2

Elementary probability

This first chapter devoted to probability theory contains the basic definitions and concepts in this field, without the formalism of *measure theory*. However, the range of problems that can be solved by using the formulas of *elementary probability* is very broad, particularly in *combinatorial analysis*. Therefore, it is necessary to do numerous exercises in order to master these basic concepts.

2.1 Random experiments

A *random experiment* is an experiment that, at least theoretically, may be repeated as often as we want and whose outcome cannot be predicted, for example, the roll of a die. Each time the experiment is repeated, an *elementary outcome* is obtained. The set of all elementary outcomes of a random experiment is called the *sample space*, which is denoted by Ω.

Sample spaces may be discrete or continuous.

(a) Discrete sample spaces. (i) Firstly, if the number of possible outcomes is finite. For example, if a die is rolled and the number that shows up is noted, then $\Omega = \{1, 2, \ldots, 6\}$.

ii) Secondly, if the number of possible outcomes is *countably infinite*, which means that there is an infinite number of possible outcomes, but these outcomes can be put in a one-to-one correspondence with the positive integers. For example, if a die is rolled until a "6" is obtained, and the number of rolls made before getting this first "6" is counted, then we have that $\Omega = \{0, 1, 2, \ldots\}$. This set is *equivalent* to the set of all natural integers $\{1, 2, \ldots\}$, because we can associate the natural number $k+1$ with each element $k = 0, 1, \ldots$ of Ω.

(b) Continuous sample spaces. If the sample space contains one or many intervals, the sample space is then *uncountably infinite*. For example, a die is rolled until a "6" is obtained and the *time* needed to get this first "6" is recorded. In this case, we have that $\Omega = \{t \in \mathbb{R} \colon t > 0\}$ [or $\Omega = (0, \infty)$].

M. Lefebvre, *Basic Probability Theory with Applications*, Springer Undergraduate Texts in Mathematics and Technology, DOI: 10.1007/978-0-387-74995-2_2,

2.2 Events

Definition 2.2.1. *An* **event** *is a set of elementary outcomes. That is, it is a subset of the sample space Ω. In particular, every elementary outcome is an event, and so is the sample space itself.*

Remarks. (i) An elementary outcome is sometimes called a *simple* event, whereas a *compound* event is made up of at least two elementary outcomes.

(ii) To be precise, we should distinguish between the elementary outcome ω, which is an element of Ω, and the *elementary event* $\{\omega\} \subset \Omega$.

(iii) The events are denoted by A, B, C, and so on.

Definition 2.2.2. *Two events, A and B, are said to be* **incompatible** *(or* **mutually exclusive***) if their intersection is empty. We then write that $A \cap B = \emptyset$.*

Example 2.2.1. Consider the experiment that consists in rolling a die and recording the number that shows up. We have that $\Omega = \{1, 2, 3, 4, 5, 6\}$. We define the events

$$A = \{1, 2, 4\}, \quad B = \{2, 4, 6\} \quad \text{and} \quad C = \{3, 5\}.$$

We have:

$$A \cup B = \{1, 2, 4, 6\}, \quad A \cap B = \{2, 4\} \quad \text{and} \quad A \cap C = \emptyset.$$

Therefore, A and C are incompatible events. Moreover, we may write that $A' = \{3, 5, 6\}$, where the symbol $'$ denotes the *complement* of the event.

To represent a sample space and some events, we often use a *Venn diagram* as in Figure 2.1. In general, for three events we have the diagram in Figure 2.2.

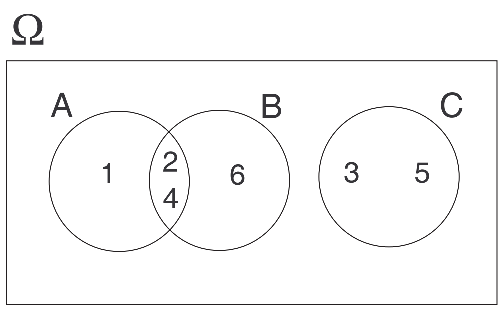

Fig. 2.1. Venn diagram for Example 2.2.1.

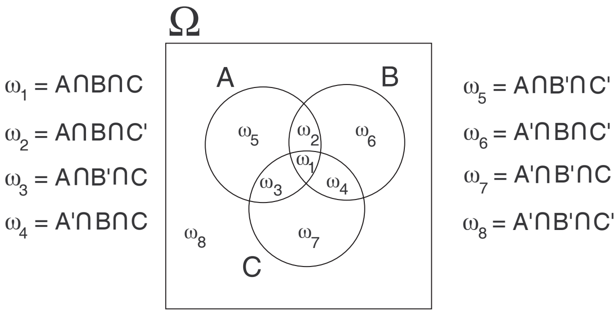

Fig. 2.2. Venn diagram for three arbitrary events.

2.3 Probability

Definition 2.3.1. *The* **probability** *of an event $A \subset \Omega$, denoted by $P[A]$, is a real number obtained by applying to A the function P that possesses the following properties:*

(i) $0 \le P[A] \le 1$;
(ii) if $A = \Omega$, *then* $P[A] = 1$;
(iii) if $A = A_1 \cup A_2 \cup \cdots \cup A_n$, *where* $A_1, \ldots, A_n$ *are incompatible events, then we may write that*

$$P[A] = \sum_{i=1}^{n} P[A_i] \quad \textit{for } n = 2, 3, \ldots, \infty.$$

Remarks. (i) Actually, we only have to write that $P[A] \ge 0$ in the definition, because we can show that

$$P[A] + P[A'] = 1,$$

which implies that $P[A] = 1 - P[A'] \le 1$.
(ii) We also have the following results:

$$P[\emptyset] = 0 \quad \text{and} \quad P[A] \le P[B] \quad \text{if } A \subset B.$$

(iii) The definition of the probability of an event is motivated by the notion of *relative frequency*. For example, suppose that the random experiment that consists in rolling a die is repeated a very large number of times, and that we wish to obtain the probability of any of the possible outcomes of this experiment, namely, the integers $1, 2, \ldots, 6$. The relative frequency of the elementary event $\{k\}$ is the quantity $f_{\{k\}}(n)$ defined by

$$f_{\{k\}}(n) = \frac{N_{\{k\}}(n)}{n},$$

where $N_{\{k\}}(n)$ is the number of times that the possible outcome k occurred among the n rolls of the die. We can write that

$$0 \leq f_{\{k\}}(n) \leq 1 \quad \text{for } k = 1, 2, \ldots, 6$$

and that

$$\sum_{k=1}^{6} f_{\{k\}}(n) = 1.$$

Indeed, we obviously have that $N_{\{k\}}(n) \in \{0, 1, \ldots, n\}$, so that $f_{\{k\}}(n)$ belongs to $[0, 1]$, and

$$\sum_{k=1}^{6} f_{\{k\}}(n) = \frac{N_{\{1\}}(n) + \cdots + N_{\{6\}}(n)}{n} = \frac{n}{n} = 1.$$

Furthermore, if A is an event containing two possible outcomes, for instance "1" and "2," then

$$f_A(n) = f_{\{1\}}(n) + f_{\{2\}}(n),$$

because the outcomes 1 and 2 cannot occur on the *same* roll of the die.

Finally, the probability of the elementary event $\{k\}$ can theoretically be obtained by taking the limit of $f_{\{k\}}(n)$ as the number n of rolls tends to infinity:

$$P[\{k\}] = \lim_{n \to \infty} f_{\{k\}}(n).$$

The probability of an arbitrary event can be expressed in terms of the relative frequency of this event, thus it is logical that the properties of probabilities more or less mimic those of relative frequencies.

Sometimes, the probability of an elementary outcome is simply equal to 1 divided by the total number of elementary outcomes. In this case, the elementary outcomes are said to be *equiprobable* (or *equally likely*). For example, if a *fair* (or *unbiased*) die is rolled, then we have that $P[\{1\}] = P[\{2\}] = \cdots = P[\{6\}] = 1/6$.

If the elementary outcomes r_i are *not* equiprobable, we can (try to) make use of the following formula:

$$P[A] = \sum_{r_i \in A} P[\{r_i\}].$$

However, this formula is only useful if we know the probability of all the elementary outcomes r_i that constitute the event A.

Now, if A and B are incompatible events, then we deduce from the third property of $P[\,\cdot\,]$ that $P[A \cup B] = P[A] + P[B]$. If A and B are not incompatible, we can show (see Figure 2.3) that

$$P[A \cup B] = P[A] + P[B] - P[A \cap B].$$

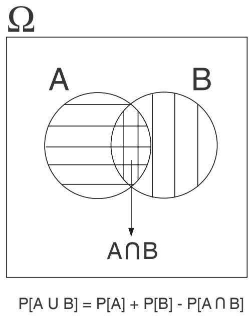

Fig. 2.3. Probability of the union of two arbitrary events.

Similarly, in the case of three arbitrary events, we have:

$$P[A \cup B \cup C] = P[A] + P[B] + P[C] - P[A \cap B] - P[A \cap C] - P[B \cap C] + P[A \cap B \cap C].$$

Example 2.3.1. The three most popular options for a certain model of new car are A: automatic transmission, B: V6 engine, and C: air conditioning. Based on the previous sales data, we may suppose that $P[A] = 0.70$, $P[B] = 0.75$, $P[C] = 0.80$, $P[A \cup B] = 0.80$, $P[A \cup C] = 0.85$, $P[B \cup C] = 0.90$, and $P[A \cup B \cup C] = 0.95$, where $P[A]$ denotes the probability that an arbitrary buyer chooses option A, and so on. Calculate the probability of each of the following events:
(a) the buyer chooses at least one of the three options;
(b) the buyer does not choose any of the three options;
(c) the buyer chooses only air conditioning;
(d) the buyer chooses exactly one of the three options.

Solution. (a) We seek $P[A \cup B \cup C] = 0.95$ (by assumption).

(b) We now seek $P[A' \cap B' \cap C'] = 1 - P[A \cup B \cup C] = 1 - 0.95 = 0.05$.

(c) The event whose probability is requested is $A' \cap B' \cap C$. We can write that

$$P[A' \cap B' \cap C] = P[A \cup B \cup C] - P[A \cup B] = 0.95 - 0.8 = 0.15.$$

(d) Finally, we want to calculate

$$\begin{aligned} &P[(A \cap B' \cap C') \cup (A' \cap B \cap C') \cup (A' \cap B' \cap C)] \\ &\qquad \overset{\text{inc.}}{=} P[A \cap B' \cap C'] + P[A' \cap B \cap C'] + P[A' \cap B' \cap C] \\ &\qquad = 3P[A \cup B \cup C] - P[A \cup B] - P[A \cup C] - P[B \cup C] \\ &\qquad = 3(0.95) - 0.8 - 0.85 - 0.9 = 0.3. \end{aligned}$$

Remarks. (i) The indication "inc." above the "=" sign means that the equality is true because of the *incompatibility* of the events. We use this type of notation often in this book to justify the passage from an expression to another.

(ii) The probability of each of the eight elementary outcomes is indicated in the diagram of Figure 2.4. First, we calculate

$$P[A \cap B] = P[A] + P[B] - P[A \cup B] = 0.7 + 0.75 - 0.8 = 0.65.$$

Likewise, we have:

$$\begin{aligned} P[A \cap C] &= 0.7 + 0.8 - 0.85 = 0.65, \\ P[B \cap C] &= 0.75 + 0.8 - 0.9 = 0.65, \\ P[A \cap B \cap C] &= P[A \cup B \cup C] - P[A] - P[B] - P[C] \\ &\quad + P[A \cap B] + P[A \cap C] + P[B \cap C] \\ &= 0.95 - 0.7 - 0.75 - 0.8 + 3(0.65) = 0.65. \end{aligned}$$

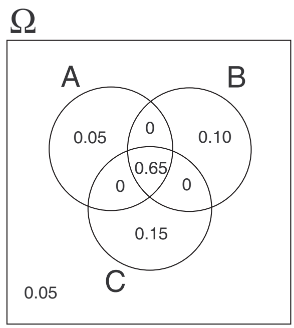

Fig. 2.4. Venn diagram for Example 2.3.1.

2.4 Conditional probability

Definition 2.4.1. *The* **conditional probability** *of event A, given that event B occurred, is defined (and denoted) by (see Figure 2.5)*

$$P[A \mid B] = \frac{P[A \cap B]}{P[B]} \quad \text{if } P[B] > 0. \tag{2.1}$$

From the above definition, we obtain the *multiplication rule*:

$$P[A \cap B] = P[A \mid B] P[B] \quad \text{if } P[B] > 0 \tag{2.2}$$

and

$$P[A \cap B] = P[B \mid A] P[A] \quad \text{if } P[A] > 0.$$

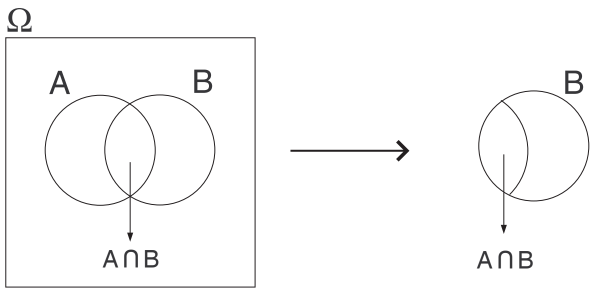

Fig. 2.5. Notion of conditional probability.

Definition 2.4.2. *Let A and B be two events such that $P[A]P[B] > 0$. We say that A and B are* **independent** *events if*

$$P[A \mid B] = P[A] \quad or \quad P[B \mid A] = P[B]. \tag{2.3}$$

We deduce from the multiplication rule that A and B are independent if and only if (iff)

$$P[A \cap B] = P[A]P[B]. \tag{2.4}$$

Actually, this equation is the definition of independence of events A and B in the general case when we can have that $P[A]P[B] = 0$. However, Definition 2.4.2 is more intuitive, whereas the general definition of independence given by Formula (2.4) is purely mathematical.

In general, the events $A_1, A_2, \ldots, A_n$ are independent iff

$$P[A_{i_1} \cap \cdots \cap A_{i_k}] = \prod_{j=1}^{k} P[A_{i_j}]$$

for $k = 2, 3, \ldots, n$, where $A_{i_l} \neq A_{i_m}$ if $l \neq m$.

Remark. If A and B are two incompatible events, then they *cannot* be independent, unless $P[A]P[B] = 0$. Indeed, in the case when $P[A]P[B] > 0$, we have:

$$P[A \mid B] = \frac{P[A \cap B]}{P[B]} = \frac{P[\emptyset]}{P[B]} = \frac{0}{P[B]} = 0 \neq P[A].$$

Example 2.4.1. A device is constituted of two components, A and B, subject to failures. The components are connected in parallel (see Figure 2.6) and are not independent. We estimate the probability of a failure of component A to be 0.2 and that of a failure of component B to be 0.8 if component A is down, and to 0.4 if component A is not down.

(a) Calculate the probability of a failure (i) of component B and (ii) of the device.

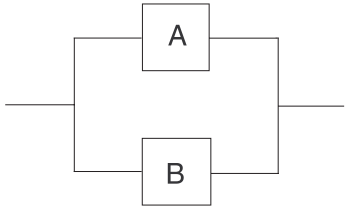

Fig. 2.6. System for part (a) of Example 2.4.1.

Solution. Let A (resp., B) be the event "component A (resp., B) is down." By assumption, we have that $P[A] = 0.2$, $P[B \mid A] = 0.8$, and $P[B \mid A'] = 0.4$.

(i) We may write (see Figure 2.7) that

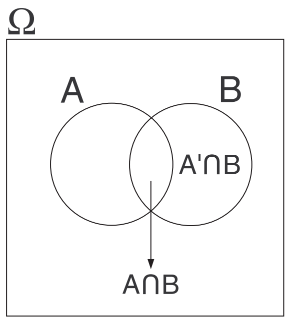

Fig. 2.7. Venn diagram for part (a) of Example 2.4.1.

$$\begin{aligned} P[B] &= P[A \cap B] + P[A' \cap B] = P[B \mid A]P[A] + P[B \mid A']P[A'] \\ &= (0.8)(0.2) + (0.4)(0.8) = 0.48. \end{aligned}$$

(ii) We seek $P[\text{Device failure}] = P[A \cap B] = P[B \mid A]P[A] = 0.16$.

(b) In order to increase the reliability of the device, a third component, C, is added to the system in such a way that components A, B, and C are connected in parallel (see Figure 2.8). The probability that component C fails is equal to 0.2, independently from the state (up or down) of components A and B. Calculate the probability that the device made up of components A, B, and C breaks down.

Solution. By assumption, $P[C] = 0.2$ and C is independent of A and B. Let F be the event "the subsystem made up of components A and B fails." We can write that

$$P[F \cap C] \overset{\text{ind.}}{=} P[A \cap B]P[C] \overset{(a)\ (ii)}{=} (0.16)(0.2) = 0.032.$$

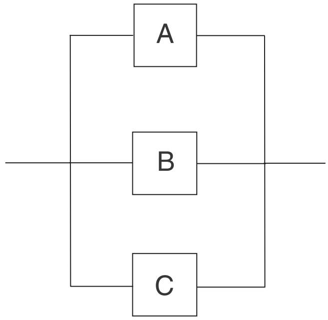

Fig. 2.8. System for part (b) of Example 2.4.1.

2.5 Total probability

Let $B_1, B_2, \ldots, B_n$ be *incompatible* and *exhaustive* events; that is, we have:

$$B_i \cap B_j = \emptyset \quad \text{if } i \neq j \quad \text{and} \quad \bigcup_{i=1}^{n} B_i = \Omega.$$

We say that the events B_i constitute a *partition* of the sample space Ω. It follows that

$$P\left[\bigcup_{i=1}^{n} B_i\right] = \sum_{i=1}^{n} P[B_i] = P[\Omega] = 1.$$

Now, let A be an arbitrary event. We can write that (see Figure 2.9)

$$P[A] = \sum_{i=1}^{n} P[A \cap B_i] = \sum_{i=1}^{n} P[A \mid B_i] P[B_i] \tag{2.5}$$

(the second equality above being valid when $P[B_i] > 0$, for $i = 1, 2, \ldots, n$).

Remark. This formula is sometimes called the *law of total probability.*

Finally, suppose that we wish to calculate $P[B_i \mid A]$, for $i = 1, \ldots, n$. We have:

$$P[B_i \mid A] = \frac{P[B_i \cap A]}{P[A]} = \frac{P[A \mid B_i] P[B_i]}{\sum_{j=1}^{n} P[A \cap B_j]} = \frac{P[A \mid B_i] P[B_i]}{\sum_{j=1}^{n} P[A \mid B_j] P[B_j]}. \tag{2.6}$$

This formula is called *Bayes' formula.*

Remark. We also have (*Bayes' rule*):

$$P[B \mid A] = \frac{P[A \mid B] P[B]}{P[A]} \quad \text{if } P[A]P[B] > 0. \tag{2.7}$$

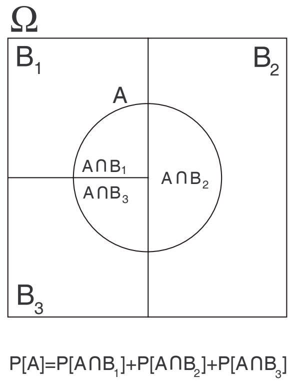

Fig. 2.9. Example of the law of total probability with $n = 3$.

Example 2.5.1. Suppose that machines M_1, M_2, and M_3 produce, respectively, 500, 1000, and 1500 parts per day, of which 5%, 6%, and 7% are defective. A part produced by one of these machines is taken at random, at the end of a given workday, and it is found to be defective. What is the probability that it was produced by machine M_3?

Solution. Let A_i be the event "the part taken at random was produced by machine M_i," for $i = 1, 2, 3$, and let D be "the part taken at random is defective." We seek

$$P[A_3 \mid D] = \frac{P[D \mid A_3]P[A_3]}{\sum_{i=1}^{3} P[D \mid A_i]P[A_i]} = \frac{(0.07)\left(\frac{1500}{3000}\right)}{(0.05)\left(\frac{1}{6}\right) + (0.06)\left(\frac{1}{3}\right) + (0.07)\left(\frac{1}{2}\right)}$$
$$= \frac{105}{190} \simeq 0.5526.$$

2.6 Combinatorial analysis

Suppose that we perform a random experiment that can be divided into two steps. On the first step, outcome A_1 *or* outcome A_2 may occur. On the second step, either of outcomes B_1, B_2, or B_3 may occur. We can use a *tree diagram* to describe the sample space of this random experiment, as in Figure 2.10.

Example 2.6.1. Tests conducted with a new breath alcohol analyzer enabled us to establish that (i) 5 times out of 100 the test proved positive even though the person subjected to the test was not intoxicated; (ii) 90 times out of 100 the test proved positive and the person tested was really intoxicated. Moreover, we estimate that 1% of the persons subjected to the test are really intoxicated. Calculate the probability that
(a) the test will be positive for the next person subjected to it;
(b) a given person is intoxicated, given that the test is positive.

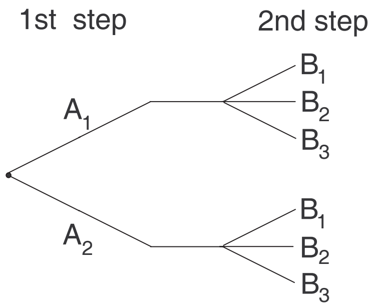

Fig. 2.10. Example of a tree diagram.

Solution. Let A be the event "the test is positive" and let E be "the person subjected to the test is intoxicated." From the above assumptions, we can construct the tree diagram in Figure 2.11, where the *marginal* probabilities of events E and E' are written above the branches, as well as the conditional probabilities of events A and A', given that E or E' occurred. Furthermore, we know by the multiplication rule that the product of these probabilities is equal to the probability of the intersections $E \cap A$, $E \cap A'$, and so on.

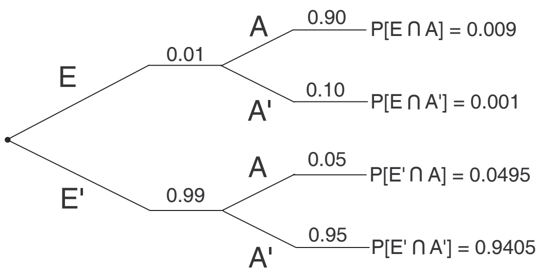

Fig. 2.11. Tree diagram in Example 2.6.1.

(a) We have:

$$P[A] = P[E \cap A] + P[E' \cap A] = 0.0585.$$

(b) We calculate

$$P[E \mid A] = \frac{P[E \cap A]}{P[A]} \overset{(a)}{=} \frac{0.009}{0.0585} \simeq 0.1538.$$

Note that this probability is very low. If we assume that 60% of the individuals subjected to the test are intoxicated (rather than 1%), then we find that $P[A]$ becomes 0.56 and $P[E \mid A] \simeq 0.9643$, which is much more reasonable. Therefore, this breath alcohol analyzer is only efficient if we use it for individuals who are suspected of being intoxicated.

Remark. In general, if a random experiment comprises k steps and if there are n_j possible outcomes on the jth step, for $j = 1, \ldots, k$, then there are $n_1 \times \cdots \times n_k$ elementary outcomes in the sample space. This is known as the *multiplication principle*.

Suppose now that we have n *distinct* objects and that we take, at random and *without* replacement, r objects among them, where $r \in \{(0,)1, \ldots, n\}$. The number of possible *arrangements* is given by

$$n \times (n-1) \times \cdots \times [n-(r-1)] = \frac{n!}{(n-r)!} := P_r^n. \tag{2.8}$$

The symbol P_r^n designates the number of *permutations* of n distinct objects taken r at a time. The *order* of the objects is important.

Remarks. (i) Reminder. We have that $n! = 1 \times 2 \times \cdots \times n$, for $n = 1, 2, 3, \ldots$, and $0! = 1$, by definition.

(ii) Taking r objects *without* replacement means that the objects are taken one at a time and that a given object cannot be chosen more than once. This is equivalent to taking the r objects all at once. In the case of sampling *with* replacement, any object can be chosen up to r times.

Example 2.6.2. If we have four different letters (for instance, a, b, c, and d), then we can form

$$P_3^4 = \frac{4!}{(4-3)!} = \frac{4!}{1!} = 24$$

different three-letter "words" if each letter is used at most once. We can use a tree diagram to draw the list of words.

Finally, if the *order* of the objects is not important, then the number of ways to take, at random and *without* replacement, r objects among n distinct objects is given by

$$\frac{n \times (n-1) \times \cdots \times [n-(r-1)]}{r!} = \frac{n!}{r!(n-r)!} := C_r^n \equiv \binom{n}{r} \tag{2.9}$$

for $r \in \{(0,)1, \ldots, n\}$. The symbol C_r^n, or $\binom{n}{r}$, designates the number of *combinations* of n distinct objects taken r at a time.

Remarks. (i) Each combination of r objects enables us to form $r!$ different permutations, because

$$P_r^r = \frac{r!}{(r-r)!} = \frac{r!}{0!} = r!.$$

(ii) Moreover, it is easy to check that $C_r^n = C_{n-r}^n$.

Example 2.6.3. Three parts are taken, at random and *without* replacement, among 10 parts, of which 2 are defective. What is the probability that at least 1 defective part is obtained?

Solution. Let F be the event "at least one part is defective among the three parts taken at random." We can write that

$$\begin{aligned} P[F] &= 1 - P[F'] = 1 - \frac{C_0^2 \cdot C_3^8}{C_3^{10}} \\ &= 1 - \frac{1 \cdot \frac{8!}{3!5!}}{\frac{10!}{3!7!}} = 1 - \frac{6 \times 7}{9 \times 10} = \frac{8}{15} = 0.5\bar{3}. \end{aligned}$$

2.7 Exercises for Chapter 2

Solved exercises

Question no. 1

We consider the following random experiment: a fair die is rolled; if (and only if) a "6" is obtained, the die is rolled a second time. How many elementary outcomes are there in the sample space Ω?

Question no. 2

Let $\Omega = \{e_1, e_2, e_3\}$, where $P[\{e_i\}] > 0$, for $i = 1, 2, 3$. How many different partitions of Ω, excluding the partition $\emptyset$, Ω can be formed?

Question no. 3

A fair die is rolled twice, independently. Knowing that an even number was obtained on the first roll, what is the probability that the sum of the two numbers obtained is equal to 4?

Question no. 4

Suppose that $P[A] = P[B] = 1/4$ and that $P[A \mid B] = P[B]$. Calculate $\mathrm{P}[A \cap B']$.

Question no. 5

A system is made up of three independent components. It operates if at least two of the three components operate. If the reliability of each component is equal to 0.95, what is the reliability of the system?

Question no. 6

Suppose that $P[A \cap B] = 1/4$, $P[A \mid B'] = 1/8$, and $P[B] = 1/2$. Calculate $P[A]$.

Question no. 7

Knowing that we obtained at least once the outcome "heads" in three independent tosses of a fair coin, what is the probability that we obtained "heads" three times?

Question no. 8

Suppose that $P[B \mid A_1] = 1/2$ and that $P[B \mid A_2] = 1/4$, where A_1 and A_2 are two equiprobable events forming a partition of Ω. Calculate $P[A_1 \mid B]$.

Question no. 9

Three horses, a, b, and c, enter in a race. If the outcome bac means that b finished first, a second, and c third, then the set of all possible outcomes is

$$\Omega = \{abc, acb, bac, bca, cab, cba\}.$$

We suppose that $P[\{abc\}] = P[\{acb\}] = 1/18$ and that each of the other four elementary outcomes has a $2/9$ probability of occurring. Moreover, we define the events

$$A = \text{“}a \text{ finishes before } b\text{”} \quad \text{and} \quad B = \text{“}a \text{ finishes before } c.\text{”}$$

(a) Do the events A and B form a partition of Ω?

(b) Are A and B independent events?

Question no. 10

Let ε be a random experiment for which there are three elementary outcomes: A, B, and C. Suppose that we repeat ε indefinitely and independently. Calculate, in terms of $P[A]$ and $P[B]$, the probability that A occurs before B.

Hints. (i) You can make use of the law of total probability.

(ii) Let D be the event “A occurs before B.” Then, we may write that

$$P[D \mid C \text{ occurs on the first repetition}] = P[D].$$

Question no. 11

Transistors are drawn at random and with replacement from a box containing a very large number of transistors, some of which are defectless and others are defective, and are tested one at a time. We continue until either a defective transistor has been obtained or three transistors in all have been tested. Describe the sample space Ω for this random experiment.

Question no. 12

Let A and B be events such that $P[A] = 1/3$ and $P[B' \mid A] = 5/7$. Calculate $P[B]$ if B is a subset of A.

Question no. 13

In a certain university, the proportion of full, associate, and assistant professors, and of lecturers is 30%, 40%, 20%, and 10% respectively, of which 60%, 70%, 90%, and 40% hold a PhD. What is the probability that a person taken at random among those teaching at this university holds a PhD?

Question no. 14

All the items in stock in a certain store bear a code made up of five letters. If the same letter is never used more than once in a given code, how many different codes can there be?

Question no. 15

A fair die is rolled twice, independently. Consider the events
$A =$ "the first number that shows up is a 6;"
$B =$ "the sum of the two numbers obtained is equal to 7;"
$C =$ "the sum of the two numbers obtained is equal to 7 or 11."

(a) Calculate $P[B \mid C]$.

(b) Calculate $P[A \mid B]$.

(c) Are A and B independent events?

Question no. 16

A commuter has two vehicles, one being a compact car and the other one a minivan. Three times out of four, he uses the compact car to go to work and the remainder of the time he uses the minivan. When he uses the compact car (resp., the minivan), he gets home before 5:30 p.m. 75% (resp., 60%) of the time. However, the minivan has air conditioning. Calculate the probability that

(a) he gets home before 5:30 p.m. on a given day;

(b) he used the compact car if he did not get home before 5:30 p.m.;

(c) he uses the minivan and he gets home after 5:30 p.m.;

(d) he gets home before 5:30 p.m. on two (independent) consecutive days and he does not use the same vehicle on these two days.

Question no. 17

Rain is forecast half the time in a certain region during a given time period. We estimate that the weather forecasts are accurate two times out of three. Mr. X goes out every day and he really fears being caught in the rain without an umbrella. Consequently, he always carries his umbrella if rain is forecast. Moreover, he even carries his umbrella one time out of three if rain is not forecast. Calculate the probability that it is raining and Mr. X does not have his umbrella.

Question no. 18

A fair die is rolled three times, independently. Let F be the event "the first number obtained is smaller than the second one, which is itself smaller than the third one." Calculate $P[F]$.

Question no. 19

We consider the set of all families having exactly two children. We suppose that each child has a 50–50 chance of being a boy. Let the events be
$A_1 =$ "both sexes are represented among the children;"
$A_2 =$ "at most one child is a girl."

(a) Are A_1 and A_2' incompatible events?

(b) Are A_1 and A_2' independent events?

(c) We also suppose that the probability that the third child of an arbitrary family is a boy is equal to 11/20 if the first two children are boys, to 2/5 if the first two children are girls, and to 1/2 in the other cases. Knowing that the third child of a given family is a boy, what is the probability that the first two are also boys?

Exercises

Question no. 1

We study the traffic (in one direction) on two roads, 1 and 2, which merge to form road 3 (see Figure 2.12). Roads 1 and 2 have the same capacity (number of lanes) and road 3 has a greater capacity than road 1 and road 2. During rush hours, the probability that the traffic is congested on road 1 (resp., road 2) is equal to 0.1 (resp., 0.3). Moreover, given that traffic is congested on road 2, it is also congested on road 1 one time out of three. We define the events

$$A, B, C = \text{“traffic is congested on roads 1, 2, 3, respectively.”}$$

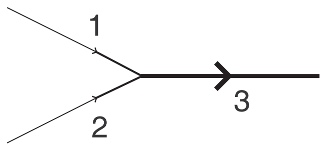

Fig. 2.12. Figure for Exercise no. 1.

(a) Calculate the probability that traffic is congested

(i) on roads 1 and 2;

(ii) on road 2, given that it is congested on road 1;

(iii) on road 1 only;

(iv) on road 2 only;

(v) on road 1 or on road 2;

(vi) neither on road 1 nor on road 2.

(b) On road 3, traffic is congested with probability

$$\begin{array}{rl} 1 & \text{if it is congested on roads 1 and 2;} \\ 0.15 & \text{if it is congested on road 2 only;} \\ 0.1 & \text{if it is neither congested on road 1 nor on road 2.} \end{array}$$

Calculate the probability that traffic is congested

(i) on road 3;

(ii) on road 1, given that it is congested on road 3.

Question no. 2

We roll a die and then we toss a coin. If we obtain "tails," then we roll the die a second time. Suppose that the die and the coin are fair. What is the probability of

(a) obtaining "heads" or a 6 on the first roll of the die;

(b) obtaining no 6s;

(c) obtaining exactly one 6;

(d) having obtained "heads," given that we obtained exactly one 6.

Question no. 3 (see Example 2.4.1)

A device is composed of two components, A and B, subject to random failures. The components are connected in parallel and, consequently, the device is down only if both components are down. The two components are not independent. We estimate that the probability of

a failure of component A is equal to 0.2;

a failure of component B is equal to 0.8 if component A is down;

a failure of component B is equal to 0.4 if component A is active.

(a) Calculate the probability of a failure
  (i) of component A if component B is down;
  (ii) of exactly one component.

(b) In order to increase the reliability of the device, a third component, C, is added in such a way that components A, B, and C are connected in parallel. The probability that component C breaks down is equal to 0.2, independently of the state (up or down) of components A and B. Given that the device is active, what is the probability that component C is down?

Question no. 4

In a factory producing electronic parts, the quality control is ensured through three tests as follows:

- each component is subjected to test no. 1;
- if a component passes test no. 1, then it is subjected to test no. 2;
- if a component passes test no. 2, then it is subjected to test no. 3;
- as soon as a component fails a test, it is returned for repair.

We define the events

$$A_i = \text{"the component fails test no. } i\text{, for } i = 1, 2, 3.\text{"}$$

From past experience, we estimate that

$$P[A_1] = 0.1, \quad P[A_2 \mid A_1'] = 0.05 \quad \text{and} \quad P[A_3 \mid A_1' \cap A_2'] = 0.02.$$

The elementary outcomes of the sample space Ω are: $\omega_1 = A_1$, $\omega_2 = A_1' \cap A_2$, $\omega_3 = A_1' \cap A_2' \cap A_3$, and $\omega_4 = A_1' \cap A_2' \cap A_3'$.

(a) Calculate the probability of each elementary outcome.

(b) Let R be the event "the component must be repaired."

(i) Express R in terms of A_1, A_2, A_3.

(ii) Calculate the probability of R.

(iii) Calculate $P[A_1' \cap A_2 \mid R]$.

(c) We test three components and we define the events

$$R_k = \text{"the } k\text{th component must be repaired, for } k = 1, 2, 3\text{" and}$$

$$B = \text{"at least one of the three components passes all three tests."}$$

We assume that the events R_k are independent.

(i) Express B in terms of R_1, R_2, R_3.

(ii) Calculate $P[B]$.

Question no. 5

Let A, B, and C be events such that $P[A] = 1/2$, $P[B] = 1/3$, $P[C] = 1/4$, and $P[A \cap C] = 1/12$. Furthermore, A and B are incompatible. Calculate $P[A \mid B \cup C]$.

Question no. 6

In a group of 20,000 men and 10,000 women, 6% of men and 3% of women suffer from a certain disease. What is the probability that a member of this group suffering from the disease in question is a man?

Question no. 7

Two tokens are taken at random and without replacement from an urn containing 10 tokens, numbered from 1 to 10. What is the probability that the larger of the two numbers obtained is 3?

Question no. 8

We consider the system in Figure 2.13. All components fail independently of each other. During a certain time period, the type A components fail with probability 0.3 and component B (resp., C) fails with probability 0.01 (resp., 0.1). Calculate the probability that the system is not down at the end of this period.

Question no. 9

A sample of size 20 is drawn (without replacement) from a lot of infinite size containing 2% defective items. Calculate the probability of obtaining at least one defective item in the sample.

Question no. 10

A lot contains 10 items, of which one is defective. The items are examined one by one, without replacement, until the defective item has been found. What is the probability that this defective item will be (a) the second item examined? (b) the ninth item examined?

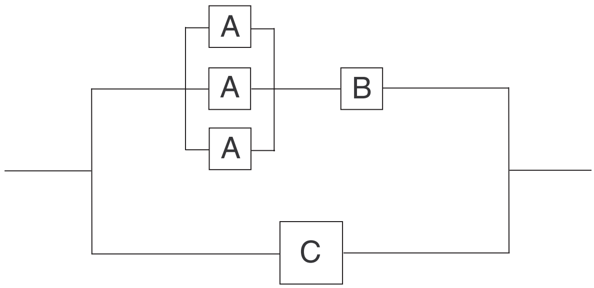

Fig. 2.13. Figure for Exercise no. 8.

Question no. 11

A bag holds two coins: a fair one and one with which we always get "heads." A coin is drawn at random and is tossed. Knowing that "heads" was obtained, calculate

(a) the probability that the fair coin was drawn;

(b) the probability of obtaining "heads" on a second toss of the same coin.

Question no. 12

The diagnosis of a physician in regard to one of her patients is unsure. She hesitates between three possible diseases. From past experience, we were able to construct the following tables:

S_i	S_1	S_2	S_3	S_4
$P[D_1 \mid S_i]$	0.2	0.1	0.6	0.4

S_i	S_1	S_2	S_3	S_4
$P[D_2 \mid S_i]$	0.2	0.5	0.5	0.3

S_i	S_1	S_2	S_3	S_4
$P[D_3 \mid S_i]$	0.6	0.3	0.1	0.2

where the D_is represent the diseases and the S_is are the symptoms. In addition, we assume that the four symptoms are incompatible, exhaustive, and equiprobable.

(a) Independently of the symptom present in the patient, what is the probability that he or she suffers from the first disease?

(b) What is the probability that the patient suffers from the second disease and presents symptom S_1?

(c) Given that the patient suffers from the third disease, what is the probability that he or she presents symptom S_2?

(d) We consider two independent patients. What is the probability that they do not suffer from the same disease, if we assume that the three diseases are incompatible?

Question no. 13

We consider a system comprising four components operating independently of each other and connected as in Figure 2.14.

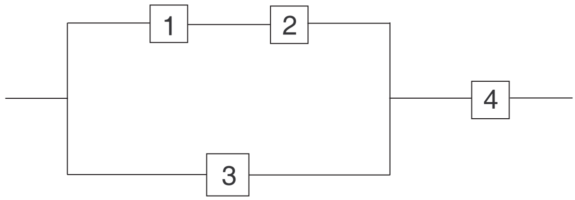

Fig. 2.14. Figure for Exercise no. 13.

The reliability of each component is supposed constant, over a certain time period, and is given by the following table:

Component	1	2	3	4
Reliability	0.9	0.95	0.95	0.99

(a) What is the probability that the system operates at the end of this time period?

(b) What is the probability that component no. 3 is down and the system still operates?

(c) What is the probability that at least one of the four components is down?

(d) Given that the system is down, what is the probability that it will resume operating if we replace component no. 1 by an identical (nondefective) component?

Question no. 14

A box contains 8 brand A and 12 brand B transistors. Two transistors are drawn at random and without replacement. What is the probability that they are both of the same brand?

Question no. 15

What is the reliability of the system shown in Figure 2.15 if the four components operate independently of each other and all have a reliability equal to 0.9 at a given time instant?

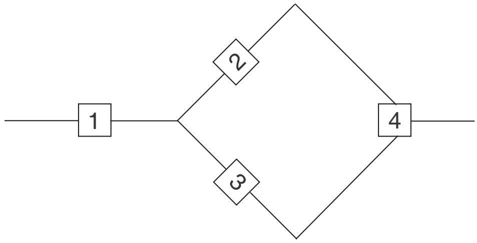

Fig. 2.15. Figure for Exercise no. 15.

Question no. 16

Let A_1 and A_2 be two events such that $P[A_1] = 1/4$, $P[A_1 \cap A_2] = 3/16$, and $P[A_2 \mid A_1'] = 1/8$. Calculate $P[A_2']$.

Question no. 17

A fair coin is tossed until either "heads" is obtained or the total number of tosses is equal to 3. Given that the random experiment ended with "heads," what is the probability that the coin was tossed only once?

Question no. 18

In a room, there are four 18-year-old male students, six 18-year-old female students, six 19-year-old male students, and x 19-year-old female students. What must be the value of x, if we want age and sex to be independent when a student is taken at random in the room?

Question no. 19

Stores S_1, S_2, and S_3 of the same company have, respectively, 50, 70, and 100 employees, of which 50%, 60%, and 75% are women. A person working for this company is taken at random. If the employee selected is a woman, what is the probability that she works in store S_3?

Question no. 20

Harmful nitrogen oxides constitute 20%, in terms of weight, of all pollutants present in the air in a certain metropolitan area. Emissions from car exhausts are responsible for 70% of these nitrogen oxides, but for only 10% of all the other air pollutants. What percentage of the total pollution for which emissions from car exhausts are responsible are harmful nitrogen oxides?

Question no. 21

Three machines, M_1, M_2, and M_3, produce, respectively, 1%, 3%, and 5% defective items. Moreover, machine M_1 produces twice as many items on an arbitrary day as machine M_2, which itself produces three times as many items as machine M_3. An item is taken at random among those manufactured on a given day, then a second item is taken at random among those manufactured by the machine that produced the first selected item. Knowing that the first selected item is defective, what is the probability that the second one is also defective?

Question no. 22

A machine is made up of five components connected as in the diagram of Figure 2.16. Each component operates with probability 0.9, independently of the other components.

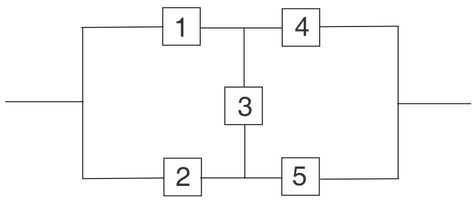

Fig. 2.16. Figure for Exercise no. 22.

(a) Knowing that component no. 1 is down, what is the probability that the machine operates?

(b) Knowing that component no. 1 is down and that the machine still operates, what is the probability that component no. 3 is active?

Question no. 23

Before being declared to conform to the technical norms, devices must pass two quality control tests. According to the data gathered so far, 75% of the devices tested in the course of a given week passed the first test. The devices are subjected to the second test, whether they pass the first test or not. We found that 80% of the devices that passed the second test had also passed the first one. Furthermore, 20% of those that failed the second test had passed the first one.

(a) What is the probability that a given device passed the second test?

(b) Find the probability that, for a given device, the second test contradicts the first one.

(c) Calculate the probability that a given device failed the second test, knowing that it passed the first one.

Question no. 24

In a certain workshop, the probability that a part manufactured by an arbitrary machine is nondefective, that is, conforms to the technical norms, is equal to 0.9. The quality control engineer proposes to adopt a procedure that classifies as nondefective with probability 0.95 the parts indeed conforming to the norms, and with only probability 0.15 those not conforming to these norms. It is decided that every part will be subjected to this quality control procedure twice, independently.

(a) What is the probability that a part having passed the procedure twice does indeed conform to the norms?

(b) Suppose that if a part fails the first control test, then it is withdrawn immediately. Let B_j be the event "a given part passed (if the case may be) the jth control test," for $j = 1, 2$. Calculate (i) $P[B_2]$ and (ii) $P[B_1' \cap B_2']$.

Question no. 25

We have 20 type I components, of which 5 are defective, and 30 type II components, of which 15 are defective.

(a) We wish to construct a system comprising 10 type I components and 5 type II components connected in series. What is the probability that the system will operate if the components are taken at random?

(b) How many different systems comprising four components connected in series, of which at least two are of type I, can be constructed, if the order of the components is taken into account?

Remarks. (i) We suppose that we can differentiate two components of the same type.

(ii) When a system is made up of components connected *in series*, then it operates if and only if every component operates.

Question no. 26

A system is made up of n components, including components A and B.

(a) Show that if the components are connected in series, then the probability that there are exactly r components between A and B is given by

$$\frac{2(n-r-1)}{(n-1)n} \quad \text{for } r \in \{0, 1, \ldots, n-2\}.$$

(b) Calculate the probability that there are exactly r components between A and B if the components are connected in circle.

(c) Suppose that $n = 5$ and that the components are connected in series. Calculate the probability of operation of the subsystem constituted of components A, B and the r components placed between them if the components operate independently of each other and all have a reliability of 0.95.

Question no. 27

A man owns a car and a motorcycle. Half the time, he uses his motorcycle to go to work. One-third of the time, he drives his car to work and, the remainder of the time, he uses public transportation. He gets home before 5:30 p.m. 75% of the time when he uses his motorcycle. This percentage is equal to 60% when he drives his car and to 2% when he uses public transportation. Calculate the probability that

(a) he used public transportation if he got home after 5:30 p.m. on a given day;

(b) he got home before 5:30 p.m. on two consecutive (independent) days and he used public transportation on exactly one of these two days.

Question no. 28

In a certain airport, a shuttle coming from the city center stops at each of the four terminals to let passengers get off. Suppose that the probability that a given passenger gets off at a particular terminal is equal to 1/4. If there are 20 passengers using the shuttle and if they occupy seats numbered from 1 to 20, what is the probability that the passengers sitting in seats nos. 1 to 4 all get off

(a) at the same stop?

(b) at different stops?

Question no. 29

A square grid consists of 289 points. A particle is at the center of the grid. Every second, it moves at random to one of the four nearest points from the one it occupies. When the particle arrives at the boundary of the grid, it is absorbed.

(a) What is the probability that the particle is absorbed after eight seconds?

(b) Let A_i be the event "the particle is at the center of the grid after i seconds." Calculate $P[A_4]$ (knowing that A_0 is certain, by assumption).

Question no. 30

Five married couples bought 10 tickets for a concert. In how many ways can they sit (in the same row) if

(a) the five men want to sit together?

(b) the two spouses in each couple want to sit together?

Multiple choice questions

Question no. 1

Two weeks prior to the most recent general election, a poll conducted among 1000 voters revealed that 48% of them intended to vote for the party in power. A survey made after the election, among the same sample of voters, showed that 90% of the persons who indeed voted for the party in power intended to vote for this party, and 10% of those who voted for another party intended (two weeks prior to the election) to vote for the party in power. Let the events be

A = "a voter, taken at random in the sample, intended to vote for the party in power;"

B = "a voter, taken at random in the sample, voted for the party in power."

(A) From the statement of the problem, we can write that $P[A] = 0.48$ and that

(a) $P[A \cap B] = 0.9$; $P[A \cap B'] = 0.1$
(b) $P[B \mid A] = 0.9$; $P[B' \mid A] = 0.1$
(c) $P[A \mid B] = 0.9$; $P[A \mid B'] = 0.1$
(d) $P[A' \cap B] = 0.9$; $P[A \cap B'] = 0.1$
(e) $P[A \mid B] = 0.9$; $P[B' \mid A] = 0.1$

(B) The probability of event B is given by

(a) 0.45 (b) 0.475 (c) 0.48 (d) 0.485 (e) 0.50 (f) 0.515

(C) Are events A and B' incompatible?

(a) yes (b) no (c) we cannot conclude from the information provided

(D) Are events A and B' independent?

(a) yes (b) no (c) we cannot conclude from the information provided

(E) Do events A and B' form a partition of the sample space Ω?

(a) yes (b) no (c) we cannot conclude from the information provided

(F) Let E be "a voter, taken at random among the 1000 voters polled, did not vote as he intended to two weeks prior to the election (in regard to the party in power)." We can write that

(a) $P[E] = P[A \mid B'] + P[A' \mid B]$
(b) $P[E] = P[A \cap B] + P[A' \cap B']$
(c) $P[E] = P[B' \mid A] + P[B \mid A']$
(d) $P[E] = P[A \cap B'] + P[A' \cap B']$
(e) $P[E] = P[A \cap B'] + P[A' \cap B]$

Question no. 2

Let A and B be two events such that

$$P[A \cap B] = P[A' \cap B] = P[A \cap B'] = p.$$

Calculate $P[A \cup B]$.

(a) p (b) $2p$ (c) $3p$ (d) $3p^2$ (e) p^3

Question no. 3

We have nine electronic components, of which one is defective. Five components are taken at random to construct a system in series. What is the probability that the system does not operate?

(a) 1/3 (b) 4/9 (c) 1/2 (d) 5/9 (e) 2/3

Question no. 4

Two dice are rolled simultaneously. If a sum of 7 or 11 is obtained, then a coin is tossed. How many elementary outcomes [of the form (die1, die2) or (die1, die2, coin)] are there in the sample space Ω?

(a) 28 (b) 30 (c) 36 (d) 44 (e) 72

Question no. 5

Let A and B be two independent events such that $P[A] < P[B]$, $P[A \cap B] = 6/25$, and $P[A \mid B] + P[B \mid A] = 1$. Calculate $P[A]$.

(a) 1/25 (b) 1/5 (c) 6/25 (d) 2/5 (e) 3/5

Question no. 6

In a certain lottery, 4 balls are drawn at random and without replacement among 25 balls numbered from 1 to 25. The player wins the grand prize if the 4 balls that she selected are drawn in the order indicated on her ticket. What is the probability of winning the grand prize?

(a) $\dfrac{1}{12,650}$ (b) $\dfrac{24}{390,625}$ (c) $\dfrac{1}{303,600}$ (d) $\dfrac{1}{390,625}$ (e) $\dfrac{1}{6,375,600}$

Question no. 7

New license plates are made up of three letters followed by three digits. If we suppose that the letters I and O are not used and that no plates bear the digits 000, how many different plates can there be?

(a) $24^3 \times 9^3$ (b) $(26 \times 25 \times 24)\,(10 \times 9 \times 8)$ (c) $24^3 \times (10 \times 9 \times 8)$
(d) $24^3 \times 999$ (e) $25^3 \times 9^3$

Question no. 8

Let $P[A \mid B] = 1/2$, $P[B'] = 1/3$, and $P[A \cap B'] = 1/4$. Calculate $P[A]$.

(a) 1/4 (b) 1/3 (c) 5/12 (d) 1/2 (e) 7/12

Question no. 9

In the lottery known as 6/49, first 6 balls are drawn at random and without replacement among 49 balls numbered from 1 to 49. Next, a seventh ball (the *bonus number*) is drawn at random among the 43 remaining balls. A woman selected what she thinks would be the six winning numbers for the next draw. What is the probability that this woman actually did not select any of the seven balls that will be drawn (including the bonus number)?

(a) $\dfrac{\binom{42}{6}}{\binom{49}{6}}$ (b) $\dfrac{\binom{42}{7}}{\binom{49}{6}}$ (c) $\dfrac{\binom{42}{6}}{\binom{49}{7}}$ (d) $\dfrac{\binom{43}{6}}{\binom{49}{7}}$ (e) $\dfrac{\binom{42}{7}}{\binom{49}{7}}$

Question no. 10

In a quality control procedure, every electronic component manufactured is subjected to (at most) three tests. After the first test, an arbitrary component is classified as either "good," "average," or "defective," and likewise after the second test. Finally, after the last test, the components are classified as either "good" or "defective." As soon as a component is classified as defective after a test, it is returned to the factory for repair. The following random experiment is performed: a component is taken at random and the result of each test it is subjected to is recorded. How many elementary outcomes are there in the sample space Ω?

(a) 3 (b) 8 (c) 11 (d) 18 (e) 21

Question no. 11

Let $P[A] = 1/3$, $P[B] = 1/2$, $P[C] = 1/4$, $P[A \mid B] = 1/2$, $P[B \mid A] = 3/4$, $P[A \mid C] = 1/3$, $P[C \mid A] = 1/4$, and $P[B \cap C] = 0$. Calculate the probability $P[A \mid B \cup C]$.

(a) 0 (b) 1/3 (c) 4/9 (d) 5/6 (e) 1

Question no. 12

A fair die is rolled twice (independently). Consider the events
$A =$ "the two numbers obtained are different;"
$B =$ "the first number obtained is a 6;"
$C =$ "the two numbers obtained are even."
Which pairs of events are the only ones comprised of independent events?

(a) no pairs (b) (A, B) (c) (A, B) and (B, C) (d) (A, B) and (A, C)
(e) the three pairs

Question no. 13

A man plays a series of games for which the probability of winning a given game, from the second one, is equal to 3/4 if he won the previous game and to 1/4 otherwise. Calculate the probability that he wins games nos. 2 and 3 consecutively if the probability that he wins the first game is equal to 1/2.

(a) 3/16 (b) 1/4 (c) 3/8 (d) 9/16 (e) 5/8

Question no. 14

A box contains two coins, one of them being fair but the other one having two "heads." A coin is taken at random and is tossed twice, independently. Calculate the probability that the fair coin was selected if "heads" was obtained twice.

(a) 1/5 (b) 1/4 (c) 1/3 (d) 1/2 (e) 3/5

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