

# Lecture 2

## Second-Order Differential Equations

Generally, second-order differential equations with variable coefficients cannot be solved in terms of the known functions. In this lecture we shall show that if one solution of the homogeneous equation is known, then its second solution can be obtained rather easily. Further, by employing the method of variation of parameters, the general solution of the nonhomogeneous equation can be constructed provided two solutions of the corresponding homogeneous equation are known.

**Homogeneous equations.** For the homogeneous linear DE of second-order with variable coefficients

$$y'' + p_1(x)y' + p_2(x)y = 0, \quad (2.1)$$

where  $p_1(x)$  and  $p_2(x)$  are continuous in  $J$ , there does not exist any method to solve it. However, the following results are well-known.

**Theorem 2.1.** There exist exactly two solutions  $y_1(x)$  and  $y_2(x)$  of (2.1) which are linearly independent (essentially different) in  $J$ , i.e., there does not exist a constant  $c$  such that  $y_1(x) = cy_2(x)$  for all  $x \in J$ .

**Theorem 2.2.** Two solutions  $y_1(x)$  and  $y_2(x)$  of (2.1) are linearly independent in  $J$  if and only if their *Wronskian* defined by

$$W(x) = W(y_1, y_2)(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} \quad (2.2)$$

is different from zero for some  $x = x_0$  in  $J$ .

**Theorem 2.3.** For the Wronskian defined in (2.2) the following Abel's identity holds:

$$W(x) = W(x_0) \exp \left( - \int_{x_0}^x p_1(t) dt \right), \quad x_0 \in J. \quad (2.3)$$

Thus, if Wronskian is zero at some  $x_0 \in J$ , then it is zero for all  $x \in J$ .

**Theorem 2.4.** If  $y_1(x)$  and  $y_2(x)$  are solutions of (2.1) and  $c_1$  and  $c_2$  are arbitrary constants, then  $c_1y_1(x) + c_2y_2(x)$  is also a solution of (2.1).

Further, if  $y_1(x)$  and  $y_2(x)$  are linearly independent, then any solution  $y(x)$  of (2.1) can be written as  $y(x) = \bar{c}_1 y_1(x) + \bar{c}_2 y_2(x)$ , where  $\bar{c}_1$  and  $\bar{c}_2$  are suitable constants.

Now we shall show that, if one solution  $y_1(x)$  of (2.1) is known (by some clever method) then we can employ *variation of parameters* to find the second solution of (2.1). For this, we let  $y(x) = u(x)y_1(x)$  and substitute this in (2.1), to get

$$(uy_1)'' + p_1(uy_1)' + p_2(uy_1) = 0,$$

or

$$u''y_1 + 2u'y_1' + uy_1'' + p_1u'y_1 + p_1uy_1' + p_2uy_1 = 0,$$

or

$$u''y_1 + (2y_1' + p_1y_1)u' + (y_1'' + p_1y_1' + p_2y_1)u = 0.$$

However, since  $y_1$  is a solution of (2.1), the above equation with  $v = u'$  is the same as

$$y_1v' + (2y_1' + p_1y_1)v = 0, \quad (2.4)$$

which is a first-order equation, and it can be solved easily provided  $y_1 \neq 0$  in  $J$ . Indeed, multiplying (2.4) by  $y_1$ , we find

$$(y_1^2v' + 2y_1'y_1v) + p_1y_1^2v = 0,$$

which is the same as

$$(y_1^2v)' + p_1(y_1^2v) = 0;$$

and hence

$$y_1^2v = c \exp\left(-\int^x p_1(t)dt\right),$$

or, on taking  $c = 1$ ,

$$v(x) = \frac{1}{y_1^2(x)} \exp\left(-\int^x p_1(t)dt\right).$$

Hence, the second solution of (2.1) is

$$y_2(x) = y_1(x) \int^x \frac{1}{y_1^2(t)} \exp\left(-\int^t p_1(s)ds\right) dt. \quad (2.5)$$

**Example 2.1.** It is easy to verify that  $y_1(x) = x^2$  is a solution of the DE

$$x^2y'' - 2xy' + 2y = 0, \quad x \neq 0.$$

For the second solution we use (2.5), to obtain

$$y_2(x) = x^2 \int^x \frac{1}{t^4} \exp\left(-\int^t \left(-\frac{2s}{s^2}\right) ds\right) dt = x^2 \int^x \frac{1}{t^4} t^2 dt = -x.$$

We note that the substitution  $w = y'/y$  converts (2.1) into a first-order nonlinear DE

$$w' + p_1(x)w + p_2(x) + w^2 = 0. \quad (2.6)$$

This DE is called *Riccati's equation*. In general it is not integrable, but if a particular solution, say,  $w_1(x)$  is known, then by the substitution  $z = w - w_1(x)$  it can be reduced to Bernoulli's equation (see Problem 1.6). In fact, we have

$$z' + w_1'(x) + p_1(x)(z + w_1(x)) + p_2(x) + (z + w_1(x))^2 = 0,$$

which is the same as

$$z' + (p_1(x) + 2w_1(x))z + z^2 = 0. \quad (2.7)$$

Since this equation can be solved easily to obtain  $z(x)$ , the solution of (2.6) takes the form  $w(x) = w_1(x) + z(x)$ .

**Example 2.2.** It is easy to verify that  $w_1(x) = x$  is a particular solution of the Riccati equation

$$w' = 1 + x^2 - 2xw + w^2.$$

The substitution  $z = w - x$  in this equation gives the Bernoulli equation

$$z' = z^2,$$

whose general solution is  $z(x) = 1/(c - x)$ ,  $x \neq c$ . Thus, the general solution of the given Riccati's equation is  $w(x) = x + 1/(c - x)$ ,  $x \neq c$ .

**Nonhomogeneous equations.** Now we shall find a particular solution of the nonhomogeneous equation

$$y'' + p_1(x)y' + p_2(x)y = r(x). \quad (2.8)$$

For this also we shall apply the method of *variation of parameters*. Let  $y_1(x)$  and  $y_2(x)$  be two solutions of (2.1). We assume  $y(x) = c_1(x)y_1(x) + c_2(x)y_2(x)$  is a solution of (2.8). Note that  $c_1(x)$  and  $c_2(x)$  are two unknown functions, so we can have two sets of conditions which determine  $c_1(x)$  and  $c_2(x)$ . Since

$$y' = c_1y_1' + c_2y_2' + c_1'y_1 + c_2'y_2$$

as a first condition we assume that

$$c_1'y_1 + c_2'y_2 = 0. \quad (2.9)$$

Thus, we have

$$y' = c_1y_1' + c_2y_2'$$

and on differentiation

$$y'' = c_1 y_1'' + c_2 y_2'' + c_1' y_1' + c_2' y_2'.$$

Substituting these in (2.8), we get

$$c_1(y_1'' + p_1 y_1' + p_2 y_1) + c_2(y_2'' + p_1 y_2' + p_2 y_2) + (c_1' y_1' + c_2' y_2') = r(x).$$

Clearly, this equation, in view of  $y_1(x)$  and  $y_2(x)$  being solutions of (2.1), is the same as

$$c_1' y_1' + c_2' y_2' = r(x). \quad (2.10)$$

Solving (2.9), (2.10), we find

$$c_1' = -\frac{r(x)y_2(x)}{\begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix}}, \quad c_2' = \frac{r(x)y_1(x)}{\begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix}};$$

and hence a particular solution of (2.8) is

$$\begin{aligned} y_p(x) &= c_1(x)y_1(x) + c_2(x)y_2(x) \\ &= -y_1(x) \int^x \frac{r(t)y_2(t)}{\begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix}} dt + y_2(x) \int^x \frac{r(t)y_1(t)}{\begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix}} dt \\ &= \int^x H(x,t)r(t)dt, \end{aligned} \quad (2.11)$$

where

$$H(x,t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1(x) & y_2(x) \end{vmatrix} \bigg/ \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix}. \quad (2.12)$$

Thus, the general solution of (2.8) is

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + y_p(x). \quad (2.13)$$

The following properties of the function  $H(x,t)$  are immediate:

- (i).  $H(x,t)$  is defined for all  $(x,t) \in J \times J$ ;
- (ii).  $\partial^j H(x,t)/\partial x^j$ ,  $j = 0, 1, 2$  are continuous for all  $(x,t) \in J \times J$ ;
- (iii). for each fixed  $t \in J$  the function  $z(x) = H(x,t)$  is a solution of the homogeneous DE (2.1) satisfying  $z(t) = 0$ ,  $z'(t) = 1$ ; and
- (iv). the function

$$v(x) = \int_{x_0}^x H(x,t)r(t)dt$$

is a particular solution of the nonhomogeneous DE (2.8) satisfying  $y(x_0) = y'(x_0) = 0$ .

**Example 2.3.** Consider the DE

$$y'' + y = \cot x.$$

For the corresponding homogeneous DE  $y'' + y = 0$ ,  $\sin x$  and  $\cos x$  are solutions. Thus, its general solution can be written as

$$\begin{aligned} y(x) &= c_1 \sin x + c_2 \cos x + \int^x \frac{\begin{vmatrix} \sin t & \cos t \\ \sin x & \cos x \end{vmatrix}}{\begin{vmatrix} \sin t & \cos t \\ \cos t & -\sin t \end{vmatrix}} \frac{\cos t}{\sin t} dt \\ &= c_1 \sin x + c_2 \cos x - \int^x (\sin t \cos x - \sin x \cos t) \frac{\cos t}{\sin t} dt \\ &= c_1 \sin x + c_2 \cos x - \cos x \sin x + \sin x \int^x \frac{1 - \sin^2 t}{\sin t} dt \\ &= c_1 \sin x + c_2 \cos x - \cos x \sin x - \sin x \int^x \sin t dt + \sin x \int^x \frac{1}{\sin t} dt \\ &= c_1 \sin x + c_2 \cos x + \sin x \int^x \frac{\operatorname{cosec} t (\operatorname{cosec} t - \cot t)}{(\operatorname{cosec} t - \cot t)} dt \\ &= c_1 \sin x + c_2 \cos x + \sin x \ln[\operatorname{cosec} x - \cot x]. \end{aligned}$$

Finally, we remark that if the functions  $p_1(x)$ ,  $p_2(x)$  and  $r(x)$  are continuous on  $J$  and  $x_0 \in J$ , then the DE (2.8) together with the *initial conditions*

$$y(x_0) = y_0, \quad y'(x_0) = y_1 \tag{2.14}$$

has a unique solution. The problem (2.8), (2.14) is called an *initial value problem*. Note that in (2.14) conditions are prescribed at the same point, namely,  $x_0$ .

## Problems

**2.1.** Given the solution  $y_1(x)$ , find the second solution of the following DEs:

- (i)  $(x^2 - x)y'' + (3x - 1)y' + y = 0 \quad (x \neq 0, 1), \quad y_1(x) = (x - 1)^{-1}$
- (ii)  $x(x - 2)y'' + 2(x - 1)y' - 2y = 0 \quad (x \neq 0, 2), \quad y_1(x) = (1 - x)$
- (iii)  $xy'' - y' - 4x^3y = 0 \quad (x \neq 0), \quad y_1(x) = \exp(x^2)$
- (iv)  $(1 - x^2)y'' - 2xy' + 2y = 0 \quad (|x| < 1), \quad y_1(x) = x.$

**2.2.** The differential equation

$$xy'' - (x + n)y' + ny = 0$$

is interesting because it has an exponential solution and a polynomial solution.

- (i) Verify that one solution is  $y_1(x) = e^x$ .  
 (ii) Show that the second solution has the form  $y_2(x) = ce^x \int^x t^n e^{-t} dt$ . Further, show that with  $c = -1/n!$ ,

$$y_2(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}.$$

Note that  $y_2(x)$  is the first  $n + 1$  terms of the Taylor series about  $x = 0$  for  $e^x$ , that is, for  $y_1(x)$ .

**2.3.** The differential equation

$$y'' + \delta(xy' + y) = 0$$

occurs in the study of the turbulent flow of a uniform stream past a circular cylinder. Verify that  $y_1(x) = \exp(-\delta x^2/2)$  is one solution. Find its second solution.

**2.4.** Let  $y_1(x) \neq 0$  and  $y_2(x)$  be two linearly independent solutions of the DE (2.1). Show that  $y(x) = y_2(x)/y_1(x)$  is a nonconstant solution of the DE

$$y_1(x)y'' + (2y_1'(x) + p_1(x)y_1(x))y' = 0.$$

**2.5.** Let the function  $p_1(x)$  be differentiable in  $J$ . Show that the substitution  $y(x) = z(x) \exp(-\frac{1}{2} \int^x p_1(t) dt)$  transforms (2.1) to the differential equation

$$z'' + \left( p_2(x) - \frac{1}{2}p_1'(x) - \frac{1}{4}p_1^2(x) \right) z = 0.$$

In particular show that the substitution  $y(x) = z(x)/\sqrt{x}$  transforms *Bessel's DE*

$$x^2 y'' + xy' + (x^2 - a^2)y = 0, \tag{2.15}$$

where  $a$  is a constant (parameter), into a simple DE

$$z'' + \left( 1 + \frac{1 - 4a^2}{4x^2} \right) z = 0. \tag{2.16}$$

**2.6.** Let  $v(x)$  be the solution of the initial value problem

$$y'' + p_1 y' + p_2 y = 0, \quad y(0) = 0, \quad y'(0) = 1$$

where  $p_1$  and  $p_2$  are constants. Show that the function

$$y(x) = \int_{x_0}^x v(x-t)r(t)dt$$

is the solution of the nonhomogeneous DE

$$y'' + p_1 y' + p_2 y = r(x)$$

satisfying  $y(x_0) = y'(x_0) = 0$ .

**2.7.** Find general solutions of the following nonhomogeneous DEs:

- (i)  $y'' + 4y = \sin 2x$
- (ii)  $y'' + 4y' + 3y = e^{-3x}$
- (iii)  $y'' + 5y' + 4y = e^{-4x}$ .

**2.8.** Verify that  $y_1(x) = x$  and  $y_2(x) = 1/x$  are solutions of

$$x^3 y'' + x^2 y' - xy = 0.$$

Use this information and the variation of parameters method to find the general solution of

$$x^3 y'' + x^2 y' - xy = x/(1+x).$$

## Answers or Hints

**2.1.** (i)  $\ln x/(x-1)$  (ii)  $(1/2)(1-x) \ln[(x-2)/x] - 1$  (iii)  $e^{-x^2}$  (iv)  $(x/2) \times \ln[(1+x)/(1-x)] - 1$ .

**2.2.** (i) Verify directly (ii) Use (2.5).

**2.3.**  $e^{-\delta x^2/2} \int^x e^{\delta t^2/2} dt$ .

**2.4.** Use  $y_2(x) = y_1(x)y(x)$  and the fact that  $y_1(x)$  and  $y_2(x)$  are solutions.

**2.5.** Verify directly.

**2.6.** Use Leibniz's formula:

$$\frac{d}{dx} \int_{\alpha(x)}^{\beta(x)} f(x, t) dt = f(x, \beta(x)) \frac{d\beta}{dx} - f(x, \alpha(x)) \frac{d\alpha}{dx} + \int_{\alpha(x)}^{\beta(x)} \frac{\partial f}{\partial x}(x, t) dt.$$

**2.7.** (i)  $c_1 \cos 2x + c_2 \sin 2x - \frac{1}{4}x \cos 2x$  (ii)  $c_1 e^{-x} + c_2 e^{-3x} - \frac{1}{2}x e^{-3x}$   
 (iii)  $c_1 e^{-x} + c_2 e^{-4x} - \frac{1}{3}x e^{-4x}$ .

**2.8.**  $c_1 x + (c_2/x) + (1/2)[(x - (1/x)) \ln(1+x) - x \ln x - 1]$ .

<http://www.springer.com/978-0-387-79145-6>

Ordinary and Partial Differential Equations  
With Special Functions, Fourier Series, and Boundary  
Value Problems

Agarwal, R.P.; O'Regan, D.

2009, XIV, 410 p. 35 illus., Softcover

ISBN: 978-0-387-79145-6