0(\Delta(B))$ onto $C_0(\Delta(A))$ satisfying $\overline{f} \to \overline{f \circ \phi}$. On the other hand, by Theorem 2.4.5, A and B are isometrically $*$-isomorphic to $C_0(\Delta(A))$ and $C_0(\Delta(B))$, respectively. It follows that A and B are isometrically $*$-isomorphic. □

Corollary 2.4.7. *Let A be a commutative C^*-algebra. For $x \in A$ consider the following conditions.*

(i) $x = x^*$.
(ii) $\sigma_A(x) \subseteq \mathbb{R}$.
(iii) $\widehat{x}$ *is real-valued.*
(iv) $x = y^*y$ *for some* $y \in A$.
(v) $\sigma_A(x) \subseteq [0, \infty)$.
(vi) $\widehat{x} \geq 0$.

Then (i), (ii), *and* (iii) *are equivalent, and so are* (iv), (v), *and* (vi).

Proof. The equivalence of (ii) and (iii) and of (v) and (vi) follows immediately from

$$\widehat{x}(\Delta(A)) \cup \{0\} = \sigma_A(x) \cup \{0\}.$$

The Gelfand homomorphism is injective and satisfies $\widehat{x^*} = \overline{\widehat{x}}$. Therefore (i) and (iii) are equivalent. If (iv) holds, then $\widehat{x} = \widehat{y^*y} = \overline{\widehat{y}}\,\widehat{y} \geq 0$. Conversely, if $\widehat{x} \geq 0$, let $f \in C_0(\Delta(A))$ be the positive square root of $\widehat{x}$. The Gelfand homomorphism being surjective, there exists $y \in A$ such that $\widehat{y} = f$. Now y satisfies $\widehat{y^*y} = \widehat{x}$ and hence $y^*y = x$. □

In the sequel we present two applications of Theorem 2.4.5. The first one (Theorem 2.4.9) is the construction of a functional calculus in which continuous functions act on elements of a commutative C^*-algebra, and the second (Theorem 2.4.12) concerns the existence of a Stone–Čech compactification for a completely regular topological space.

We know that in general the spectrum of an element in a Banach algebra may become larger upon passing to a subalgebra. We need that for C^*-algebras this cannot happen as we observe next.

Lemma 2.4.8. *Let* A *be a commutative* C^**-algebra with identity* e *and* B *a* C^**-subalgebra of* A *containing* e*. Then* $\sigma_A(x) = \sigma_B(x)$ *for each* $x \in B$.

Proof. It suffices to show that if $y \in B$ is invertible in A, then y is already invertible in B. Let $y \in B \cap G(A)$ and note first that $y^* \in G(A)$ since

$$(y^{-1})^* y^* = (yy^{-1})^* = e^* = ee^* = (e^*e)^* = e^{**} = e.$$

Thus $yy^* \in G(A)$ and, by Theorem 2.2.5, $\widehat{yy^*}(\Delta(A)) = \sigma_A(yy^*)$. On the other hand, by Lemma 2.4.4,

$$\widehat{yy^*}(\Delta(A)) = \{\varphi(yy^*) : \varphi \in \Delta(A)\} = \{|\varphi(y)|^2 : \varphi \in \Delta(A)\}.$$

Hence $\sigma_A(yy^*) \subseteq [0, \infty)$, so that $\rho_A(yy^*) = \mathbb{C} \backslash \sigma_A(yy^*)$ is connected. Theorem 1.2.12 now yields

$$\sigma_B(yy^*) = \sigma_A(yy^*).$$

Therefore, yy^* is invertible in B and hence so is y. □

Theorem 2.4.9. *Let A be a commutative C^*-algebra with identity e and $x \in A$. Let $A(x)$ denote the smallest C^*-subalgebra of A containing x and e. There exists a unique isometric $*$-isomorphism*

$$\phi : C(\sigma_A(x)) \to A(x), \quad f \to f(x)$$

with the property that ϕ maps the constant function 1 onto e and the function $\lambda \to \lambda$ onto x.

Proof. Because $\sigma_A(x) = \sigma_{A(x)}(x)$ by Lemma 2.4.8, we can assume that $A = A(x)$. This means that the set of all polynomials in x, x^*, and e is dense in A. Let $f \in C(\sigma_A(x))$ denote the function $f(\lambda) = \lambda$, and suppose that ϕ_1 and ϕ_2 are isometric $*$-isomorphisms from $C(\sigma_A(x))$ onto $A = A(x)$ with $\phi_j(1_{\sigma_A(x)}) = e$ and $\phi_j(f) = x$. Then

$$\phi_j^{-1}(x^*) = \overline{\phi_j^{-1}(x)} = \bar{f}, \quad j = 1, 2,$$

so that ϕ_1^{-1} and ϕ_2^{-1} coincide on all polynomials in x, x^* and e. Since $A = A(x)$ and ϕ_1^{-1} and ϕ_2^{-1} are continuous, we conclude that $\phi_1^{-1} = \phi_2^{-1}$.

To prove the existence of ϕ, we show first that $\varphi \to \varphi(x)$ defines a homeomorphism between $\Delta(A)$ and $\sigma_A(x)$. Every $\varphi \in \Delta(A)$ is determined by its value at x since φ is continuous, A is generated by x, x^*, and e, and $\varphi(x^*) = \overline{\varphi(x)}$ and $\varphi(e) = 1$. Thus $\varphi \to \varphi(x)$ is injective. On the other hand, $\widehat{x}(\Delta(A)) = \sigma_A(x)$ by Theorem 2.2.5. Clearly, the map $\varphi \to \varphi(x)$ from $\Delta(A)$ onto $\sigma_A(x)$ is continuous, and hence it is a homeomorphism since $\Delta(A)$ is compact and $\sigma_A(x)$ is a Hausdorff space. Let ψ denote the associated isometric $*$-isomorphism between $C(\sigma_A(x))$ and $C(\Delta(A))$; that is,

$$\psi(g)(\varphi) = g(\varphi(x)), \quad g \in C(\sigma_A(x)), \quad \varphi \in \Delta(A).$$

By Theorem 2.4.5 the Gelfand homomorphism $y \to \widehat{y}$ is an isometric $*$-isomorphism from $A = A(x)$ onto $C(\Delta(A))$. Composing its inverse with ψ, we obtain an isometric $*$-isomorphism $\phi : C(\sigma_A(x)) \to A = A(x)$ given by

$$\phi(g) = y \ \text{ if and only if } \ \widehat{y}(\varphi) = g(\varphi(x)) \text{ for all } \varphi \in \Delta(A).$$

Then ϕ has the required properties since $1_{\sigma_A(x)}(\varphi(x)) = 1 = \widehat{e}(\varphi)$ and $f(\varphi(x)) = \varphi(x) = \widehat{x}(\varphi)$ for all $\varphi \in \Delta(A)$. □

Remark 2.4.10. Returning to Example 2.4.2, let T be a normal operator in a Hilbert space H and $A(T)$ the closed subalgebra of $\mathcal{B}(H)$ generated by T, T^* and the identity operator I on H. According to the preceding theorem, there is a unique isometric $*$-isomorphism from $C(\sigma(T))$ onto $A(T)$ which maps the function $f(\lambda) = \lambda$ to T and the constant one function to I. This result can be used to derive the spectral theorem for normal operators in Hilbert spaces. Because of this, Theorem 2.4.9 is often referred to as the *abstract spectral theorem*.

For the second application of Theorem 2.4.5 mentioned above we first recall some notions from topology.

Definition 2.4.11. Let X be a Hausdorff space. A pair (Y, β), consisting of a compact Hausdorff space Y and a mapping $\beta : X \to Y$, is called a *Stone–Čech compactification* of X, if the following conditions are satisfied.

(i) $\beta(X)$ is dense in Y, and $\beta : X \to \beta(X)$ is a homeomorphism.
(ii) Every $f \in C^b(X)$ extends continuously to Y in the sense that there exists $\widetilde{f} \in C(Y)$ such that $\widetilde{f}(\beta(x)) = f(x)$ for all $x \in X$.

Of course, $\widetilde{f}$ is then uniquely determined since $\beta(X)$ is dense in Y.

Suppose now that X possesses a Stone–Čech compactification (Y, β). Then given a closed subset E of X and $x \in X \setminus E$, there exists $f \in C^b(X)$ such that $f|_E = 0$ and $f(x) \neq 0$. In fact, if C is a closed subset of Y with $C \cap \beta(X) = \beta(E)$, then $\beta(x) \notin C$, and hence by Urysohn's lemma we find $g \in C(Y)$ such that $g(\beta(x)) \neq 0$ and $g|_C = 0$. Now, $f = g \circ \beta \in C^b(X)$ has the desired properties. A Hausdorff space X for which $C^b(X)$ shares this separation property is called *complelety regular*.

Stone and Čech proved that every completely regular space admits a Stone–Čech compactification, which is uniquely determined up to homeomorphisms. We conclude this section by showing that the existence of a Stone–Čech compactification can be obtained as an application of Gelfand's theory.

Theorem 2.4.12. *Let X be a completely regular topological space. Let $Y = \Delta(C^b(X))$ and define $\beta : X \to Y$ by $\beta(x) = \varphi_x$, where φ_x denotes the evaluation of functions in $C(Y)$ at x. Then (Y, β) is a Stone–Čech compactification of X.*

Proof. $C^b(X)$ is a commutative C^*-algebra with identity. Therefore, $Y = \Delta(C^b(X))$ is compact, and the Gelfand homomorphism $f \to \widehat{f}$ is an isometric $*$-isomorphism from $C^b(X)$ onto $C(Y)$. The map

$$\beta : X \to Y, \quad x \to \varphi_x$$

is one-to-one because given distinct points x_1 and x_2 in X, the complete regularity of X guarantees the existence of some $f \in C^b(X)$ with

$$\varphi_{x_1}(f) = f(x_1) \neq f(x_2) = \varphi_{x_2}(f).$$

Condition (ii) of Definition 2.4.11 is satisfied with $\widetilde{f} = \widehat{f}$ since, by definition of β,

$$\widehat{f}(\beta(x)) = \widehat{f}(\varphi_x) = \varphi_x(f) = f(x)$$

for all $f \in C^b(X)$ and $x \in X$.

To verify that $\beta : X \to \beta(X)$ is a homeomorphism, for $x_0 \in X, \epsilon > 0$, and $f_1, \ldots, f_n \in C^b(X)$ consider the sets

$$V = \{x \in X : |f_i(x) - f_i(x_0)| < \epsilon, 1 \leq i \leq n\} \subseteq X$$

and

$$U = \{\varphi_x \in \beta(X) : |\varphi_x(f_i) - \varphi_{x_0}(f_i)| < \epsilon, 1 \leq i \leq n\} \subseteq \beta(X).$$

Then $V = \beta^{-1}(U)$ and V is open in X. These sets U form an open basis for the relative topology on $\beta(X) \subseteq Y = \Delta(C^b(X))$. Hence β is continuous, and for β to be open it suffices to show that such sets V form a basis for the topology on X. For that, let W be an open subset of X containing x_0. Then, since X is completely regular, there exists $f \in C^b(X)$ such that $f(x_0) \neq 0$ and $f|_{X \setminus W} = 0$. It follows that

$$x_0 \in \{x \in X : |f(x) - f(x_0)| < |f(x_0)|\} \subseteq W.$$

To complete the proof of the theorem it remains to show that $\beta(X)$ is dense in Y. Assuming the contrary, there exists $g \in C(Y)$ such that $g \neq 0$, but $g|_{\beta(X)} = 0$. The Gelfand homomorphism maps $C^b(X)$ onto $C(Y)$. Thus we find $f \in C^b(X)$ such that $\widehat{f} = g$. Then

$$0 = g(\varphi_x) = \widehat{f}(\varphi_x) = \varphi_x(f) = f(x)$$

for all $x \in X$. However, $f = 0$ implies $g = 0$. This contradiction shows that $\beta(X)$ is dense in Y. □

2.5 The uniform algebras $P(X)$ and $R(X)$

The next two sections centre around elaborating the Gelfand representation of certain algebras of continuous functions on compact spaces.

Definition 2.5.1. Let X be a compact Hausdorff space. A closed subalgebra A of $C(X)$, equipped with the $\| \cdot \|_\infty$-norm, is called a *uniform algebra* if A separates the points of X and contains the constant functions.

In Example 1.1.2 we have already introduced, for X a compact subset of $\mathbb{C}$, the uniform algebras $P(X), R(X)$, and $A(X)$. The definitions in the more general case of a compact subset of $\mathbb{C}^n$ are analogous. Instead of polynomials, rational functions, and holomorphic functions in one variable we simply have to take such functions in n complex variables.

Remark 2.5.2. If A is a uniform algebra on X then, because X is compact and $\Delta(A)$ is a Hausdorff space, the mapping $\phi : x \to \varphi_x$, where $\varphi_x(f) = f(x)$ for $f \in A$, is a homeomorphism of X onto its range $\phi(X) \subseteq \Delta(A)$. In general, however, $\phi(X)$ is a proper subset of $\Delta(A)$.

Our goal is to determine the structure spaces of $P(X)$, $R(X)$, and $A(X)$. In this section we treat $P(X)$ and $R(X)$ for $X \subseteq \mathbb{C}^n$ and in the next section $A(X)$ for $X \subseteq \mathbb{C}$. Moreover, we study the problem of when equality holds for any of the inclusions $P(X) \subseteq R(X)$ and $R(X) \subseteq A(X)$.

Example 2.5.3. Let $\mathbb{D} = \{z \in \mathbb{C} : |z| \leq 1\}$ and $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$, the boundary of $\mathbb{D}$.

(1) The algebra $P(\mathbb{D})$ is generated by the function $f(z) = z, z \in \mathbb{D}$. Now, $\sigma_{P(\mathbb{D})}(f) = \mathbb{D}$. In fact, if $|\lambda| > 1$, then the function

$$z \to \frac{1}{\lambda - f(z)} = \frac{1}{\lambda} \frac{1}{1 - \frac{z}{\lambda}} = \frac{1}{\lambda} \sum_{n=0}^{\infty} \left(\frac{z}{\lambda}\right)^n$$

is a uniform limit of polynomials on $\mathbb{D}$, and hence the function $\lambda - f$ is invertible in $P(\mathbb{D})$. Thus, by Lemma 2.3.3, the mapping $z \to \varphi_z$, where $\varphi_z(g) = g(z)$ for $g \in P(\mathbb{D})$, is a homeomorphism between $\mathbb{D}$ and the structure space of $P(\mathbb{D})$.

By the maximum modulus principle, the mapping $r : g \to g|_{\mathbb{T}}$ is an isometric isomorphism from $P(\mathbb{D})$ onto $P(\mathbb{T})$. It follows that $\Delta(P(\mathbb{T})) = \mathbb{D}$ via the mapping $z \to \varphi_z, \varphi_z(h) = r^{-1}(h)(z)$ for $h \in P(\mathbb{T})$.

(2) We claim that $P(\mathbb{D}) = A(\mathbb{D}) \neq C(\mathbb{D})$. Since the function $z \to \overline{z}$ fails to be holomorphic, $A(\mathbb{D}) \neq C(\mathbb{D})$. To show that $P(\mathbb{D}) = A(\mathbb{D})$, let $f \in A(\mathbb{D})$ and for $0 < t < 1$, define f_t by $f_t(z) = f(tz)$. Then f_t is a holomorphic function on $\{z \in \mathbb{C} : |z| < 1/t\}$, and $f_t \to f$ uniformly on $\mathbb{D}$ as $t \to 1$ because f is uniformly continuous on $\mathbb{D}$. Finally, f_t admits a power series representation and hence can be uniformly approximated by polynomials on $\mathbb{D}$. Thus f is a uniform limit of polynomials on $\mathbb{D}$, as required.

Definition 2.5.4. Let X be a compact subset of $\mathbb{C}^n$. The *polynomially convex hull*, $\widehat{X}_p$, of X is the set

$$\widehat{X}_p = \{z \in \mathbb{C}^n : |p(z)| \leq \|p|_X\|_\infty \text{ for all polynomials } p\}.$$

Then, by Definition 2.3.4, X is polynomially convex if and only if $X = \widehat{X}_p$. The *rational convex hull* $\widehat{X}_r$ of X is the set of all $z \in \mathbb{C}^n$ such that

$$|p(z)| \leq |q(z)| \cdot \left\| \frac{p}{q}|_X \right\|_\infty$$

for all polynomials p and q with $q \neq 0$ on X. Finally, X is said to be *rationally convex* if $\widehat{X}_r = X$.

We continue with some simple observations concerning $\widehat{X}_p$ and $\widehat{X}_r$.

Remark 2.5.5. (1) Clearly, $X \subseteq \widehat{X}_r \subseteq \widehat{X}_p$. In particular, if X is polynomially convex, then it is rationally convex.

(2) Each compact subset of $\mathbb{C}$ is rationally convex. Indeed, if $z_0 \in \mathbb{C} \setminus X$, then $q(z) = z - z_0$ satisfies $1 > 0 = |q(z_0)| \cdot \|(1/q)|_X\|_\infty$. On the other hand,

recall that X is polynomially convex if and only if $\mathbb{C}\setminus X$ is connected (Theorem 2.3.7).

(3) Both $\widehat{X}_p$ and $\widehat{X}_r$ are compact. To verify this, since these sets are closed and $\widehat{X}_r \subseteq \widehat{X}_p$, it is enough to show that $\widehat{X}_p$ is bounded. Now, with $p_j(z) = z_j, 1 \leq j \leq n$, for every $z \in \widehat{X}_p$,

$$\|z\|^2 = \sum_{j=1}^{n} |p_j(z)|^2 \leq \sum_{j=1}^{n} \|p_j|_X\|_\infty^2.$$

Lemma 2.5.6. *For any compact subset X of $\mathbb{C}^n$,*

$$\widehat{X}_r = \{z \in \mathbb{C}^n : p(z) \in p(X)\, for\, every\, polynomial\ p\}.$$

Proof. Let $z \in \mathbb{C}^n$ and suppose that there is a polynomial p such that $p(z) \notin p(X)$. Then $q(w) = p(w) - p(z)$ is non-zero on X and

$$1 > 0 = |q(z)| \cdot \left\| \frac{1}{q}|_X \right\|_\infty ,$$

so that $z \notin \widehat{X}_r$. Conversely, if $z \notin \widehat{X}_r$, then there are polynomials p and q, with $q \neq 0$ on X, such that

$$|p(z)| > |q(z)| \cdot \left\| \frac{p}{q}|_X \right\|_\infty .$$

In particular, $p(z) \neq 0$. If $q(z) = 0$, we are done since $0 \notin q(X)$. Otherwise, replacing p by $g = q(z)p(z)^{-1}p$, we get that $g(z) = q(z)$ and

$$\left\| \frac{g}{q}|_X \right\|_\infty = \left| \frac{q(z)}{p(z)} \right| \cdot \left\| \frac{p}{q}|_X \right\|_\infty < 1.$$

Then the polynomial $f = q - g$ satisfies $f(z) = 0$ and $0 \notin f(X)$, for if $x \in X$ and $f(x) = 0$, then $(g/q)(x) = 1$ contradicting $\|(g/q)|_X\|_\infty < 1$. □

We can now work out the Gelfand representation of $P(X)$ and $R(X)$.

Theorem 2.5.7. *Let X be a compact subset of $\mathbb{C}^n$.*

(i) *The restriction map $\phi : f \to f|_X$ is an isometric isomorphism from $P(\widehat{X}_p)$ onto $P(X)$. Moreover, for $x \in \widehat{X}_p$, define $\varphi_x : P(X) \to \mathbb{C}$ by*

$$\varphi_x(f) = \phi^{-1}(f)(x), \quad f \in P(X).$$

Then $x \to \varphi_x$ is a homeomorphism from $\widehat{X}_p$ onto $\Delta(P(X))$.

(ii) *The map $\phi : f \to f|_X$ is an isometric isomorphism from $R(\widehat{X}_r)$ onto $R(X)$, and $x \to \varphi_x$, where*

$$\varphi_x(f) = \phi^{-1}(f)(x), \quad f \in R(X), \quad x \in \widehat{X}_r,$$

is a homeomorphism between $\widehat{X}_r$ and $\Delta(R(X))$.

Proof. (i) The map $q|_{\widehat{X}_p} \to q|_X$ takes the dense subalgebra of $P(\widehat{X}_p)$ consisting of all polynomial functions on $\widehat{X}_p$ homomorphically onto the corresponding subalgebra of $P(X)$. This map preserves the norm since

$$|q(z)| \leq \|q|_X\|_\infty$$

for all polynomials q and all $z \in \widehat{X}_p$. It follows that ϕ is an isometric isomorphism from $P(\widehat{X}_p)$ onto $P(X)$. For each $x \in \widehat{X}_p, \varphi_x$ as defined above belongs to $\Delta(P(X))$, and the mapping $x \to \varphi_x, \widehat{X}_p \to \Delta(P(X))$ is injective. It is continuous since

$$x \to \varphi_x(f) = \phi^{-1}(f)(x)$$

is a continuous function on $\widehat{X}_p$ for every $f \in P(X)$. Hence $x \to \varphi_x$ maps $\widehat{X}_p$ homeomorphically onto its image in $\Delta(P(X))$. It remains to show that given $\varphi \in \Delta(P(X))$, there exists $x \in \widehat{X}_p$ such that $\varphi = \varphi_x$. To that end, let $x_j = \varphi(p_j|_X)$, where $p_j(z) = z_j, 1 \leq j \leq n$. We claim that $x = (x_1, \ldots, x_n) \in \widehat{X}_p$ and $\varphi = \varphi_x$. For any polynomial q,

$$q(x) = q(\varphi(p_1|_X), \ldots, \varphi(p_n|_X)) = \varphi(q|_X),$$

and hence $|q(x)| \leq \|q|_X\|_\infty$. This proves $x \in \widehat{X}_p$, and $\varphi = \varphi_x$ follows from

$$\varphi_x(p_j|_X) = p_j(x) = x_j = \varphi(p_j|_X),$$

$1 \leq j \leq n$, since the functions $p_j|_X$ generate $P(X)$.

(ii) is proved in very much the same way as (i). Note first that if $f = (p/q)|_{\widehat{X}_r}$, where p and q are polynomials with $q \neq 0$ on $\widehat{X}_r$, then $\|f\|_\infty = \|f|_X\|_\infty$ since for each $x \in \widehat{X}_r$,

$$|p(x)| \leq |q(x)| \cdot \left\| \frac{p}{q}|_X \right\|_\infty .$$

Consequently, $f \to f|_X$ maps the dense subalgebra of rational functions in $R(\widehat{X}_r)$ homomorphically and isometrically onto a dense subalgebra of $R(X)$. This yields the first statement in (ii).

Clearly, for each $x \in \widehat{X}_r$,

$$\varphi_x(f) = \phi^{-1}(f)(x), \quad f \in R(X),$$

defines an element of $\Delta(R(X))$, and the mapping $x \to \varphi_x, \widehat{X}_r \to \Delta(R(X))$ is injective and continuous. What is left to be shown is that every $\varphi \in \Delta(R(X))$ is of the form $\varphi = \varphi_x$ for some $x \in \widehat{X}_r$. Given φ, as in (i) define $x = (x_1, \ldots, x_n) \in \mathbb{C}^n$ by

$$x_j = \varphi(p_j|_X), \quad 1 \leq j \leq n.$$

Now, for every polynomial q,

$$q(x) = q(\varphi(p_1|_X), \dots, \varphi(p_n|_X)) = \varphi(q|_X) = \widehat{q|_X}(\varphi) \in \sigma_{R(X)}(q|_X) = q(X).$$

According to Lemma 2.5.6 this shows that $x \in \widehat{X}_r$. Finally, $\varphi_x(p_j|_X) = x_j = \varphi(p_j|_X)$ implies $\varphi_x(p|_X) = \varphi(p|_X)$ and hence

$$\varphi_x\left(\frac{p}{q}|_X\right) = \varphi_x(p|_X)\varphi_x(q|_X)^{-1} = \varphi(p|_X)\varphi(q|_X)^{-1} = \varphi\left(\frac{p}{q}|_X\right)$$

for all polynomials p and q with $q \neq 0$ on X. It follows that $\varphi = \varphi_x$. $\square$

In the proof of part (i) of Theorem 2.5.7, for surjectivity of the map $x \to \varphi_x$ from $\widehat{X}_p$ to $\Delta(P(X))$ we could alternatively have appealed to the proof of Theorem 2.3.6. We now obtain the following approximation result.

Theorem 2.5.8. *If X is a compact subset of $\mathbb{C}^n$, then $P(X) = R(X)$ if and only if $\widehat{X}_p = \widehat{X}_r$. In particular, for a compact subset X of $\mathbb{C}$, $P(X) = R(X)$ if and only if $\mathbb{C} \setminus X$ is connected.*

Proof. Suppose first that $P(X) = R(X)$, and let $x \in \widehat{X}_p$. Then the function $\varphi_x : f \to (\phi^{-1}f)(x)$, where ϕ is as in part (i) of Theorem 2.5.7, defines an element of $\Delta(P(X)) = \Delta(R(X))$. By Theorem 2.5.7(ii), $\varphi_x = \varphi_y$ for some $y \in \widehat{X}_r$. It follows that

$$q(y) = \varphi_y(q|_X) = \varphi_x(q|_X) = q(x)$$

for all polynomials q, so that $x = y$. This shows $\widehat{X}_p \subseteq \widehat{X}_r$ and hence $\widehat{X}_p = \widehat{X}_r$ (Remark 2.5.2).

Conversely, let $\widehat{X}_p = \widehat{X}_r$. To prove $R(X) \subseteq P(X)$ it suffices to show that if q is a polynomial such that $q(z) \neq 0$ for all $z \in X$, then $q|_X$ is invertible in $P(X)$. Now, by Lemma 2.5.6, q has no zero on $\widehat{X}_r$. Since $\widehat{X}_r = \widehat{X}_p = \Delta(P(X))$, this implies $\varphi(q|_X) \neq 0$ for every $\varphi \in \Delta(P(X))$. Therefore $q|_X$ is contained in no maximal ideal of $P(X)$, and therefore is invertible in $P(X)$.

Finally, suppose that $n = 1$. If $\mathbb{C} \setminus X$ is connected, then X is polynomially convex (Theorem 2.3.7) and hence $\widehat{X}_r = \widehat{X}_p$. Conversely, if $\widehat{X}_p = \widehat{X}_r$ then, because every compact subset of $\mathbb{C}$ is rationally convex (Remark 2.5.5), X is polynomially convex and hence $\mathbb{C} \setminus X$ is connected. $\square$

Next we show an interesting result about generation of $R(X)$.

Theorem 2.5.9. *If X is a compact subset of $\mathbb{C}^n$, then $R(X)$ is generated by $n + 1$ elements.*

Proof. The set of $n + 1$ generators we produce consists of the coordinate functions $p_j(z) = z_j, z \in X, 1 \leq j \leq n$, and an additional function f which has to be constructed. Notice first that since $P(X)$ contains a countable dense subset, there exists a sequence of polynomials $q_m, m \in \mathbb{N}$, such that $q_m \neq 0$ on X and the set

$$\left\{ \frac{p}{q_m}\Big|_X : m \in \mathbb{N}, p \text{ a polynomial} \right\}$$

is dense in $R(X)$. Let $g_m = q_m|_X$, and by induction define positive real numbers $c_m, m \in \mathbb{N}$, so that

$$c_m \left\| g_m^{-1} \right\|_\infty < 2^{-m} \text{ and } c_m \left\| g_m^{-1} g_k \right\|_\infty < 2^{-m} c_k$$

for $1 \leq k \leq m-1$. Then the series $\sum_{k=1}^\infty c_k g_k^{-1}(z)$ converges uniformly on X and hence defines an element f of $R(X)$. We claim that A, the unital closed subalgebra of $R(X)$ generated by f and all the $p_j, 1 \leq j \leq n$, coincides with $R(X)$.

The set of functions of the form $(pg_m^{-1})|_X$, $m \in \mathbb{N}$, p a polynomial, is dense in $R(X)$. Therefore it is enough to show that $g_m^{-1} \in A$ for every $m \in \mathbb{N}$. Let

$$f_m = \sum_{k=m}^\infty c_k g_k^{-1} \in R(X).$$

Next, observe that, for each $m \in \mathbb{N}$, $g_m^{-1} \in A$ provided that $f_m \in A$. Indeed, this can be seen as follows. If $f_m \in A$, then $f_m g_m \in A$ and, by the choice of c_k,

$$\begin{aligned} \left\| f_m g_m - c_m \right\|_\infty &= \left\| \sum_{k=m+1}^\infty c_k g_m g_k^{-1} \right\|_\infty \leq \sum_{k=m+1}^\infty c_k \left\| g_m g_k^{-1} \right\|_\infty \\ &\leq c_m \sum_{k=m+1}^\infty 2^{-k} < c_m. \end{aligned}$$

Thus $f_m g_m$ is invertible in A, and hence so is g_m. It now follows by induction that $f_m \in A$ for all $m \in \mathbb{N}$. Indeed, $f_1 = f \in A$, and supposing that $f_1, \ldots, f_m \in A$, by the preceding paragraph, $g_1^{-1}, \ldots, g_m^{-1} \in A$. It follows that

$$f_{m+1} = f - \sum_{k=1}^m c_k g_k^{-1} \in A.$$

This finishes the proof of the theorem. □

It is worth pointing out that we have not proved that $R(X)$ admits a system of $n+1$ generators, each of which is a rational function. In fact, this strengthened version is false, as can already be seen in the plane: if X is a compact subset of $\mathbb{C}$ and $\mathbb{C} \backslash X$ has infinitely many connected components, then $R(X)$ cannot be generated by a finite family of rational functions (Exercise 2.12.41) even though it is doubly generated as a Banach algebra.

A nice geometric consequence of Theorem 2.5.9 and the previous results is the following

Corollary 2.5.10. *Every rationally convex compact subset of* $\mathbb{C}^n$ *is homeomorphic to some polynomially convex subset of* $\mathbb{C}^{n+1}$.

Proof. If X is a compact subset of $\mathbb{C}^n$ and rationally convex, then $X = \Delta(R(X))$ by Theorem 2.5.7. On the other hand, $R(X)$ is generated by $n+1$ elements $f_1, \ldots, f_{n+1}$, and hence, by Lemma 2.3.3, $\Delta(R(X))$ is homeomorphic to the joint spectrum

$$\sigma_{R(X)}(f_1, \ldots, f_{n+1}) \subseteq \mathbb{C}^{n+1},$$

which is polynomially convex by Theorem 2.3.6. □

We proceed by constructing a compact subset X of $\mathbb{C}$ with empty interior such that $R(X) \neq C(X)$. This example is usually called *Swiss cheese*, a label which becomes apparent from the construction.

Example 2.5.11. As before, let $\mathbb{D}$ denote the closed unit disc. We are going to show the existence of a sequence of closed discs $\Delta_j, j \in \mathbb{N}$, of radii $r_j > 0$ with the following properties.

(1) $\Delta_j \subseteq \mathbb{D}^\circ = \{z \in \mathbb{C} : |z| < 1\}$ and $\Delta_j \cap \Delta_k = \emptyset$ for $j \neq k$.
(2) $\sum_{j=1}^\infty r_j < 1$.
(3) $\mathbb{D} \setminus \bigcup_{j=1}^\infty \Delta_j^\circ$ has an empty interior.

Let $y_1, y_2, \ldots$ be a numbering of the countable set of complex numbers $\alpha + i\beta \in \mathbb{D}^\circ$ with α, β rational. We construct by induction on n a sequence $(\Delta_n)_n$ of closed discs such that (1) holds for $1 \leq j \leq k \leq n$, $0 < r_j < 2^{-j}$ for $1 \leq j \leq n$ and

$$y_1, \ldots, y_n \in \bigcup_{j=1}^n \Delta_j = \overline{\bigcup_{j=1}^n \Delta_j^\circ}.$$

For $y \in \mathbb{C}$ and $r > 0$, let $B(y, r)$ denote the closed disc of radius r around y. Choose $0 < r_1 < \frac{1}{2}$ such that $\Delta_1 = B(y_1, r_1) \subseteq \mathbb{D}^\circ$. Suppose that $\Delta_1, \ldots, \Delta_n$ with the required properties have been found. Then $y_m \notin \bigcup_{j=1}^n \Delta_j$ for some $m \geq n+1$. Indeed, otherwise $y_k \in \bigcup_{j=1}^n \Delta_j$ for all k and hence, because the set $\{y_k : k \in \mathbb{N}\}$ is dense in $\mathbb{D}^\circ$, $\mathbb{D}^\circ = \bigcup_{j=1}^n \Delta_j$, which is impossible. Let m be minimal such that $y_m \notin \bigcup_{j=1}^n \Delta_j$, and choose $0 < r_{n+1} < 2^{-(n+1)}$ such that $\Delta_{n+1} = B(y_m, r_{n+1})$ satisfies

$$\Delta_{n+1} \subseteq \mathbb{D}^\circ \text{ and } \Delta_{n+1} \cap \left(\bigcup_{j=1}^n \Delta_j \right) = \emptyset.$$

This finishes the inductive step. It is obvious that the sequence $(\Delta_j)_j$ has properties (1) and (2).

Now, let $X = \mathbb{D} \setminus \bigcup_{j=1}^\infty \Delta_j^\circ$. Then X has empty interior because $y_n \in \overline{\bigcup_{j=1}^n \Delta_j^\circ}$ for each n. So (3) holds also.

To prove that $R(X) \neq C(X)$, we construct a bounded linear functional l on $C(X)$ such that $l \neq 0$ and $l|_{R(X)} = 0$. Let z_j denote the centre of $\Delta_j, j \in \mathbb{N}$, and let $\Gamma_j, j \in \mathbb{N}$, be the curve defined by

$$\Gamma_j(t) = z_j + r_j e^{it}, \quad t \in [0, 2\pi].$$

Moreover, define Γ_0 by

$$\Gamma_0(t) = e^{-it}, \quad t \in [0, 2\pi].$$

For $f \in C(X)$, let

$$l(f) = \sum_{j=0}^{\infty} \int_{\Gamma_j} f(z)dz.$$

Note that since $\sum_{j=1}^{\infty} r_j < \infty$ and

$$\left| \int_{\Gamma_j} f(z)dz \right| = \left| r_j \int_0^{2\pi} f(z_j + r_j e^{it}) i e^{it} dt \right| \leq 2\pi r_j \|f\|_\infty,$$

the above series converges absolutely, and therefore l defines a bounded linear functional on $C(X)$.

Now, $\int_{\Gamma_0} \bar{z} dz = -2\pi i$ and

$$\int_{\Gamma_j} \bar{z} dz = i r_j \int_0^{2\pi} (\overline{z_j} + r_j e^{-it}) e^{it} dt = 2\pi i r_j^2$$

for $j \geq 1$. Thus, by property (2),

$$l(z \to \bar{z}) = 2\pi i \left(\sum_{j=1}^{\infty} r_j^2 - 1 \right) \neq 0.$$

It remains to show that $l|_{R(X)} = 0$.

To that end, let p and q be complex polynomials such that $q(z) \neq 0$ for all $z \in X$. Then $q \neq 0$ on some open neighbourhood V of X. Let $X_n = \mathbb{D} \setminus \bigcup_{j=1}^{n} \Delta_j^\circ$, so that $X_{n+1} \subseteq X_n$ for all n and $X = \bigcap_{n=1}^{\infty} X_n$. It follows that $X_n \subseteq V$ for all $n \geq n_0$ for some $n_0 \in \mathbb{N}$. We want to apply Cauchy's integral formula to the holomorphic function $f = (p/q)|_V$ and the closed curves $\Gamma_0, \Gamma_1, \ldots, \Gamma_n$ in V, $n \geq n_0$. For every point $z \notin \Gamma_j([0, 2\pi])$, let $w(\Gamma_j, z)$ denote the winding number of Γ_j with respect to z. If $z \in \mathbb{C} \setminus V$, then either $z \notin \mathbb{D}$ and hence $w(\Gamma_j, z) = 0$ for all $j \in \mathbb{N}_0$, or $z \in \mathbb{D}$. In the latter case, $z \in \bigcup_{j=1}^{n} \Delta_j^\circ$ because $\mathbb{D} \setminus \bigcup_{j=1}^{n} \Delta_j^\circ = X_n \subseteq V$ for $n \geq n_0$, and therefore $z \in \Delta_j^\circ$ for exactly one $j \in \mathbb{N}$. This implies that

$$\sum_{k=0}^{n} w(\Gamma_k, z) = w(\Gamma_j, z) + w(\Gamma_0, z) = 1 - 1 = 0.$$

Thus we have seen that $\Sigma_{k=0}^{n} w(\Gamma_k, z) = 0$ for all $z \in \mathbb{C} \setminus V$ and $n \geq n_0$. A version of Cauchy's integral formula (see [23, p. 206]) now yields that

$$\sum_{j=0}^{n} \int_{\Gamma_j} f(z)dz = 0$$

for all $n \geq n_0$ and hence $l(f|_X) = 0$. Since l is continuous, it follows that $l|_{R(X)} = 0$.

The next theorem holds more generally for compact subsets of $\mathbb{C}$ of Lebesgue measure zero and in this generality is referred to as the *Hartogs–Rosenthal theorem.*

Theorem 2.5.12. *Let X be a countable compact subset of $\mathbb{C}$. Then $P(X) = C(X)$.*

Proof. We first observe that $\mathbb{C} \setminus X$ is connected. To see this, let $z_1, z_2 \in \mathbb{C} \setminus X$. Since X is countable, there is a ray L emanating from z_1 which does not intersect X. For any point $z \in L$, let $\overline{z, z_2}$ denote the line segment connecting z and z_2. Again, because X is countable, one of them, say $\overline{z, z_2}$, misses X. So z_1 and z_2 are connected in $\mathbb{C} \setminus X$ by $\overline{z_1, z} \cup \overline{z, z_2}$. Theorem 2.5.5 now shows that $P(X) = R(X)$.

It remains to show that $R(X) = C(X)$. Let $\mu \in C(X)^*$, that is, a bounded regular Borel measure on X, and suppose that μ is nonzero and nevertheless annihilates $R(X)$. Note that, for every $z \in \mathbb{C} \setminus X$, the function $w \to 1/(w - z)$ belongs to $R(X)$ and hence $\int_X 1/(w - z)d\mu(w) = 0$. Since $\operatorname{supp} \mu$ is countable and compact, at least one of the points of $\operatorname{supp} \mu$ is open in $\operatorname{supp} \mu$. So there exist z_0 and an open disc U centered at z_0 of radius $R > 0$ such that $\mu(\{z_0\}) \neq 0$ and $U \cap \operatorname{supp} \mu = \{z_0\}$. Since X is countable, we find $0 < r < R$ such that the path $\gamma(t) = z_0 + re^{it}, t \in [0, 2\pi]$, does not meet X. An easy application of Fubini's theorem shows that

$$\int_\gamma \left(\int_X \frac{1}{w - z} d\mu(w) \right) dz = \int_X \left(\int_\gamma 1/(w - z)dz \right) d\mu(w).$$

Now the left-hand side of this equation is zero since $X \cap \gamma[0, 2\pi] = \emptyset$ and $\int_X 1/(w - z)d\mu(w) = 0$ for every $z \notin X$. On the other hand, the right-hand side is nonzero. To see this, note first that if $w \in \operatorname{supp} \mu$, then either $w \notin U$ or $w = z_0$. In the first case, $\int_\gamma 1/(w - z)dz = 0$, whereas $\int_\gamma 1/(w - z)dz = -2\pi i$ in the second case. It follows that

$$\int_X \left(\int_\gamma \frac{1}{w - z} dz \right) d\mu(w) = -2\pi i \mu(\{z_0\}) \neq 0.$$

This contradiction shows that there is no nonzero $\mu \in C(X)^*$ annihilating $R(X)$. Thus $R(X) = C(X)$ by the Hahn–Banach theorem. $\square$

Since a countable compact subset of $\mathbb{C}$ has empty interior, Theorem 2.5.12 is a very special case of Mergelyan's theorem which states that if X is a compact subset of $\mathbb{C}$ such that $\mathbb{C} \setminus X$ is connected, then $P(X) = A(X)$. It is also worth pointing out that $R(X) = C(X)$ holds more generally whenever X is totally disconnected. In fact, this follows from Corollary 3.5.6 because every compact subset X of $\mathbb{C}$ is rationally convex (Remark 2.5.5) and therefore homeomorphic to $\Delta(R(X))$ (Theorem 2.5.7). However, the proof of Corollary 3.5.6 relies on Shilov's idempotent theorem. A somewhat surprising consequence of Theorem 2.5.12 is the following corollary.

Corollary 2.5.13. *Let X be a countable compact Hausdorff space and let A be a closed subalgebra of $C(X)$. Then A is self-adjoint.*

Proof. Let $f \in A$. Then $f(X) \cup \{0\}$ is a countable compact subset of $\mathbb{C}$. By the preceding theorem there exists a sequence of polynomials $p_n, n \in \mathbb{N}$, such that $p_n(z) \to \overline{z}$ uniformly on $f(X) \cup \{0\}$. Let $q_n = p_n - p_n(0)$. Then each q_n is a polynomial without constant term and $q_n(z) \to \overline{z}$ uniformly on $f(X)$. Thus $q_n(f(x)) \to \overline{f(x)}$ uniformly on X. Since q_n is without constant term, $q_n \circ f \in A$. This proves that $\overline{f} \in A$. □

In concluding this section we present a theorem (Theorem 2.5.15 below), which is usually referred to as *Wermer's maximality theorem*. The proof requires the following lemma.

Lemma 2.5.14. *Let X be a compact Hausdorff space and A a uniform algebra on X. If f and g are functions in A such that $\|1 + f + \overline{g}\|_\infty < 1$, then $f + g$ is invertible in A.*

Proof. Let $h = f + g$ and $c = \|1 + \operatorname{Re} h\|_\infty$. Since $\|1 + f + \overline{g}\|_\infty < 1$ and hence $\|1 + \overline{f} + g\|_\infty < 1$, we have

$$\|1 + \operatorname{Re} h\|_\infty = \frac{1}{2}\|1 + f + \overline{g} + 1 + \overline{f} + g\|_\infty < 1.$$

Thus, for all $x \in X, |1 + \operatorname{Re} h(x)| \leq c < 1$. This means that $h(x)$ lies in the left half-plane for all x, which suggests that, for small $\epsilon > 0, 1 + \epsilon\, h(x)$ lies in the unit disc for all x. In fact,

$$\begin{aligned} |1 + \epsilon h(x)|^2 &= 1 + 2\epsilon \operatorname{Re} h(x) + \epsilon^2 |h(x)|^2 \\ &\leq 1 + 2\epsilon(c - 1) + \epsilon^2 \|h\|_\infty^2, \end{aligned}$$

for all $x \in X$. Since $c < 1$, it follows that $\|1 + \epsilon h\|_\infty < 1$ for sufficiently small $\epsilon > 0$. This ϵh is invertible (Lemma 1.2.6) and hence so is h. □

Theorem 2.5.15. *Let A be a uniform algebra on the unit circle $\mathbb{T}$ such that $P(\mathbb{T}) \subseteq A$. Then either $A = P(\mathbb{T})$ or $A = C(\mathbb{T})$.*

Proof. For $h \in C(\mathbb{T})$ and $k \in \mathbb{Z}$, let

$$c_k(h) = \frac{1}{2\pi} \int_0^{2\pi} h(e^{it}) e^{-ikt} dt,$$

the k-th Fourier coefficient of h. Then $h \in P(\mathbb{T})$ if and only if $c_k(h) = 0$ for all $k < 0$.

Now suppose that $A \neq P(\mathbb{T})$. Then there exists $h \in A$ with $c_k(h) \neq$ for some $k < 0$. Without loss of generality we can assume that $c_{-1}(h) = 1$. Indeed, the function g defined by $g(z) = h(z) z^{-(k+1)}$ belongs to A since $A \supseteq P(\mathbb{T})$ and $-(k+1) \geq 0$ and $c_{-1}(g) = c_k(h) \neq 0$. Choose a trigonometric polynomial r with $\|h - r\|_\infty < \frac{1}{2}$ and define $s \in C(\mathbb{T})$ by

$$s(z) = r(z) + (1 - c_{-1}(r)) z^{-1}.$$

Then $c_{-1}(s) = c_{-1}(r) + (1 - c_{-1}(r)) = 1$ and

$$\begin{aligned} \|s - h\|_\infty &\leq \|r - h\|_\infty + |1 - c_{-1}(r)| = \|r - h\|_\infty + |c_{-1}(h - r)| \\ &\leq 2\|h - r\|_\infty < 1. \end{aligned}$$

Thus s is of the form

$$s(z) = \sum_{k=-N}^{-2} c_k(s) z^k + z^{-1} + \sum_{k=0}^{N} c_k(s) z^k$$

for some $N \in \mathbb{N}$. It follows that

$$\begin{aligned} zs(z) &= \sum_{k=-N}^{-2} c_k(s) z^{k+1} + 1 + z \sum_{k=0}^{N} c_k(s) z^k \\ &= \overline{z}\, \overline{p(z)} + 1 + z q(z), \end{aligned}$$

where p and q are polynomials in z. Since $\|s - h\|_\infty < 1$, we obtain that

$$\|1 + z(q - h) + \overline{z}\,\overline{p}\|_\infty = \|zs - zh\|_\infty = \|s - h\|_\infty < 1.$$

Since $q - h \in A$ and $p \in A$, Lemma 2.5.14 shows that the function

$$z \to z(q(z) - h(z)) + zp(z) = z(q - h + p)(z)$$

is invertible in A. So the function $z \to z$ is invertible in A, and hence A contains all the functions $z \to z^m, m \in \mathbb{Z}$. Because the linear combinations of these functions are dense in $C(\mathbb{T})$, we conclude that $A = C(\mathbb{T})$. $\square$

2.6 The structure space of $A(X)$

Let X be a compact subset of $\mathbb{C}$. Recall that $A(X)$ is the closed subalgebra of $C(X)$ consisting of all functions in $C(X)$ which are holomorphic on the interior X° of X. Our aim is to work out the structure space of $A(X)$. Since $A(X)$ is a uniform algebra, the mapping $x \to \varphi_x$, where φ_x is the point evaluation $\varphi_x(f) = f(x)$, $f \in A(X)$, at x, is an embedding of the compact set X into $\Delta(A(X))$. As might be expected, this map is actually surjective, but this is much harder to prove than the corresponding fact for $R(X)$. To establish this result, we need a sequence of preparatory lemmas.

In passing, we mention that in some special cases we already know that $\Delta(A(X)) = X$.

Remark 2.6.1. Clearly, $R(X) \subseteq A(X)$, and $A(X) = C(X)$ whenever $X^\circ = \emptyset$. There exist sets X such that $R(X)$ is strictly contained in $A(X)$. An example is provided by the so-called Swiss cheese (Example 2.5.11) which was obtained by deleting countable many disjoint open discs from the closed unit disc in an appropriate way. On the other hand, $P(\mathbb{D}) = A(\mathbb{D}) \neq C(\mathbb{D})$ (Example 2.5.3).

In the sequel, $\lambda(M)$ denotes the Lebesgue measure of a Borel subset M of $\mathbb{C}$.

Lemma 2.6.2. *Let X be a Borel subset of $\mathbb{C}$. Then, for any $z \in \mathbb{C}$,*

$$\int_X \frac{1}{|x-z|} dx \leq 2(\pi\lambda(X))^{1/2}.$$

In particular, the functions $x \to 1/(x-z)$, $z \in \mathbb{C}$, are integrable on compact subsets of $\mathbb{C}$.

Proof. Nothing has to be shown if $\lambda(X) = 0$ or $\lambda(X) = \infty$. Thus we can assume that $0 < \lambda(X) < \infty$. Let $R = \pi^{-1/2}\lambda(X)^{1/2}$, $S = \{x \in \mathbb{C} : |x-z| \leq R\}$ and, for any $\varepsilon > 0$, $S_\epsilon = \{x \in \mathbb{C} : \varepsilon \leq |x-z| \leq R\}$. Then, introducing polar coordinates, we get

$$\begin{aligned}\int_S \frac{1}{|x-z|} dx &= \lim_{\epsilon \to 0} \int_{S_\epsilon} \frac{1}{|x-z|} dx = \lim_{\epsilon \to 0} \int_\epsilon^R \int_0^{2\pi} d\varphi\, dr \\ &= \lim_{\epsilon \to 0} 2\pi(R-\epsilon) = 2\pi R \\ &= 2(\pi\lambda(X))^{1/2}.\end{aligned}$$

It therefore suffices to show that

$$\int_X \frac{1}{|x-z|} dx \leq \int_S \frac{1}{|x-z|} dx.$$

To that end, note first that $\lambda(X) = \pi R^2 = \lambda(S)$ and $X = (X \cap S) \cup (X \setminus S)$ and hence

$$\lambda(X \setminus S) = \lambda(X) - \lambda(X \cap S) = \lambda(S) \setminus \lambda(S \cap X) = \lambda(S \setminus X).$$

Now $1/|x-z| \geq 1/R$ on $S \setminus X$ and $1/|x-z| \leq 1/R$ on $X \setminus S$. It follows that

$$\begin{aligned}
\int_X \frac{1}{|x-z|} dx &= \int_{X \cap S} \frac{1}{|x-z|} dx + \int_{X \setminus S} \frac{1}{|x-z|} dx \\
&\leq \int_{X \cap S} \frac{1}{|x-z|} dx + \frac{1}{R} \lambda(X \setminus S) \\
&= \int_{X \cap S} \frac{1}{|x-z|} dx + \frac{1}{R} \lambda(S \setminus X) \\
&\leq \int_{X \cap S} \frac{1}{|x-z|} dx + \int_{S \setminus X} \frac{1}{|x-z|} dx \\
&= \int_S \frac{1}{|x-z|} dx,
\end{aligned}$$

as required. □

Lemma 2.6.3. *Let K be a compact subset of $\mathbb{C}$ and g a bounded Borel measurable function on K. Define a function f on $\mathbb{C}$ by*

$$f(z) = \int_K \frac{g(x)}{(x-z)} dx.$$

Then f vanishes at infinity and f is holomorphic on $\mathbb{C} \setminus K$ and continuous everywhere.

Proof. First of all, the integral exists for all $z \in \mathbb{C}$ because the function $x \to 1/(x-z)$ is integrable on compact sets (Lemma 2.6.2) and g is bounded.

If $R > 0$ and the distance from z to K is $\geq R$, then

$$|f(z)| \leq \frac{1}{R} \int_K |g(x)| dx \leq \frac{\lambda(K)}{R} \|g\|_\infty.$$

This shows that f vanishes at infinity.

Next, for $z, z_0 \in \mathbb{C} \setminus K$ with $z \neq z_0$ we have

$$\frac{f(z) - f(z_0)}{z - z_0} = \frac{1}{z - z_0} \int_K \left(\frac{g(x)}{x-z} - \frac{g(x)}{x - z_0} \right) dx = \int_K \frac{g(x)}{(x-z)(x-z_0)} dx.$$

Since $z_0 \notin K$, the function $x \to g(x)/(x-z)(x-z_0)$ converges uniformly on K, as $z \to z_0$, with limit $g(x)/(x-z_0)^2$. Therefore, as $z \to z_0$,

$$\frac{f(z) - f(z_0)}{z - z_0} \to \int_K \frac{g(x)}{(x-z_0)^2} dx.$$

Thus f is holomorphic on $\mathbb{C} \setminus K$.

It remains to show that f is continuous at all points of K. We fix $R > 0$ so that $K \subseteq U = \{x \in \mathbb{C} : |x| < R/2\}$ and prove that f is continuous on U. Since the function $x \to 1/x$ is integrable on any compact subset of $\mathbb{C}$ (Lemma 2.6.2) and $C_c(\mathbb{C})$ is dense in $L^1(\mathbb{C})$, given $\epsilon > 0$, there exists $h \in C_c(\mathbb{C})$ such that

$$\int_{|x|\leq R} \left| h(x) - \frac{1}{x} \right| dx \leq \epsilon.$$

For $y \in U$ we then have

$$\begin{aligned}
\left| f(y) - \int_K g(u)h(u-y)du \right| &\leq \int_K |g(u)| \cdot \left| \frac{1}{u-y} - h(u-y) \right| du \\
&\leq \|g\|_\infty \int_{K-y} \left| \frac{1}{u} - h(u) \right| du \\
&\leq \|g\|_\infty \int_{|u|\leq R} \left| \frac{1}{u} - h(u) \right| du \\
&\leq \epsilon \|g\|_\infty.
\end{aligned}$$

As h is uniformly continuous, there exists $\delta > 0$ such that, for all $x, y \in \mathbb{C}$, $|h(x) - h(y)| \leq \epsilon$ whenever $|x - y| \leq \delta$. For $x, y \in U$ with $|x - y| \leq \delta$ it follows that

$$\begin{aligned}
|f(x) - f(y)| &\leq 2\epsilon \, \|g\|_\infty + \int_K |g(u)| \cdot |h(u-x) - h(u-y)| du \\
&\leq \epsilon \|g\|_\infty (2 + \lambda(K)).
\end{aligned}$$

This shows that f is (uniformly) continuous on U. □

Lemma 2.6.4. *Let X and K be compact subsets of $\mathbb{C}$ and let $f \in A(X)$. Extend f to all of $\mathbb{C}$ by setting $f(x) = 0$ for all $x \in \mathbb{C} \setminus X$, and define h on $\mathbb{C}$ by*

$$h(z) = \int_K \frac{f(x) - f(z)}{x - z} dx.$$

Then h is continuous on $\mathbb{C}$ and holomorphic on X°.

Proof. Since $x \to f(x) - f(z)$ is a bounded Borel measurable function on $\mathbb{C}$, $h(z)$ is defined for all $z \in \mathbb{C}$ and h is a continuous function (Lemma 2.6.3). Therefore, to show that h is holomorphic on X°, by Morera's theorem it is enough to verify that $\int_\gamma h(z)dz = 0$ for every triangle path γ which together with its interior is contained in X°. For that, fix γ, let Γ denote the trace of γ, and note first that the function

$$(x, z) \to \frac{f(x) - f(z)}{x - z}$$

is a Borel function on $K \times \Gamma$ satisfying

$$\int_\gamma \left(\int_K \frac{|f(x) - f(z)|}{|x - z|} \, dx \right) dz < \infty$$

(Lemma 2.6.2). Thus we can apply Fubini's theorem to conclude that

$$\int_\gamma h(z) dz = \int_\gamma \left(\int_K \frac{f(x) - f(z)}{x - z} \, dx \right) dz = \int_K \left(\int_\gamma \frac{f(x) - f(z)}{x - z} dz \right) dx.$$

Now the inner integral along γ is zero for all $x \in \mathbb{C} \setminus \Gamma$. In fact, this is so for every $x \in \mathbb{C} \setminus X^\circ$ since then the function $z \to (x - z)^{-1}(f(x) - f(z))$ is holomorphic on X° , whereas for each $x \in X^\circ \setminus \Gamma$,

$$\int_\gamma \frac{f(x) - f(z)}{x - z} \, dz = 2\pi i \left(f(x) - \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z - x} \, dz \right) = 0$$

by the Cauchy integral formula. Since $K \cap \Gamma$ has Lebesgue measure zero, it follows that $\int_\gamma h(z) dz = 0$, as required. □

The preceding three lemmas together lead to the following approximation result which is the main tool to prove that $\Delta(A(A)) = X$.

Lemma 2.6.5. *Let X be a compact subset of $\mathbb{C}$ and let $z_0 \in X$ and $f \in A(X)$. Then there exists a sequence $(f_n)_n$ in $A(X)$ such that*

$$f(z) - f(z_0) - (z - z_0) f_n(z) \to 0$$

uniformly on X as $n \to \infty$.

Proof. Replacing X by $X - z_0$ and f by $f - f(z_0)$, we can assume that $z_0 = 0$ and $f(z_0) = 0$. Extend f to all of $\mathbb{C}$ by setting $f(x) = 0$ for $x \in \mathbb{C} \setminus X$. For $n \in \mathbb{N}$, let $K_n = \{x \in \mathbb{C} : |x| \leq 1/n\}$ and define f_n on $\mathbb{C}$ by

$$f_n(z) = \frac{n^2}{\pi} \int_{K_n} \frac{f(x) - f(z)}{x - z} \, dx.$$

Then each f_n is continuous on $\mathbb{C}$ and holomorphic on X° (Lemma 2.6.4).

Since $\lambda(K_n) = \pi/n^2$, we have for all $z \in \mathbb{C}$,

$$z f_n(z) - f(z) = \frac{n^2}{\pi} \int_{K_n} \left(z \frac{f(x) - f(z)}{x - z} - f(z) \right) dx.$$

We need to estimate the integral on the right. For $r > 0$, let

$$M(r) = \sup\{|f(z)| : z \in X, |z| \leq r\}.$$

With this notation, for all $z \in \mathbb{C}$, it follows that

$$(1) \qquad |z f_n(z) - f(z)| \leq \frac{n^2}{\pi} \left(|z| M\left(\frac{1}{n}\right) + \frac{1}{n} |f(z)| \right) \int_{K_n} \frac{1}{|x - z|} \, dx.$$

Now, if $|z| > 1/n$, then $|x - z| \geq |z| - 1/n$ for all $x \in K_n$ and hence

$$\frac{n^2}{\pi} \int_{K_n} \frac{1}{|x-z|} dx \leq \frac{1}{|z| - 1/n}. \tag{2}$$

On the other hand, if $q \geq 1$ and $|z| \leq q/n$, then by Lemma 2.6.2

$$\int_{K_n} \frac{1}{|x-z|}\, dx = \int_{K_n - z} \frac{1}{|x|}\, dx \leq \int_{|x| \leq 2q/n} \frac{1}{|x|} dx \leq 2\pi \frac{2q}{n},$$

and hence, for all such z,

$$\frac{n^2}{\pi} \int_{K_n} \frac{1}{|x-z|}\, dx \leq 4qn. \tag{3}$$

Now let $\epsilon > 0$ be given and choose $q \in \mathbb{N}$, $q > 1$, such that $(q-1)\epsilon > \|f\|_\infty$. If $|z| \geq q/n > 1/n$, then combining (1) and (2) yields

$$\begin{aligned} |zf_n(z) - f(z)| &\leq \frac{|z| M(\frac{1}{n}) + \frac{1}{n} |f(z)|}{|z| - \frac{1}{n}} \\ &\leq \left(1 + \frac{\frac{1}{n}}{|z| - \frac{1}{n}}\right) M\left(\frac{1}{n}\right) + \frac{1}{n} \|f\|_\infty \frac{1}{|z| - \frac{1}{n}} \\ &\leq \frac{q}{q-1} M\left(\frac{1}{n}\right) + \|f\|_\infty \frac{1}{q-1} \\ &\leq \frac{q}{q-1} M\left(\frac{1}{n}\right) + \epsilon. \end{aligned}$$

Similarly, if $|z| < q/n$ then combining (1) and (3) gives

$$\begin{aligned} |zf_n(z) - f(z)| &\leq 4qn \left(\frac{q}{n} M\left(\frac{1}{n}\right) + \frac{1}{n} M\left(\frac{q}{n}\right)\right) \\ &= 4q \left(qM\left(\frac{1}{n}\right) + M\left(\frac{q}{n}\right)\right). \end{aligned}$$

However, $M(r) \to 0$ as $r \to 0$ since f is continuous on X and $f(0) = 0$. It follows that $zf_n(z) - f(z) \to 0$ uniformly on X. □

Theorem 2.6.6. *Let X be a compact subset of $\mathbb{C}$. Then the mapping $x \to \varphi_x$, where $\varphi_x(f) = f(x)$ for all $f \in A(X)$, is a homeomorphism between X and $\Delta(A(X))$. With this identification of $\Delta(A(X))$ and X, the Gelfand homomorphism of $A(X)$ is the identity.*

Proof. We only have to show that given $\varphi \in \Delta(A(X))$, there exists $x \in X$ such that $\varphi(f) = f(x)$ for all $f \in A(X)$.

Let $x = \varphi(\mathrm{id}_X)$. Then $x \in X$ since, for every $\lambda \in \mathbb{C} \setminus X$, the function $z \to 1/(\lambda - z)$ belongs to $A(X)$ and therefore

$$\lambda \notin \sigma_{A(X)}(\mathrm{id}_X) = \widehat{\mathrm{id}_X}(\Delta(A(X))).$$

Now, for any $f \in A(X)$, by Lemma 2.6.5 there exists a sequence $(f_n)_n$ in $A(X)$ such that

$$f(z) - f(x) - (z - x)f_n(z) \to 0$$

uniformly on X. This implies that

$$\varphi(f) - f(x) = \lim_{n\to\infty} (\varphi(\mathrm{id}_X) - x)\varphi(f_n) = 0,$$

as was to be shown.

2.7 The Gelfand representation of $L^1(G)$

In commutative harmonic analysis the central object of study is the L^1-algebra of a locally compact Abelian group. In this section we present its Gelfand representation. Thus, in the sequel, G always denotes a locally compact Abelian group and $L^1(G)$ the convolution algebra of integrable functions on G.

To begin with, we introduce the dual group of G which turns out to be canonically identifiable with $\Delta(L^1(G))$.

Definition 2.7.1. A *character* α of G is a continuous homomorphism from G into the circle group $\mathbb{T}$. Clearly, the pointwise product of two characters is again a character and so is α^{-1} defined by $\alpha^{-1}(x) = \overline{\alpha(x)}$ for all $x \in G$. Thus $\widehat{G}$, the set of all characters of G, forms a group, the *dual group* of G.

We proceed to show that there is a bijection between $\widehat{G}$ and $\Delta(L^1(G))$.

Theorem 2.7.2. *For $\alpha \in \widehat{G}$, let $\varphi_\alpha : L^1(G) \longrightarrow \mathbb{C}$ be defined by*

$$\varphi_\alpha(f) = \int_G f(x)\overline{\alpha(x)}dx, \quad f \in L^1(G).$$

Then $\varphi_\alpha \in \Delta(L^1(G))$ and the mapping $\alpha \to \varphi_\alpha$ is a bijection from $\widehat{G}$ onto $\Delta(L^1(G))$.

Proof. Of course, φ_α is a linear functional. For $f, g \in C_c(G)$, Fubini's theorem and the invariance of Haar measure yield

$$\begin{aligned}
\varphi_\alpha(f * g) &= \int_G \overline{\alpha(x)} \int_G f(y)g(y^{-1}x)dydx \\
&= \int_G \int_G f(y)\overline{\alpha(x)}g(y^{-1}x)dxdy \\
&= \int_G \int_G f(y)\overline{\alpha(yx)}g(x)dxdy \\
&= \int_G \int_G \overline{\alpha(x)}g(x)\overline{\alpha(y)}f(y)dxdy \\
&= \varphi_\alpha(f)\varphi_\alpha(g).
\end{aligned}$$

Since $|\varphi_\alpha(f)| \leq \|f\|_1$ for all $f \in L^1(G)$, this formula even holds for all $f, g \in L^1(G)$. Moreover, φ_α is nonzero since for any nonnegative function f in $C_c(G)$, $f \neq 0$, we have

$$\varphi_\alpha(\alpha f) = \int_G f(x)|\alpha(x)|^2 dx > 0.$$

This shows that $\varphi_\alpha \in \Delta(L^1(G))$. Moreover, the map $\alpha \to \varphi_\alpha$ is injective. Indeed, if $\alpha, \beta \in \widehat{G}$ are such that

$$0 = \varphi_\alpha(f) - \varphi_\beta(f) = \int_G f(x)\big(\overline{\alpha(x)} - \overline{\beta(x)}\big)dx$$

for all $f \in L^1(G)$, then $\alpha = \beta$ because $L^1(G)^* = L^\infty(G)$ and α and β are continuous functions.

It remains to show that given $\varphi \in \Delta(L^1(G))$, there exists $\alpha \in \widehat{G}$ such that $\varphi = \varphi_\alpha$. To that end, choose $g \in L^1(G)$ such that $\varphi(g) = 1$ and observe that since $\varphi \in L^1(G)^*$, there exists $\chi \in L^\infty(G)$ such that $\varphi(f) = \int_G f(x)\chi(x)dx$ for all $f \in L^1(G)$. The function

$$(x, y) \to \chi(x)f(y)g(y^{-1}x)$$

belongs to $L^1(G \times G)$, and hence Fubini's theorem implies that

$$\begin{aligned}
\varphi(f) = \varphi(f * g) &= \int_G \chi(x)\left(\int_G f(y)g(y^{-1}x)dy\right)dx \\
&= \int_G f(y)\left(\int_G g(y^{-1}x)\chi(x)dx\right)dy \\
&= \int_G f(y)\varphi(L_y g)dy
\end{aligned}$$

for all $f \in L^1(G)$. Now, define $\alpha : G \to \mathbb{C}$ by $\alpha(y) = \overline{\varphi(L_y g)}$, $y \in G$. The function α is continuous because the map $y \to L_y g$ from G into $L^1(G)$ is continuous and

$$|\alpha(x) - \alpha(y)| = |\varphi(L_x g - L_y g)| \leq \|L_x g - L_y g\|_1$$

for all $x, y \in G$. From $g * L_{xy} g = L_x g * L_y g$ it follows that

$$\begin{aligned}
\alpha(xy) &= \overline{\varphi(L_{xy} g)} = \overline{\varphi(g)}\,\overline{\varphi(L_{xy} g)} = \overline{\varphi(g * L_{xy} g)} \\
&= \overline{\varphi(L_x g * L_y g)} = \overline{\varphi(L_x g)\varphi(L_y g)} \\
&= \alpha(x)\alpha(y).
\end{aligned}$$

We claim that $|\alpha(x)| = 1$ for all $x \in G$. For that, notice that

$$|\alpha(y)| = |\varphi(L_y g)| \leq \|L_y g\|_1 = \|g\|_1$$

for all $y \in G$, and hence, by the multiplicativity of α,

$$|\alpha(x)|^n = |\alpha(x^n)| \leq \|g\|_1$$

for all $n \in \mathbb{Z}$. Since $\alpha(e) = \overline{\varphi(g)} = 1$, we conclude that $|\alpha(x)| = 1$ for every $x \in G$. This shows that $\alpha \in \widehat{G}$ and $\varphi_\alpha = \varphi$. $\square$

After identifying $\Delta(L^1(G))$ as a set with $\widehat{G}$, our next purpose is to describe the Gelfand topology on $\widehat{G}$ in terms of G itself rather than $L^1(G)$.

Lemma 2.7.3. *Let $f \in L^1(G)$ and $\alpha \in \widehat{G}$.*

(i) *For all $x \in G$, $(f * \alpha)(x) = \alpha(x)\widehat{f}(\alpha) = \widehat{L_{x^{-1}}f}(\alpha)$. In particular, $\widehat{L^1(G)}$ is invariant under multiplication with functions of the form $\alpha \to \alpha(x)$, $x \in G$.*

(ii) *If $g \in L^1(G)$ is defined by $g(x) = \alpha(x)f(x)$, then $\widehat{g} = L_\alpha \widehat{f}$. In particular, $\widehat{L^1(G)} \subseteq C_0(\widehat{G})$ is translation invariant.*

(iii) *$\widehat{f^*} = \overline{\widehat{f}}$ and $\widehat{L^1(G)} \subseteq C_0(\widehat{G})$ is norm-dense in $C_0(\widehat{G})$.*

Proof. (i) $f * \alpha$ is a continuous function and

$$(f * \alpha)(x) = \int_G f(y)\alpha(y^{-1}x)dy = \alpha(x)\widehat{f}(\alpha)$$

for all $x \in G$. On the other hand,

$$(f * \alpha)(x) = \int_G f(xy)\overline{\alpha(y)}dy = \widehat{L_{x^{-1}}f}(\alpha).$$

(ii) For all $\beta \in \widehat{G}$, we have

$$\widehat{g}(\beta) = \int_G f(x)\overline{\beta(x)}\alpha(x)dx = \widehat{f}(\alpha^{-1}\beta) = L_\alpha \widehat{f}(\beta),$$

so that $L_\alpha \widehat{f} = \widehat{g} \in \widehat{L^1(G)}$.

(iii) For each $\alpha \in \widehat{G}$, we have

$$\widehat{f^*}(\alpha) = \int_G \overline{f(x^{-1})\alpha(x)}dx = \int_G \overline{f(x)}\alpha(x)dx = \overline{\widehat{f}(\alpha)},$$

so that $\widehat{f^*} = \overline{\widehat{f}}$. Thus $\widehat{L^1(G)}$ is a self-adjoint subalgebra of $C_0(\widehat{G})$ which strongly separates the points of $\widehat{G}$ and therefore is dense in $(C_0(\widehat{G}), \|\cdot\|_\infty)$ by the Stone–Weierstrass theorem. $\square$

Lemma 2.7.4. *Let $f \in L^1(G)$ and $\epsilon > 0$ and let σ denote the Gelfand topology on $\widehat{G}$. Then there exists a neighbourhood W of e in G with the following property. If $y, x \in G$ and $\beta, \alpha \in \widehat{G}$ are such that $y \in Wx$, $\varphi_\alpha(f) = 1$, and $\beta \in U(\alpha, f, L_x f, \epsilon/3)$, then*

$$|\beta(y) - \alpha(x)| < \epsilon.$$

In particular, the function $(x, \alpha) \to \alpha(x)$ is continuous on $G \times (\widehat{G}, \sigma)$.

Proof. For arbitrary $y, x \in G$ and $\beta, \alpha \in \widehat{G}$ such that $\widehat{f}(\alpha) = 1$ we obtain from Lemma 2.7.3,

$$\begin{aligned}|\beta(y) - \alpha(x)| &\leq |\overline{\beta(y)} - \overline{\beta(y)}\widehat{f}(\beta)| + |\overline{\beta(y)}\widehat{f}(\beta) - \overline{\beta(x)}\widehat{f}(\beta)| \\ &\quad + |\overline{\beta(x)}\widehat{f}(\beta) - \overline{\alpha(x)}\widehat{f}(\alpha)| \\ &= |1 - \widehat{f}(\beta)| + |\widehat{L_y f}(\beta) - \widehat{L_x f}(\beta)| + |\widehat{L_x f}(\beta) - \widehat{L_x f}(\alpha)| \\ &\leq |\widehat{f}(\beta) - \widehat{f}(\alpha)| + \|L_y f - L_x f\|_1 + |\widehat{L_x f}(\beta) - \widehat{L_x f}(\alpha)|.\end{aligned}$$

Now let W be a neighbourhood of e such that $\|L_s f - L_t f\|_1 < \epsilon/3$ whenever $t^{-1}s \in W$. For all $y \in Wx$ and $\beta \in U(\alpha, f, L_x f, \epsilon/3)$ it then follows that $|\beta(y) - \alpha(x)| < \epsilon$.

For the last statement of the lemma we only have to recall that given $\alpha \in \widehat{G}$, there exists $f \in L^1(G)$ such that $\widehat{f}(\alpha) = 1$. □

We now consider the compact open topology τ on $\widehat{G}$. A τ-neighbourhood basis of $\alpha_0 \in \widehat{G}$ is formed by the collection of sets

$$V(\alpha_0, K, \epsilon) = \{\alpha \in \widehat{G} : |\alpha(x) - \alpha_0(x)| < \epsilon \text{ for all } x \in K\},$$

where $\epsilon > 0$ and K is any compact subset of G. Then $(\widehat{G}, \tau)$ is a topological group since $V(\alpha_0, K, \epsilon)^{-1} = V(\alpha_0^{-1}, K, \epsilon)$ and

$$V(\alpha_0, K, \epsilon)V(\beta_0, K, \epsilon) \subseteq V(\alpha_0\beta_0, K, 2\epsilon).$$

In fact, the latter inclusion follows from

$$\begin{aligned}|\alpha\beta(x) - \alpha_0\beta_0(x)| &\leq |\alpha(x)(\beta(x)) - \beta_0(x))| + |\beta_0(x)(\alpha(x) - \alpha_0(x))| \\ &\leq |\beta(x) - \beta_0(x)| + |\alpha(x) - \alpha_0(x)|.\end{aligned}$$

Theorem 2.7.5. *On $\widehat{G}$ the Gelfand topology and the compact open topology coincide.*

Proof. Let 1_G denote the trivial character of G. Note that, for $\alpha \in \widehat{G}$, $\delta > 0$, and $f_1, \ldots, f_n \in L^1(G)$, we have

$$\alpha\, U(1_G, f_1, \ldots, f_n, \delta) = U(\alpha, f_1\alpha, \ldots, f_n\alpha, \delta).$$

In fact, for $\beta \in \widehat{G}$ and $f \in L^1(G)$,

$$\begin{aligned}\varphi_\beta(f\alpha) - \varphi_\alpha(f\alpha) &= \int_G f(x)\alpha(x)(\overline{\beta(x)} - \overline{\alpha(x)})dx \\ &= \int_G f(x)(\overline{\alpha^{-1}\beta(x)} - 1)dx \\ &= \varphi_{\alpha^{-1}\beta}(f) - \varphi_{1_G}(f).\end{aligned}$$

Hence we only have to verify that every τ-neighbourhood of 1_G contains a σ-neighbourhood of 1_G and vice versa.

Let $V(1_G, K, \delta)$ be given and choose $f \in L^1(G)$ such that $\int_G f(x)dx = 1$. By Lemma 2.7.4, there exists a neighbourhood W of e in G such that if $x, y \in G$ satisfy $y \in Wx$ and if $\alpha \in U(1_G, f, L_x f, \delta/3)$, then $|\alpha(y) - 1| < \delta$. Because K is compact, we find $x_1, \dots, x_r \in K$ so that $K \subseteq \bigcup_{j=1}^r W x_j$. It follows that

$$U(1_G, f, L_{x_1} f, \dots, L_{x_r} f, \delta/3) \subseteq V(1_G, K, \delta).$$

Conversely, let $U(1_G, f_1, \dots, f_n, \delta)$ be given. We can assume that $f_j \neq 0$ for all $j = 1, \dots, n$. For every j, choose $g_j \in C_c(G)$ with $\|f_j - g_j\|_1 < \delta/4$. Set

$$K = \bigcup \{\operatorname{supp} g_j : 1 \leq j \leq n\}$$

and

$$\epsilon = \frac{\delta}{2} \min\{\|f_j\|^{-1} : 1 \leq j \leq n\}.$$

We claim that

$$V(1_G, K, \epsilon) \subseteq U(1_G, f_1, \dots, f_n, \delta).$$

Indeed, if $\alpha \in V(1_G, K, \epsilon)$ then, for each $j = 1, \dots, n$,

$$\begin{aligned}
|\varphi_\alpha(f_j) - \varphi_{1_G}(f_j)| &\leq \int_K |f_j(x)| \cdot |\alpha(x) - 1| dx \\
&\quad + \int_{G \setminus K} |f_j(x)| \cdot |\alpha(x) - 1| dx \\
&< \epsilon \|f_j\|_1 + 2 \int_{G \setminus K} |f_j(x)| dx \\
&= \varepsilon \|f_j\|_1 + 2 \int_{G \setminus K} |f_j(x) - g_j(x)| dx \\
&\leq \epsilon \|f_j\|_1 + 2 \|f_j - g_j\|_1 \\
&\leq \delta.
\end{aligned}$$

This completes the proof. □

Since $(\widehat{G}, \sigma)$ is locally compact and $(\widehat{G}, \tau)$ is a topological group, Theorem 2.7.5 in particular shows that $\widehat{G}$ is a locally compact group. Identifying $\Delta(L^1(G))$ as a topological space with $\widehat{G}$, the Gelfand representation of $L^1(G)$ is the mapping $f \to \widehat{f}$, where $\widehat{f} \in C_0(\widehat{G})$ is defined by

$$\widehat{f}(\alpha) = \int_G f(x) \overline{\alpha(x)} dx, \quad \alpha \in \widehat{G}.$$

We now present a number of simple examples of dual groups, such as $\widehat{\mathbb{R}}$, $\widehat{\mathbb{Z}}$, and $\widehat{\mathbb{T}}$.

Example 2.7.6. (1) The dual group of the real line $\mathbb{R}$ is topologically isomorphic to $\mathbb{R}$. In fact, for each $y \in \mathbb{R}$, define a character α_y of $\mathbb{R}$ by $\alpha_y(x) = \exp(2\pi ixy)$, $x \in \mathbb{R}$. Then the map $y \to \alpha_y$ from $\mathbb{R}$ into $\widehat{\mathbb{R}}$ is injective and every character of $\mathbb{R}$ arises in this way (see Exercise 2.12.29). In addition, $y \to \alpha_y$ is a homeomorphism. Now,

$$\varphi_{\alpha_y}(f) = \int_{\mathbb{R}} f(x) \exp(-2\pi ixy)dx = \widehat{f}(y).$$

Thus, after identifying $\widehat{\mathbb{R}}$ with $\mathbb{R}$, the Gelfand homomorphism of $L^1(\mathbb{R})$ agrees with the Fourier transformation.

(2) In Example 2.2.10 we have already determined the Gelfand representation of $l^1(\mathbb{Z})$. Implicit in the arguments given there is the fact that the map $z \to \alpha_z$, where $\alpha_z(n) = z^n$ for $n \in \mathbb{Z}$, is a homeomorphism between $\mathbb{T}$ and the dual group $\widehat{\mathbb{Z}}$.

(3) It follows from $\widehat{\mathbb{Z}} = \mathbb{T}$ and the duality theorem for locally compact Abelian groups (a proof of which we present in Theorem A.5.2) that $\widehat{\mathbb{T}}$ is isomorphic to $\mathbb{Z}$. However, this can be seen directly as follows. First, for every $n \in \mathbb{Z}$, the function $z \to z^n$ is a character of $\mathbb{T}$. To show that every character α of $\mathbb{T}$ is of this form, consider the functions f_k on $\mathbb{T}$ defined by $f_k(z) = z^k$ $(k \in \mathbb{Z})$. By the Weierstrass approximation theorem the linear span of these functions f_k is dense in $C(\mathbb{T})$ and hence in $L^1(\mathbb{T})$. Thus $\widehat{f_k}(\alpha) \neq 0$ for at least one k. On the other hand, for arbitrary $k, l \in \mathbb{Z}$,

$$f_k * f_l(z) = \int_{\mathbb{T}} t^k (t^{-1}z)^l dt = z^k = f_k(z)$$

if $k = l$ and $= 0$ otherwise. Thus $\widehat{f_k}(\alpha)\widehat{f_l}(\alpha) = \widehat{f_k}(\alpha)$ if $l = k$ and $\widehat{f_k}(\alpha)\widehat{f_l}(\alpha) = 0$ otherwise. This implies that $\widehat{f_k}(\alpha) = 1$ for exactly one k and $\widehat{f_l}(\alpha) = 0$ for all $l \in \mathbb{Z}$, $l \neq k$. Now, because

$$\widehat{f_l}(\alpha_k) = \int_{\mathbb{T}} z^{l-k} dz = \delta_{kl}$$

for all $l \in \mathbb{Z}$, we obtain that $\alpha = \alpha_k$, as was to be shown. Finally, the relation $\widehat{f_l}(\alpha_k) = \delta_{kl}$ $(k, l \in \mathbb{Z})$ also shows that the Gelfand topology on $\widehat{\mathbb{T}}$ is discrete. Thus $\widehat{\mathbb{T}}$ is topologically isomorphic to $\mathbb{Z}$, and identifying $\widehat{\mathbb{T}}$ with $\mathbb{Z}$, we have

$$\widehat{f}(n) = \int_{\mathbb{T}} f(z) z^{-n} dz$$

for $f \in L^1(\mathbb{T})$ and $n \in \mathbb{Z}$.

(4) Let G_1 and G_2 be two locally compact Abelian groups and $G = G_1 \times G_2$ their direct product. It is not difficult to show that the map

$$(\alpha_1, \alpha_2) \to \alpha,\ \alpha(x_1, x_2) = \alpha_1(x_1)\alpha_2(x_2) \quad (\alpha_j \in \widehat{G}_j, x_j \in G_j, j = 1, 2)$$

furnishes a topological isomorphism from $\widehat{G}_1 \times \widehat{G}_2$ to $\widehat{G}$. Therefore, combining the cases (1), (2), and (3), the Gelfand representation of $L^1(G)$ can be explicitly given for groups of the form $\mathbb{R}^m \times \mathbb{Z}^n \times \mathbb{T}^r$, $m, n, r \in \mathbb{N}_0$.

Our next goal is to show that $L^1(G)$ is semisimple. To achieve this opens the opportunity to introduce the regular representation of $L^1(G)$ and the group C^*-algebra of G. Both are needed anyway in Chapter 4 in our approach to establish regularity of $L^1(G)$.

To start with, recall that for $f \in L^1(G)$ and $g \in C_c(G)$, the convolution product $f * g$ is given by

$$(f * g)(x) = \int_G f(y) g(y^{-1}x) dy$$

for every $x \in G$, and $f * g$ is a continuous function. For $g, h \in C_c(G)$ and $f \in L^1(G)$, using Fubini's theorem and Hölder's inequality, we get

$$\begin{aligned}
\left| \int_G (f * g)(x) h(x) dx \right| &= \left| \int_G \int_G f(y) g(y^{-1}x) h(x) dy dx \right| \\
&= \left| \int_G \int_G f(y) g(x) h(yx) dx dy \right| \\
&\leq \int_G |f(y)| \int_G |g(x) L_{y^{-1}} h(x)| dx dy \\
&\leq \int_G |f(y)| \cdot \|L_{y^{-1}} h\|_2 \|g\|_2 dy \\
&\leq \|f\|_1 \|g\|_2 \|h\|_2.
\end{aligned}$$

Since $C_c(G)$ is dense in $L^2(G)$ it follows that the map

$$h \to \int_G h(x)(f * g)(x) dx$$

extends to a bounded linear functional on $L^2(G)$ the norm of which is at most $\|f\|_1 \|g\|_2$. Since $L^2(G)^* = L^2(G)$, we conclude that $f * g \in L^2(G)$ and $\|f * g\|_2 \leq \|f\|_1 \|g\|_2$ (see also Proposition A.4.7). Thus the linear mapping $g \to f * g$ from $C_c(G)$ into $L^2(G)$ extends uniquely to a bounded linear transformation $\lambda_f : L^2(G) \to L^2(G)$ and $\|\lambda_f\| \leq \|f\|_1$.

Theorem 2.7.7. *The mapping $\lambda : f \to \lambda_f$ from $L^1(G)$ into $\mathcal{B}(L^2(G))$ is an injective $*$-homomorphism.*

Proof. It is clear that λ is linear. For $f_1, f_2 \in L^1(G)$ and $g \in C_c(G)$,

$$\lambda_{f_1 * f_2}(g) = f_1 * (f_2 * g) = \lambda_{f_1}(\lambda_{f_2}(g)).$$

Thus λ is a homomorphism. Moreover, for $f \in L^1(G)$ and $g, h \in C_c(G)$,

$$\begin{aligned}\langle\lambda_f^*(g),h\rangle &= \langle g,\lambda_f(h)\rangle = \int_G g(x)\int_G \overline{f(y)h(y^{-1}x)}dydx\\ &= \int_G\int_G \overline{f(y^{-1})}g(x)\overline{h(yx)}dxdy\\ &= \int_G\int_G f^*(y)g(y^{-1}x)\overline{h(x)}dxdy = \int_G (f^* * g)(x)\overline{h(x)}dx\\ &= \langle\lambda_{f^*}(g),h\rangle.\end{aligned}$$

This proves that λ is a $*$-homomorphism.

Finally, λ is injective. Indeed, if $f \in L^1(G)$ is such that $0 = \lambda_f(g) = f * g$ for all $g \in C_c(G)$, then $f = 0$ since $C_c(G)$ contains an approximate identity for $L^1(G)$. □

Definition 2.7.8. The $*$-homomorphism $\lambda : f \to \lambda_f$ from $L^1(G)$ into $\mathcal{B}(L^2(G))$ is called the *regular representation* of $L^1(G)$ on the Hilbert space $L^2(G)$. Let $C^*(G)$ denote the closure of $\lambda(L^1(G))$ in $\mathcal{B}(L^2(G))$. Then, by Theorem 2.7.7, $C^*(G)$ is a commutative C^*-algebra, the so-called *group C^*-algebra* of G.

Every commutative C^*-algebra is semisimple and λ is injective (Theorem 2.7.7). Thus we conclude the following

Corollary 2.7.9. *$L^1(G)$ is semisimple.*

We now turn to the interesting and likewise important question of when the Gelfand homomorphism $L^1(G) \to C_0(\widehat{G})$ is surjective. Clearly, this is the case if G is finite because then $\widehat{L^1(G)}$ is a finite-dimensional dense linear subspace of $C_0(\widehat{G})$. To establish the converse, we first show that surjectivity forces G to be discrete.

Lemma 2.7.10. *Let G be a locally compact Abelian group, and suppose that the Gelfand homomorphism $\Gamma : f \to \widehat{f}$ from $L^1(G)$ into $C_0(\widehat{G})$ is surjective. Then G has to be discrete.*

Proof. Let $\Gamma^* : M(\widehat{G}) = (C_0(\widehat{G}))^* \to L^1(G)^* = L^\infty(G)$ denote the dual mapping of Γ. Since Γ is surjective, it is an isomorphism of Banach spaces and hence Γ^* is also an isomorphism. For $\mu \in M(\widehat{G})$, define its inverse Fourier–Stieltjes transform $\check{\mu}$ on G by $\check{\mu}(x) = \int_G \overline{\alpha(x)}d\mu(\alpha)$. Then, using Fubini's theorem, for any $\mu \in M(\widehat{G})$ and $f \in L^1(G)$,

$$\begin{aligned}\langle\Gamma^*(\mu),f\rangle = \langle\mu,\Gamma(f)\rangle &= \int_{\widehat{G}} \widehat{f}(\alpha)d\mu(\alpha)\\ &= \int_{\widehat{G}}\left(\int_G f(x)\overline{\alpha(x)}dx\right)d\mu(\alpha)\\ &= \int_G f(x)\left(\int_{\widehat{G}}\overline{\alpha(x)}d\mu(\alpha)\right)dx\\ &= \int_G f(x)\check{\mu}(x)dx.\end{aligned}$$

It follows that $\Gamma^*(\mu) = \check{\mu}$ locally almost everywhere for every $\mu \in M(\widehat{G})$. Now, using the facts that the function $(x, \alpha) \to \alpha(x)$ is continuous on $G \times \widehat{G}$ (Lemma 2.7.4) and that

$$|\check{\mu}(x) - \check{\mu}(y)| \leq \int_{\widehat{G}} |\alpha(x) - \alpha(y)| d|\mu|(\alpha),$$

it is easily verified that $\check{\mu}$ is continuous. Since Γ^* is onto, this means that every function in $L^\infty(G)$ is equal locally almost everywhere to a continuous function. However, this implies that G is discrete. Indeed, let U be an open, relatively compact subset of G which is not dense in G and let g be a continuous function on G which equals the characteristic function of U locally almost everywhere. Then $g(x) = 1$ for $x \in \overline{U}$, whereas $g(x) = 0$ for $x \in G \setminus U$. It follows that U is closed in G and since this holds for any such set U, we conclude that G is discrete. □

Lemma 2.7.11. *Let G be a compact Abelian group and let X be an infinite subgroup of $\widehat{G}$. Then there exists $f \in C(G)$ such that* $\operatorname{supp}\widehat{f} \subseteq X$ *and* $\sum\limits_{\chi \in X} |\widehat{f}(\chi)| = \infty$.

Proof. The key step in achieving the existence of such a function f is to find a sequence $(g_n)_n$ of continuous functions on G with the following properties.

(1) $\|g_n\|_\infty \leq 2^{(n+1)/2}$.
(2) The range of $\widehat{g}_n$ is contained in $\{-1, 0, 1\}$.
(3) $|\widehat{g}_n|$ is the characteristic function of some subset X_n of X having precisely 2^n elements.

Suppose first that such a sequence $(g_n)_n$ exists. Then, because X is infinite and all X_n are finite, we can inductively define a sequence of characters $\chi_1, \chi_2, \ldots$ in X such that, with $\chi_0 = 1_G$, the sets $\chi_n^{-1} X_n, n \in \mathbb{N}_0$, are pairwise disjoint. Then, let $f_n = 2^{-n/2} \chi_n g_n, n \in \mathbb{N}_0$, so that the range of $\widehat{f_n}$ is contained in $\{-2^{-2/n}, 0, 2^{-n/2}\}$ and $2^{n/2}|\widehat{f}_n|$ is the characteristic function of the set $A_n = \chi_n X_n$ which contains exactly 2^n elements. Since $\|f_n\|_\infty \leq 2^{-n/2}\|g_n\|_\infty \leq 2^{1/2}$, we can define a continuous function f on G by setting

$$f(x) = \sum_{n=0}^{\infty} 2^{-n/2} f_n(x).$$

Now, if $\chi \notin \bigcup_{n=0}^\infty A_n$, then

$$\widehat{f}(\chi) = \sum_{n=0}^{\infty} 2^{-n/2} \widehat{f}_n(\chi) = 0,$$

whereas, if $\chi \in \bigcup_{n=0}^\infty A_n$, then $\chi \in A_n$ for exactly one n and hence

$$|\widehat{f}(\chi)| = 2^{-n/2} |\widehat{f}_n(\chi)| = 2^{-n}.$$

It follows that $\operatorname{supp} \widehat{f} \subseteq X$ and

$$\sum_{\chi \in X} |\widehat{f}(\chi)| = \sum_{n=0}^{\infty} \left(\sum_{\chi \in A_n} |\widehat{f}(\chi)| \right) = \sum_{n=0}^{\infty} 2^{-n} |A_n| = \infty,$$

as required.

To start the construction of the sequence $(g_n)_n$, first let $\chi_1, \chi_2, \ldots$ be any sequence of characters of G which are specified later. Define sequences $g_0, g_1, \ldots$ and $h_0, h_1, \ldots$ inductively in $C(G)$ by $g_0 = h_0 = 1_G$ and

$$g_{n+1} = g_n + \chi_{n+1} h_n \text{ and } h_{n+1} = g_n - \chi_{n+1} h_n.$$

It is straightforward to verify that

$$|g_{n+1}(x)|^2 + |h_{n+1}(x)|^2 = 2(|g_n(x)|^2 + |h_n(x)|^2)$$

for all $x \in G$ and $n \in \mathbb{N}_0$. Hence the supremum norm of $|g_{n+1}|^2 + |h_{n+1}|^2$ bounded by $\leq 2^{n+2}$ whenever the supremum norm of $|g_n|^2 + |h_n|^2$ is bounded by $\leq 2^{n+1}$. Suppose that the ranges of both $\widehat{g}_n$ and $\widehat{h}_n$ are contained in $\{-1, 0, 1\}$. Then the same is true of $\widehat{g}_{n+1}$ and $\widehat{h}_{n+1}$ provided that χ_{n+1} has the property that

$$\operatorname{supp} \widehat{g}_n \cap \chi_{n+1}(\operatorname{supp} \widehat{h}_n) = \emptyset.$$

Moreover, if χ_{n+1} has this property and $|\widehat{g}_n|$ and $|\widehat{h}_n|$ are the characteristic functions of sets E_n and F_n, respectively, such that E_n and F_n each contain precisely 2^n elements, then $|\widehat{g}_{n+1}|$ is the characteristic function of $E_n \cup \chi_{n+1} F_n$, which has 2^{n+1} elements, and similarly for $|\widehat{h}_{n+1}|$.

It is now obvious fairly how the sequence $\chi_1, \chi_2, \ldots$ has to be chosen. Since $\int_G \chi(x) dx = 0$ for every $\chi \neq 1_G$ (Exercise 2.12.30), we have $\widehat{g}_0 = \widehat{h}_0 = \delta_{1_G}$. Hence χ_1 may be any nontrivial character from X. Suppose that $\chi_1, \ldots, \chi_n \in X$ have been chosen such that $g_1, \ldots, g_n$ and $h_1, \ldots, h_n$ have the above properties. Then we simply have to select $\chi_{n+1} \in X$ so that $E_n \cap \chi_{n+1}^{-1} F_n = \emptyset$, and this is possible since E_n and F_n are finite and X is infinite. This completes the construction of a sequence $(g_n)_n$ with properties (1), (2), and (3) above. □

The functions f_n constructed in the proof of Lemma 2.7.11 are analogues of the Rudin-Shapiro trigonometric polynomials on the circle group (see [72, p. 33, Exercise 6]).

Theorem 2.7.12. *Let G be a locally compact Abelian group. Then the Gelfand homomorphism $\Gamma : L^1(G) \to C_0(\widehat{G})$ is surjective if and only if G is finite.*

Proof. Suppose that Γ is surjective. Then G is discrete by Lemma 2.7.10. Towards a contradiction, assume that G is infinite. For each $x \in G$, define a character χ_x of $\widehat{G}$ by $\chi_x(\alpha) = \alpha(x)$. Then $x \to \chi_x$ is a bijection between

G and the subgroup $X = \{\chi_x : x \in G\}$ of the dual group $\widehat{\widehat{G}}$ of $\widehat{G}$. Since $\widehat{G}$ is compact and X is infinite, by Lemma 2.7.11 there exists $f \in C(\widehat{G})$ such that $\operatorname{supp}\widehat{f} \subseteq X$ and $\sum_{\chi\in X} |\widehat{f}(\chi)| = \infty$. Since Γ is surjective, there exists $g \in L^1(G)$ with $\widehat{g} = f$. Now recall that $\int_{\widehat{G}} \chi(\alpha)d\alpha = 0$ for every $\chi \in \widehat{\widehat{G}} \setminus \{1_{\widehat{G}}\}$. It follows that

$$\begin{aligned}
\widehat{f}(\chi_x) &= \int_{\widehat{G}} \overline{\chi_x(\alpha)} \left(\sum_{y\in G} g(y)\overline{\alpha(y)} \right) d\alpha \\
&= \sum_{y\in G} g(y) \left(\int_{\widehat{G}} \overline{\chi_x(\alpha)}\, \overline{\chi_y(\alpha)} d\alpha \right) \\
&= \sum_{y\in G} g(y) \left(\int_{\widehat{G}} \overline{\chi_{xy}(\alpha)} d\alpha \right) \\
&= g(x^{-1})
\end{aligned}$$

for every $x \in G$. Thus

$$\sum_{x\in G} |g(x^{-1})| = \sum_{x\in G} |\widehat{f}(\chi_x)| = \sum_{\chi\in X} |\widehat{f}(\chi)| = \infty.$$

This contradiction shows that G must be finite. □

To prove Theorem 2.7.12, it is possible to avoid the use of Lemma 2.7.11 and instead only apply Lemma 2.7.10 and the Pontryagin duality theorem. However, we prefer not to utilise the duality theorem although in Appendix A.5 we have presented a proof of it, based on the Plancherel theorem. In addition, we feel the construction performed in the proof of Lemma 2.7.11 is of independent interest.

2.8 Beurling algebras $L^1(G,\omega)$

Let G be a locally compact Abelian group and ω a weight function on G. In Section 1.3 we have introduced the associated Beurling algebra $L^1(G,\omega)$. Extending some of the results of the preceding section, we now describe the structure space $\Delta(L^1(G,\omega))$ of $L^1(G,\omega)$ in terms of so-called ω-bounded generalised characters of G. These generalized characters can be identified explicitly when G is either the additive group of real numbers or the group of integers. We also show that $L^1(G,\omega)$ is always semisimple.

Definition 2.8.1. An *ω-bounded generalised character* on G is a continuous homomorphism α from G into the multiplicative group $\mathbb{C}^\times$ of nonzero complex numbers satisfying $|\alpha(x)| \leq \omega(x)$ for all $x \in G$. Let $\widehat{G}(\omega)$ denote the set of all such ω-bounded generalised characters on G equipped with the topology of uniform convergence on compact subsets of G.

It is clear from the very definition of $\widehat{G}(\omega)$ that $\widehat{G}$ is contained in $\widehat{G}(\omega)$ if and only if $\omega(x) \geq 1$ for all $x \in G$. Our first result is the analogue of Theorem 2.7.2.

Theorem 2.8.2. *Let G be a locally compact Abelian group and ω a weight on G. For $\alpha \in \widehat{G}(\omega)$, define $\varphi_\alpha : L^1(G,\omega) \to \mathbb{C}$ by*

$$\varphi_\alpha(f) = \int_G f(x)\overline{\alpha(x)}dx, \quad f \in L^1(G,\omega).$$

Then $\varphi_\alpha \in \Delta(L^1(G,\omega))$, and the map $\alpha \to \varphi_\alpha$ is a bijection between $\widehat{G}(\omega)$ and $\Delta(L^1(G,\omega))$.

Proof. It is straightforward to show that φ_α is a nonzero homomorphism and that, since $C_c(G) \subseteq L^1(G,\omega)$, the map $\alpha \to \varphi_\alpha$ is injective (compare the proof of Theorem 2.7.2).

To show that every $\varphi \in \Delta(L^1(G,\omega))$ equals φ_α for some $\alpha \in \widehat{G}(\omega)$, we proceed in a similar manner as in the proof of Theorem 2.7.2. Choose $g \in C_c(G)$ such that $\varphi(g) = 1$ and define $\alpha : G \to \mathbb{C}$ by $\alpha(y) = \overline{\varphi(L_y g)}, y \in G$. Then α is continuous because the map $x \to L_x g$ from G into $L^1(G,\omega)$ is continuous (Lemma 1.3.6) and

$$|\alpha(x) - \alpha(y)| = |\varphi(L_x g - L_y g)| \leq \|L_x g - L_y g\|_{1,\omega}.$$

For all $y \in G$, using Lemma 1.3.6,

$$|\alpha(y)| = |\varphi(L_y g)| \leq \|L_y g\|_{1,\omega} \leq \omega(y)\|g\|_{1,\omega}.$$

Moreover, since φ is a homomorphism, we have $\alpha(xy) = \alpha(x)\alpha(y)$ for all $x, y \in G$ (compare the proof of Theorem 2.7.2) and therefore

$$|\alpha(y)| = |\alpha(y^n)|^{1/n} \leq \omega(y^n)^{1/n}\|g\|_{1,\omega}^{1/n} \leq \omega(y)\|g\|_{1,\omega}^{1/n}$$

for all $y \in G$ and $n \in \mathbb{N}$. It follows that $|\alpha(y)| \leq \omega(y)$ for all $y \in G$. This shows that $\alpha \in \widehat{G}(\omega)$.

Finally, for any $f \in C_c(G)$,

$$\begin{aligned}\varphi(f) = \varphi(g * f) &= \varphi\left(x \to \int_G f(y)L_y g(x)dy\right)\\ &= \int_G f(y)\varphi(L_y g)dy = \int_G f(y)\overline{\alpha(y)}dy\\ &= \varphi_\alpha(f).\end{aligned}$$

Since φ and φ_α are continuous, we conclude that $\varphi = \varphi_\alpha$. □

Remark 2.8.3. Suppose that the weight ω on G satisfies $\lim_{n\to\infty} \omega(x^n)^{1/n} = 1$ for all $x \in G$. Then $\widehat{G} = \widehat{G}(\omega)$. In fact, the condition implies that $\omega(x) \geq 1$

for all $x \in G$ and hence $\widehat{G} \subseteq \widehat{G}(\omega)$. Conversely, let $\alpha \in \widehat{G}(\omega)$. We have seen in the proof of Theorem 2.8.2 that

$$|\alpha(x)| \leq \lim_{n\to\infty} \omega(x^n)^{1/n}$$

and hence $|\alpha(x)| \leq 1$ for all $x \in G$. Since α is multiplicative, this implies that $|\alpha(x)| = 1$ for all $x \in G$. Therefore, $\widehat{G}(\omega) \subseteq \widehat{G}$.

Lemma 2.8.4. *Let $f \in L^1(G,\omega)$, $x \in G$, and $\epsilon > 0$. Then there exist a neighbourhood W of e in G and $\delta > 0$ with the following property. If $y \in G$ and $\beta, \alpha \in \widehat{G}(\omega)$ are such that $y \in Wx$, $\varphi_\alpha(f) = 1$ and $\beta \in U(\alpha, f, L_x f, \delta)$, then*

$$|\beta(y) - \alpha(x)| < \epsilon.$$

In particular, the function $(x,\alpha) \to \alpha(x)$ on $G \times \widehat{G}(\omega)$ is continuous.

Proof. Note first that $\widehat{L_z f}(\gamma) = \overline{\gamma(z)}\widehat{f}(\gamma)$ for all $z \in G$ and $\gamma \in \widehat{G}(\omega)$ since γ is multiplicative. For arbitrary $y, x \in G$ and $\beta, \alpha \in \widehat{G}(\omega)$ such that $\varphi_\alpha(f) = 1$, as in the proof of Lemma 2.7.4 we get

$$\begin{aligned}|\beta(y) - \alpha(x)| &\leq |\beta(y)| \cdot |1 - \widehat{f}(\beta)| + \|L_y f - L_x f\|_{1,\omega} + |\widehat{L_x f}(\beta) - \widehat{L_x f}(\alpha)| \\ &\leq \omega(y)|\widehat{f}(\beta) - \widehat{f}(\alpha)| + \|L_y f - L_x f\|_{1,\omega} + |\widehat{L_x f}(\beta) - \widehat{L_x f}(\alpha)|.\end{aligned}$$

Now, fix a compact neighbourhood K of x and let

$$C = \max\{1, \sup\{\omega(t) : t \in K\}\} < \infty$$

(Lemma 1.3.3). Let $\delta = \epsilon(3C)^{-1}$ and let W be a neighbourhood of e such that $Wx \subseteq K$ and

$$\|L_y f - L_x f\|_{1,\omega} < \delta$$

for all $y \in Wx$ (Lemma 1.3.6). Then, if $y \in Wx$ and $\beta \in U(\alpha, f, L_x f, \delta)$, the above estimate shows that $|\beta(y) - \alpha(x)| < \epsilon$. □

Because weight functions are only locally bounded, in contrast to the case of $L^1(G)$ we cannot expect that W and δ in the preceding lemma can be chosen independently of x.

Theorem 2.8.5. *On $\widehat{G}(\omega) = \Delta(L^1(G,\omega))$ the Gelfand topology coincides with the topology of uniform convergence on compact subsets of G.*

Proof. Let g be a bounded measurable function on G with compact support, K say. Then $g \in L^1(G,\omega)$ (Lemma 1.3.5) and for any $\alpha, \beta \in \widehat{G}(\omega)$,

$$\begin{aligned}|\varphi_\alpha(g) - \varphi_\beta(g)| &\leq \int_K |g(x)| \cdot |\alpha(x) - \beta(x)| dx \\ &\leq \|g\|_\infty |K| \sup_{x\in K} |\alpha(x) - \beta(x)|,\end{aligned}$$

where $|K|$ denotes Haar measure of K. Now, given $f \in L^1(G, \omega)$ and $\epsilon > 0$, there exists such a function g satisfying $\|g - f\|_{1,\omega} \leq \epsilon$. It follows that

$$\begin{aligned} |\varphi_\alpha(f) - \varphi_\beta(f)| &\leq 2\|f - g\|_{1,\omega} + |\varphi_\alpha(g) - \varphi_\beta(g)| \\ &\leq 2\epsilon + \|g\|_\infty |K| \sup_{x \in K} |\alpha(x) - \beta(x)|. \end{aligned}$$

This shows that the Gelfand topology on $\widehat{G}(\omega)$ is coarser than the topology of uniform convergence on compact subsets of G.

Conversely, let $\alpha \in \widehat{G}(\omega)$, a compact subset K of G, and $\epsilon > 0$ be given. Let

$$V(\alpha, K, \epsilon) = \{\beta \in \widehat{G}(\omega) : |\beta(x) - \alpha(x)| < \epsilon \text{ for all } x \in K\}$$

and choose $f \in L^1(G, \omega)$ such that $\widehat{f}(\alpha) = 1$. By Lemma 2.8.4, for every $x \in K$ there exist a neighbourhood W_x of e in G and $\delta_x > 0$ with the following property: If $y \in W_x x$ and $\beta \in U(\alpha, f, L_x f, \delta_x)$, then $|\beta(y) - \alpha(x)| < \epsilon$. Since K is compact, there exist $x_1, \ldots, x_n \in K$ such that $K \subseteq \bigcup_{j=1}^n W_{x_j} x_j$. Let $\delta = \min\{\delta_{x_1}, \ldots, \delta_{x_n}\}$. Then

$$U(\alpha, f, L_{x_1} f, \ldots, L_{x_n} f, \delta) \subseteq V(\alpha, K, 2\epsilon).$$

Indeed, if β is in the set on the left side and $x \in K$, then $x \in W_{x_j} x_j$ for some $j \in \{1, \ldots, n\}$ and $\beta \in U(\alpha, f, L_{x_j} f, \delta_{x_j})$ and therefore

$$|\beta(x) - \alpha(x)| \leq |\beta(x) - \alpha(x_j)| + |\alpha(x_j) - \alpha(x)| < 2\epsilon.$$

This shows that the Gelfand topology on $\widehat{G}(\omega)$ is finer than the topology of uniform convergence on compact subsets of G. □

Identifying $\Delta(L^1(G, \omega))$ as a topological space with $\widehat{G}(\omega)$, the Gelfand representation of $L^1(G, \omega)$ is given by the map $f \to \widehat{f}(\alpha)$, where $\widehat{f}(\alpha) = \int_G f(x)\overline{\alpha(x)}dx$ for $\alpha \in \widehat{G}(\omega)$.

We now determine $\Delta(L^1(G, \omega))$ for G equal to $\mathbb{R}$ or to $\mathbb{Z}$.

Lemma 2.8.6. *Let ω be a weight function on $\mathbb{R}$ and define nonnegative real numbers R_+ and R_- by*

$$R_+ = \inf\{\omega(t)^{1/t} : t > 0\} \text{ and } R_- = \sup\{\omega(-t)^{-1/t} : t > 0\}.$$

Then $0 < R_- \leq R_+$, and every $z \in \mathbb{C}$ satisfying $-\ln R_+ \leq \operatorname{Re} z \leq -\ln R_-$ defines an element φ_z of $\Delta(L^1(\mathbb{R}, \omega))$ by

$$\varphi_z(f) = \int_{\mathbb{R}} f(t) e^{-zt} dt, \quad f \in L^1(\mathbb{R}, \omega).$$

Proof. We show first that

$$R_+ = \lim_{t \to \infty} \omega(t)^{1/t}.$$

To see this, let $\epsilon > 0$ and choose $t_0 > 0$ such that $\omega(t_0)^{1/t_0} \leq R_+ + \epsilon$. Write any $t > 0$ as $t = mt_0 + s$, where $m \in \mathbb{N}_0$ and $0 \leq s < t_0$. Then

$$\begin{aligned}\omega(t)^{1/t} &\leq \omega(mt_0)^{1/t}\omega(s)^{1/t} \leq \omega(t_0)^{1/t_0}(\omega(t_0)^{-s/t_0})^{1/t}\omega(s)^{1/t}.\\ &\leq (R_+ + \epsilon)(\omega(t_0)^{-s/t_0})^{1/t}\omega(s)^{1/t}.\end{aligned}$$

This inequality shows that $\omega(t)^{1/t}$ converges, as $t \to \infty$, with limit R_+. Similarly, it is shown that

$$R_- = \lim_{t\to\infty} \omega(-t)^{-1/t}.$$

Since $\omega(0) \leq \omega(-t)\omega(t)$ for all $t \in \mathbb{R}$, we obtain

$$0 < R_- = \lim_{t\to\infty} \omega(0)^{1/t}\omega(-t)^{-1/t} \leq \lim_{t\to\infty} \omega(t)^{1/t} = R_+.$$

Now, let $z \in \mathbb{C}$ be such that $-\ln R_+ \leq \operatorname{Re} z \leq -\ln R_-$. We claim that $|e^{-zt}| \leq \omega(t)$ for all $t \in \mathbb{R}$. For this, notice that by definition of R_+,

$$\exp(-t\operatorname{Re} z) \leq \exp(t\ln R_+) \leq \exp(t\ln(\omega(t)^{1/t})) = \omega(t)$$

for all $t > 0$. Similarly, for all $t < 0$,

$$\exp(-t\operatorname{Re} z) \leq \exp(t\ln R_-) \leq \exp(t\ln(\omega(t)^{1/t})) = \omega(t).$$

Thus $|e^{-zt}| = \exp(-t\operatorname{Re} z) \leq \omega(t)$ for all $t \in \mathbb{R}$ and hence the integral

$$\int_{\mathbb{R}} f(t)e^{-zt}dt$$

converges absolutely for each $f \in L^1(\mathbb{R},\omega)$. Therefore, we can define a bounded linear functional φ_z on $L^1(\mathbb{R},\omega)$ by

$$\varphi_z(f) = \int_{\mathbb{R}} f(t)e^{-tz}dt.$$

It is then easily verified that $\varphi_z(f * g) = \varphi_z(f)\varphi_z(g)$ for all $f, g \in L^1(\mathbb{R},\omega)$. Hence $\varphi_z \in \Delta(L^1(\mathbb{R},\omega))$. □

Proposition 2.8.7. *Let ω be any weight on $\mathbb{R}$ and let R_+ and R_- be as in Lemma 2.8.6. Let S_ω be the vertical strip in the complex plane defined by*

$$S_\omega = \{z \in \mathbb{C} : -\ln R_+ \leq \operatorname{Re} z \leq -\ln R_-\}.$$

Then the map $z \to \varphi_z$, where φ_z is as in Lemma 2.8.6, is a homeomorphism from S_ω onto $\Delta(L^1(\mathbb{R},\omega))$.

Proof. It is clear that the map $z \to \varphi_z$ from S_ω into $\Delta(L^1(\mathbb{R},\omega))$ is injective. We show that every $\varphi \in \Delta(L^1(\mathbb{R},\omega))$ arises in this manner. To see this, recall first from Theorem 2.8.2 that there exists a continuous function $\gamma : \mathbb{R} \to \mathbb{C}$ satisfying $\gamma(t+s) = \gamma(t)\gamma(s)$ and $0 < |\gamma(t)| \leq \omega(t)$ for all $t, s \in \mathbb{R}$ and

$$\varphi(f) = \int_{\mathbb{R}} f(t)\overline{\gamma(t)}dt$$

for all $f \in L^1(\mathbb{R}, \omega)$. The functional equation $\gamma(t+s) = \gamma(t)\gamma(s)$ and the continuity of γ imply that there exists $w \in \mathbb{C}$ such that

$$\gamma(t) = e^{iwt}$$

for all $t \in \mathbb{R}$ (Exercise 2.12.29). If $w = a + ib$ with $a, b \in \mathbb{R}$, then $|\gamma(t)| \leq \omega(t)$ implies that $e^{-bt} \leq \omega(t)$ for all $t \in \mathbb{R}$. Since

$$e^{-b} = (e^{-bn})^{1/n} \leq \omega(n)^{1/n} \to R_+$$

as $n \to \infty$, we get $-b \leq \ln R_+$. Similarly

$$e^{-b} = (e^{(-b)(-n)})^{-1/n} = |\gamma(-n)|^{-1/n} \geq \omega(-n)^{-1/n} \to R_-$$

as $n \to \infty$, whence $-b \geq \ln R_-$. Thus $-\ln R_+ \leq b \leq -\ln R_-$ and hence $b + ia \in S_\omega$ and $\varphi = \varphi_{b+ia}$.

By Theorem 2.8.5, the map $\alpha \to \varphi_\alpha$ is a homeomorphism between $\widehat{\mathbb{R}}(\omega)$ and $\Delta(L^1(\mathbb{R}, \omega))$. On the other hand, the map $z \to \alpha_z$, where $\alpha_z(t) = \overline{e^{zt}}$ for $t \in \mathbb{R}$, from S_ω to $\widehat{\mathbb{R}}(\omega)$ is bijective and obviously a homeomorphism. Combining these two facts shows that $z \to \varphi_z$ is a homeomorphism from S_ω onto $\Delta(L^1(\mathbb{R}, \omega))$. □

The formula of Lemma 2.8.6 is reminiscent of the Laplace transform. In fact, $\varphi_z(f)$ is nothing but the Laplace transform of f at $z \in S_\omega$. We now turn to the group of integers.

Proposition 2.8.8. *Let ω be a weight function on $\mathbb{Z}$ and define positive real numbers R_+ and R_- by*

$$R_+ = \inf\{w(n)^{1/n} : n \in \mathbb{N}\} \ \ \textit{and} \ \ R_- = \sup\{w(m)^{1/m} : m \in -\mathbb{N}\}.$$

Then there is a homeomorphism from the annulus

$$K(R_-, R_+) = \{z \in \mathbb{C} : R_- \leq |z| \leq R_+\}.$$

onto $\Delta(l^1(\mathbb{Z}, \omega))$ given by $z \to \varphi_z$, where

$$\varphi_z(f) = \sum_{n=-\infty}^{\infty} f(n)z^n, \ \ f \in l^1(\mathbb{Z}, \omega).$$

Proof. The following formulae can be verified in very much the same manner as the spectral radius formula (Lemma 1.2.5):

$$R_+ = \lim_{n\to\infty} w(n)^{1/n} \ \ \text{and} \ \ R_- = \lim_{n\to\infty} w(-n)^{-1/n}.$$

For the reader's convenience we nevertheless include the proof of the second. Let $\varepsilon > 0$ be given and choose $k \in \mathbb{N}$ such that $\omega(-k)^{-1/k} > R_- - \varepsilon$. Write $n \in \mathbb{N}$ in the form $n = p(n)k + q(n)$, where $p(n) \in \mathbb{N}_0$ and $0 \le q(n) < k$. Then

$$\frac{p(n)}{n} = \frac{1}{k}\left(1 - \frac{q(n)}{n}\right) \to \frac{1}{k}$$

as $n \to \infty$. Since $\omega(r+s) \le \omega(r)\omega(s)$ for all $r, s \in \mathbb{Z}$, we have $\omega(-n) \le \omega(-k)^{p(n)}\omega(-q(n))$ and hence, for all $n \in \mathbb{N}$,

$$\omega(-n)^{-1/n} \ge \omega(-k)^{-p(n)/n}\omega(-q(n))^{-1/n}.$$

The right hand side converges to $\omega(-k)^{-1/k}$ as $n \to \infty$. Thus $\omega(-n)^{-1/n} > R_- - \varepsilon$ eventually and therefore $R_- = \lim_{n\to\infty} \omega(-n)^{-1/n}$. Now, the inequality

$$\omega(n)^{1/n}\omega(-n)^{1/n} \ge \omega(0)^{1/n} \ge 1$$

implies that $\omega(n)^{1/n} \ge \omega(-n)^{-1/n}$ for all $n \in \mathbb{N}$. It follows that

$$R_- = \lim_{n\to\infty} \omega(-n)^{-1/n} \le \lim_{n\to\infty} \omega(n)^{1/n} = R_+.$$

For $z \in K(R_-, R_+)$ and $f \in l^1(\mathbb{Z},\omega)$, by definition of R_+ and R_-, we have

$$\begin{aligned}
\sum_{n\in\mathbb{Z}} |f(n)| \cdot |z|^n &= |f(0)| + \sum_{n=1}^{\infty} |f(n)| \cdot |z|^n + \sum_{n=1}^{\infty} |f(-n)| \cdot |z|^{-n} \\
&\le |f(0)| + \sum_{n=1}^{\infty} |f(n)|\omega(n) + \sum_{n=1}^{\infty} |f(-n)|(\omega(-n)^{-1/n})^{-n} \\
&\le \sum_{n\in\mathbb{Z}} |f(n)|\omega(n) \\
&= \|f\|_{1,\omega}.
\end{aligned}$$

Thus, for every $z \in K(R_-, R_+)$ we can define a bounded linear functional on $l^1(\mathbb{Z},\omega)$ by

$$\varphi_z(f) = \sum_{n\in\mathbb{Z}} f(n)z^n.$$

Then, for $f, g \in l^1(\mathbb{Z},\omega)$,

$$\begin{aligned}
\varphi_z(f * g) &= \sum_{n\in\mathbb{Z}} z^n \left(\sum_{m\in\mathbb{Z}} f(n-m)g(m)\right) \\
&= \sum_{m\in\mathbb{Z}} g(m)z^m \left(\sum_{n\in\mathbb{Z}} f(n-m)z^{n-m}\right) \\
&= \varphi_z(f)\varphi_z(g).
\end{aligned}$$

So $\varphi_z \in \Delta(l^1(\mathbb{Z},\omega))$ and since $\varphi_z(\delta_1) = z$. The map $z \to \varphi_z$ from $K(R_-, R_+)$ into $\Delta(l^1(\mathbb{Z},\omega))$ is injective, and the map is continuous since $z \to \varphi_z(\delta_m) = z^m$ is continuous for each m. Conversely, let $\varphi \in \Delta(l^1(\mathbb{Z},\omega))$ and set $z = \varphi(\delta_1)$. Then, for all $n \in \mathbb{N}$,

$$|z|^n = |\varphi(\delta_n)| \leq \|\delta_n\|_{1,\omega} = \omega(n),$$

and hence $|z| \leq \inf\{\omega(n)^{1/n} : n \in \mathbb{N}\} = R_+$. Similarly, it is shown that $|z| \geq R_-$. Since $\varphi(\delta_n) = z^n$ for all $n \in \mathbb{Z}$ and the finite linear combinations of the Dirac functions δ_n, $n \in \mathbb{Z}$, are dense in $l^1(\mathbb{Z},\omega)$, continuity of φ implies that $\varphi = \varphi_z$. Thus $z \to \varphi_z$ is a continuous bijection between the compact space $K(R_-, R_+)$ and the Hausdorff space $\Delta(l^1(\mathbb{Z},\omega))$ and hence is a homeomorphism. □

Propositions 2.8.7 and 2.8.8 in particular show that $L^1(\mathbb{R},\omega)$ and $l^1(\mathbb{Z},\omega)$ are semisimple for any weight ω. Our intention is to establish semisimplicity of $L^1(G,\omega)$ for arbitrary locally compact Abelian groups. We start with the following dichotomy.

Lemma 2.8.9. *$L^1(G,\omega)$ is either semisimple or radical.*

Proof. Assume that $L^1(G,\omega)$ is not radical, and fix any $\varphi \in \Delta(L^1(G,\omega))$. By Theorem 2.8.2, there exists a continuous function $\gamma : G \to \mathbb{C}$ satisfying $\gamma(xy) = \gamma(x)\gamma(y)$, $0 < |\gamma(x)| \leq \omega(x)$ for all $x, y \in G$ and

$$\varphi(f) = \int_G f(x)\overline{\gamma(x)}dx$$

for all $f \in L^1(G,\omega)$. For each $\alpha \in \widehat{G}$, define $\psi_\alpha \in L^1(G,\omega)^*$ by

$$\psi_\alpha(f) = \int_G f(x)\overline{\alpha(x)\gamma(x)}dx.$$

Then $\psi_\alpha \in \Delta(L^1(G,\omega))$ since $\alpha\gamma \in \widehat{G}(\omega)$. Now, let f be an element of the radical of $L^1(G,\omega)$. Then $f\overline{\gamma} \in L^1(G)$ and

$$\widehat{f\overline{\gamma}}(\alpha) = \psi_\alpha(f) = 0$$

for all $\alpha \in \widehat{G}$. Since $L^1(G)$ is semisimple (Corollary 2.7.9), it follows that $f\overline{\gamma} = 0$ and hence $f = 0$ almost everywhere since $\gamma(x) \neq 0$ for all $x \in G$. This shows that $L^1(G,\omega)$ is semisimple. □

Theorem 2.8.10. *Let G be a locally compact Abelian group and ω a weight on G. Then the Beurling algebra $L^1(G,\omega)$ is semisimple.*

Proof. By virtue of Lemma 2.8.9, it suffices to show that $L^1(G,\omega)$ is not radical. We construct a function $f \in L^1(G,\omega)$ such that

$$r_{L^1(G,\omega)}(f) = \lim_{n\to\infty} \|f^n\|_{1,\omega}^{1/n} > 0,$$

where f^n denotes the n-fold convolution product of f.

Choose a relatively compact symmetric neighbourhood U of the identity e of G and let $f = 1_U$, the characteristic function of U. Then

$$M = \sup\{\omega(x) : x \in U\} < \infty$$

since ω is locally bounded, and $\omega(x) \le M^n$ for all $x \in U^n$. Since $f \in L^1(G, \omega)$ and $\omega(x) \ge \frac{\omega(e)}{\omega(x^{-1})}$ for all $x \in G$, it follows that

$$\begin{aligned}\|f^n\|_{1,\omega} &= \int_G |f^n(x)|\omega(x)dx \\ &\ge \omega(e) \int_G |1_U^n(x)| \frac{1}{\omega(x^{-1})} dx \\ &\ge \frac{\omega(e)}{M^n} \|1_U^n\|_1.\end{aligned}$$

This inequality implies that

$$\begin{aligned}r_{L^1(G,\omega)}(f) = \lim_{n\to\infty} \|f^n\|_{1,\omega}^{1/n} &\ge \frac{1}{M} \lim_{n\to\infty} \|1_U^n\|^{1/n} \\ &= \frac{1}{M} r_{L^1(G)}(1_U),\end{aligned}$$

and hence $r_{L^1(G,\omega)}(f) > 0$, as required. □

2.9 The Fourier algebra of a locally compact group

In this section we present a class of semisimple commutative Banach algebras which is currently a matter of intensive study, the *Fourier algebras* $A(G)$ of locally compact groups G. When G is Abelian, $A(G)$ can be shown to be isometrically isomorphic to $L^1(\widehat{G})$. We introduce $A(G)$ and determine its structure space.

Let G be an arbitrary locally compact group. For functions f and g in $L^2(G)$, the function $f * \check{g} : G \to \mathbb{C}$ is defined by

$$f * \check{g}(x) = \int_G f(xy)g(y)dy.$$

Then $f * \check{g} \in C_0(G)$ and $\|f * \check{g}\|_\infty \le \|f\|_2 \|g\|_2$. Since the mappings $f \to f * \check{g}$ and $g \to f * \check{g}$ from $L^2(G)$ into $C_0(G)$, respectively, are linear and continuous, there is a unique continuous linear map ϕ from the projective tensor product $L^2(G)\widehat{\otimes}_\pi L^2(G)$ into $C_0(G)$ satisfying $\phi(f \otimes g) = f * \check{g}$ for all f and g in $L^2(G)$).

Definition 2.9.1. Let $A(G)$ denote the range of the map

$$\phi : L^2(G)\widehat{\otimes}_\pi L^2(G) \to C_0(G),$$

and endow $A(G)$ with the quotient norm from $L^2(G)\widehat{\otimes}_\pi L^2(G)$. Then $A(G)$ becomes a Banach space.

Since $C_0(G)$ is dense in $L^2(G)$, $C_c(G) \otimes C_c(G)$ is dense in $L^2(G)\widehat{\otimes}_\pi L^2(G)$ and hence $\pi(C_c(G) \otimes C_c(G))$ is dense in $A(G)$. So $A(G) \cap C_c(G)$ is dense in $A(G)$.

Theorem 2.9.2. *With pointwise multiplication, $A(G)$ is a Banach algebra.*

Proof. Let $f_1, f_2, g_1, g_2 \in C_c(G)$. We first show that

$$\phi(f_1 \otimes g_1)\phi(f_2 \otimes g_2) \in A(G)$$

and that

$$\|\phi(f_1 \otimes g_1)\phi(f_2 \otimes g_2)\| \leq \|\phi(f_1 \otimes g_1)\| \cdot \|\phi(f_2 \otimes g_2)\|.$$

To that end, for $y \in G$, define functions F_y and G_y on G by

$$F_y(x) = f_1(xy)f_2(x) \text{ and } G_y(x) = g_1(xy)g_2(x).$$

Then $F_y, G_y \in C_c(G)$ and the map $y \to F_y \otimes G_y$ vanishes outside the compact subset

$$C = (\operatorname{supp} f_2)^{-1} \operatorname{supp} f_1 \,\cap\, (\operatorname{supp} g_2)^{-1} \operatorname{supp} g_1$$

of G. Moreover, the map $y \to F_y \otimes G_y$ from G into $L^2(G)\widehat{\otimes}_\pi L^2(G)$ is continuous. Indeed, for $y, y_0 \in G$,

$$\begin{aligned}\pi(F_y \otimes G_y - F_{y_0} \otimes G_{y_0}) \leq{} & \|F_y - F_{y_0}\|_2(\|G_y - G_{y_0}\|_2 + \|G_{y_0}\|_2) \\ & + \|F_{y_0}\|_2 \|G_y - G_{y_0}\|_2\end{aligned}$$

and

$$\|F_y - F_{y_0}\|_2 \leq \|f_2\|_\infty \|R_y f_1 - R_{y_0} f_1\|_2,$$

and similarly for G_y. Thus the vector-valued integral

$$H = \int_C (F_y \otimes G_y)dy = \int_G (F_y \otimes G_y)dy$$

exists and defines an element of $L^2(G)\widehat{\otimes}_\pi L^2(G)$. Then

$$\phi(H) = \phi(f_1 \otimes g_1)\phi(f_2 \otimes g_2).$$

Indeed, for each $x \in G$, we have

$$
\begin{aligned}
\phi(f_1 \otimes g_1)(x)\phi(f_2 \otimes g_2)(x) &= \int_G f_1(xy)g_1(y)dy \int_G f_2(xz)g_2(z)dz \\
&= \int_G \int_G f_1(xzy)g_1(zy)f_2(xz)g_2(z)dzdy \\
&= \int_G \left(\int_G F_y(xz)G_y(z)dz \right) dy \\
&= \int_G \phi(F_y \otimes G_y)(x)dy \\
&= \phi(H)(x).
\end{aligned}
$$

This shows that $[\phi(L^2(G) \otimes L^2(G))]^2 \subseteq A(G)$. We now need to estimate the integral $\int_C \pi(F_y \otimes G_y)dy$. Note first that

$$
\begin{aligned}
\int_C \|F_y\|_2^2 dy &= \int_C \left(\int_C |f_1(xy)f_2(x)|^2 dx \right) dy \\
&= \int_C |f_2(x)|^2 \left(\int_C |f_1(xy)|^2 dy \right) dx \\
&= \|f_1\|_2^2 \|f_2\|_2^2,
\end{aligned}
$$

and similarly $\int_G \|G_y\|_2^2 dy = \|g_1\|_2^2 \|q_2\|_2^2$. Thus, by the Cauchy-Schwarz inequality,

$$
\begin{aligned}
\int_C \pi(F_y \otimes G_y)dy &= \int_C \|F_y\|_2 \|G_y\|_2 dy \\
&\leq \left(\int_C \|F_y\|_2^2 dy \right)^{1/2} \left(\int_C \|G_y\|_2^2 dy \right)^{1/2} \\
&= \|f_1\|_2 \|f_2\|_2 \|g_1\|_2 \|g_2\|_2 \\
&= \pi(f_1 \otimes g_1)\pi(f_2 \otimes g_2).
\end{aligned}
$$

Combining this estimate with the above formula for $\phi(H)$ gives

$$
\|\phi(f_1 \otimes g_1)\phi(f_2 \otimes g_2)\|_{A(G)} \leq \|\phi(f_1 \otimes g_1)\|_{A(G)} \|\phi(f_2 \otimes g_2)\|_{A(G)}.
$$

Thus multiplication on $\phi(L^2(G) \otimes L^2(G))$ is continuous. This implies that $A(G)$ is closed under multiplication and the norm on $A(G)$ is submultiplicative. □

The following lemma will be used to determine $\Delta(A(G))$ and also later in Chapter 5.

Lemma 2.9.3. *Let $a \in G$ and $f \in A(G)$ such that $f(a) = 0$. Then, given $\epsilon > 0$, there exists $h \in A(G) \cap C_c(G)$ vanishing in a neighbourhood of a such that $\|h - f\|_{A(G)} \leq \epsilon$.*

Proof. Notice first that, since $A(G) \cap C_c(G)$ is dense in $A(G)$, without loss of generality we can assume that $f \neq 0$, f has compact support and $\epsilon \leq \|f\|_\infty$ and $\epsilon < 1$. Let

$$W = \{y \in G : \|f - R_y f\|_{A(G)} \leq \epsilon\}.$$

Then W is a compact neighbourhood of e in G. Choose an open neighbourhood V of e such that $V \subseteq W$ and $\sup\{|f(ay)| : y \in V\} \leq \epsilon$, and choose a compact neighbourhood U of e such that $U \subseteq V$ and $|U| \geq |V|(1-\epsilon)$. Now, define functions u, g and h by setting $u = |U|^{-1} 1_U, g = 1_{aV} f$ and

$$h = (f - g) * \check{u} \in A(G).$$

Then h has compact support since W is compact and f has compact support. For any $x \in G$,

$$h(x) = |U|^{-1} \int_U f(xy)[1 - 1_{aV}(xy)]dy.$$

It follows that, if $x \in G$ satisfies $a^{-1}xU \subseteq V$, then $h(x) = 0$. Thus h vanishes in a neighbourhood of a. Moreover,

$$\|u\|_2 = |U|^{-1/2} \leq |V|^{-1/2} \left(\frac{1}{1-\epsilon}\right)^{1/2},$$

$$\|g\|_2 = \left(\int_{aV} |f(y)|^p dy\right)^{1/2} \leq \epsilon |V|^{1/2},$$

and

$$\begin{aligned} \|f - f * \check{u}\|_{A(G)} &= \left\| f - |U|^{-1} \int_U (R_y f) dy \right\|_{A(G)} \\ &\leq \sup_{y \in U} \|f - R_y f\|_{A(G)} \leq \epsilon. \end{aligned}$$

Combining all these estimates, we obtain

$$\|f - h\|_{A(G)} \leq \|f - f * \check{u}\|_{A(G)} + \|g\|_2 \|\check{u}\|_2 \leq \epsilon + \epsilon \left(\frac{1}{1-\epsilon}\right)^{1/2}.$$

This finishes the proof. □

Theorem 2.9.4. *Let G be a locally compact group. For $x \in G$, let $\varphi_x : A(G) \to \mathbb{C}$ denote the evaluation at x. Then the map $x \to \varphi_x$ is a homeomorphism from G onto $\Delta(A(G))$.*

Proof. It is obvious that $\varphi_x \in \Delta(A(G))$ and that the map $x \to \varphi_x$ is injective. Now let $\varphi \in \Delta(A(G))$ be given and suppose that $\varphi \neq \varphi_x$ for all $x \in G$. Then, for each $x \in G$ there exists $f_x \in A(G)$ such that $\varphi(f_x) = 1$, but $\varphi_x(f_x) = 0$.

By Lemma 2.9.3, every $g \in A(G)$ vanishing at x is the limit of a sequence $(g_n)_n$ in $A(G)$ with the property that each g_n vanishes in a neighbourhood of x. Therefore we can assume that f_x vanishes in a neighbourhood V_x of x.

Since $A(G) \cap C_c(G)$ is dense in $A(G)$, there exists $f_0 \in C_c(G) \cap A(G)$ such that $\varphi(f_0) = 1$. Choose $x_1, \ldots, x_n \in \operatorname{supp} f_0$ such that

$$\operatorname{supp} f_0 \subseteq \bigcup_{j=1}^{n} V_{x_j}$$

and let

$$f = f_0 f_{x_1} \cdot \ldots \cdot f_{x_n} \in A(G).$$

Then $f(x) = 0$ for every $x \in G$, whereas

$$\varphi(f) = \varphi(f_0) \prod_{j=1}^{n} \varphi(f_{x_j}) = 1.$$

This contradiction shows that $\varphi = \varphi_x$ for some $x \in G$.

Finally, since the subalgebra $A(G)$ of $C_0(G)$ strongly separates the points of G, by Proposition 2.2.14 the topology on G coincides with the weak topology defined by the set of functions $x \to f(x) = \varphi_x(f), f \in A(G)$. Thus the map $x \to \varphi_x$ from G to $\Delta(A(G))$ is a homeomorphism. $\square$

Of course, after identifying $\Delta(A(G))$ with G, the Gelfand homomorphism of $A(G)$ is nothing but the identity mapping. In particular, $A(G)$ is semisimple.

We close this section with a straightforward result which, in the terminology of Chapter 4, implies that $A(G)$ is regular.

Lemma 2.9.5. *Let G be a locally compact group, K a compact subset of G and U an open subset of G such that $U \supseteq K$. Then there exists $u \in A(G) \cap C_c(G)$ with the following properties: $0 \leq u \leq 1$, $u(x) = 1$ for all $x \in K$ and $u(x) = 0$ for all $x \in G \setminus U$.*

Proof. Since K is compact, there exists a compact symmetric neighbourhood V of the identity such that $KV^2 \subseteq U$. Let

$$u(x) = |V|^{-1}(1_{KV} * \check{1_V})(x) = |V|^{-1} \cdot |xV \cap KV|.$$

Then $0 \leq u \leq 1$. If $x \in K$, then $|xV \cap KV| = |xV| = |V|$, so that $u(x) = 1$, whereas if $x \notin KV^2$, then $xV \cap KV = \emptyset$ and hence $u(x) = 0$. Thus $\operatorname{supp} u \subseteq KV^2$, which is compact. In particular, $u(x) = 0$ for all $x \in G \setminus U$. $\square$

2.10 The algebra of almost periodic functions

In Theorem 2.4.12 we have seen that the Stone–Čech compactification $\beta(X)$ of a completely regular topological space X arises as the structure space of the commutative C^*-algebra $C^b(X)$. In this section we study, for G a locally compact group, a certain C^*-subalgebra of $C^b(G)$, the algebra $AP(G)$ of almost periodic functions on G, and show that $\Delta(AP(G))$ is homeomorphic to the Bohr compactification of G.

Let G be a topological group. A complex-valued bounded continuous function f on G is called *left almost periodic* (respectively, *right almost periodic*) if the set $C_f = \{L_x f : x \in G\}$ (respectively, the set $D_f = \{R_y f : y \in G\}$) is relatively compact in $(C^b(G), \|\cdot\|_\infty)$. Let $AP(G)$ denote the set of all left almost periodic functions on G.

Example 2.10.1. Let G be a compact group. Then $AP(G) = C(G)$. In fact, for $f \in C(G)$ the map $x \to L_x f$ from G into $C(G)$ is continuous because f is uniformly continuous and

$$\|L_x f - L_y f\|_\infty = \sup_{t \in G} |f(x^{-1}t) - f(y^{-1}t)|.$$

So C_f is a continuous image of the compact group G, hence compact.

Lemma 2.10.2. *$AP(G)$ is a closed $*$-subalgebra of $C^b(G)$.*

Proof. It is clear that $C_{f+g} \subseteq C_f + C_g, C_{\alpha f} = \alpha C_f$ and $C_{fg} \subseteq C_f C_g$ for $f, g \in C^b(G)$ and $\alpha \in \mathbb{C}$. Thus $AP(G)$ is a subalgebra of $C^b(G)$. Also $f \in AP(G)$ implies that $\overline{f} \in AP(G)$. It remains to show that $AP(G)$ is closed in $C^b(G)$.

Let $f \in \overline{AP(G)}$. Since C_f is bounded in $C^b(G)$, by the Arzela–Ascoli theorem it suffices to verify that C_f is equicontinuous. To that end, let $x \in G$ and $\epsilon > 0$ be given. Choose $g \in AP(G)$ such that $\|f - g\|_\infty \leq \epsilon/3$. Since C_g is equicontinuous, there is a neighbourhood V of x such that $|L_a g(y) - L_a g(x)| \leq \epsilon/3$ for all $a \in G$ and $y \in V$. If follows that

$$\begin{aligned} |L_a f(y) - L_a f(x)| &\leq |L_a f(y) - L_a g(y)| + |L_a g(y) - L_a g(x)| \\ &\quad + |L_a g(x) - L_a f(x)| \\ &\leq 2\|f - g\|_\infty + |L_a g(y) - L_a g(x)| \leq \epsilon \end{aligned}$$

for all $y \in V$ and $a \in G$. So C_f is equicontinuous. $\square$

Since $AP(G)$ is a unital commutative C^*-algebra, Theorem 2.4.5 implies the following

Corollary 2.10.3. *Let $\Delta(AP(G))$ denote the structure space of $AP(G)$. Then the Gelfand homomorphism is an isometric $*$-isomorphism from $AP(G)$ onto $C(\Delta(AP(G)))$.*

Each $x \in G$ defines an element $\varphi_x \in \Delta(AP(G))$ by $\varphi_x(f) = f(x), f \in AP(G)$.

Lemma 2.10.4. *The mapping $\phi : x \to \varphi_x$ from G into $\Delta(AP(G))$ is continuous and has dense range.*

Proof. Because $\varDelta(AP(G))$ carries the w^*-topology and the functions $x \to \varphi_x(f) = f(x)$, $f \in AP(G)$, are continuous on G, it follows that ϕ is continuous.

Suppose that there exists a nonempty open subset U of $\varDelta(AP(G))$ such that $U \cap \phi(G) = \emptyset$. Then, by Urysohn's lemma there exists $g \in C(\varDelta(AP(G)))$ with $g \neq 0$ and $g|_{\varDelta(AP(G))\setminus U} = 0$. By Corollary 2.10.3, $g = \widehat{f}$ for some $f \in AP(G)$. But then

$$f(x) = \varphi_x(f) = \widehat{f}(\varphi_x) = g(\varphi_x) = 0$$

for all $x \in G$, contradicting $f \neq 0$. Thus $\phi(G)$ is dense in $\varDelta(AP(G))$. □

Our aim is to introduce a group structure on $\varDelta(AP(G))$ which makes $\varDelta(AP(G))$ a compact group and ϕ a group homomorphism. Of course, the mapping ϕ is in general not injective and it is not clear at all that the families of points in G which cannot be separated by $AP(G)$ are cosets of some normal subgroup of G and that therefore ϕ defines a group structure on $\phi(G) \subseteq \varDelta(AP(G))$. Moreover, supposing that this problem can be satisfactorily settled, there remains the question of extending the group structure on $\phi(G)$ to the whole of $\varDelta(AP(G))$. To handle these problems requires us to consider two-sided translates of $f \in AP(G)$ and to show that actually such an f is also right almost periodic.

Lemma 2.10.5. *Let $f \in AP(G)$ and $\epsilon > 0$. Then there exist finitely many $a_1, \ldots, a_n \in G$ with the following property. For every $a \in G$ there exists some $j \in \{1, \ldots, n\}$ such that*

$$|f(xay) - f(xa_jy)| < \epsilon$$

for all $x, y \in G$.

Proof. There exist $b_1, \ldots, b_m \in G$ such that the set $\{L_{b_j}f : 1 \leq j \leq m\}$ forms an $\epsilon/4$-net for C_f. Let $\varGamma$ be the finite set of all mappings γ from $\{1, \ldots, m\}$ to itself with the property that there exists $a_\gamma \in G$ such that

$$\|L_{b_{\gamma(i)}}f - L_{b_i a_\gamma}f\|_\infty < \frac{\epsilon}{4}$$

for $i = 1, \ldots, m$. For each $\gamma \in \varGamma$, choose such an a_γ. Now, given any $a \in G$, by the choice of $b_1, \ldots, b_m$ for every $1 \leq i \leq m$ there exists some $j(i) \in \{1, \ldots, m\}$ such that

$$\|L_{b_i a}f - L_{b_{j(i)}}f\|_\infty < \frac{\epsilon}{4}.$$

So $i \to j(i)$ defines an element of $\varGamma$. It follows that for every $a \in G$ we find some $\gamma \in \varGamma$ such that

$$\|L_{b_i a}f - L_{b_i a_\gamma}f\|_\infty < \frac{\epsilon}{2}$$

for all $1 \leq i \leq m$. Since for every $x \in G$ there exists b_i so that $\|L_x f - L_{b_i} f\|_\infty < \epsilon/4$, we obtain that

$$\begin{aligned}\|L_{xa}f - L_{xa_\gamma}f\|_\infty &\leq \|L_{xa}f - L_{b_ia}f\|_\infty + \|L_{b_ia}f - L_{b_ia_\gamma}f\|_\infty \\ &\quad + \|L_{b_ia_\gamma}f - L_{xa_\gamma}f\|_\infty \\ &< \frac{\epsilon}{4} + \frac{\varepsilon}{2} + \frac{\epsilon}{4} = \epsilon.\end{aligned}$$

Thus we have seen that for every $a \in G$ there exists some a_γ such that

$$|f(xay) - f(xa_\gamma y)| \leq \|L_{xa}f - L_{xa_\gamma}f\|_\infty < \epsilon$$

for all $x, y \in G$. Now, enumerate $\{a_\gamma : \gamma \in \Gamma\}$ as $\{a_1, \ldots, a_n\}$. □

Corollary 2.10.6. *Let $f \in AP(G)$ and $\epsilon > 0$. Then there exist $a_1, \ldots, a_n \in G$ such that the functions $L_{a_i}R_{a_j}f, 1 \leq i, j \leq n$, form an ϵ-net for the set of all two-sided translates $L_aR_bf, a, b \in G$.*

Proof. Choose $0 < \delta < \epsilon/2$. By Lemma 2.10.5, there exist $a_1, \ldots, a_n \in G$ with the property that for any $a \in G$ there is $j \in \{1, \ldots, n\}$ such that, for all $x, y \in G$, $|f(xay) - f(xa_jy)| < \delta$. Thus, given $a, b \in G$, there exist i and j such that

$$|f(at) - f(a_it)| < \delta \text{ and } |f(sb) - f(sa_j)| < \delta$$

for all $s, t \in G$. It follows that, for all $x \in G$,

$$\begin{aligned}|f(axb) - f(a_ixa_j)| &\leq |f(axb) - f(a_ixb)| + |f(a_ixb) - f(a_ixa_j)| \\ &< 2\delta,\end{aligned}$$

whence $\|L_aR_bf - L_{a_i}R_{a_j}f\|_\infty \leq 2\delta < \epsilon$. □

Corollary 2.10.7. *Retain the notation of Corollary 2.10.6. If x and y are elements of G such that*

$$|f(a_ixa_j) - f(a_iya_j)| < \epsilon$$

for all $1 \leq i, j \leq n$, then

$$|f(axb) - f(ayb)| < 3\epsilon$$

for all $a, b \in G$.

Proof. Given $a, b \in G$, by Corollary 2.10.6 there exist i and j such that

$$\|L_aR_bf - L_{a_i}R_{a_j}f\|_\infty < \epsilon.$$

Combining with the presumed inequality, we get

$$\begin{aligned}|f(axb) - f(ayb)| &\leq \|L_aR_bf - L_{a_i}R_{a_j}f\|_\infty \\ &\quad + |f(a_ixa_j) - f(a_iya_j)| \\ &\quad + \|L_{a_i}R_{a_j}f - L_aR_bf\|_\infty \\ &< 3\epsilon,\end{aligned}$$

as claimed. □

It follows from Corollary 2.10.6 that every left almost periodic function is automatically right almost periodic. Therefore, in the sequel we simply call the functions in $AP(G)$ *almost periodic* rather than left almost periodic.

Another consequence of Corollary 2.10.6 is that every almost periodic function is uniformly continuous. Now, on every noncompact locally compact group G one can construct a bounded continuous function which fails to be uniformly continuous (Exercise 2.12.55). Thus $AP(G)$ is a proper subalgebra of $C^b(G)$ whenever G is a noncompact locally compact group.

Let $\varphi, \psi \in \Delta(AP(G))$. For neighbourhoods U of φ and V of ψ in $\Delta(AP(G))$, let

$$\Delta_{U,V} = \{\varphi_{xy} : x, y \in G \text{ such that } \varphi_x \in U \text{ and } \varphi_y \in V\}.$$

Then $\Delta_{U,V} \neq \emptyset$ since $\phi(G)$ is dense in $\Delta(AP(G))$. Let $\mathcal{U}$ and $\mathcal{V}$ be the set of all neighbourhoods of φ and ψ, respectively. Because $\Delta_{U_1,V_1} \subseteq \Delta_{U_2,V_2}$ whenever $U_1 \subseteq U_2$ and $V_1 \subseteq V_2$, the collection of all closed subsets $\overline{\Delta}_{U,V}$ of $\Delta(AP(G))$, where $U \in \mathcal{U}$ and $V \in \mathcal{V}$, has the finite intersection property. $\Delta(AP(G))$ being compact, it follows that the set

$$\Delta_{\varphi,\psi} := \bigcap \{\overline{\Delta}_{U,V} : U \in \mathcal{U}, V \in \mathcal{V}\}$$

is nonempty.

We shall see soon (Corollary 2.10.9) that $\Delta_{\varphi,\psi}$ is a singleton for any two elements φ, ψ of$\Delta(AP(G))$. Since $\varphi_{xy} \in \Delta_{\varphi_x,\varphi_y}$ it follows in particular that $\Delta_{\varphi_x,\varphi_y} = \{\varphi_{xy}\}$ for all $x, y \in G$.

Lemma 2.10.8. *Let $\alpha, \beta \in \Delta(AP(G))$ and $f \in AP(G)$. Let $\epsilon > 0$ and let $\{L_{x_1}f, \ldots, L_{x_n}f\}$ be an ϵ-net for C_f and $\{R_{y_1}f, \ldots, R_{y_m}f\}$ an ϵ-net for D_f. Define neighbourhoods U and V of α and β, respectively, by*

$$U = U(\alpha, R_{y_1}f, \ldots, R_{y_m}f, \epsilon) \text{ and } V = U(\beta, L_{x_1}f, \ldots, L_{x_n}f, \epsilon).$$

If $x, a, y, b \in G$ are such that $\varphi_x, \varphi_a \in U$ and $\varphi_y, \varphi_b \in V$, then

$$|\varphi_{xy}(f) - \varphi_{ab}(f)| < 8\epsilon.$$

Proof. Choose $j \in \{1, \ldots, n\}$ and $k \in \{1, \ldots, m\}$ such that

$$\|L_{x^{-1}}f - L_{x_j}f\|_\infty < \epsilon \text{ and } \|R_b f - R_{y_k}f\|_\infty < \epsilon.$$

Then we have

$$\begin{aligned}
|\varphi_{xy}(f) - \varphi_{ab}(f)| &\leq |f(xy) - f(xb)| + |f(xb) - f(ab)| \\
&= |L_{x^{-1}}f(y) - L_{x^{-1}}f(b)| + |R_b f(x) - R_b f(a)| \\
&\leq |L_{x^{-1}}f(y) - L_{x_j}f(y)| + |L_{x_j}f(y) - L_{x_j}f(b)| \\
&\quad + |L_{x_j}f(b) - L_{x^{-1}}f(b)| + |R_b f(x) - R_{y_k}f(x)| \\
&\quad + |R_{y_k}f(x) - R_{y_k}f(a)| + |R_{y_k}f(a) - R_b f(a)| \\
&\leq 2\|L_{x^{-1}}f - L_{x_j}f\|_\infty + |L_{x_j}f(x) - L_{x_j}f(a)| \\
&\quad + 2\|R_b f - R_{y_k}f\|_\infty + |R_{y_k}f(y) - R_{y_k}f(b)\| \\
&\leq 4\epsilon + |\varphi_y(L_{x_j}f) - \varphi_b(L_{x_j}f)| + |\varphi_x(R_{y_k}f) - \varphi_a(R_{y_k}f)|.
\end{aligned}$$

Now, since $\varphi_x, \varphi_a \in U$ and $\varphi_y, \varphi_b \in V$,

$$|\varphi_x(R_{y_k} f) - \varphi_a(R_{y_k} f)| < 2\epsilon \text{ and } |\varphi_y(L_{x_j} f) - \varphi_b(L_{x_j} f)| < 2\epsilon.$$

It follows that $|\varphi_{xy}(f) - \varphi_{ab}(f)| < 8\epsilon$. □

Corollary 2.10.9. *For each pair of elements φ, ψ of $\Delta(AP(G)), \Delta_{\varphi,\psi}$ is a singleton.*

Proof. Let $\alpha, \beta \in \Delta_{\varphi,\psi}$ and $f \in AP(G)$. We show that $|\alpha(f) - \beta(f)| < \delta$ for each $\delta > 0$. Fix δ and let $\epsilon = \delta/24$. Let U and V be defined as in Lemma 2.10.8. By definition of $\Delta_{\varphi,\psi}$ there exist $x, a, y, b \in G$ such that $\varphi_x, \varphi_a \in U, \varphi_y, \varphi_b \in V$, and

$$|\alpha(f) - \varphi_{xy}(f)| < \frac{\delta}{3} \text{ and } |\beta(f) - \varphi_{ab}(f)| < \frac{\delta}{3}.$$

From Lemma 2.10.8 we now infer that

$$\begin{aligned}|\alpha(f) - \beta(f)| &\leq |\alpha(f) - \varphi_{xy}(f)| + |\varphi_{xy}(f) - \varphi_{ab}(f)| \\ &\quad + |\varphi_{ab}(f) - \beta(f)| \\ &\leq \frac{\delta}{3} + \frac{\delta}{3} + 8\epsilon = \delta,\end{aligned}$$

as required. □

Now we are able to introduce a group structure on $\Delta(AP(G))$.

Theorem 2.10.10. *Let G be a topological group. For $\varphi, \psi \in \Delta(AP(G))$, let $\varphi\psi$ denote the unique element of $\Delta_{\varphi,\psi}$. Then the assignment*

$$(\varphi, \psi) \to \varphi\psi,\ \Delta(AP(G)) \times \Delta(AP(G)) \to \Delta(AP(G))$$

turns $\Delta(AP(G))$ into a compact group. Furthermore, $\varphi_x\varphi_y = \varphi_{xy}$ for $x, y \in G$.

Proof. The last statement is clear since $\Delta_{\varphi_x,\varphi_y} = \{\varphi_{xy}\}$. We show next that multiplication on $\Delta(AP(G))$ is continuous. Let α and β be two elements of $\Delta(AP(G))$. It suffices to show that given $\delta > 0$ and $f_1, \ldots, f_n \in AP(G)$, there exist neighbourhoods U of α and V of β in $\Delta(AP(G))$, respectively, such that

$$|\varphi\psi(f_j) - \alpha\beta(f_j)| < \delta$$

for all $\varphi \in U$ and $\psi \in V$ and $j = 1, \ldots, n$.

Let $\epsilon = \delta/10$ and for any $\rho \in \Delta(AP(G))$ let

$$W_\rho = \{\gamma \in \Delta(AP(G)) : |\gamma(f_j) - \rho(f_j)| < \epsilon \text{ for } 1 \leq j \leq n\}.$$

For each $j = 1, \ldots, n$, Lemma 2.10.8 provides neighbourhoods U_j of α and V_j of β such that

$$|\varphi_{xy}(f_j) - \varphi_{ab}(f_j)| < 8\epsilon$$

whenever $x, a, y, b \in G$ are such that $\varphi_x, \varphi_a \in U_j$ and $\varphi_y, \varphi_b \in V_j$. Let $U = \cap_{j=1}^n U_j$ and $V = \bigcap_{j=1}^n V_j$. Since $\alpha\beta \in \overline{\Delta}_{U,V}$, we have $\Delta_{U,V} \cap W_{\alpha\beta} \neq \emptyset$, and hence there exist $a, b \in G$ such that $\varphi_a \in U, \varphi_b \in V$ and $\varphi_{ab} \in W_{\alpha\beta}$. Now, let $\varphi \in U$ and $\psi \in V$ be arbitrary. Then $\Delta_{U,V} \cap W_{\varphi\psi} \neq \emptyset$ and hence there exist $x, y \in G$ such that $\varphi_x \in U, \varphi_y \in V$, and $\varphi_{xy} \in W_{\varphi\psi}$. Therefore we have

$$|\varphi_{ab}(f_j) - \alpha\beta(f_j)| < \epsilon \text{ and } |\varphi_{xy}(f_j) - \varphi\psi(f_j)| < \epsilon$$

for $j = 1, \ldots, n$. Because $\varphi_x, \varphi_a \in U$ and $\varphi_y, \varphi_b \in V$, $|\varphi_{xy}(f_j) - \varphi_{ab}(f_j)| < 8\epsilon$ for $j = 1, \ldots, n$. Combining these inequalities gives

$$\begin{aligned} |\varphi\psi(f_j) - \alpha\beta(f_j)| &\leq |\varphi\psi(f_j) - \varphi_{xy}(f_j)| + |\varphi_{xy}(f_j) - \varphi_{ab}(f_j)| \\ &\quad + |\varphi_{ab}(f_j) - \alpha\beta(f_j)| < 10\epsilon \\ &= \delta. \end{aligned}$$

Thus multiplication on $\Delta(AP(G))$ is continuous.

It remains to show the existence and continuity of inverses in $\Delta(AP(G))$. Let $\varphi \in AP(G)$ and let $(x_\alpha)_\alpha$ be a net in G such that $\varphi_{x_\alpha} \to \varphi$ in $\Delta(AP(G))$. We show that the net $(\varphi_{x_\alpha^{-1}})_\alpha$ converges to some element of $\Delta(AP(G))$ and that the limit does not depend on the choice of the net $(x_\alpha)_\alpha$ in G but only on the fact that $\varphi_{x_\alpha} \to \varphi$.

Let $f \in AP(G)$ and $\epsilon > 0$. By Corollary 2.10.6 there exist $a_1, \ldots, a_n \in G$ such that the functions $L_{a_i} R_{a_j} f, 1 \leq i, j \leq n$, form an $\epsilon/3$-net for the set of all two-sided translates $L_a R_b f, a, b \in G$. Define a neighbourhood U of φ in $\Delta(AP(G))$ by

$$U = \{\psi \in \Delta(AP(G)) : |\psi(L_{a_i} R_{a_j} f) - \varphi(L_{a_i} R_{a_j} f)| < \epsilon/3, 1 \leq i, j \leq n\}.$$

If x and y are elements of G such that $\varphi_x, \varphi_y \in U$, then

$$|f(a_i x a_j) - f(a_i y a_j)| < \epsilon/3, \quad 1 \leq i, j \leq n,$$

and hence, by Corollary 2.10.7,

$$|f(axb) - f(ayb)| < \epsilon$$

for all $a, b \in G$. Taking $a = x^{-1}$ and $b = y^{-1}$ this becomes $|f(y^{-1}) - f(x^{-1})| < \epsilon$. This shows that, for each $f \in AP(G)$, the net

$$(\varphi_{x_\alpha^{-1}}(f))_\alpha = (f(x_\alpha^{-1}))_\alpha$$

forms a Cauchy net in $\mathbb{C}$ and that

$$\lim_\alpha \varphi_{x_\alpha^{-1}}(f) = \lim_\beta \varphi_{y_\beta^{-1}}(f),$$

where $(y_\beta)_\beta$ is another net in G such that $\varphi_{y_\beta} \to \varphi$ in $\Delta(AP(G))$. Thus we can define a map $\varphi^{-1} : AP(G) \to \mathbb{C}$ by

$$\varphi^{-1}(f) = \lim_\alpha \varphi_{x_\alpha^{-1}}(f), \quad f \in AP(G),$$

by taking $(x_\alpha)_\alpha$ to be any net in G such that $\varphi_{x_\alpha} \to \varphi$. It is clear that $\varphi^{-1} \in \Delta(AP(G))$ and that $\varphi_x^{-1} = \varphi_{x^{-1}}$ for every $x \in G$. Since multiplication in $\Delta(AP(G))$ is continuous and $\varphi_{ab} = \varphi_a \varphi_b$ for all $a, b \in G$, it follows that

$$\varphi\varphi^{-1} = \lim_\alpha \varphi_{x_\alpha} \cdot \lim_\alpha \varphi_{x_\alpha^{-1}} = \lim_\alpha (\varphi_{x_\alpha x_\alpha^{-1}}) = \varphi_e.$$

Consequently, $\Delta(AP(G))$ is a group and φ^{-1} is the inverse of φ.

Finally, the map $\varphi \to \varphi^{-1}$ from $\Delta(AP(G))$ into $\Delta(AP(G))$ is continuous. To see this, let $\psi \in \Delta(AP(G))$, $f \in AP(G)$, and $\delta > 0$. Define $g \in AP(G)$ by $g(x) = f(x^{-1})$. If $\varphi \in \Delta(AP(G))$ and $x, y \in G$ are such that

$$|\varphi(g) - \psi(g)| < \delta, \quad |\varphi_x(g) - \varphi(g)| < \delta \text{ and } |\varphi_y(g) - \psi(g)| < \delta,$$

then

$$|\varphi_{x^{-1}}(f) - \varphi_{y^{-1}}(f)| = |\varphi_x(g) - \varphi_y(g)| < 3\delta$$

and hence

$$|\varphi^{-1}(f) - \psi^{-1}(f)| \leq |\varphi^{-1}(f) - \varphi_{x^{-1}}(f)| + |\varphi_{y^{-1}}(f) - \psi^{-1}(f)| + 3\delta.$$

As we have shown above, $\varphi_{x^{-1}} \to \varphi^{-1}$ and $\varphi_{y^{-1}} \to \psi^{-1}$ whenever $\varphi_x \to \varphi$ and $\varphi_y \to \psi$. Hence it follows that $\varphi \to \varphi^{-1}$ is continuous. □

We have thus achieved making $\Delta(AP(G))$ a compact group having the following properties.

(1) The map $\phi : G \to \Delta(AP(G))$ is a homomorphism with dense range.
(2) A bounded continuous function f on G is almost periodic if and only if there exists a function $\widehat{f} \in C(\Delta(AP(G)))$ such that $f(x) = \widehat{f}(\phi(x))$ for all $x \in G$.

We remark next that properties (1) and (2) determine the compact group $\Delta(AP(G))$ up to topological isomorphism.

Remark 2.10.11. Let $\Delta = \Delta(AP(G))$ and suppose that Δ' is a second compact group and $\phi' : G \to \Delta'$ is a homomorphism satisfying the analogous properties (1) and (2). Then $\widehat{f'} \to \widehat{f}$ is an algebraic isomorphism of $C(\Delta')$ onto $C(\Delta)$. Let $\delta : \Delta \to \Delta'$ be the associated homeomorphism; that is, $\delta(\varphi)(\widehat{f'}) = \varphi(\widehat{f})$ for $\varphi \in \Delta$ and $f \in AP(G)$. Then

$$\delta(\phi(x))(\widehat{f'}) = \phi(x)(\widehat{f}) = f(x) = \phi'(x)(\widehat{f'})$$

for all $x \in G$ and $f \in AP(G)$. Thus δ extends the homomorphism $\phi' \circ \phi^{-1} : \phi(G) \to \phi'(G)$. Because δ is a homeomorphism and $\phi(G)$ is dense in Δ, it follows that δ is a topological isomorphism.

Definition 2.10.12. The topological group G is said to be *almost periodic* if the homomorphism $\phi : G \to \Delta(AP(G))$ is injective. Even though in general ϕ need not be injective, $\Delta(AP(G))$ is called the *Bohr* or *almost periodic compactification of* G and usually denoted $b(G)$.

In the remainder of this section we use the notation $b(G)$ in order to emphasize the fact that $b(G)$ is a compact group rather than just the structure space of the algebra $AP(G)$.

We now turn to locally compact Abelian groups. Of course, in that case a major portion of the analysis in this section is superfluous. However, for such G, considerably more can be said about $AP(G)$, and $b(G)$ can be identified in terms of G only.

Let $T(G)$ denote the linear subspace of $C^b(G)$ consisting of all finite linear combinations of characters of G. Functions in $T(G)$ are called *trigonometric polynomials.* Since $\widehat{G}$ is a group, $T(G)$ is a subalgebra of $C^b(G)$. For $\chi \in \widehat{G}$ and $x \in G$ we have $L_x\chi(y) = \overline{\chi(x)}\chi(y)$. Thus

$$C_\chi = \{\chi(x)\chi : x \in G\} \subseteq \mathbb{T} \cdot \chi,$$

which is a compact subset of $C^b(G)$. This implies that $T(G) \subseteq AP(G)$.

Theorem 2.10.13. *Let G be a locally compact Abelian group. The Gelfand isomorphism $f \to \widehat{f}$ from $AP(G)$ onto $C(b(G))$ maps $\widehat{G}$ onto $\widehat{b(G)}$ and hence $T(G)$ onto $T(b(G))$. Moreover, $T(G)$ is norm dense in $AP(G)$.*

Proof. It suffices to show that if $\gamma \in \widehat{G}$, then $\widehat{\gamma} \in \widehat{b(G)}$, and that every character of $b(G)$ arises in this way. For $x, y \in G$, we have

$$\widehat{\gamma}(\varphi_x\varphi_y) = \widehat{\gamma}(\varphi_{xy}) = \gamma(xy) = \gamma(x)\gamma(y) = \widehat{\gamma}(\varphi_x)\widehat{\gamma}(\varphi_y).$$

Since $\widehat{\gamma}$ is continuous on $b(G)$ and $\phi(G)$ is dense in $b(G)$, we conclude that $\widehat{\gamma} \in \widehat{b(G)}$.

Conversely, if $\chi \in \widehat{b(G)}$ then $\chi \circ \phi \in \widehat{G}$ since ϕ is a continuous homomorphism from G into $b(G)$. By the first part of the proof $\widehat{\chi \circ \phi} \in \widehat{b(G)}$. The two characters χ and $\widehat{\chi \circ \phi}$ of $b(G)$ agree on the dense subset $\phi(G)$, whence $\chi = \widehat{\chi \circ \phi}$.

Because the Gelfand homomorphism of $AP(G)$ onto $C(b(G))$ is isometric and, as we have just seen, maps $T(G)$ onto $T(b(G))$. Thus for the last statement of the theorem it is enough to observe that $T(b(G))$ is norm dense in $C(b(G))$. Now, if H is a compact Abelian group, then $T(H)$ is $*$-subalgebra of $C(H)$ which strongly separates the points of H. Thus $T(H)$ is dense in $C(H)$ by the Stone–Weierstrass theorem. $\square$

Corollary 2.10.14. *Let G be a locally compact Abelian group, and let $\widehat{G}_d$ denote the algebraic group $\widehat{G}$ endowed with the discrete topology. Then the discrete dual group $\widehat{b(G)}$ of $b(G)$ is isomorphic to $\widehat{G}_d$.*

Proof. Being the dual group of the compact group $b(G)$, $\widehat{b(G)}$ is discrete. By Theorem 2.10.13, the Gelfand homomorphism of $AP(G)$ maps $\widehat{G}$ onto $\widehat{b(G)}$ and this map is obviously a group isomorphism. Thus $\widehat{G}_d$ is isomorphic to $\widehat{b(G)}$. □

Employing the Pontryagin duality theorem for locally compact Abelian groups, Corollary 2.10.14 can be rephrased as follows. The group $b(G)$ is topologically isomorphic to the dual group of $\widehat{G}_d$ since it is topologically isomorphic to the dual group of $\widehat{b(G)}$.

2.11 Structure spaces of tensor products

The purpose of this section is to determine the structure space of the tensor product of two commutative Banach algebras and to investigate its semisimplicity. For the basic theory of tensor products of Banach algebras we refer to Section 1.5. We remind the reader that ϵ denotes the injective tensor norm.

Lemma 2.11.1. *Let A and B be commutative Banach algebras and let γ be an algebra cross-norm on $A \otimes B$ such that $\gamma \geq \epsilon$. Given $\varphi \in \Delta(A)$ and $\psi \in \Delta(B)$, there is a unique element of $\Delta(A \widehat{\otimes}_\gamma B)$, denoted $\varphi \widehat{\otimes}_\gamma \psi$, such that*

$$(\varphi \widehat{\otimes}_\gamma \psi)(x \otimes y) = \varphi(x)\psi(y)$$

for all $x \in A$ and $y \in B$. Furthermore, the mapping

$$\Delta(A) \times \Delta(B) \to \Delta(A \widehat{\otimes}_\gamma B),\ (\varphi, \psi) \to \varphi \widehat{\otimes}_\gamma \psi$$

is a bijection.

Proof. Let $\varphi \in \Delta(A)$ and $\psi \in \Delta(B)$ and recall first that there is a unique homomorphism $\omega : A \otimes B \to \mathbb{C}$ such that $\omega(x \otimes y) = \varphi(x)\psi(y)$ for all $x \in A$ and $y \in B$. By definition of ϵ and since $\gamma \geq \epsilon$, for any $x_1, \ldots, x_n \in A$ and $y_1, \ldots, y_n \in B$, we have

$$\left|\omega\left(\sum_{j=1}^n x_j \otimes y_j\right)\right| = \left|\sum_{j=1}^n \varphi(x_j)\psi(y_j)\right| \leq \epsilon\left(\sum_{j=1}^n x_j \otimes y_j\right) \leq \gamma\left(\sum_{j=1}^n x_j \otimes y_j\right).$$

Thus ω is continuous with respect to γ and therefore extends uniquely to an element of $\Delta(A \otimes_\gamma B)$, denoted $\varphi \widehat{\otimes}_\gamma \psi$.

The mapping $(\varphi, \psi) \to \varphi \widehat{\otimes}_\gamma \psi$ is injective. To verify this, let $\varphi_1, \varphi_2 \in \Delta(A)$ and $\psi_1, \psi_2 \in \Delta(B)$ such that $\varphi_1 \widehat{\otimes}_\gamma \psi_1 = \varphi_2 \widehat{\otimes}_\gamma \psi_2$. Fix $b \in B$ such that $\psi_1(b) = 1$. Then, for all $x \in A$,

$$\varphi_1(x) = \varphi_1(x)\psi_1(b) = (\varphi_1 \otimes \psi_1)(x \otimes b) = (\varphi_2 \otimes \psi_2)(x \otimes b) = \varphi_2(x)\psi_2(b).$$

Now, since φ_1 and φ_2 are non-zero homomorphisms, this equation implies that $\psi_2(b) = 1$. Hence $\varphi_1 = \varphi_2$, and this in turn yields that $\psi_1 = \psi_2$.

It remains to show that given $\rho \in \Delta(A \widehat{\otimes}_\gamma B)$, there exist $\varphi \in \Delta(A)$ and $\psi \in \Delta(B)$ such that $\rho(x \otimes y) = \varphi(x)\psi(y)$ for all $x \in A$ and $y \in B$. Choose $a \in A$ and $b \in B$ such that $\rho(a \otimes b) = 1$, and define $\varphi : A \to \mathbb{C}$ and $\psi : B \to \mathbb{C}$ by

$$\varphi(x) = \rho(xa \otimes b) \text{ and } \psi(y) = \rho(a \otimes yb).$$

Clearly, φ and ψ are linear maps and

$$\varphi(x)\psi(y) = \rho(xa^2 \otimes yb^2) = \rho(x \otimes y)\rho(a^2 \otimes b^2) = \rho(x \otimes y)$$

for all $x \in A$ and $y \in B$. In particular, both φ and ψ are nonzero. Finally, for $x_1, x_2 \in A$,

$$\begin{aligned}\varphi(x_1x_2) &= \rho(x_1x_2a \otimes b) = \rho(x_1x_2a \otimes b)\rho(a \otimes b)\\ &= \rho((x_1a \otimes b)(x_2a \otimes b)) = \rho(x_1a \otimes b)\rho(x_2a \otimes b)\\ &= \varphi(x_1)\varphi(x_2),\end{aligned}$$

and similarly, $\psi(y_1y_2) = \psi(y_1)\psi(y_2)$ for all $y_1, y_2 \in B$. Thus φ and ψ have all the required properties. □

Theorem 2.11.2. *Let A and B be commutative Banach algebras and let γ be an algebra cross-norm on $A \otimes B$ such that $\gamma \geq \epsilon$. Then the mapping*

$$\Delta(A) \times \Delta(B) \to \Delta(A \widehat{\otimes}_\gamma B),\ (\varphi, \psi) \to \varphi \widehat{\otimes}_\gamma \psi$$

is a homeomorphism.

Proof. As to continuity, it suffices to show that for each $c \in A \widehat{\otimes}_\gamma B$, the function $(\varphi, \psi) \to (\varphi \widehat{\otimes}_\gamma \psi)(c)$ is continuous on $\Delta(A) \times \Delta(B)$. For $c \in A \otimes B$, say $c = \sum_{j=1}^n a_j \otimes b_j$, $a_j \in A, b_j \in B, 1 \leq j \leq n$, this follows at once from the equation

$$(\varphi \widehat{\otimes}_\gamma \psi)(c) = \sum_{j=1}^n \varphi(a_j)\psi(b_j).$$

Now, let $z \in A \widehat{\otimes}_\gamma B$ be arbitrary. Since $\|\varphi \widehat{\otimes}_\gamma \psi\| \leq 1$, the function $(\varphi, \psi) \to (\varphi \widehat{\otimes}_\gamma \psi)(z)$ is a uniform limit on $\Delta(A) \times \Delta(B)$ of functions $(\varphi, \psi) \to (\varphi \widehat{\otimes}_\gamma \psi)(c)$, $c \in A \otimes B$, and therefore is continuous.

For openness, it is enough to prove that the mappings $\varphi \widehat{\otimes}_\gamma \psi \to \varphi$ and $\varphi \widehat{\otimes}_\gamma \psi \to \psi$ from $\Delta(A \widehat{\otimes}_\gamma B)$ into $\Delta(A)$ and $\Delta(B)$, respectively, are continuous. To show that the map $\varphi \widehat{\otimes}_\gamma \psi \to \varphi$ is continuous, we check that, for each $a \in A$, the function

$$F_a : \Delta(A \widehat{\otimes}_\gamma B) \to \mathbb{C},\ \varphi \widehat{\otimes}_\gamma \psi \to \varphi(a)$$

is continuous. Fix $a \in A$ and for every $\rho = \varphi \widehat{\otimes}_\gamma \psi \in \Delta(A \widehat{\otimes}_\gamma B)$ select $a_\rho \in A$ and $b_\rho \in B$ such that $\rho(a_\rho \otimes b_\rho) = 1$. Then

$$F_a(\rho) = \varphi(a)\varphi(a_\rho)\psi(b_\rho) = \varphi(aa_\rho)\psi(b_\rho) = \rho(aa_\rho \otimes b_\rho).$$

Now let $(\rho_\alpha)_\alpha$ be a net in $\Delta(A \widehat{\otimes}_\gamma B)$ converging to some $\rho \in \Delta(A \widehat{\otimes}_\gamma B)$. Then

$$\begin{aligned} \rho_\alpha(a_\rho \otimes b_\rho)\rho_\alpha(aa_{\rho_\alpha} \otimes b_{\rho_\alpha}) &= \rho_\alpha(aa_\rho \otimes b_\rho)\rho_\alpha(a_{\rho_\alpha} \otimes b_{\rho_\alpha}) \\ &= \rho_\alpha(aa_\rho \otimes b_\rho) \\ &\to \rho(aa_\rho \otimes b_\rho). \end{aligned}$$

Since $\rho_\alpha(a_\rho \otimes b_\rho) \to \rho(a_\rho \otimes b_\rho) = 1$, we conclude that

$$F_a(\rho_\alpha) = \rho_\alpha(aa_{\rho_\alpha} \otimes b_{\rho_\alpha}) \to \rho(aa_\rho \otimes b_\rho) = F_a(\rho).$$

Thus F_a is a continuous function. Similarly, the map $\varphi \widehat{\otimes}_\gamma \psi \to \psi$ from $\Delta(A \widehat{\otimes}_\gamma B)$ to $\Delta(B)$ is continuous. □

As the reader will have observed, the last slightly more technical part of the preceding proof can be omitted when A and B are unital. Indeed, in this case the map $(\varphi, \psi) \to \varphi \widehat{\otimes}_\gamma \psi$ is a continuous bijection from the compact space $\Delta(A) \times \Delta(B)$ to the Hausdorff space $\Delta(A \widehat{\otimes}_\gamma B)$ and hence is a homeomorphism.

Corollary 2.11.3. *Let A, B and γ be as before. If $A \widehat{\otimes}_\gamma B$ is semisimple, then so are A and B.*

Proof. Let $a \in A$ such that $\widehat{a} = 0$. Fix any nonzero $b \in B$. Then

$$\widehat{a \otimes b}(\varphi \widehat{\otimes}_\gamma \psi) = \varphi(a)\psi(b) = 0$$

for all $\varphi \in \Delta(A)$ and $\psi \in \Delta(B)$. Since every $\rho \in \Delta(A \widehat{\otimes}_\gamma B)$ is of the form $\rho = \varphi \widehat{\otimes}_\gamma \psi$ for some $\varphi \in \Delta(A)$ and $\psi \in \Delta(B)$, we get that $\widehat{a \otimes b} = 0$. Because $A \widehat{\otimes}_\gamma B$ is semisimple, it follows that $a \otimes b = 0$ and hence $a = 0$. So A is semisimple, and similarly for B. □

Remark 2.11.4. The converse to Corollary 2.11.3 is false. In fact, Milne [89] has shown that the following two conditions are equivalent.

(i) The projective tensor product of any two semisimple commutative Banach algebras is semisimple.
(ii) Every Banach space has the approximation property.

However, as first shown by Enflo [31], there are Banach spaces which don't share the approximation property (for all this, compare [114]). In this context compare also Theorem 2.11.6 below and Appendix A.2.

Elements of $\Delta(B)$ give rise to certain continuous homomorphisms from $A \widehat{\otimes}_\gamma B$ onto A. These homomorphisms are extremely useful when dealing with tensor products.

Lemma 2.11.5. *Let A and B be commutative Banach algebras and let γ be an algebra cross-norm on $A \otimes B$ such that $\gamma \geq \epsilon$. Let $\psi \in \Delta(B)$. Then there is a unique continuous homomorphism $\phi_\psi : A \widehat{\otimes}_\gamma B \to A$ such that $\phi_\psi(a \otimes b) = \psi(b)a$ for all $a \in A$ and $b \in B$.*

Proof. The map $A \times B \to A, (a, b) \to \psi(b)a$ is bilinear. Hence there is a unique linear map $\phi_\psi : A \otimes B \to A$ satisfying $\phi_\psi(a \otimes b) = \psi(b)a$ for all $a \in A$ and $b \in B$. Now, let $a_1, \ldots, a_n \in A$ and $b_1, \ldots, b_n \in B$. Then, with $x = \sum_{j=1}^n a_j \otimes b_j$,

$$\begin{aligned}\|\phi_\psi(x)\| &= \left\| \sum_{j=1}^n \psi(b_j)a_j \right\| = \sup \left\{ \left| f\left(\sum_{j=1}^n \psi(b_j)a_j \right) \right| : f \in A_1^* \right\} \\ &\leq \sup \left\{ \left| \sum_{j=1}^n f(a_j)g(b_j) \right| : f \in A_1^*, g \in B_1^* \right\} = \epsilon(x) \\ &\leq \gamma(x).\end{aligned}$$

Thus ϕ_ψ is norm decreasing for the norm γ on $A \otimes B$ and therefore extends uniquely to a continuous linear map, also denoted ϕ_ψ, from $A \widehat{\otimes}_\gamma B$ to A. Finally, ϕ_ψ is a homomorphism since

$$\begin{aligned}\phi_\psi((a \otimes b)(a' \otimes b')) &= \phi_\psi(aa' \otimes bb') = \psi(bb')aa' \\ &= (\psi(b)a)(\psi(b')a') \\ &= \phi_\psi(a \otimes b)\phi_\psi(a' \otimes b')\end{aligned}$$

for $a, a' \in A$ and $b, b' \in B$. □

Of course, starting with $\varphi \in \Delta(A)$, we obtain an analogous homomorphism $\phi_\varphi : A \widehat{\otimes}_\gamma B \to B$. We proceed with two applications of Lemma 2.11.5 which concern the projective tensor product. The first one settles the important question of when $A \widehat{\otimes}_\pi B$ is semisimple.

Theorem 2.11.6. *Let A and B be commutative Banach algebras. Then the projective tensor product $A \widehat{\otimes}_\pi B$ is semisimple if and only if the following two conditions are satisfied.*

(i) *A and B are semisimple.*

(ii) *The natural homomorphism $A \widehat{\otimes}_\pi B \to A \widehat{\otimes}_\epsilon B$ is injective.*

Proof. Suppose first that $A \widehat{\otimes}_\pi B$ is semisimple. Then A and B are semisimple by Corollary 2.11.3. Let ϕ be the natural homomorphism from $A \widehat{\otimes}_\pi B$ into $A \widehat{\otimes}_\epsilon B$ and let $c = \sum_{j=1}^\infty a_j \otimes b_j$, where $\sum_{j=1}^\infty \|a_j\| \cdot \|b_j\| < \infty$, be an element of $A \widehat{\otimes}_\pi B$ such that $\phi(c) = 0$. Then $\sum_{j=1}^\infty f(a_j)g(b_j) = 0$ for all $f \in A^*$ and $g \in B^*$. In particular,

$$(\varphi \widehat{\otimes}_\pi \psi)(c) = \sum_{j=1}^\infty \varphi(a_j)\psi(b_j) = 0$$

for all $\varphi \in \Delta(A)$ and $\psi \in \Delta(B)$. Since $A \widehat{\otimes}_\pi B$ is semisimple and every element of $\Delta(A \widehat{\otimes}_\pi B)$ is of the form $\varphi \widehat{\otimes}_\pi \psi$, it follows that $c = 0$. Thus ϕ is injective.

Conversely, suppose that conditions (i) and (ii) hold. Let c be an element in the radical of $A \widehat{\otimes}_\pi B$, say $c = \sum_{j=1}^\infty a_j \otimes b_j$. Since A is semisimple, from Lemma 2.11.5 we get that $\sum_{j=1}^\infty \psi(b_j)a_j = 0$ for all $\psi \in \Delta(B)$. This implies that, for every $f \in A^*$ and all $\psi \in \Delta(B)$,

$$0 = f\left(\sum_{j=1}^\infty \psi(b_j)a_j\right) = \sum_{j=1}^\infty \psi(b_j)f(a_j) = \psi\left(\sum_{j=1}^\infty f(a_j)b_j\right).$$

Since B is semisimple, it follows that $\sum_{j=1}^\infty f(a_j)b_j = 0$ for every $f \in A^*$. This in turn gives

$$0 = g\left(\sum_{j=1}^\infty f(a_j)b_j\right) = \sum_{j=1}^\infty f(a_j)g(b_j) = (f \widehat{\otimes}_\epsilon g)(c)$$

for every $f \in A^*$ and $g \in B^*$. Now, condition (ii) yields $c = 0$. So $A \widehat{\otimes}_\pi B$ is semisimple. □

Proposition 2.11.7. *Let A and B be commutative Banach algebras. Then $A \widehat{\otimes}_\pi B$ is unital if and only if both A and B are unital.*

Proof. It is apparent that if e_A and e_B are identities of A and B, respectively, then $e_A \otimes e_B$ is an identity of $A \widehat{\otimes}_\pi B$.

Conversely, let $\sum_{j=1}^\infty a_j \otimes b_j$, where $a_j \in A$ and $b_j \in B$, represent an identity for $A \widehat{\otimes}_\pi B$. Then

$$a \otimes b = \sum_{j=1}^\infty a_j a \otimes b_j b$$

for all $a \in A$ and $b \in B$. Since $A \widehat{\otimes}_\pi B$ is unital, $\Delta(A \widehat{\otimes}_\pi B) \neq \emptyset$ and hence $\Delta(A)$ and $\Delta(B)$ are both nonempty by Theorem 2.11.2. Choose $\psi \in \Delta(B)$ and $b \in B$ with $\psi(b) = 1$, and let $\phi_\psi : A \widehat{\otimes}_\pi B \to A$ be the homomorphism of Lemma 2.11.5. Then

$$\begin{aligned} a = \phi_\psi(a \otimes b) &= \phi_\psi\left(\sum_{j=1}^\infty a_j a \otimes b_j b\right) \\ &= \sum_{j=1}^\infty \phi_\psi(a_j a \otimes b_j b) = \sum_{j=1}^\infty \psi(b_j b)a_j a \\ &= \left(\sum_{j=1}^\infty \psi(b_j)a_j\right) a \end{aligned}$$

for all $a \in A$. Thus $\sum_{j=1}^\infty \psi(b_j)a_j$ is an identity for A. Similarly, it is shown that B is unital. □

To conclude this section, let A be a semisimple commutative Banach algebra and G a locally compact Abelian group. Since L^1-spaces do have the approximation property, we could use Theorem 2.11.6 and Corollary 2.7.9 to deduce that $L^1(G,A) = L^1(G)\,\widehat{\otimes}_\pi A$ is semisimple. However, this can be shown directly without appealing to Theorem 2.11.6 as follows.

Theorem 2.11.8. *Let G be a locally compact Abelian group and A a semisimple commutative Banach algebra. Then $L^1(G,A) = L^1(G)\,\widehat{\otimes}_\pi A$ is semisimple.*

Proof. Let $\phi : L^1(G)\,\widehat{\otimes}_\pi A \to L^1(G,A)$ be the isometric isomorphism satisfying $\phi(f\otimes a)(x) = f(x)a$ for all $f \in L^1(G)$ and $a \in A$ and almost all $x \in G$ (Proposition 1.5.4). For $\alpha \in \widehat{G}$, let φ_α be the corresponding element of $\Delta(L^1(G))$ and recall that $\Delta(L^1(G)\,\widehat{\otimes}_\pi A) = \Delta(L^1(G)) \times \Delta(A)$ (Theorem 2.11.2). Let $f \in L^1(G), \alpha \in \widehat{G}, a \in A$ and $\psi \in \Delta(A)$. Then

$$\begin{aligned}
(\varphi_\alpha \widehat{\otimes}_\pi \psi)(f\otimes a) &= \widehat{f}(\alpha)\psi(a) = \psi(a)\int_G f(x)\overline{\alpha(x)}dx \\
&= \int_G \overline{\alpha(x)}\psi(f(x)a)dx \\
&= \int_G \overline{\alpha(x)}\psi(\phi(f\otimes a)(x))dx \\
&= \psi(\phi(f\otimes a))^\frown(\alpha).
\end{aligned}$$

By linearity and continuity, this implies

$$(\varphi_\alpha \widehat{\otimes}_\pi \psi)(u) = \widehat{\psi(\phi(u))}(\alpha)$$

for all $u \in L^1(G)\,\widehat{\otimes}_\pi A, \alpha \in \widehat{G}$ and $\psi \in \Delta(A)$. Since $L^1(G)$ is semisimple, this equation shows that if $u \in L^1(G)\,\widehat{\otimes}_\pi A$ is such that $\widehat{u} = 0$, then $\psi(\phi(u)) = 0$ for all $\psi \in \Delta(A)$. Thus $\phi(u) = 0$ since A is semisimple and hence $u = 0$ as ϕ is injective. □

2.12 Exercises

Exercise 2.12.1. The following example shows that the Gleason–Kahane–Zelazko theorem (Theorem 2.1.2) fails to hold for real Banach algebras. Let $A = C^{\mathbb{R}}([0,1])$ be the algebra of all real valued continuous functions on $[0,1]$ with the supremum norm. Define $\varphi : A \to \mathbb{R}$ by $\varphi(f) = \int_0^1 f(t)dt$. Show that $\varphi(f) \neq 0$ whenever f is invertible, but φ is not multiplicative.

Exercise 2.12.2. Find an example of a real commutative Banach algebra with identity which does not admit a nonzero real multiplicative linear functional.

Exercise 2.12.3. Let V denote the Volterra integral operator on $L^2[0,1]$ defined by

$$Vf(s) = \int_0^s f(t)dt, \;\; f \in L^2[0,1], s \in [0,1],$$

and let A be the closed subalgebra of $\mathcal{B}(L^2[0,1])$ generated by V. Show that A has precisely one maximal ideal.

Exercise 2.12.4. For $1 \leq p < \infty$, consider the non-unital commutative Banach algebra $l^p(\mathbb{N})$. Identify the maximal modular ideals of $l^p(\mathbb{N})$. Show that $l^p(\mathbb{N})$ has maximal ideals which are not modular.

Exercise 2.12.5. Let A be a non-unital commutative Banach algebra and M a maximal ideal of A. Show that M is modular if and only if M has codimension one and does not contain A^2.

Exercise 2.12.6. Let A be the algebra of entire functions in the complex plane endowed with the norm $\|f\| = \sup\{|f(z)| : |z| = 1\}$. Then A is a non-complete commutative normed algebra. Prove that A contains maximal ideals of infinite codimension.

Exercise 2.12.7. Let A be the algebra of all continuously differentiable functions $f : [0,1] \to \mathbb{C}$ with pointwise multiplication and the norm $\|f\| = \|f\|_\infty + \|f'\|_\infty$. Let

$$I = \{f \in A : f(0) = f'(0) = 0\}.$$

Show that A/I is a two-dimensional algebra which has a one-dimensional radical. Thus A is an example of a semisimple commutative Banach algebra which admits a non-semisimple quotient.

Exercise 2.12.8. Find examples showing that Corollaries 2.1.10 and 2.1.11 are no longer true without assuming semisimplicity.

Exercise 2.12.9. Let A be a commutative Banach algebra and $\Gamma : A \to \Gamma(A) \subseteq C_0(\Delta(A))$ its Gelfand homomorphism. Show that Γ is a topological isomorphism (if and) only if there exists $c > 0$ such that $\|a^2\| \geq c\,\|a\|^2$ for all $a \in A$.

Exercise 2.12.10. Let A be a semisimple commutative Banach algebra with norm $\|\cdot\|$, and let B be a subalgebra of A which is a Banach algebra with some norm $|\cdot|$. Show that there exists a constant $c > 0$ such that $\|x\| \leq c|x|$ for all $x \in B$.

Exercise 2.12.11. Let A and B be commutative Banach algebras and $A \oplus B$ their direct sum with the norm $\|(a,b)\| = \max(\|a\|, \|b\|)$. Show that there is a canonical homeomorphism between $\Delta(A \oplus B)$ and the topological disjoint union of $\Delta(A)$ and $\Delta(B)$.

Exercise 2.12.12. Let A be a unital commutative Banach algebra, and let I_1 and I_2 be nontrivial closed ideals of A such that $A = I_1 \oplus I_2$. Show that $\Delta(A)$ is not connected.

Exercise 2.12.13. Let A be a semisimple commutative Banach algebra and $\widehat{A} = \{\widehat{a} : a \in A\}$. Let $\phi : \Delta(A) \to \Delta(A)$ be a homeomorphism. We say that *ϕ is induced from a homomorphism* $h : A \to A$ if $\phi(\varphi)(x) = \varphi(h(x))$ for all $x \in A$ and $\varphi \in \Delta(A)$.

(i) Prove that ϕ is induced from a homomorphism $h : A \to A$ if and only if $f \in \widehat{A}$ implies $f \circ \phi \in \widehat{A}$.

(ii) Find an analogous condition on ϕ which is equivalent to ϕ being induced from an automorphism of A.

Exercise 2.12.14. In Exercise 2.12.13, take $A = l^1(\mathbb{Z})$ and identify $\Delta(A)$ with $\mathbb{T}$. Conclude that a homeomorphism $\phi : \mathbb{T} \to \mathbb{T}$ is induced from a homomorphism of A if and only if $\phi \in \widehat{A}$.

Exercise 2.12.15. Let A and B be commutative Banach algebras and let $h : A \to B$ be a homomorphism with dense range. Show that

$$h^* : \Delta(B) \to \Delta(A),\ \ h^*(\psi)(a) = \psi(h(a)),$$

$a \in A, \psi \in \Delta(B)$, defines an injective continuous mapping from $\Delta(B)$ into $\Delta(A)$. If B is unital, then h^* maps $\Delta(B)$ homeomorphically onto $h^*(\Delta(B))$.

Exercise 2.12.16. Construct examples of semisimple commutative Banach algebras A and B and a homomorphism $h : A \to B$ with dense range such that the corresponding mapping $h^* : \Delta(B) \to \Delta(A)$ (see Exercise 2.12.15)

(i) is not surjective,

(ii) not a homeomorphism onto its range.

Exercise 2.12.17. Let X and Y be nonempty compact Hausdorff spaces and $\phi : C(X) \to C(Y)$ a unital homomorphism, and let $\phi^* : \Delta(C(Y)) \to \Delta(C(X))$ be the map $\varphi \to \varphi \circ \phi$. Show

(i) ϕ^* is injective if and only if ϕ is surjective.

(ii) ϕ^* is surjective if and only if ϕ is injective.

Exercise 2.12.18. Let X be a compact Hausdorff space and let A be a uniform algebra on X. Let $\varphi : A \to \mathbb{C}$ be a homomorphism. Show that there exists a probability measure μ on X such that $\varphi(f) = \int_X f(x)d\mu(x)$ for all $f \in A$.

Exercise 2.12.19. Let A and B be semisimple and unital commutative Banach algebras. Let ϕ be a linear map of A onto B. Prove that ϕ is an algebra isomorphism between A and B if and only if $\sigma_B(\phi(x)) = \sigma_A(x)$ for all $x \in A$.

Exercise 2.12.20. Let $A \subseteq A_1 \subseteq A_2$ be commutative Banach algebras with norms $\|\cdot\|, \|\cdot\|_1$ and $\|\cdot\|_2$ respectively. Assume that A is dense in A_1 and in A_2 in their respective norms and that $\Delta(A) = \Delta(A_2)$ (that is, every element of $\Delta(A)$ is continuous with respect to $\|\cdot\|_2$). Show that $\Delta(A_1) = \Delta(A_2)$.

Exercise 2.12.21. Let A be a semisimple and faithful commutative Banach algebra. For any $T \in M(A)$, let f_T denote the continuous function on $\Delta(A)$ satisfying $\widehat{Tx}(\varphi) = f_T(\varphi)\widehat{x}(\varphi)$ for all $\varphi \in \Delta(A)$ (Proposition 2.2.16). Show that the mapping $T \to f_T$ is a continuous isomorphism from $M(A)$ onto the subalgebra

$$B = \{f \in C^b(\Delta(A)) : f \cdot \widehat{x} \in \widehat{A} \text{ for all } x \in A\}$$

of $C^b(\Delta(A))$.

Exercise 2.12.22. Let A be a semisimple commutative Banach algebra, $T : A \to A$ a bounded linear operator and T^* the adjoint of T. Prove that $T \in M(A)$ if and only if for each $\varphi \in \Delta(A)$ there exists a constant $c(\varphi)$ such that $T^*(\varphi) = c(\varphi)\varphi$.

Exercise 2.12.23. Let A be a commutative Banach algebra such that $\Delta(A)$ is infinite. Prove that there exists $x \in A$ such that $\sigma_A(x)$ is infinite.
(Hint: Let $\varphi_n \in \Delta(A)$, $n \in \mathbb{N}$, such that $\varphi_n \neq \varphi_m$ for $n \neq m$. For $m, n \in \mathbb{N}$, $n \neq m$, let

$$V_{m,n} = \{x \in A : \varphi_m(x) \neq \varphi_n(x)\}$$

and show that $V_{m,n}$ is dense in A. Conclude that $\cap\{V_{m,n} : m \neq n\} \neq \emptyset$).

Exercise 2.12.24. Consider the disc algebra $A(\mathbb{D})$ and view $\mathbb{D} = \Delta(A(\mathbb{D}))$ as a subset of $A(\mathbb{D})^*$. Show that the topology on $\mathbb{D}$ induced by the norm topology of $A(\mathbb{D})^*$ coincides with the complex plane topology on $\mathbb{D}^\circ$ and with the discrete topology on $\mathbb{T}$.
(Hint: For the first part of the assertion, use Schwarz' lemma which states that if $f : \mathbb{D}^\circ \to \mathbb{D}$ is a holomorphic function vanishing at $z_0 \in \mathbb{D}^\circ$, then $|f(z)| \leq |z - z_0|/|1 - z\,\overline{z_0}|$ for all $z \in \mathbb{D}^\circ$.)

Exercise 2.12.25. Let A be a closed subalgebra of $C(\mathbb{D})$ satisfying the following two conditions:

(1) The function $z \to z$ belongs to A.

(2) For every $f \in A$, $\|f\|_\infty = \|f|_\mathbb{T}\|_\infty$.

Then $A \subseteq A(\mathbb{D})$. To prove this, proceed as follows.

(i) Apply Wermer's maximality theorem (Theorem 2.5.15) to conclude that $A|_\mathbb{T} = \{f|_\mathbb{T} : f \in A\}$ is equal to either $P(\mathbb{T})$ or $C(\mathbb{T})$.

(ii) By (2), every $g \in A|_\mathbb{T}$ extends uniquely to some $\widetilde{g} \in A$. Consider the homomorphism $g \to \widetilde{g}(0)$ from $A|_\mathbb{T}$ to $\mathbb{C}$ to exclude the possibility that $A|_\mathbb{T} = C(\mathbb{T})$.

(iii) Show that if $f \in A$ and $g \in A(\mathbb{D})$ are such that $f|_\mathbb{T} = g|_\mathbb{T}$, then $f = g$.

Exercise 2.12.26. Let A be a commutative Banach algebra with identity e. Prove that the following two conditions are equivalent.

(i) For $x, y \in A, \exp x = \exp y$ implies that $x - y = (2k\pi i)e$ for some $k \in \mathbb{Z}$.

(ii) $\Delta(A)$ is connected.

(Hint: Show that the equation $\exp x = e$ has no nonzero solution in the radical of A and that it has solutions different from $(2k\pi i)e, k \in \mathbb{Z}$, if and only if $\Delta(A)$ is not connected.)

Exercise 2.12.27. Let $l^p(\mathbb{N})$ be as in Exercise 1.6.9. Determine $\Delta(l^p(\mathbb{N}))$.

Exercise 2.12.28. Let $\mathrm{Lip}_\alpha[0,1]$ be the Banach algebra of Lipschitz functions of order α (see Exercise 1.6.11). For $t \in [0,1]$, let $\varphi_t(f) = f(t)$, $f \in \mathrm{Lip}_\alpha[0,1]$. Show that the map $t \to \varphi_t$ is a homeomorphism of $[0,1]$ onto $\Delta(\mathrm{Lip}_\alpha[0,1])$. (Hint: If $f \in \mathrm{Lip}_\alpha[0,1]$ is such that $f(t) \neq 0$ for all $t \in [0,1]$, then $\frac{1}{f} \in \mathrm{Lip}_\alpha[0,1]$.)

Exercise 2.12.29. Let γ be a continuous homomorphism of $\mathbb{R}$ into the multiplicative group $\mathbb{C}^\times$ of nonzero complex numbers.

(i) Show that γ is differentiable and satisfies the differential equation

$$\gamma'(t) = \gamma(0)\gamma(t),\ t \in \mathbb{R}.$$

(Hint: There exists $c > 0$ such that $\int_0^c \gamma(s)ds \neq 0$, and then

$$\gamma(t) = \left(\int_0^c \gamma(s)ds\right)^{-1} \int_0^{c+t} \gamma(s)ds$$

for all $t \in \mathbb{R}$.)

(ii) Deduce that there exists $z \in \mathbb{C}$ such that $\gamma(t) = e^{zt}$ for all $t \in \mathbb{R}$.

Exercise 2.12.30. Let G be a compact Abelian group and let α and β be distinct characters of G. Show the *orthogonality relation* $\int_G \alpha(x)\overline{\beta(x)}dx = 0$. (Hint: For $\gamma \in \widehat{G} \setminus \{1_G\}$, choose $x_0 \in G$ such that $\gamma(x_0) \neq 1$ and observe that $\int_G \gamma(x)dx = \gamma(x_0)\int_G \gamma(x)dx$).

Exercise 2.12.31. Let G be a compact Abelian group with normalized Haar measure and let $1 \leq p < \infty$. With convolution, $L^p(G)$ is a commutative Banach algebra. For $\chi \in \widehat{G}$ and $f \in L^p(G)$, let

$$\varphi_\chi(f) = \widehat{f}(\chi) = \int_G f(x)\overline{\chi(x)}dx.$$

Show that the map $\chi \to \varphi_\chi$ is a bijection between $\widehat{G}$ and $\Delta(L^p(G))$ and that $\Delta(L^p(G))$ is discrete.

Exercise 2.12.32. In Theorem 2.7.12 it was shown that, for a locally compact Abelian group G, the Gelfand transform $f \to \widehat{f}$ from $L^1(G)$ to $C_0(\widehat{G})$ is onto (equivalently, the norms $f \to \|f\|_1$ and $f \to \|\widehat{f}\|_\infty$ are equivalent) only when G is finite. For the real line, one can explicitly construct a sequence of functions $f_n \in L^1(\mathbb{R})$, $n \in \mathbb{N}$, such that $\|f_n\|_1 = 1$ for all n, whereas $\|\widehat{f_n}\|_\infty \to 0$ as $n \to \infty$. In fact, normalizing Lebesgue measure so that $[0,1]$ has measure one, define f_n by

$$f_n(t) = \frac{1}{\sqrt{\pi}} \exp(-(1+ni)t^2),$$

$t \in \mathbb{R}$, and show that this sequence has the stated properties.
(Hint: Use the formula

$$\int_{\mathbb{R}} \exp(-ist - zt^2)dt = \left(\frac{\pi}{z}\right)^{1/2} \exp\left(-\frac{s^2}{4z}\right),$$

which holds for all $s \in \mathbb{R}$ and all $z \in \mathbb{C}$ with $\mathrm{Re}z > 0$.)

Exercise 2.12.33. Let $f \in C(\mathbb{T}) \subseteq L^1(\mathbb{T})$. Show that the following conditions are equivalent.

(i) $f \in P(\mathbb{T})$.
(ii) There exists $g \in A(\mathbb{D})$ such that $g|_{\mathbb{T}} = f$.
(iii) $\widehat{f}(-n) = 0$ for all $n \in \mathbb{N}$.

Exercise 2.12.34. Let μ denote Lebesgue measure on the unit interval [0,1] and let $L^\infty(\mu)$ be the space of equivalence classes modulo sets of measure zero of complex valued essentially bounded measurable functions on [0,1]. With the essential supremum norm, pointwise multiplication and $f \to \overline{f}$, $L^\infty(\mu)$ is a unital commutative C^*-algebra. Let $\Delta = \Delta(L^\infty(\mu))$ and $L^\infty(\mu) \to C(\Delta)$, $f \to \widehat{f}$ the Gelfand isomorphism. Observe that $\widehat{f} \to \int_0^1 f(t)d\mu(t)$ is a bounded linear functional of norm one on $C(\Delta)$. By the Riesz representation theorem there is a regular probability measure $\widehat{\mu}$ on Δ satisfying

$$\int_\Delta \widehat{f}(\varphi)d\widehat{\mu}(\varphi) = \int_0^1 f(t)d\mu(t)$$

for all $f \in L^\infty(\mu)$.

(i) Show that $\widehat{\mu}(U) > 0$ for every nonempty open subset U of Δ.

(ii) Show that for every nonempty open subset U of Δ there exists $f_U \in L^\infty(\mu)$ such that $\widehat{f_U} = 1_U$ $\widehat{\mu}$-almost everywhere.

Exercise 2.12.35. Retain the setting and notation of Exercise 2.12.34. Prove that Δ is extremally disconnected, that is, the closure of every open subset of Δ is open.
(Hint: Let U be an open subset of Δ and $f \in L^\infty(\mu)$ such that $\widehat{f} = 1_U$ $\widehat{\mu}$-almost everywhere. Deduce from continuity of $\widehat{f}$ that $\widehat{f}$ takes only the values 0 and 1.)

Exercise 2.12.36. Let A be a commutative Banach $*$-algebra. Then A is called symmetric if $\varphi(x^*) = \overline{\varphi(x)}$ for all $x \in A$ and $\varphi \in \Delta(A)$. Prove that the following two conditions are equivalent.

(i) A is symmetric.

(ii) $-1 \notin \sigma_A(x^*x)$ for every $x \in A$.

(Hint: Without loss of generality, assume that A has an identity e. For (ii) $\Rightarrow$ (i), use (ii) to show that if x is a selfadjoint element of A and α and β are real numbers with $\beta \neq 0$, then $(\alpha + i\beta)e - x$ is invertible.)

Exercise 2.12.37. Let $\mathbb{D} = \{z \in \mathbb{C} : |z| \leq 1\}$, the closed unit disc, and $A = P(\mathbb{D})$. For $f \in A$, define f^* by $f^*(z) = \overline{f(\overline{z})}$.

(i) Show that $f \to f^*$ is an involution on A.

(ii) Does this involution turn A into a C^*-algebra?

(iii) Is $\sigma_A(f^*f) \subseteq \mathbb{R}$ for all $f \in A$?

(iv) Which $\varphi \in \Delta(A)$ satisfy $\varphi(f^*f) \geq 0$ for all $f \in A$?

(v) Does there exist an involution $f \to \widetilde{f}$ on A such that $\varphi(\widetilde{f}) = \overline{\varphi(f)}$ for all $f \in A$ and $\varphi \in \Delta(A)$?

Exercise 2.12.38. Let G be a locally compact Abelian group such that $G \neq \{e\}$. Construct a function $f \in L^1(G)$ such that $\|f^* * f\| \neq \|f\|^2$, thereby showing that $\|\cdot\|_1$ fails to be a C^*-norm.

(Hint: In case G has at least three elements e, a and b, choose a compact symmetric neighbourhood V of e with the property that the three sets V, aV and bV are pairwise disjoint and consider the function $f = 1_V + i1_{aV} + 1_{bV}$.)

Exercise 2.12.39. Let A be C^*-algebra with identity e. Then e is an extreme point of the unit ball $A_1 = \{a \in A : \|a\|_1 \leq 1\}$. To prove this, proceed as follows.

(i) Suppose that $e = \frac{1}{2}(a + b)$, $a, b \in A_1$. Show that there exist selfadjoint elements x and y of A_1 such that $e = \frac{1}{2}(x + y)$ and $xy = yx$.

(ii) Let B be the closed subalgebra of A generated by x, y and e. Apply Theorem 2.4.5 and show that $\widehat{x}(\varphi) = \widehat{y}(\varphi)$ for all $\varphi \in \Delta(B)$.

Exercise 2.12.40. Let $S^n = \{x \in \mathbb{R}^{n+1} : \|x\| = 1\}$, the unit sphere in $\mathbb{R}^{n+1}$, $n \geq 1$. Use the Stone–Weierstrass theorem to show that $C(S^n)$ admits a system of $n + 1$ generators.

(Remark: Using cohomology theory, one can prove that $C(S^n)$ cannot admit a system of less than $n + 1$ generators.)

Exercise 2.12.41. Let X be a compact subset of $\mathbb{C}$ and suppose that $\mathbb{C} \setminus X$ has infinitely many connected components. Prove that $R(X)$ cannot be generated by finitely many rational functions.

Exercise 2.12.42. For $0 < r < R < \infty$ let $K(r, R)$ denote the compact annulus

$$K(r, R) = \{z \in \mathbb{C} : r \leq |z| \leq R\}.$$

Prove that the uniform algebra $A(K(r, R))$ is generated by the two functions $z \to z$ and $z \to 1/z$.

Exercise 2.12.43. Let A be a commutative Banach algebra and let $\varphi_1, \dots, \varphi_n$ be distinct elements of $\Delta(A)$. Show that the mapping

$$x \to (\varphi_1(x), \dots, \varphi_n(x))$$

maps A onto $\mathbb{C}^n$.

Exercise 2.12.44. Let K be a compact subset of $\mathbb{C}^n$, $n \in \mathbb{N}$. Show that there exist a unital commutative Banach algebra A and elements $x_1, \dots, x_n$ of A such that

$$K = \sigma_A(x_1, \dots, x_n).$$

Why does this not contradict Theorem 2.3.6?

Exercise 2.12.45. Let A be a commutative Banach algebra with identity e and let $x_1, \dots, x_n \in A$. Let Λ denote the set of all $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n$ with the property that for any $y_1, \dots, y_n \in A$, the element $\sum_{j=1}^n y_j(\lambda e - x_j)$ is not invertible in A. Prove that

$$\Lambda = \sigma_A(x_1, \dots, x_n).$$

(Hint: Let $\lambda = (\lambda_1, \dots, \lambda_n) \in \Lambda$. To show that $\lambda \in \sigma_A(x_1, \dots, x_n)$, observe that the set of all elements $\sum_{j=1}^n y_j(\lambda e - x_j)$, $y_j \in A$, either equals A or is a proper ideal of A and hence is contained in a maximal ideal.)

Exercise 2.12.46. Prove that the two-dimensional torus

$$T = \{(z, w) \in \mathbb{C}^2 : |z| = |w| = 1\}$$

in $\mathbb{C}^2$ has as its polynomially convex hull the 4-dimensional bicylinder $\mathbb{D} \times \mathbb{D}$. (Remark: The question of whether there is any relation between the topological dimension of a compact subset of $\mathbb{C}^n$ and the topological dimension of its polynomially convex hull has been a matter of some interest.)

Exercise 2.12.47. Consider the following subset

$$Y = \left\{(z, w) \in \mathbb{C}^2 : z \neq 0, w = \frac{1}{z}\right\}$$

of $\mathbb{C}^2$. Show that for every compact subset X of Y, the polynomially convex hull $\widehat{X}_p$ of X is contained in Y.

Exercise 2.12.48. In Proposition 2.8.8, consider the following choices of w:

(i) $w(n) = 2^n$ for all $n \in \mathbb{Z}$;
(ii) $w(n) = 2^n$ for $n \geq 0$ and $w(n) = 1$ for $n < 0$;
(iii) $w(n) = 1 + 2^n$ for all $n \in \mathbb{Z}$;
(iv) $w(n) = 1 + 2^n$ for $n \geq 0$ and $w(n) = 1$ for $n < 0$.

For which of these choices is $K(R_-, R_+)$ a circle? For which of them is $\widehat{l^1(\mathbb{Z}, w)}$ closed under complex conjugation?

Exercise 2.12.49. Determine the structure space of the Beurling algebra $l^1(\mathbb{Z}, \omega)$ for the following weights ω:

(i) $\omega(n) = e^{|n|}$, $n \in \mathbb{Z}$.

(ii) $\omega_\alpha(n) = (1 + |n|)^\alpha$, $n \in \mathbb{Z}$, $0 < \alpha < \infty$.

Exercise 2.12.50. Let ω be a continuous weight function on $\mathbb{R}^+$. Then the limit $\lim_{t\to\infty} \omega(t)^{1/t}$ exists and is equal to $\rho = \inf\{\omega(t)^{1/t} : t > 0\}$. Suppose that $\rho > 0$ and let

$$S = \{z \in \mathbb{C} : \mathrm{Re} z \geq - \ln \rho\}.$$

The purpose of this exercise is to determine, by analogy with Beurling algebras on $\mathbb{R}$ (Proposition 2.8.7), the structure space of the convolution algebra $L^1(\mathbb{R}^+, \omega)$.

(i) For $z \in S$, show that

$$\varphi_z(f) = \int_0^\infty f(t) e^{-zt} dt,\ f \in L^1(\mathbb{R}^+, \omega),$$

defines an element of $\Delta(L^1(\mathbb{R}^+, \omega))$.

(ii) Prove that every element of $\Delta(L^1(\mathbb{R}^+, \omega))$ is of the form φ_z for some $z \in S$.

(iii) Deduce that $L^1(\mathbb{R}^+, \omega)$ is semisimple.

(iv) Show that the map $z \to \varphi_z$ is a homeomorphism from the halfplane S onto $\Delta(L^1(\mathbb{R}^+, \omega))$.

Exercise 2.12.51. Let ω and ρ be as in the preceding exercise and assume that $\rho = 0$. Show that then $L^1(\mathbb{R}^+, \omega)$ is radical. An example of such a radical weight is $\omega(t) = \exp(-t^2), t \in \mathbb{R}^+$.

Exercise 2.12.52. Let $\mathbb{T}$ be the multiplicative group of complex numbers of absolute value one with normalized Haar measure. Recall that the Fourier transform of $f \in C(\mathbb{T})$ on $\widehat{\mathbb{T}} = \mathbb{Z}$ is defined by $\widehat{f}(n) = \int_{\mathbb{T}} f(z) z^{-n} dz$, $n \in \mathbb{Z}$. Prove that $f \to \widehat{f}$ furnishes an isometric isomorphism of the Fourier algebra $A(\mathbb{T})$ to $l^1(\mathbb{Z})$.

Exercise 2.12.53. Show that the Fourier algebra $A(\mathbb{Z})$ of the group of integers is isometrically isomorphic to $L^1(\mathbb{T})$.

Exercise 2.12.54. Let G be a locally compact group and $A(G)$ the Fourier algebra of G as studied in Section 2.9. Exploit Lemmas 2.9.3 and 2.9.5 (with $K = \{e\}$) to establish the existence of a net $(u_\alpha)_\alpha$ in $A(G)$ with the following properties:

(1) $\|u_\alpha\|_{A(G)} = u_\alpha(e) = 1$ for all α;

(2) $\|v u_\alpha\|_{A(G)} \to 0$ for every $v \in A(G)$ with $v(e) = 0$.

Exercise 2.12.55. Let G be a noncompact locally compact group. Show that there exists a bounded continuous function on G which fails to be uniformly continuous (and hence is not almost periodic).

Exercise 2.12.56. Let G be a locally compact group and

$$N = \{x \in G : f(x) = f(e) \text{ for all } f \in AP(G)\}.$$

Using only the fact that $f \in AP(G)$ implies that $L_a R_b f \in AP(G)$ for all $a, b \in G$, give a direct proof that N is a normal subgroup of G and that, for $x, y \in G$, $f(x) = f(y)$ for all $f \in AP(G)$ if and only if $y^{-1}x \in N$.

Exercise 2.12.57. Let G be a locally compact Abelian group and let $P : L^\infty(G) \to L^\infty(G)$ be a norm-bounded projection such that $P(L_x f) = L_x(P(f))$ for all $f \in L^\infty(G)$ and $x \in G$. Show that P maps $AP(G)$ into $AP(G)$ and that there exists a finite measure μ on the Bohr compactification $b(G)$ such that $P(f) = f * \mu$ for all $f \in AP(G) = C(b(G))$.

Exercise 2.12.58. Let A be a commutative Banach algebra and let D be a continuous derivation of A. The *Singer–Wermer theorem* states that $Dx \in \operatorname{rad}(A)$ for every $x \in A$. In particular, there are no nonzero continuous derivations on a semisimple commutative Banach algebra.

Prove the Singer–Wermer theorem as follows. For $\varphi \in \Delta(A)$ and $x \in A$, consider the function $z \to \varphi(\exp(zD)x)$. Show that this is a bounded holomorphic function in the entire complex plane (note that $x \to \varphi(\exp(zD)x)$ is a multiplicative linear functional on A). Conclude that $\varphi(Dx) = 0$.

Let A be a commutative Banach algebra and $\varphi \in \Delta(A)$. A linear functional D on A is called a *point derivation at* φ if $D(ab) = \varphi(a)D(b) + \varphi(b)D(a)$ for all $a, b \in A$.

Exercise 2.12.59. Show that there is a nonzero continuous point derivation on $\mathrm{Lip}_\alpha[0,1]$ at every $t \in [0,1]$.
(Hint: Let $(t_n)_n \subseteq [0,1]$ be a sequence such that $t_n \to t$ and $t_n \neq t$ for all n. Define $l_n \in (\mathrm{Lip}_\alpha[0,1])^*$ by

$$l_n(f) = \frac{f(t_n) - f(t)}{|t_n - t|^\alpha},$$

and let l be a w^*-accumulation point of the sequence $(l_n)_n$ in $(\mathrm{Lip}_\alpha[0,1])^*$.)

Exercise 2.12.60. Let $t \in [0,1]$ and let I and J be the closed ideals in $C^n[0,1]$ defined by

$$I = \{f \in C^n[0,1] : f(t) = 0\} \text{ and } J = \{f \in C^n[0,1] : f(t) = f'(t) = 0\}.$$

It follows from Taylor's formula that I^2 is dense in J. Let D be a continuous point derivation of $C^n[0,1]$ at t, that is,

$$D(fg) = f(t)D(g) + g(t)D(f)$$

for all $f, g \in C^n[0,1]$. Show that $D(J) = \{0\}$ and hence D is of the form $D(f) = \alpha f(t) + \beta f'(t)$ for some $\alpha, \beta \in \mathbb{C}$. Conclude that $D(f) = \beta f'(t)$ for all $f \in C^n[0,1]$.

Exercise 2.12.61. Let $A \subseteq C(X)$ and $B \subseteq C(Y)$ be uniform algebras. Let

$$A \otimes_u B = \{f \in C(X \times Y) : f(\cdot, y) \in A \text{ for all } y \in Y \\ \text{and } f(x, \cdot) \in B \text{ for all } x \in X\}.$$

Show that $A \otimes_u B$ is a uniform algebra on $X \times Y$ and that $\Delta(A \otimes_u B)$ can be canonically identified with $\Delta(A) \times \Delta(B)$. $A \otimes_u B$ is called the *uniform tensor product* (slice product) of A and B.

Exercise 2.12.62. Let A and B be commutative Banach algebras such that A is semisimple and B is finite dimensional. Prove, without using the fact that B has the approximation property, that $A \widehat{\otimes}_\pi B$ is semisimple.
(Hint: Let $\Delta(B) = \{\psi_1, \ldots, \psi_m\}$ and choose $b_j \in \cap\{\ker \psi_k : k \neq j\}$ such that $\psi_j(b_j) = 1$. Then $b_1, \ldots, b_m$ form a basis of B.)

Exercise 2.12.63. Let G and H be discrete Abelian groups with dual groups $\widehat{G}$ and $\widehat{H}$. Prove that the Gelfand homomorphism maps $l^1(G \times H)$ into the projective tensor product $C(\widehat{G}) \widehat{\otimes}_\pi C(\widehat{H})$.

2.13 Notes and references

Theorem 2.1.2, characterizing multiplicative linear functionals on (not necessarily commutative) Banach algebras, has been established independently by Gleason [44] and Kahane and Zelazko [64] using analytic tools. The fairly elementary algebraic proof given here was found by Roitman and Sternfeld [109] and the preliminary Lemma 2.1.1 is due to Zelazko [141]. There exist an extensive theory and a wealth of interesting examples of radical commutative Banach algebras. These play a fundamental role in the investigation of automatic continuity problems (see [25] for a comprehensive account). We have confined ourselves to including just two illustrative examples. The continuity results Corollaries 2.1.10 and 2.1.12 and the uniqueness of norm property, Corollary 2.1.11, trace back to Rickart [106]. Corollaries 2.1.11 and 2.1.12 hold as well for non-commutative semisimple Banach algebras. This follows from Johnson's theorem [61] stating that if A and B are Banach algebras with B semisimple, then every homomorphism from A onto B is continuous. For a short proof of Johnson's theorem, see [101].

The Gelfand representation is the pioneering work of Gelfand. All the basic results presented in Section 2.2 appeared first in [38] and [40] and are nowadays part of any book on Banach algebras. Also, the examples and immediate applications of Gelfand's theory given in Section 2.2 are standard.

Many commutative Banach algebras are generated by finitely many elements. If $a_1, \ldots, a_n$ generate A, then $\Delta(A)$ is canonically homeomorphic to the joint spectrum of $a_1, \ldots, a_n$, which is a compact subset of $\mathbb{C}^n$. It is therefore an important issue to identify the compact subsets of $\mathbb{C}^n$ arising in this manner as joint spectra. Theorem 2.3.6, which states that these are exactly the polynomial convex subsets of $\mathbb{C}^n$, was shown by Shilov, as was Theorem 2.3.7, which says that a compact subset of $\mathbb{C}$ is polynomially convex if and only if its complement is connected [121, 123]. The problem of a topological characterisation of polynomial convex subsets of $\mathbb{C}^n$ for $n \geq 2$ is open. For more details and partial results we refer the reader to [126].

C^*-algebras were first studied by Gelfand and Naimark in their fundamental paper [39]. Theorem 2.4.5, which is usually referred to as the commutative Gelfand-Naimark theorem and which identifies the commutative C^*-algebras as precisely the uniform algebras $C_0(X)$, where X is a locally compact Hausdorff space, as well as the continuous functional calculus (Theorem 2.4.9) can be found in [39]. Let X be a completely regular topological space. The introduction of the Stone–Čech compactification $\beta(X)$ as the structure space of the commutative C^*-algebra $C^b(X)$ (Theorem 2.4.12) is for instance given in [126] and [36].

There is a vast literature on uniform algebras, in particular on $P(X)$, $R(X)$ and $A(X)$, where X is a compact subset of $\mathbb{C}^n$. We refer the reader to the monographs by Stout [126], Gamelin [36] and Leibowitz [78] concerning much more detailed material. Equality to hold at any position in the chain of inclusions $P(X) \subseteq R(X) \subseteq A(X) \subseteq C(X)$ can be interpreted as a result in qualitative approximation theory and is therefore of interest beyond Banach algebra theory. Samples of such results are Theorem 2.5.8 and Theorem 2.5.12, the former being a major step towards Mergelyan's theorem which asserts that if X is a compact subset of $\mathbb{C}$, then $P(X) = A(X)$ precisely when $\mathbb{C} \setminus X$ is connected. Except for $n = 1$, there are no topological characterisations of those compact subsets of $\mathbb{C}^n$ which arise as structure spaces of algebras $P(X)$ and $R(X)$ (Theorem 2.5.7). Examples of compact subsets X of $\mathbb{C}$ with empty interior for which $R(X) \neq C(X)$ have been given by several authors. The example we have presented in Section 2.5 is basically due to Mergelyan [87], somewhat modified by McKissick [85] (see also [73]). The maximality theorem, Theorem 2.5.15, was found by Wermer [136]. Lemma 2.5.14 and the simple proof of Theorem 2.5.15 based on it was discovered by Cohen [22]. The related result displayed in Exercise 2.12.25 was shown by Rudin [112].

Theorem 2.6.6 is due to Arens [4] and can also be found in [126] and [36]. It is worth pointing out that when X is a compact subset of $\mathbb{C}^n$ for some $n > 1$, then $\Delta(A(X))$ need not be homeomorphic with a subset of $\mathbb{C}^n$ [126].

The convolution algebras $L^1(G)$ of locally compact Abelian groups, which are the central object of study in commutative harmonic analysis, form a large and extremely important class of commutative Banach algebras. The fact that the structure space of $L^1(G)$ identifies canonically with the dual group $\widehat{G}$ of G, endowed with the topology of uniform convergence of characters on compact subsets of G (Theorems 2.7.2 and 2.7.5) is classical. We refer to [54], [105], and [113]. Note that for G the group of real numbers, the Gelfand transform is nothing but the Fourier transform. To show semisimplicity of $L^1(G)$, we have exploited the left regular representation of $L^1(G)$ on $L^2(G)$ and the semisimplicity of commutative C^*-algebras. A highly non-trivial fact is that the Gelfand homomorphism of $L^1(G)$ into $C_0(\widehat{G})$ is surjective only when G is finite (Theorem 2.7.12). There are approaches to Theorem 2.7.12 different from the one chosen here, either using the Pontryagin duality theorem or some other tools none of which we want to employ in this context (see [28, Theorem B.4.6], [34], and [45]).

Beurling algebras behave in many respects similarly to L^1-algebras. For instance, Theorems 2.8.2 and 2.8.5 exposing the Gelfand representation of $L^1(G,\omega)$, parallel Theorems 2.7.3 and 2.7.5. Some technical complications, however, arise from the facts that weights are only locally bounded and that the set of ω-bounded generalized characters is less handy than the dual group $\widehat{G}$. The concrete realizations of $\Delta(L^1(\mathbb{R},\omega))$ and $\Delta(l^1(\mathbb{Z},\omega))$ by means of a vertical strip and an annulus in the complex plane, respectively, are classical [41, Chapter III]. The elementary proof of semisimplicity of $L^1(G,\omega)$ given here (Theorem 2.8.10) is due to Bhatt and Dedania [16].

The Fourier algebra $A(G)$ of a locally compact group G was introduced by Eymard [32] as the predual of the group von Neumann algebra $VN(G)$. The realization of $A(G)$ which we have taken as the definition and all the basic results, such as Theorem 2.9.4 and Lemma 2.9.5, are contained in [32]. Our presentation follows the one in [25]. Eymard has also shown that $A(G)$ is isometrically isomorphic to $L^1(\widehat{G})$ when G is Abelian. This is one of the reasons why the large class of Fourier algebras currently attracts a lot of attention within the theory of commutative Banach algebras. A result of Leptin [79] says that $A(G)$ has a bounded approximate identity if and only if G is a so-called amenable group. One of the many open questions is whether $A(G)$ always possesses an (unbounded) approximate identity.

The Bohr or almost periodic compactification $b(G)$ of a locally compact Abelian group G originated from a paper by Bohr [18] who was the first to study almost periodic functions on the real line. Discussions of the subject under various different aspects can be found in the monographs by Hewitt and Ross [54, 55], Loomis [81], and Weil [134]. In particular, the fairly elementary proof showing that one-sided almost periodic functions are necessarily two-sided almost periodic is due to Loomis. In Section 2.10 we have established the existence and properties of $b(G)$ by applying Gelfand's theory to the commutative C^*-algebra of almost periodic functions.

Tensor products of commutative Banach algebras have been investigated by several authors. Theorem 2.11.2, which canonically identifies $\Delta(A\widehat{\otimes}_\gamma B)$ with the product space $\Delta(A)\times\Delta(B)$, was independently shown by Tomiyama [128] and Gelbaum [37], following earlier work of Hausner [48, 49] and G.P. Johnson [62] on $L^1(G,A)$ and $C(X,A)$. The more subtle question of when the projective tensor product $A\,\widehat{\otimes}_\pi B$ is semisimple was addressed in [128], where Theorem 2.11.6 can be found. The fact that condition (ii) of Theorem 2.11.6 need not be satisfied and consequently the projective tensor product of two semisimple commutative Banach algebras need not be semisimple, was discovered by Milne [89] by exploiting the existence of Banach spaces which don't share the approximation property [31]. Note in this context that Theorem 2.11.6 contradicts Corollary 1 of [77].

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