

How Does One Cut a Triangle? I

You are familiar with a picture of three midlines drawn in an arbitrary triangle T (see Figure 2.1).

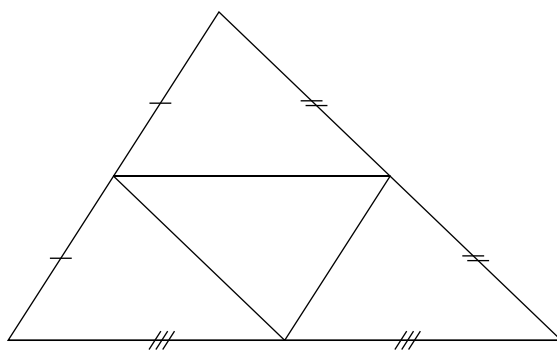


Figure 2.1

The midlines cut T into four triangles congruent to each other. You probably know that this construction can be easily generalized to all perfect squares:

For every positive integer n , every triangle T can be cut into n^2 congruent triangles.

All we have to do is partition each side of T into n segments of equal length (see Figure 2.2) and connect the corresponding marks of partitions by lines parallel to the sides of the triangle T (you should prove that we indeed get n^2 congruent triangles).

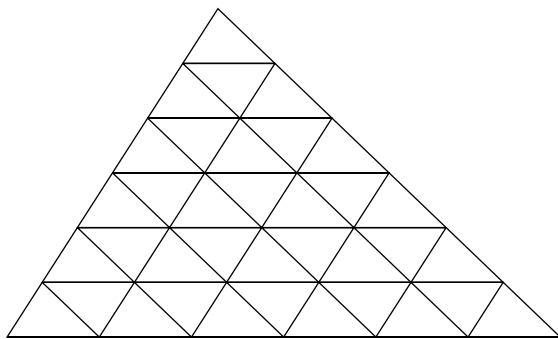


Figure 2.2

We arrive at our first major problems:

- (1) Are there any other numbers n besides the perfect squares such that every triangle can be cut into n triangles congruent to each other?
- (2) What would be the answer if we relax the requirement of congruency to similarity?

In other words:

Grand Problem I. Find all positive integers n such that every triangle can be cut into n triangles congruent to each other.

Grand Problem II. Find all positive integers n such that every triangle can be cut into n triangles similar to each other.

It appears that if anything, the first problem is easier than the second. The opposite, in fact, is true. We have all the tools necessary to solve Grand Problem II here. Problem I will have to wait until we build some additional instruments in the next chapter.

Let us *try* to solve Grand Problem II.

Figure 2.2 shows that every triangle can be cut into n^2 triangles similar to each other. By combining all $(n - 1)^2$ triangles located above the lowest cut parallel to the base, we get Figure 2.3 from Figure 2.2.

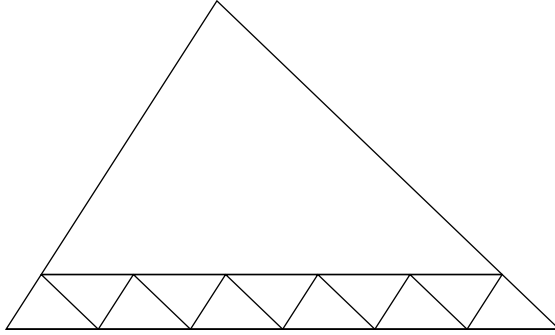


Figure 2.3

Figure 2.3 shows a partition of a triangle into $n^2 - (n - 1)^2 + 1$ triangles similar to each other, but

$$n^2 - (n - 1)^2 + 1 = 2n.$$

Therefore, every triangle can be cut into any *even* number of similar triangles greater or equal than 4.

What about odd numbers? Well, let us start with the partition of Figure 2.1 and cut one of the four triangles into $2n$ similar triangles, as in Figure 2.3. We will get $2n + 3$ similar triangles (Figure 2.4). Not only is $2n + 3$ an odd number, but every positive odd number greater than 5 can be expressed (prove it) in the form $2n + 3$, where n is an integer greater than 1.

Thus far we have proven that every triangle can be cut into n triangles similar to each other for all n except the first three prime numbers

$$n = 2, 3, 5,$$

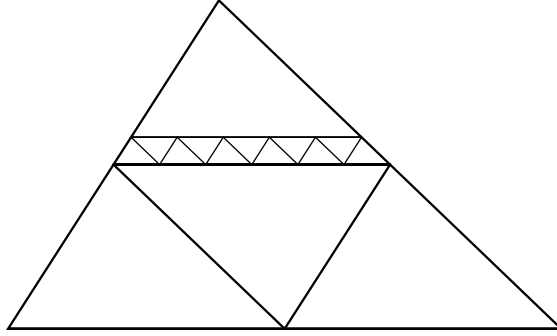


Figure 2.4

for which our constructions in Figures 2.3 and 2.4 do not work.

Will something else work?

An insightful reader may have sensed that we introduced integral independence in Chapter 1 for a reason. Indeed, *assume that a triangle T with integrally independent angles A, B, C is cut into a number of triangles with angles α, β, γ , similar to each other* (see Figure 2.5).

The partition of the triangle T , of course, induces the partitions of its angles A, B, C into the sums of angles congruent to α, β, γ (see Figure 2.5):

$$\begin{aligned} A &= a_{11}\alpha + a_{12}\beta + a_{13}\gamma \\ B &= a_{21}\alpha + a_{22}\beta + a_{23}\gamma \\ C &= a_{31}\alpha + a_{32}\beta + a_{33}\gamma, \end{aligned} \tag{8}$$

where all the coefficients a_{ij} are non-negative integers.

For $j = 1, 2, 3$ we denote $\Sigma_j = a_{1j} + a_{2j} + a_{3j}$ (in other words, Σ_1, Σ_2 , and Σ_3 are the sums of columns in the system of equalities (8)). Then by adding up the three equalities of (8) and remembering that $A + B + C = \pi$, we get:

$$\pi = \Sigma_1\alpha + \Sigma_2\beta + \Sigma_3\gamma. \tag{9}$$

Let us consider two cases:

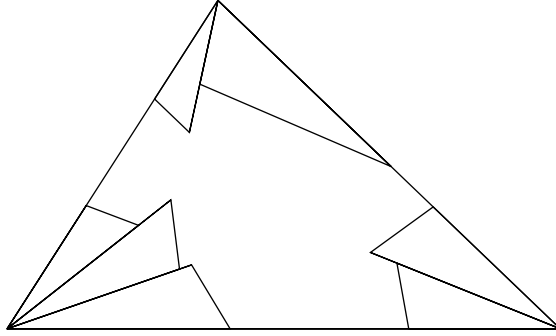


Figure 2.5

Case 1. At least one of the numbers $\Sigma_1, \Sigma_2, \Sigma_3$, say Σ_3 , is equal to zero.

$\Sigma_3 = 0$ implies $a_{13} = a_{23} = a_{33} = 0$ (since all a_{ij} are non-negative), and system (8) simplifies to coincide with system (6) in Chapter 1. Due to the statement of Problem 1.3.8, (6) implies that the numbers A, B, C are integrally dependent, which contradicts our assumption.

Therefore, the following case takes place:

Case 2. Each of the numbers $\Sigma_1, \Sigma_2, \Sigma_3$ is positive. Then $\Sigma_1\alpha + \Sigma_2\beta + \Sigma_3\gamma \geq \alpha + \beta + \gamma = \pi$, and due to equality (9), we conclude that

$$\Sigma_1 = \Sigma_2 = \Sigma_3 = 1.$$

This equality implies (show how!) that up to a permutation of symbols α, β, γ , system (8) simplifies to look as follows:

$$\begin{aligned} A &= \alpha \\ B &= \beta \\ C &= \gamma. \end{aligned} \tag{10}$$

The equalities of (10) have two very important consequences:

- (1) triangles of the partition are not only similar to each other; they are similar to the original triangle T ;

- (2) the angles of the triangle T were in fact not split; they were cut off (see Figure 2.6).

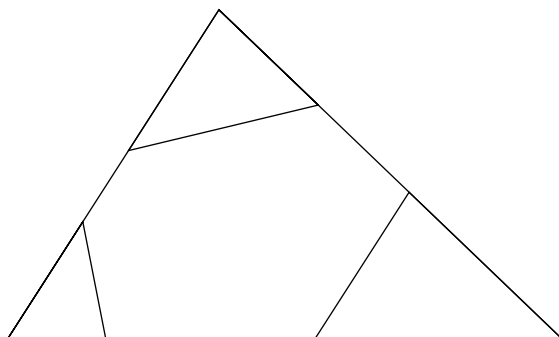


Figure 2.6

We did not solve our Grand Problem II (yet!). We did prove a powerful result, not very important by itself but instrumental in solving Grand Problem II and useful in other problems (such as Grand Problem I!).

In mathematics such results are called *lemmas*. I prefer to call them *tools*.

Tool 2.1. If a triangle T with integrally independent angles is cut into a number of triangles T_n similar to each other, then:

- (1) each triangle T_n is similar to T ;

and

- (2) the angles of T are not split, but rather cut off (as in Figure 2.6).

Tool 2.1 immediately proves that a triangle T with integrally independent angles cannot be cut into 2 or 3 similar triangles; indeed, Figure 2.6 shows that we must get at least 4 pieces.

Assume that a triangle with integrally independent angles is cut into 5 similar triangles. The middle piece of the partition in Figure

2.6 must be at most a quadrilateral, since by Tool 2.1 we already cut off the angles of T three triangles similar to T .

Thus, the middle piece is a triangle or a quadrilateral (see Figures 2.7 and 2.8).

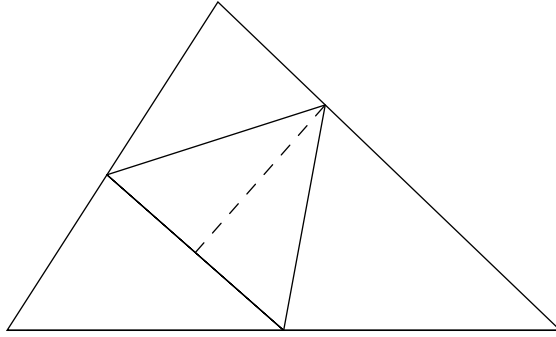


Figure 2.7

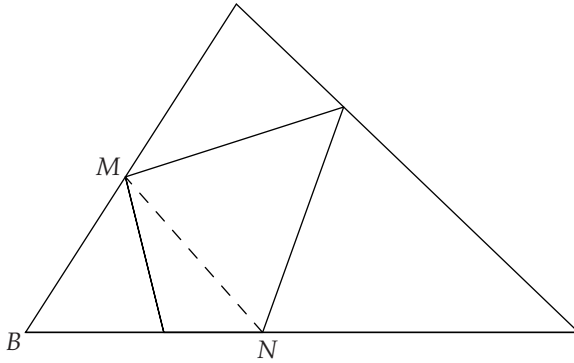


Figure 2.8

We need a little extra tool, which you can easily prove on your own:

Tool 2.2. If a triangle is cut into two triangles T_1 and T_2 similar to each other, then T_1 and T_2 are right triangles.

Now, if the middle piece were a triangle, as in Figure 2.7, then it must have been cut into two triangles T_1 and T_2 similar to each other. By Tool 2.2, T_1 and T_2 are right triangles. This means that by Tool 2.1, all five triangles of the partition, as well as the original triangle T , are right triangles. But the angles of the right triangle are integrally dependent (see Problem 1.3.2), which contradicts our assumption that the angles of T are integrally independent.

If the middle piece were a quadrilateral as in Figure 2.8, then it must have been cut into two triangles by one of its diagonals. But this diagonal (MN in Figure 2.8) cuts off the original triangle T a triangle (MNB in Figure 2.8) that is split into two triangles similar to each other. From here we travel to a contradiction by exactly the same way as in the previous case (when the middle piece was a triangle).

You, my reader, may think that Grand II is solved. In fact we “only” proved that a triangle T with integrally independent angles can not be cut into 2, 3, or 5 triangles similar to each other. But what if such a triangle T does not exist?

Cheer up: it exists! Moreover, you proved it in your home entertainment, Problem 1.3.3: the angles of measure $\sqrt{2}^\circ$, $\sqrt{3}^\circ$, and $(180 - \sqrt{2} - \sqrt{3})^\circ$ are integrally independent.

Grand II is done. *Every triangle can be cut into any number n of triangles similar to each other, except the first three primes: 2, 3, and 5.* \square

I created Grand II in April 1970, when I served as one of the judges of the Soviet Union National Mathematical Olympiad. The judges liked the problem. They selected the critical part of it for the juniors (ninth graders) competition:

Can every triangle be cut into five triangles similar to each other?

Then came the meeting to approve the problems with the Chairman of the Organizing Committee, Andrej Nikolaevich Kolmogorov, one of the greatest mathematicians of the twentieth century.

Kolmogorov quietly listened to the presentation of all the problems and their solutions, and then said:

"I would only like to replace the problem about five similar triangles."

"Why, Andrej Nikolaevich?" asked the head judge for the ninth grade, Yuri Ionin.

"It is too difficult. I am not sure I would have solved it," replied Kolmogorov.

The reason was found valid, and the problem was replaced. Of course, I was not thrilled about it. And yet, in a way, I was satisfied. The great Kolmogorov thought Grand II was a difficult problem. "What would he think then about Grand I?" I exclaimed to myself.

Nineteen years later, Grand II has finally made it into the Mathematical Olympiad of the International Summer Institute at Oakdale, Long Island, New York. The competitors included some remarkable high school students from the Soviet Union, France, Switzerland, and the United States. The winner, Vania Arzhantsev, a senior from Kiev and a candidate to the Soviet team for the 1990 International Mathematical Olympiad in Beijing, was the only one to substantially advance in the problem.

His proof has an important consequence: integral independence of angles is not a necessary condition. He used a triangle with the angles 1° , 6° , and 173° as an example of a triangle that cannot be cut into five triangles similar to each other. I will share details of this solution with you in Chapter 11.

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