Preface

The aim of this book is to present the basic modern aspects of nonlinear analysis and then to illustrate their use in different applied problems.

Nonlinear analysis was born from the need to deal with nonlinear equations which arise in various problems of science, engineering, and economics and which are often notoriously difficult to solve. On a theoretical level, nonlinear analysis is a remarkable mixture of various areas of mathematics such as topology, measure theory, functional analysis, nonsmooth analysis, and multivalued analysis. On an applied level, nonlinear analysis provides the necessary tools to formulate and study realistic and accurate models describing various phenomena in different areas of physical sciences, engineering and economics. For this reason, the theoretically inclined nonmathematician (physicist, engineer, or economist), needs to have a working knowledge of at least some of the basic aspects of nonlinear analysis. This knowlegde can help him build good models for the phenomena he studies, study them in detail and extract from them important information which is crucial to the design process. As a consequence, nonlinear analysis has acquired an interdisciplinary character and is a prerequisite for many nonmathematicians, who wish to investigate their problems in detail with the greatest possible generality. This leads to a continuously increasing need for books that survey this large area of mathematical analysis and present its applications.

There should be no misunderstanding. The subject is vast, it touches many different areas of mathematics and its applications cover several other fields in science and engineering. In this volume, we make an effort to present the basic theoretical aspects and the main applications of nonlinear analysis. Of course the treatment is not exhaustive; such a project would require several volumes. Nevertheless, we believe that we touch the main parts of the theory and of the applications. Mathematicians and nonmathematicians alike can find in this volume material that covers their interests and can be useful in their research and/or teaching.

Chapter 1 begins with the calculus of smooth and nonsmooth functions. We present the Gâteaux and Fréchet derivatives and develop their calculus in full detail. In the direction of nonsmooth functions, first we deal with convex functions for which we develop a duality theory and a theory of subdifferentiation. Subsequently, we generalize to locally Lipschitz functions (Clarke's theory). We also introduce and study related geometrical concepts (such as tangent and normal cones) for various kinds of sets. Finally we investigate a kind of variational convergence of functions, known as Γ-convergence, which is suitable in the stability (sensitivity) analysis of variational problems.

In Chapter 2, we use the tools of the previous chapter in order to study extremal and optimal control problems. We begin with a detailed study of the notion of lower semicontinuity of functions. Next we examine constrained minimization problems and develop the method of Lagrange multipliers. This leads to mimimax theorems, saddle points and the theory of KKM-multimaps. Section 2.4 deals with some modern aspects of the direct method, which involve the so-called variational principles, central among them being the so-called "Ekeland variational principle". The last two sections deal with the calculus of variations and optimal control. In optimal control, we focus on existence theorems, relaxation and the necessary conditions for optimality (Pontryagin's maximum principle).

Chapter 3 deals with some important families of nonlinear maps and examines their uses in fixed point theory. We start with compact and Fredholm operators which are the natural generalizations of finite rank maps. Subsequently we pass to operators of monotone type and to accretive operators. Monotone operators exhibit remarkable surjectivity properties which play a central role in the existence theory of nonlinear boundary value problems. Accretive operators are closely related to the generation theory of linear and nonlinear semigroups. We investigate this connection. Then we introduce the Brouwer degree (finite-dimensional) and the Leray–Schauder and Browder–Skrypnik degrees (the latter for operators of monotone type) which are infinite-dimensional. Having these degree maps, we can move to the fixed point theory. We deal with metric fixed points, topological fixed points and investigate the interplay between order and fixed point theory.

Chapter 4 presents the main aspects of critical point theory which is the basic tool in the so-called "variational method" in the study of nonlinear boundary value problems. We start with minimax theorems describing the critical values of a C^1-functional. Then we present the Ljusternik–Schnirelmann theory for multiple critical points of nonlinear homogeneous maps. This way we have all the necessary tools to develop the spectral properties of the Laplacian and of the p-Laplacian (under Dirichlet, Neumann and periodic boundary conditions). Then using the Lagrange multipliers method we deal with abstract eigenvalue problems. Finally we present some basic notions and results from bifurcation theory.

Chapter 5 uses the tools developed in Chapters 3 and 4 in order to study nonlinear boundary value problems (involving ordinary differential equations and elliptic partial differential equations). First we illustrate the variational method based on the minimax principles of critical point theory and then we present the method of upper and lower solutions and the degree-theoretic method. Subsequently we consider nonlinear eigenvalue problems, for which we produce constant sign and nodal (sign changing) solutions. Then we prove maximum and comparison principles involving the Laplacian and p-Laplacian differential operators. Finally we deal with periodic Hamiltonian systems. We consider the problem of prescribed minimal period and the problem of a prescribed energy level. For both we prove existence theorems.

In Chapter 6, we deal with the properties of maps which have as values sets (multifunctions or set-valued maps). We introduce and study their continuity (Section 6.1) and measurability (Section 6.2) properties. Then for such multifunctions (continuous or measurable), we investigate whether they admit continuous or measurable selectors (Michael's theorem and Kuratowski–Ryll Nardzewski, and the Yankov–von Neuman–Aumann selection theorems). This leads to the study of the sets of integrable selectors of a multifunction, which in turn permits a detailed set-valued integration. The notion of decomposability (an effective substitute of convexity) plays a central role in this direction. Then we prove fixed point theorems for multifunctions and also study Carathéodory multifunctions. Finally we introduce and study various notions of convergence of sets that arise naturally in applications.

In Chapter 7, we consider applications to problems of mathematical economics. We consider both static and dynamic models. We start with the static model of an exchange economy. Assuming that perfect competition prevails, which is modelled by a continuum (nonatomic measure space) of agents. We prove a "core Walras equivalence theorem" and we also establish the existence of Walras allocations. We then turn our attention to growth models (dynamic models). First we deal with an infinite horizon, discrete-time, multisector growth model and we establish the existence of optimal programs for both discounted and undiscounted models. For the latter, we use the notion of "weak maximality". Then we determine the asymptotic properties of optimal programs via weak and strong turnpike theorems. We then examine uncertain growth models and optimal programs for both nonstationary discounted and stationary undiscounted models. We also characterize them using a price system. Continuous-time discounted models are then considered, and finally we characterize choice behavior consistent with the "Expected Utility Hypothesis".

Chapter 8 deals with deterministic and stochastic games, which provide a substantial amount of generalization of some of the notions considered in the previous chapter. We start with noncooperative n-players games, for which we introduce the notion of "Nash equilibrium". We show the existence of such equilibria. Then we consider cooperative n-players games, for which we define the notion of "core" and show its nonemptiness. We continue with random games with a continuum of players and an infinite-dimensional strategy space. For such games, we prove the existence of "Cournot–Nash equilibria". We also study corresponding Bayesian games. Subsequently, using the formalism of dynamic programming, we consider stochastic, 2-player, zero-sum games. Finally, using approximate subdifferentials for convex function, we produce approximate Nash equilibria for noncooperative games with noncompact strategy sets.

Chapter 9 studies how information can be incorporated as a variable in various decision models (in particular in ones with asymmetric information structure). First we present the mathematical framework, which will allow the analytical treatment of the notion of information. For this purpose, we define two comparable metric topologies, which we study in detail. Then we examine the ex-post view and the ex-ante view, in the modelling of systems with uncertainty. In both cases we establish the continuity of the model in the information variable. Subsequently, we introduce a third mode of convergence of information and study prediction sequences. We also study games with incomplete information and games with a general state space and an unbounded cost function.

The final chapter (Chapter 10) deals with evolution equations and the mathematical tools associated with them. These tools are developed in the first section and central among them is the notion of "evolution triple". We consider semilinear evolutions, which we study using the semigroup method. We then move on to nonlinear evolutions. We consider evolutions driven by subdifferential operators (this class of problems incorporates variational inequalities) and problems with operators of monotone type, defined within the framework of an evolution triple. The first class is treated using nonlinear semigroup theory, while the second requires Galerkin approximations. We conclude with an analogous study of second-order evolutions.

The treatment of all subjects is rigorous and every chapter ends with an extensive survey of the literature.

We hope that both mathematicians and nonmathematicians alike, will find some interesting and useful for their needs in this volume.

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