Preface

For every mathematician, ring theory and K-theory are intimately connected: algebraic K-theory is largely the K-theory of rings. At first sight, polytopes, by their very nature, must appear alien to surveyors of this heartland of algebra.

But in the presence of a discrete structure, polytopes define affine monoids, and, in their turn, affine monoids give rise to monoid algebras. Their spectra are the building blocks of toric varieties, an area that has developed rapidly in the last four decades.

From a purely systematic viewpoint, "monoids" should therefore replace "polytopes" in the title of the book. However, such a change would conceal the geometric flavor that we have tried to preserve through all chapters.

Before delving into a description of the contents we would like to mention three general features of the book: (1) the exhibiting of interactions of convex geometry, ring theory, and K-theory is not the only goal; we present some of the central results in each of these fields; (2) the exposition is of constructive (i. e., algorithmic) nature at many places throughout the text—there is no doubt that one of the driving forces behind the current popularity of combinatorial geometry is the quest for visualization and computation; (3) despite the large amount of information from various fields, we have strived to keep the polytopal perspective as the major organizational principle.

We start with polyhedra and triangulations (Chapter 1), which is the minimum of the structures, considered in the book. Important aspects of this chapter are the duality of cones and regular subdivisions of polytopes.

The interaction of convex bodies with the lattice of integer points quickly leads to normal affine monoids and Hilbert bases of rational cones, dealt with in Chapter 2. In the same chapter we discuss naive geometric characterizations of Hilbert bases that had been an open problem until a counterexample was found in 1998. Nevertheless there remain difficult problems in this area.

Chapter 3 presents one of the pearls in the theory of lattice polytopes—the Knudsen–Mumford theorem on unimodular triangulations of high multiples of lattice polytopes. It dates back to the origins of toric geometry in the early 1970s. Being conceived in the context of semistable reductions, it remained largely unnoticed by the combinatorial world. The situation changed when Sturmfels found the link between (unimodular) triangulations and Gröbner bases. In an attempt to develop an effective variant of the Knudsen–Mumford result, we derive dimensionally uniform polynomial bounds for unimodularly covered multiples of lattice polytopes, together with a companion result for rational cones.

Part II puts the affine monoids in their natural habitat of monoid algebras, where they have settled since the days of Gordan and Hilbert. Chapter 4 develops the basic theory. Already here one encounters divisor class groups and Picard groups—algebraic invariants outside the category of bare monoids. The discussion of seminormal monoid rings, prepared on the monoid level in Chapter 2, points forward to K-theory in Part III.

In Chapter 5 it is shown that an affine monoid ring remembers its combinatorial genesis: the underlying monoid is uniquely determined by the algebra structure of its monoid ring. In the same chapter we compute the group of graded automorphism of a normal affine monoid ring. Informally, these linear groups relate to the general linear groups in the same way that arbitrary lattice polytopes relate to unimodular simplices. The basic tools in this chapter come from the theory of linear algebraic groups, and Borel's theorem on maximal tori is used in a crucial way.

The origin of combinatorial commutative algebra is Stanley's very successful attempt to base the enumerative theory of linear diophantine systems and the Ehrhart theory of lattice polytopes on Hilbert functions of graded rings. A key theorem is Hochster's result that normal affine monoid algebras are Cohen–Macaulay. Our treatment in Chapter 6 is based on a minimum of homological algebra. For lack of space we have omitted local cohomology; in view of the extensive treatment in the books by Bruns and Herzog [72] and Miller and Sturmfels [261] this omission seems acceptable.

Chapter 7 develops Gröbner basis theory of toric ideals. It moves regular subdivisions of lattice polytopes center stage. Via degrees of (unimodular) triangulations and initial ideals it is related to the Koszul property, another major theme of the chapter. We do not go into finite free resolutions of toric rings, as this topic is extensively covered in [261].

Part III is devoted to K-theory. While the previous chapters consider affine monoid rings as graded objects, we now turn to projective modules that are invisible in the essentially local structure imposed by the grading—should nontrivial such modules exist at all. By a theorem of the second author, nontrivial (nongraded) projective modules do indeed not exist over affine monoid rings, provided the weakest necessary condition, namely seminormality, is satisfied. The proof of this result, based on the Quillen–Suslin theorem, is the main subject of Chapter 8. It is accompanied by a discussion of several related results, in particular the K_0-regularity of affine monoid rings.

In sharp contrast to the behavior of K_0, the K_1-groups of affine monoid rings, are much larger than the K_1-groups of their rings of coefficients, at least in a very large class. This is proved in Chapter 9, together with several accompanying topics, like the action of the general linear group on unimodular rows, and a survey of the nilpotence of higher K-groups over affine monoid rings. All the K-theoretical background for Chapters 8 and 9, except several short digressions into higher K-theory, is found in the classical books by Bass [19] and Milnor [262].

In the book there are two places of paradigmatic shift: from polytopes to rings in Chapter 4 and from rings to nonaffine varieties in Chapter 10. In the latter we have restricted ourselves to invariants reminiscent of those studied in the preceding chapters.

Namely, in the first part we survey Grothendieck groups, Chow groups, and intersection theory, and their rich interaction with each other via Grothendieck–Riemann–Roch and with the combinatorial structure of toric varieties. In full treatment, these topics would constitute a book of its own. Our survey, apart from original papers, draws from such well-known sources as Fulton [133], [134], and Oda [275].

In the second part of Chapter 10 two topics that extend the K-theory of Chapters 8 and 9 directly are highlighted by a full discussion: the existence of simplicial projective toric varieties with "huge" K_0-groups, and the triviality of equivariant vector bundles on a representation space of an abelian group.

Each chapter is accompanied by exercises. Their degrees of difficulty vary considerably, ranging from mere tests on the basic notions to guided tours into research. Some of the exercises will be used in the text. For such we have included extensive hints or pointers to the literature.

This book has many facets and draws from many sources. A priori it had been clear to us that a completely self-contained treatment was impossible. The theory of polyhedra and affine monoids is developed from scratch, while for ring theory and K-theory, let alone algebraic geometry, some prerequisites are necessary. For ring theory we expect some fluency, acquired by a study of an introductory text like Atiyah and Macdonald [10]. For projective modules (over the type of rings of our interest) we recommend Lam's book [237]. Sources for the background material in K-theory, toric varieties, and intersection theory have been mentioned above. At many places we have inserted small introductions to the concepts needed, citing the main theorems on which we build. This is also true for Chapter 10, but the reader needs a certain acquaintance with algebraic geometry, in the context of the first three chapters of Hartshorne [178].

The idea of writing this book came up in 2001 during the preparation of notes for the lecture series given by the first author at the summer school on toric geometry in Grenoble 2000 [64]. The actual implementation started in Spring 2003 when both authors were visiting Mathematical Sciences Research Institute in Berkeley for a semester-long program in commutative algebra. Over the years work on the book was supported by grants from DFG, DAAD, NSF and by the RiP program of the Mathematisches Forschungsinstitut Oberwolfach. The Universität Osnabrück and San Francisco State University hosted the respective authors several times. We thank all these institutions for their generous support.

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