2

Numbers

2.1 Integers

You remember the definition of a prime number. On p. 7, we defined a prime number and formulated the Fundamental Theorem of Arithmetic. Numerous beautiful results can be presented here, but I will limit myself to problems illustrating some ideas that require practically no knowledge of number theory.

2.1. Prove that for any integer n, $n^5 - 5n^3 + 4n$ is divisible by 120.

Solution. First of all, let us decompose $P(n) = n^5 - 5n^3 + 4n$ and 120 into factors:

$$\begin{aligned} P(n) &= n(n^4 - 5n^2 + 4) \\ &= n(n^2 - 1)(n^2 - 4) \\ &= (n-2)(n-1)n(n+1)(n+2); \end{aligned}$$

$$120 = 2^3 \cdot 3 \cdot 5.$$

Since for any integer n, $P(n)$ is a product of five consecutive integers, and one of any five consecutive integers is a multiple of 5, $P(n)$ is divisible by 5.

Similarly, out of any three consecutive integers, one is a multiple of 3; therefore, $P(n)$ is divisible by 3 for any integer n.

Out of any four consecutive integers, one is a multiple of 4, plus one more is even. Therefore, $P(n)$ is divisible by $4 \times 2 = 8$ for any integer n.

A. Soifer, *Mathematics as Problem Solving*, DOI: 10.1007/978-0-387-74647-0_2,

Due to the Fundamental Theorem of Arithmetic, $P(n)$ is divisible by $23 \times 3 \times 5 = 120$. $\square$

As a 7th-grader, I faced this problem at the Moscow Mathematical Olympiad in the spring of 1962, which offered us four problems and four hours to solve them. It took me no time to solve the other three problems. After much effort, I finally conquered this problem. The divisibility of the expression by 8 was, of course, the difficulty to overcome.

Given a quadratic equation $ax^2 + bx + c = 0$; the number $D = b^2 - 4ac$ is called the *discriminant* of the equation.

2.2. Is there an integer x such that $x^2 + x + 3$ is a multiple of 121?

Solution I Assume that

$$x^2 + x + 3 = 121k,$$

where x and k are integers. We then have a quadratic equation in x:

$$x^2 + x + (3 - 121k) = 0.$$

In order for a solution to be an integer (remember, the problem asks whether an integer x exists!), the discriminant of the equation has to be a perfect square:

$$4 \times 121k - 11 = n^2$$

where n is an integer; that is,

$$n^2 = 11(4k \times 11 - 1).$$

This means that n^2 is divisible by 11 but not by 11^2. On the other hand, due to Problem 1.14, since n^2 is divisible by 11, n is divisible by 11 as well, which in turn implies that n^2 is divisible by 112. This is a contradiction.

Therefore, there is no integer x such that $x^2 + x + 3$ is a multiple of 121. $\square$

Solution II $(x^2 + x + 3)$ is divisible by 121 if and only if $4(x^2 + x + 3)$ is divisible by 121, but

$$4(x^2 + x + 3) = (2x + 1)^2 + 11.$$

If for some integers x and k,

$$(2x+1)^2 + 11 = 11^2 \times k,$$

then

$$(2x+1)^2 = 11(11k-1).$$

Just as in the first solution, the contradiction is derived from the fact that a square, namely $(2x+1)^2$, is divisible by 11 but is not divisible by 11^2. □

2.3. Find all integral solutions of the equation

$$x^2 + y^2 + x + y = 3.$$

Solution I Let $L(x, y) = x^2+y^2+x+y = x(x+1)+y(y+1)$. For any integer x, $x(x+1)$ is even as the product of two consecutive integers. Similarly, $y(y+1)$ is even. Thus for any integers x, y, $L(x, y)$ is even and therefore not equal to 3.

The solution set is empty. □

Solution II By multiplying both sides of the given equation by 4, we get:

$$(4x^2 + 4x) + (4y^2 + 4y) = 12,$$

or

$$(2x+1)^2 + (2y+1)^2 = 14.$$

On the other hand, a direct check shows that 14 is not the sum of two squares of integers. □

Problems

2.4. Prove that for any integer n, $n^5 - n$ is divisible by 30.

2.5. Prove that for a prime p greater than 3, $p^2 - 1$ is divisible by 24.

2.6. Prove that for any primes p and q, each greater than 3, $p^2 - q^2$ is divisible by 24.

2.7. Can $4p + 1$ be a prime number if both p and $2p + 1$ are primes and $p > 3$?

2.8. Prove that the remainder upon dividing any prime number by 30 is again a prime.

2.9. Find the integer solutions to

$$15x^2 - 7y^2 = 9.$$

2.10. Prove that for any positive integer n, $10^n + 18n - 1$ is divisible by 27.

2.2 Rational and Irrational Numbers

We have already met rational and irrational numbers in Chapter 1, Section 1.3 (Problems 1.11, 1.13, and 1.15). As you know, rational numbers can be presented in the form m/n, where m and n are integers and $n \neq 0$. But how do we recognize whether a number given as a decimal fraction is rational or irrational? Rational numbers are terminal or infinite repeating decimal fractions.

2.11. Prove that the number

$$A = 0.101001000\ldots,$$

where the number of zeros between units increases by one, is irrational.

Solution. Assume that A is a repeating fraction, i.e., after the first k digits, the same sequence of n digits (we'll call it *period*) repeats. Since the number of consecutive zeros in the decimal representation of A is increasing, we can find $2n + k$ consecutive zeros, but this implies that all n digits of the period are zeros. Therefore, in the decimal decomposition of A we get only zeros from some point on.

However, this contradicts the definition of A, which allows us to find a digit one further right than any given digit of the decimal representation of A. □

2.12. The numbers a, b, and $\sqrt{a} + \sqrt{b}$ are rational. Prove that the numbers $\sqrt{a}$ and $\sqrt{b}$ are rational as well.

Solution. The numbers a and b are rational; therefore, $(a + b)$ is rational. The numbers $(a + b)$ and $(\sqrt{a} + \sqrt{b})$ are rational, thus $\sqrt{a} - \sqrt{b} = \frac{a+b}{\sqrt{a}+\sqrt{b}}$ is rational. Now we can see that

$$\sqrt{a} = \tfrac{1}{2}\left[(\sqrt{a} + \sqrt{b}) + (\sqrt{a} - \sqrt{b})\right]$$

is rational; so is $\sqrt{b} = (\sqrt{a} + \sqrt{b}) - \sqrt{a}$. □

2.13. Prove that $1 + \sqrt{5}$ cannot be written as a sum of squares of numbers of the form $a + b\sqrt{5}$ with rational a and b.

Solution. Let us first note that if for integers x_1, x_2, y_1, y_2,

$$x_1 + y_1\sqrt{5} = x_2 + y_2\sqrt{5},$$

then $x_1 = x_2$ and $y_1 = y_2$. Indeed, otherwise we would get $x_1 \neq x_2$ and $y_1 \neq y_2$, $\sqrt{5} = \frac{x_1 - x_2}{y_2 - y_1}$, with an irrational left side and a rational right side.

Now let us assume that

$$1 + \sqrt{5} = (a_1 + b_1\sqrt{5})^2 + (a_2 + b_2\sqrt{5})^2 + \cdots + (a_n + b_n\sqrt{5})^2.$$

Due to the uniqueness proven above, we can conclude that

$$1 - \sqrt{5} = (a_1 - b_1\sqrt{5})^2 + (a_2 - b_2\sqrt{5})^2 + \cdots + (a_n - b_n\sqrt{5})^2.$$

But $1 - \sqrt{5} < 0$, while the right side is nonnegative (as a sum of squares) — a contradiction. □

2.14. Let p/q, where p and q are integers and their greatest common divisor is 1, be a solution of the algebraic equation

$$a_0x^n + a_1x^{n-1} + \cdots + a_n = 0$$

with integral coefficients a_i. Prove that p is a divisor of a_n and q is a divisor of a_0.

Solution. Essentially, we are given the equality

$$a_0\frac{p^n}{q^n} + a_1\frac{p^{n-1}}{q^{n-1}} + \cdots + a_{n-1}\frac{p}{q} + a_n = 0.$$

Therefore,

$$a_0 p^n = q(-a_1 p^{n-1} - \cdots - a_n q^{n-1}),$$

i.e., q is a divisor of $a_0 p^n$. Since $\gcd(p, q) = 1$, this implies that q is a divisor of a_0.

Similarly,

$$a_n q^n = p(-a_{n-1} q^{n-1} - \cdots - a_0 p^{n-1}),$$

and thus p is a divisor of a_n. □

The statement of Problem 2.14 has a very important consequence:

Corollary 2.1. *Any rational solution of the equation*

$$x^n + a_1 x^{n-1} + \cdots + a_n = 0$$

with integral coefficients is an integer.

Prove it!

Problems

2.15. The number A is given as a decimal fraction:

$$A = 0.10000000001\ldots,$$

where units occupy the first, tenth, hundredth, thousandth, etc., positions after the dot, and zeros everywhere else. Prove that

(a) A is an irrational number;
(b) A^2 is an irrational number.

2.16. Prove that for any integer n,

$$\frac{n}{3} + \frac{n^2}{2} + \frac{n^3}{6}$$

is an integer.

2.17. Prove that for any positive integer n,

$$\frac{n}{6} + \frac{n^2}{2} + \frac{n^3}{3}$$

is an integer.

2.18. Prove that $\sqrt[3]{2}$ cannot be written in the form $p + q\sqrt{r}$, where p, q, and r are rational numbers.

2.19. Solve Problem 1.38 without the use of mathematical induction, i.e., prove that for any positive integer n,

$$\frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}.$$

Springer
http://www.springer.com/978-0-387-74646-3

Mathematics as Problem Solving
Soifer, A.
2009, XVIII, 106 p., Softcover
ISBN: 978-0-387-74646-3