Chapter 2
Legendre-Fenchel Duality on Phase Space

We start by recalling the basic concepts and relevant tools of convex analysis that will be used throughout the book. In particular, we review the Fenchel-Legendre duality and its relationship with subdifferentiability. The material of the first four sections is quite standard and does not include proofs, which we leave and recommend to the interested reader. They can actually be found in most books on convex analysis, such as those of Brézis [26], Ekeland and Temam [47], Ekeland [46], and Phelps [130].

Our approach to evolution equations and partial differential systems, however, is based on convex calculus on "phase space" $X \times X^*$, where X is a reflexive Banach space and X^* is its dual. We shall therefore consider Lagrangians on $X \times X^*$ that are convex and lower semicontinuous in both variables. All elements of convex analysis will apply, but the calculus on state space becomes much richer for many reasons, not the least of which is the possibility of introducing associated Hamiltonians, which are themselves Legendre conjugates but in only one variable.

Another reason for the rich structure will become more evident in the next chapter where the abundance of natural automorphisms on phase space and their interplay with the Legendre transform becomes an essential ingredient of our self-dual variational approach.

2.1 Basic notions of convex analysis

Definition 2.1. A function $\varphi : X \to \mathbf{R} \cup \{+\infty\}$ on a Banach space X is said to be:

1. *lower semicontinuous (weakly lower semicontinuous)* if, for every $r \in \mathbf{R}$, its *epigraph* $\mathrm{Epi}(\varphi) := \{(x,r) \in X \times \mathbf{R}; \varphi(x) \le r\}$ is closed for the norm topology (resp., weak topology) of $X \times \mathbf{R}$, which is equivalent to saying that whenever (x_α) is a net in X that converges strongly (resp., weakly) to x, then $f(x) \le \liminf_\alpha f(x_\alpha)$.
2. *convex* if, for every $r \in \mathbf{R}$, its *epigraph* $\mathrm{Epi}(\varphi)$ is a convex subset of $X \times \mathbf{R}$, which is equivalent to saying that $f(\lambda x + (1-\lambda y) \le \lambda f(x) + (1-\lambda) f(y)$ for any $x, y \in X$ and $0 \le \lambda \le 1$.

N. Ghoussoub, *Self-dual Partial Differential Systems and Their Variational Principles*,
Springer Monographs in Mathematics, DOI 10.1007/978-0-387-84897-6_2,

3. *proper* if its effective domain (i.e., the set $\mathrm{Dom}(\varphi) = \{x \in X; \varphi(x) < +\infty\}$) is nonempty, the effective domain being convex whenever φ is convex.

We shall denote by $\mathscr{C}(X)$ the class of convex lower semicontinuous functions on a Banach space X.

Operations on convex lower semicontinuous functions

Consider φ and ψ to be two functions in $\mathscr{C}(X)$. Then,

1. The functions $\varphi + \psi$ and $\lambda\varphi$ when $\lambda \geq 0$ are also in $\mathscr{C}(X)$.
2. The function $x \to \max\{\varphi(x), \psi(x)\}$ is in $\mathscr{C}(X)$.
3. The inf-convolution $x \to \varphi \star \psi(x) := \inf\{\varphi(y) + \psi(x-y); y \in X\}$ is convex. If φ and ψ are bounded below, then $\varphi \star \psi$ is in $\mathscr{C}(X)$ and $\mathrm{Dom}(\varphi \star \psi) = \mathrm{Dom}(\varphi) + \mathrm{Dom}(\psi)$. Moreover, $\varphi \star \psi$ is continuous at a point $x \in X$ if either φ or ψ is continuous at x.
4. If $\rho \in \mathscr{C}(\mathbf{R})$, then $x \to \rho(\|x\|_X)$ is in $\mathscr{C}(X)$.

Convex functions enjoy various remarkable properties that make them agreeable to use in variational problems. We now summarize some of them.

Proposition 2.1. *If $\varphi : X \to \mathbf{R} \cup \{+\infty\}$ is a convex function on a Banach space X, then:*

1. *φ is lower semicontinuous if and only if it is weakly lower semicontinuous, in which case it is the supremum of all continuous affine functions below it.*
2. *If φ is a proper convex lower semicontinuous function on X, then it is continuous on the interior D of its effective domain, provided it is nonempty.*

We shall often use the immediate implication stating that any convex lower semicontinuous function that is finite on the unit ball of X is necessarily continuous. However, one should keep in mind that there exist continuous and convex functions on Hilbert space that are not bounded on the unit ball [130].

2.2 Subdifferentiability of convex functions

Definition 2.2. Let $\varphi : X \to \mathbf{R} \cup \{+\infty\}$ be a convex lower semicontinuous function on a Banach space X. Define the *subdifferential $\partial\varphi$ of φ* to be the following set-valued function: If $x \in \mathrm{Dom}(\varphi)$, set

$$\partial\varphi(x) = \{p \in X^*; \langle p, y - x\rangle \leq \varphi(y) - \varphi(x) \text{ for all } y \in X\}, \tag{2.1}$$

and if $x \notin \mathrm{Dom}(\varphi)$, set $\partial\varphi(x) = \emptyset$.

The subdifferential $\partial\varphi(x)$ is a closed convex subset of the dual space X^*. It can, however, be empty even though $x \in \mathrm{Dom}(\varphi)$, and we shall write

$$\mathrm{Dom}(\partial\varphi) = \{x \in X; \partial\varphi(x) \neq \emptyset\}. \tag{2.2}$$

An application of the celebrated Bishop-Phelps theorem due to Brondsted and Rockafellar (see [130]) however yields the following useful result.

Proposition 2.2. *Let φ be a proper convex lower semicontinuous function on X. Then,*

1. $\mathrm{Dom}(\partial\varphi)$ *is dense in* $\mathrm{Dom}(\varphi)$.
2. *Moreover, $\partial\varphi(x) \neq \emptyset$ at any point x in the interior of* $\mathrm{Dom}(\varphi)$ *where φ is continuous.*

If $x \in \mathrm{Dom}(\varphi)$, we define the more classical notion $d^+\varphi(x)$ of a "right-derivative" at x as

$$\langle d^+\varphi(x), y\rangle := \lim_{t\to 0^+} \tfrac{1}{t}\big(\varphi(x+ty) - \varphi(x)\big) \text{ for any } y \in X. \tag{2.3}$$

The relationship between the two types of derivatives is given by

$$p \in \partial\varphi(x) \text{ if and only if } \langle p, y\rangle \leq \langle d^+\varphi(x), y\rangle \text{ for any } y \in X. \tag{2.4}$$

Now φ is said to be *Gâteaux-differentiable* at a point $x \in \mathrm{Dom}(\varphi)$ if there exists $p \in X^*$, which will be denoted by $D_G\varphi(x)$ such that

$$\langle p, y\rangle = \lim_{t\to 0} \tfrac{1}{t}\big(\varphi(x+ty) - \varphi(x)\big) \text{ for any } y \in X. \tag{2.5}$$

It is then easy to see the following relationship between the two notions.

Proposition 2.3. *Let φ be a convex function on X.*

1. *If φ is Gâteaux-differentiable at a point $x \in \mathrm{Dom}(\varphi)$, then $\partial\varphi(x) = \{D_G\varphi(x)\}$.*
2. *Conversely, if φ is continuous at $x \in \mathrm{Dom}(\varphi)$, and if the subdifferential of φ at x is single valued, then $\partial\varphi(x) = \{D_G\varphi(x)\}$.*

Subdifferentials satisfy the following calculus.

Proposition 2.4. *Let φ and ψ be in $\mathscr{C}(X)$ and $\lambda \geq 0$. We then have the following properties:*

1. *$\partial(\lambda\varphi)(x) = \lambda\partial\varphi(x)$ and $\partial\varphi(x) + \partial\psi(x) \subset \partial(\varphi+\psi)(x)$ for any $x \in X$.*
2. *Moreover, equality $\partial\varphi(x) + \partial\psi(x) = \partial(\varphi+\psi)(x)$ holds at a point $x \in \mathrm{Dom}(\varphi) \cap \mathrm{Dom}(\psi)$, provided either φ or ψ is continuous at x.*
3. *If $A : Y \to X$ is a bounded linear operator from a Banach space Y into X, and if φ is continuous at some point in $R(A) \cap \mathrm{Dom}(\varphi)$, then $\partial(\varphi \circ A)(y) = A^*\partial\varphi(Ay)$ for every point $y \in Y$.*

As a set-valued map, the subdifferential has the following useful properties.

Definition 2.3. A subset G of $X \times X^*$ is said to be

1. *monotone*, provided

$$\langle x - y, p - q\rangle \geq 0 \text{ for every } (x, p) \text{ and } (y, p) \text{ in } G. \tag{2.6}$$

2. *maximal monotone* if it is maximal in the family of monotone subsets of $X \times X^*$ ordered by set inclusion, and
3. *cyclically monotone*, provided that for any finite number of points $(x_i, p_i)_{i=0}^n$ in G with $x_0 = x_n$, we have

$$\sum_{k=1}^{n} \langle p_k, x_k - x_{k-1} \rangle \geq 0. \tag{2.7}$$

A set-valued map $T : X \to 2^{X^*}$ is then said to be *monotone* (resp., *maximal monotone*) (resp., *cyclically monotone*), provided its graph $G(T) = \{(x,p) \in X \times X^*; p \in T(x)\}$ is *monotone* (resp., *maximal monotone*) (resp., *cyclically monotone*).

The following result was established by Rockafellar. See for example [130].

Theorem 2.1. *Let $\varphi : X \to \mathbf{R} \cup \{+\infty\}$ be a proper convex and lower semicontinuous functional on a Banach space X. Then, its differential map $x \to \partial \varphi(x)$ is a maximal cyclically monotone map.*

Conversely, if $T : X \to 2^{X^*}$ is a maximal cyclically monotone map with a nonempty domain, then there exists a proper convex and lower semicontinuous functional on X such that $T = \partial \varphi$.

2.3 Legendre duality for convex functions

Let $\varphi : X \to \mathbf{R} \cup \{+\infty\}$ be any function. Its Fenchel-Legendre dual is the function φ^* on X^* given by

$$\varphi^*(p) = \sup\{\langle x, p \rangle - \varphi(x); x \in X\}. \tag{2.8}$$

Proposition 2.5. *Let $\varphi : X \to \mathbf{R} \cup \{+\infty\}$ be a proper function on a reflexive Banach space. The following properties then hold:*

1. *φ^* is a proper convex lower semicontinuous function from X^* to $\mathbf{R} \cup \{+\infty\}$.*
2. *$\varphi^{**} := (\varphi^*)^* : X \to \mathbf{R} \cup \{+\infty\}$ is the largest convex lower semicontinuous function below φ. Moreover, $\varphi = \varphi^{**}$ if and only if φ is convex and lower semicontinuous on X.*
3. *For every $(x,p) \in X \times X^*$, we have $\varphi(x) + \varphi^*(p) \geq \langle x, p \rangle$, and the following are equivalent:*

   i) $\varphi(x) + \varphi^*(p) = \langle x, p \rangle$,
   ii) $p \in \partial \varphi(x)$,
   iii) $x \in \partial \varphi^*(p)$.

Proposition 2.6. *Legendre duality satisfies the following rules:*

1. *$\varphi^*(0) = -\inf_{x \in X} \varphi(x)$.*
2. *If $\varphi \leq \psi$, then $\varphi^* \geq \psi^*$.*

3. *We have* $(\inf_{i\in I}\varphi_i)^* = \sup_{i\in I}\varphi_i^*$ *and* $(\sup_{i\in I}\varphi_i)^* \le \inf_{i\in I}\varphi_i^*$ *whenever* $(\varphi_i)_{i\in I}$ *is a family of functions on* X.
4. *For every* $\lambda > 0$, $(\lambda\varphi)^*(p) = \lambda\varphi^*(\frac{1}{\lambda}p)$.
5. *For every* $\alpha \in R$, $(\varphi+\alpha)^* = \varphi^* - \alpha$.
6. *For a fixed* $a \in X$, *we have, for every* $p \in X^*$, $\varphi_a^*(p) = \varphi^*(p) + \langle a, p\rangle$, *where* $\varphi_a(x) := \varphi(x-a)$.
7. *If* ρ *is an even function in* $\mathscr{C}(\mathbf{R})$, *then the Legendre transform of* $\varphi(x) = \rho(\|u\|_X)$ *is* $\varphi^*(p) = \rho^*(\|p\|_{X^*})$. *In particular, if* $\varphi(x) = \frac{1}{\alpha}\|x\|_X^\alpha$, *then* $\varphi^*(p) = \frac{1}{\beta}\|p\|_{X^*}^\beta$, *where* $\frac{1}{\alpha} + \frac{1}{\beta} = 1$.
8. *If* φ *and* ψ *are proper functions, then* $(\varphi \star \psi)^* = \varphi^* + \psi^*$.
9. *Conversely, if* $\mathrm{Dom}(\varphi) - \mathrm{Dom}(\psi)$ *contains a neighborhood of the origin, then* $(\varphi+\psi)^* = \varphi^* \star \psi^*$.
10. *Let* $A : D(A) \subset X \to Y$ *be a linear operator with a closed graph, and let* $\varphi : Y \to \mathbf{R}\cup\{+\infty\}$ *be a proper function in* $\mathscr{C}(Y)$. *Then, the dual of the function* φ_A *defined on* X *as* $\varphi_A(x) = \varphi(Ax)$ *if* $x \in D(A)$ *and* $+\infty$ *otherwise, is*

$$\varphi_A^*(p) = \inf\{\varphi^*(q); A^*q = p\}.$$

11. *Let* $h(x) := \inf\{F(x_1,x_2); x_1,x_2 \in X, x = \frac{1}{2}(x_1+x_2)\}$, *where* F *is a function on* $X \times X$. *Then,* $h^*(p) = F^*(\frac{p}{2},\frac{p}{2})$ *for every* $p \in X^*$.
12. *Let* g *be the function on* $X \times X$ *defined by* $g(x_1,x_2) = \|x_1 - x_2\|^2$. *Then,* $g^*(p_1,p_2) = \frac{1}{4}\|p_1\|^2$ *if* $p_1 + p_2 = 0$ *and* $+\infty$ *otherwise.*

The following lemma will be useful in Chapter 5. It can be used to interpolate between convex functions, and is sometimes called the *proximal average*.

Lemma 2.1. *Let* $f_1, f_2 : X \to \mathbf{R}\cup\{+\infty\}$ *be two convex lower semicontinuous functions on a reflexive Banach space* X. *The Legendre dual of the function* h *defined for* $X \in X$ *by*

$$h(x) := \inf\left\{\frac{1}{2}f_1(x_1) + \frac{1}{2}f_2(x_2) + \frac{1}{8}\|x_1 - x_2\|^2; x_1,x_2 \in X, x = \frac{1}{2}(x_1+x_2)\right\}$$

is given by the function h^* *defined for* $p \in X^*$ *by*

$$h^*(p) = \inf\left\{\frac{1}{2}f_1^*(p_1) + \frac{1}{2}f_2^*(p_2) + \frac{1}{8}\|p_1 - p_2\|^2; p_1,p_2 \in X^*, p = \frac{1}{2}(p_1+p_2)\right\}.$$

Proof. Note that

$$h(x) := \inf\left\{F(x_1,x_2); x_1,x_2 \in X, x = \frac{1}{2}(x_1+x_2)\right\},$$

where F is the function on $X \times X$ defined as $F(x_1,x_2) = g_1(x_1,x_2) + g_2(x_1,x_2)$ with

$$g_1(x_1,x_2) = \tfrac{1}{2}f_1(x_1) + \tfrac{1}{2}f_2(x_2) \quad \text{and} \quad g_2(x_1,x_2) = \tfrac{1}{8}\|x_1 - x_2\|^2.$$

It follows from rules (10) and (7) in Proposition 2.6 that

$$h^*(p) = F^*\left(\frac{p}{2}, \frac{p}{2}\right) = (g_1+g_2)^*\left(\frac{p}{2}, \frac{p}{2}\right) = g_1^* \star g_2^*\left(\frac{p}{2}, \frac{p}{2}\right).$$

It is easy to see that

$$g_1^*(p_1, p_2) = \frac{1}{2} f_1^*\left(\frac{p_1}{2}\right) + \frac{1}{2} f_2^*\left(\frac{p_2}{2}\right),$$

while rule (11) of Proposition 2.6 gives that

$$g_2^*(p_1, p_2) = 2\|p_1\|^2 \quad \text{if } p_1 + p_2 = 0 \quad \text{and} \quad +\infty \text{ otherwise.}$$

It follows that

$$\begin{aligned}
h^*(p) &= g_1^* \star g_2^*\left(\frac{p}{2}, \frac{p}{2}\right) \\
&= \inf\left\{\frac{1}{2} f_1^*\left(\frac{p_1}{2}\right) + \frac{1}{2} f_2^*\left(\frac{p_2}{2}\right) + 2\left\|\frac{p}{2} - \frac{p_1}{4}\right\|^2; p_1, p_2 \in X^*, p = p_1 + p_2\right\} \\
&= \inf\left\{\frac{1}{2} f_1^*(q_1) + \frac{1}{2} f_2^*(q_2) + 2\left\|\frac{p}{2} - \frac{q_1}{2}\right\|^2; q_1, q_2 \in X^*, p = \frac{1}{2}(q_1+q_2)\right\} \\
&= \inf\left\{\frac{1}{2} f_1^*(q_1) + \frac{1}{2} f_2^*(q_2) + \frac{1}{8}\|q_2 - q_1\|^2; q_1, q_2 \in X^*, p = \frac{1}{2}(q_1+q_2)\right\}.
\end{aligned}$$

The following theorem can be used to prove rule (8) in Proposition 2.6. It will also be needed in what follows.

Theorem 2.2 (Fenchel and Rockafellar). *Let φ and ψ be two convex functions on a Banach space X such that φ is continuous at some point $x_0 \in \mathrm{Dom}(\varphi) \cap \mathrm{Dom}(\psi)$. Then,*

$$\inf_{x\in X}\{\varphi(x) + \psi(x)\} = \max_{p\in X^*}\{-\varphi^*(-p) - \psi^*(p)\}. \tag{2.9}$$

The theorem above holds, for example, whenever $\mathrm{Dom}(\varphi) - \mathrm{Dom}(\psi)$ contains a neighborhood of the origin or more generally if the set $\mathrm{IntDom}(\varphi) \cap \mathrm{Dom}(\psi)$ is nonempty.

The following simple lemma will be used often throughout this text. Its proof is left as an exercise.

Lemma 2.2. *If $\varphi : X \mapsto \mathbf{R} \cup \{+\infty\}$ is a proper convex and lower semicontinuous functional on a Banach space X such that $-A \le \varphi(y) \le \frac{B}{\alpha}\|y\|_Y^\alpha + C$ with $A \ge 0$, $C \ge 0$, $B > 0$, and $\alpha > 1$, then for every $p \in \partial\varphi(y)$*

$$\|p\|_{X^*} \le \left\{\alpha B^{\frac{\beta}{\alpha}}(\|y\|_X + A + C) + 1\right\}^{\alpha-1}. \tag{2.10}$$

2.4 Legendre transforms of integral functionals

Let Ω be a Borel subset of $\mathbf{R}^n$ with finite Lebesgue measure, and let X be a separable reflexive Banach space. Consider a bounded below function $\varphi : \Omega \times X \to \mathbf{R} \cup \{+\infty\}$ that is measurable with respect to the σ-field generated by the products of Lebesgue sets in Ω and Borel sets in X. We can associate to φ a functional Φ defined on $L^\alpha(\Omega, X)$ ($1 \le \alpha \le +\infty$) via the formula

$$\Phi(x) = \int_\Omega \varphi(\omega, x(\omega))\, d\omega,$$

where $x \in L^\alpha(\Omega, X)$. We now relate the Legendre transform and subdifferential of φ as a function of its second variable on X to the Legendre transform and subdifferential of Φ as a function on $L^\alpha(\Omega, X)$. We shall use the following obvious notation. For $\omega \in \Omega$, $x \in X$, and $p \in X^*$,

$$\varphi^*(\omega, p) = \varphi(\omega, \cdot)^*(p) \quad \text{and} \quad \partial\varphi(\omega, x) = \partial\varphi(\omega, \cdot)(x).$$

The following proposition summarizes the relations between the function φ and "its integral" Φ. A proof can be found in [46].

Proposition 2.7. *Assume X is a reflexive and separable Banach space, that $1 \le \alpha \le +\infty$, $\frac{1}{\alpha} + \frac{1}{\beta} = 1$, and that $\varphi : \Omega \times X \to \mathbf{R} \cup \{+\infty\}$ is jointly measurable such that $\int_\Omega |\varphi^*(\omega, \bar{p}(\omega))| d\omega < \infty$ for some $\bar{p} \in L^\beta(\Omega, X)$, which holds in particular if φ is bounded below on $\Omega \times X$.*

1. *If the function $\varphi(\omega, \cdot)$ is lower semicontinuous on X for almost every $\omega \in \Omega$, then Φ is lower semicontinuous on $L^\alpha(\Omega, X)$.*
2. *If $\varphi(\omega, \cdot)$ is convex on X for almost every $\omega \in \Omega$, then Φ is convex on $L^\alpha(\Omega, X)$.*
3. *If $\varphi(\omega, \cdot)$ is convex and lower semicontinuous on X for almost every $\omega \in \Omega$, and if $\Phi(\bar{x}) < +\infty$ for some $\bar{x} \in L^\infty(\Omega, X)$, then the Legendre transform of Φ on $L^\beta(\Omega, X)$ is given by*

$$\Phi^*(p) = \textstyle\int_\Omega \varphi^*(\omega, p(\omega))\, d\omega \quad \text{for all } p \in L^\beta(\Omega, X). \tag{2.11}$$

4. *If $\int_\Omega |\varphi(\omega, \bar{x}(\omega))| d\omega < \infty$ and $\int_\Omega |\varphi^*(\omega, \bar{p}(\omega))| d\omega < \infty$ for some $\bar{x}$ and $\bar{p}$ in $L^\infty(\Omega, X)$, then for every $x \in L^\alpha(\Omega, X)$ we have*

$$\partial\Phi(x) = \Big\{ p \in L^\beta(\Omega, X);\, p(\omega) \in \partial\varphi(\omega, x(\omega)) \text{ a.e.} \Big\}. \tag{2.12}$$

Exercises 2.A. Legendre transforms of energy functionals

1. Review and prove all the statements in Sections 2.1 to 2.4.
2. Let Ω be a bounded smooth domain in $\mathbf{R}^n$, and define on $L^2(\Omega)$ the convex lower semicontinuous functional

$$\varphi(u) = \begin{cases} \frac{1}{2}\int_\Omega |\nabla u|^2 & \text{on} \quad H_0^1(\Omega) \\ +\infty & \text{elsewhere.} \end{cases} \tag{2.13}$$

Show that its Legendre-Fenchel conjugate for the L^2-duality is $\varphi^*(v) = \frac{1}{2}\int_\Omega |\nabla(-\Delta)^{-1}v|^2 dx$ and that its subdifferential $\partial\varphi = -\Delta$ with domain $H_0^1(\Omega) \cap H^2(\Omega)$.

3. Consider the Hilbert space $H^{-1}(\Omega)$ equipped with the norm induced by the scalar product $\langle u, v\rangle_{H^{-1}(\Omega)} = \int_\Omega u(-\Delta)^{-1}v dx$. For $m \geq \frac{n-2}{n+2}$, we have $L^{m+1}(\Omega) \subset H^{-1}$, and so we may consider the functional

$$\varphi(u) = \begin{cases} \frac{1}{m+1}\int_\Omega |u|^{m+1} & \text{on} \quad L^{m+1}(\Omega) \\ +\infty & \text{elsewhere.} \end{cases} \tag{2.14}$$

Show that its Legendre-Fenchel conjugate is $\varphi^*(v) = \frac{m}{m+1}\int_\Omega |(-\Delta)^{-1}v|^{\frac{m+1}{m}} dx$ with subdifferential $\partial\varphi(u) = -\Delta(u^m)$ on $D(\partial\varphi) = \{u \in L^{m+1}(\Omega); u^m \in H_0^1(\Omega)\}$.

4. If $0 < m < 1$, then $(-\Delta)^{-1}u$ does not necessarily map $L^{m+1}(\Omega)$ into $L^{\frac{m+1}{m}}$, and so we consider the space X defined as

$$X = \{u \in L^{m+1}(\Omega); (-\Delta)^{-1}u \in L^{\frac{m+1}{m}}(\Omega)\}$$

equipped with the norm $\|u\|_X = \|u\|_{m+1} + \|(-\Delta)^{-1}u\|_{\frac{m+1}{m}}$. Show that the functional $\varphi(u) = \frac{1}{m+1}\int_\Omega |u|^{m+1}$ is convex and lower semicontinuous on X with Legendre-Fenchel transform equal to

$$\varphi^*(v) = \begin{cases} \frac{m}{m+1}\int_\Omega |(-\Delta)^{-1}v|^{\frac{m+1}{m}} dx & \text{if} \quad (-\Delta)^{-1}v \in L^{\frac{m+1}{m}}(\Omega) \\ +\infty & \text{otherwise.} \end{cases} \tag{2.15}$$

2.5 Legendre transforms on phase space

Let X be a reflexive Banach space. Functions $L : X \times X^* \to \mathbf{R} \cup \{+\infty\}$ on phase space $X \times X^*$ will be called *Lagrangians*, and we shall consider the class $\mathcal{L}(X)$ of those Lagrangians that are proper convex and lower semicontinuous (in both variables). The Legendre-Fenchel dual (in both variables) of L is defined at $(q, y) \in X^* \times X$ by

$$L^*(q, y) = \sup\{\langle q, x\rangle + \langle y, p\rangle - L(x, p); x \in X, p \in X^*\}.$$

The (partial) domains of a Lagrangian L are defined as

$$\mathrm{Dom}_1(L) = \{x \in X; L(x, p) < +\infty \text{ for some } p \in X^*\}$$

and

$$\mathrm{Dom}_2(L) = \{p \in X^*; L(x, p) < +\infty \text{ for some } x \in X\}.$$

To each Lagrangian L on $X \times X^*$, we can define its corresponding Hamiltonian $H_L : X \times X \to \bar{\mathbf{R}}$ (resp., co-Hamiltonian $\tilde{H}_L : X^* \times X^* \to \overline{\mathbf{R}}$) by

$$H_L(x, y) = \sup\{\langle y, p\rangle - L(x, p); p \in X^*\} \text{ and } \tilde{H}_L(p, q) = \sup\{\langle y, p\rangle - L(y, q); y \in X\},$$

which is the Legendre transform in the second variable (resp., first variable). Their domains are

$$\begin{aligned}\mathrm{Dom}_1(H_L) :&= \{x \in X; H_L(x,y) > -\infty \text{ for all } y \in X\} \\ &= \{x \in X; H_L(x,y) > -\infty \text{ for some } y \in X\}\end{aligned}$$

and

$$\begin{aligned}\mathrm{Dom}_2(\tilde{H}_L) :&= \{q \in X^*; \tilde{H}_L(p,q) > -\infty \text{ for all } p \in X^*\} \\ &= \{q \in X^*; \tilde{H}_L(p,q) > -\infty \text{ for some } p \in X^*\}.\end{aligned}$$

It is clear that $\mathrm{Dom}_1(L) = \mathrm{Dom}_1(H_L)$ and $\mathrm{Dom}_2(L) = \mathrm{Dom}_2\tilde{H}_L$.

Remark 2.1. To any pair of proper convex lower semicontinuous functions φ and ψ on a Banach space X, one can associate a Lagrangian on state space $X \times X^*$ via the formula $L(x,p) = \varphi(x) + \psi^*(p)$. Its Legendre transform is then $L^*(p,x) = \psi(x) + \varphi^*(p)$. Its Hamiltonian is $H_L(x,y) = \psi(y) - \varphi(x)$ if $x \in \mathrm{Dom}(\varphi)$ and $-\infty$ otherwise, while its co-Hamiltonian is $\tilde{H}_L(p,q) = \varphi^*(p) - \psi^*(q)$ if $q \in \mathrm{Dom}(\psi^*)$ and $-\infty$ otherwise. The domains are then $\mathrm{Dom}_1 H_L := \mathrm{Dom}(\varphi)$ and $\mathrm{Dom}_2(\tilde{H}_L) := \mathrm{Dom}(\psi^*)$. These Lagrangians will be the building blocks of the variational approach developed in this book.

Operations on Lagrangians

We define on the class of Lagrangians $\mathscr{L}(X)$ the following operations:

Scalar multiplication: If $\lambda > 0$ and $L \in \mathscr{L}(X)$, define the Lagrangian $\lambda \cdot L$ on $X \times X^*$ by

$$(\lambda \cdot L)(x,p) = \lambda^2 L\left(\frac{x}{\lambda}, \frac{p}{\lambda}\right).$$

Addition: If $L, M \in \mathscr{L}(X)$, define the sum $L \oplus M$ on $X \times X^*$ by:

$$(L \oplus M)(x,p) = \inf\{L(x,r) + M(x,p-r); r \in X^*\}.$$

Convolution: If $L, M \in \mathscr{L}(X)$, define the convolution $L \star M$ on $X \times X^*$ by

$$(L \star M)(x,p) = \inf\{L(z,p) + M(x-z,p); z \in X\}.$$

Right operator shift: If $L \in \mathscr{L}(X)$ and $\Gamma : X \to X^*$ is a bounded linear operator, define the Lagrangian L_Γ on $X \times X^*$ by

$$L_\Gamma(x,p) := L(x, -\Gamma x + p).$$

Left operator shift: If $L \in \mathscr{L}(X)$ and if $\Gamma : X \to X^*$ is an invertible operator, define the Lagrangian ${}_\Gamma L$ on $X \times X^*$ by

$${}_\Gamma L(x,p) := L(x - \Gamma^{-1}p, \Gamma x).$$

Free product: If $\{L_i; i \in I\}$ is a finite family of Lagrangians on reflexive Banach spaces $\{X_i; i \in I\}$, define the Lagrangian $L := \Sigma_{i \in I} L_i$ on $(\Pi_{i \in I} X_i) \times (\Pi_{i \in I} X_i^*)$ by

$$L((x_i)_i,(p_i)_i) = \Sigma_{i\in I} L_i(x_i,p_i).$$

Twisted product: If $L \in \mathscr{L}(X)$ and $M \in \mathscr{L}(Y)$, where X and Y are two reflexive spaces, then for any bounded linear operator $A : X \to Y^*$, define the Lagrangian $L \oplus_A M$ on $(X \times Y) \times (X^* \times Y^*)$ by

$$(L \oplus_A M)((x,y),(p.q)) := L(x, A^*y + p) + M(y, -Ax + q).$$

Antidualization of convex functions: If φ, ψ are convex functions on $X \times Y$ and if A is any bounded linear operator $A : X \to Y^*$, define the Lagrangian $\varphi \oplus_A \psi$ on $(X \times Y) \times (X^* \times Y^*)$ by

$$\varphi \oplus_A \psi((x,y),(p,q)) = \varphi(x,y) + \psi^*(A^*y + p, -Ax + q).$$

Remark 2.2. The convolution operation defined above should not be confused with the standard convolution for L and M as convex functions in both variables. Indeed, it is easy to see that in the case where $L(x,p) = \varphi(x) + \varphi^*(p)$ and $M(x,p) = \psi(x) + \psi^*(p)$, addition corresponds to taking

$$(L \oplus M)(x,p) = (\varphi + \psi)(x) + \varphi^* \star \psi^*(p),$$

while convolution reduces to

$$(L \star M)(x,p) = (\varphi \star \psi)(x) + (\varphi^* + \psi^*)(p).$$

Proposition 2.8. *Let X be a reflexive Banach space. Then,*

1. $(\lambda \cdot L)^* = \lambda \cdot L^*$ *for any* $L \in \mathscr{L}(X)$ *and any* $\lambda > 0$.
2. $(L \oplus M)^* \leq L^* \star M^*$ *and* $(L \star M)^* \leq L^* \oplus M^*$ *for any* $L, M \in \mathscr{L}(X)$.
3. *If M is a basic Lagrangian of the form $\varphi(Ux) + \psi^*(V^*p)$, where ψ is continuous on X and U,V are two automorphisms of X, then $(L \star M)^* = L^* \oplus M^*$ for any $L \in \mathscr{L}(X)$.*
4. *If $L, M \in \mathscr{L}(X)$ are such that* $\mathrm{Dom}_2(L^*) - \mathrm{Dom}_2(M^*)$ *contains a neighborhood of the origin, then $(L \star M)^* = L^* \oplus M^*$.*
5. *If $L, M \in \mathscr{L}(X)$ are such that* $\mathrm{Dom}_1(L) - \mathrm{Dom}_1(M)$ *contains a neighborhood of the origin, then $(L \oplus M)^* = L^* \star M^*$.*
6. *If $L \in \mathscr{L}(X)$ and $\Gamma : X \to X^*$ is a bounded linear operator, then $(L_\Gamma)^*(p,x) = L^*(\Gamma^* x + p, x)$.*
7. *If $L \in \mathscr{L}(X)$ and if $\Gamma : X \to X^*$ is an invertible operator, then $({}_\Gamma L)^*(p,x) = L^*(-\Gamma^* x, (\Gamma^{-1})^* p + x)$.*
8. *If $\{L_i; i \in I\}$ is a finite family of Lagrangians on reflexive Banach spaces $\{X_i; i \in I\}$, then*
$$(\Sigma_{i\in I} L_i)^*((p_i)_i,(x_i)_i) = \Sigma_{i\in I} L_i^*(p_i,x_i).$$
9. *If $L \in \mathscr{L}(X)$ and $M \in \mathscr{L}(Y)$, where X and Y are two reflexive spaces, then for any bounded linear operator $A : X \to Y^*$, we have*

$$(L \oplus_A M)^*((p,q),(x,y)) = L^*(A^*y + p, x) + M^*(-Ax + q, y).$$

10. If φ and ψ are convex functions on $X \times Y$ and A is any bounded linear operator $A : X \to Y^*$, then the Lagrangian L defined on $(X \times Y) \times (X^* \times Y^*)$ by $L((x,y),(p,q)) = \varphi(x,y) + \psi^*(A^*y + p, -Ax + q)$ has a Legendre transform

$$L^*((p,q),(x,y)) = \psi(x,y) + \varphi^*(A^*y + p, -Ax + q).$$

Proof. (1) is obvious.

To prove (2) fix $(q,y) \in X^* \times X$ and use the formula $(\varphi \star \psi)^* \le \varphi^* + \psi^*$ in one variable on the functions $\varphi(p) = L(z,p)$ and $\psi(p) = M(v,p)$ to write

$$\begin{aligned}
(L \star M)^*(q,y) &= \sup\{\langle q,x\rangle + \langle y,p\rangle - L(z,p) - M(x-z,p); (z,x,p) \in X \times X \times X^*\} \\
&= \sup\{\langle q,v+z\rangle + \langle y,p\rangle - L(z,p) - M(v,p); (z,v,p) \in X \times X \times X^*\} \\
&\le \sup_{(z,v)\in X\times X} \{\langle q,v+z\rangle + \sup\{\langle y,p\rangle - L(z,p) - M(v,p); p \in X^*\}\} \\
&\le \sup_{(z,v)\in X\times X} \Big\{\langle q,v+z\rangle + \inf_{w\in X}\{\sup_{p_1\in X^*}(\langle w,p_1\rangle - L(z,p_1)) \\
&\qquad\qquad + \sup_{p_2\in X^*}(\langle y-w,p_2\rangle - M(v,p_2))\}\Big\} \\
&\le \inf_{w\in X}\Big\{ \sup_{(z,p_1)\in X\times X^*} \{\langle q,z\rangle + \langle w,p_1\rangle - L(z,p_1))\} \\
&\qquad\qquad + \sup_{(v,p_2)\in X\times X^*} \{\langle q,v\rangle + \langle y-w,p_2\rangle - M(v,p_2)\Big\} \\
&= \inf_{w\in X}\{L^*(q,w) + M^*(q,y-w)\} \\
&= (L^* \oplus M^*)(q,y).
\end{aligned}$$

For (3), assume that $M(x,p) = \varphi(Ux) + \psi^*(V^*p)$, where φ and ψ are convex continuous functions and U and V are automorphisms of X. Fix $(q,y) \in X^* \times X$ and write

$$\begin{aligned}
(L \star M)^*(q,y) &= \sup\{\langle q,x\rangle + \langle y,p\rangle - L(z,p) - M(x-z,p); (z,x,p) \in X \times X \times X^*\} \\
&= \sup\{\langle q,v+z\rangle + \langle y,p\rangle - L(z,p) - M(v,p); (z,v,p) \in X \times X \times X^*\} \\
&= \sup_{p\in X^*}\Big\{\langle y,p\rangle + \sup_{(z,v)\in X^2}\{\langle q,v+z\rangle - L(z,p) - \varphi(Uv)\} - \psi^*(V^*p)\Big\} \\
&= \sup_{p\in X^*}\Big\{\langle y,p\rangle + \sup_{z\in X}\{\langle q,z\rangle - L(z,p)\} \\
&\qquad\qquad + \sup_{v\in X}\{\langle q,v\rangle - \varphi(Uv)\} - \psi^*(V^*p)\Big\} \\
&= \sup_{p\in X^*}\Big\{\langle y,p\rangle + \sup_{z\in X}\{\langle q,z\rangle - L(z,p)\} + \varphi^*((U^{-1})^*q) - \psi^*(V^*p)\Big\}
\end{aligned}$$

$$
\begin{aligned}
&= \sup_{p\in X^*}\sup_{z\in X}\{\langle y,p\rangle + \langle q,z\rangle - L(z,p) - \psi^*(V^*p)\} + \varphi^*((U^{-1})^*q)\\
&= (L+T)^*(q,y) + \varphi^*((U^{-1})^*q),
\end{aligned}
$$

where $T(z,p) := \psi^*(V^*p)$ for all $(z,p)\in X\times X^*$. Note now that

$$
T^*(q,y) = \sup_{z,p}\{\langle q,z\rangle + \langle y,p\rangle - \psi^*(V^*p)\} = \begin{cases} +\infty & \text{if } q\neq 0,\\ \psi((V^{-1}y) & \text{if } q=0,\end{cases}
$$

in such a way that by using the duality between sums and convolutions in both variables, we get

$$
\begin{aligned}
(L+T)^*(q,y) &= \operatorname{conv}(L^*,T^*)(q,y)\\
&= \inf_{r\in X^*, z\in X}\{L^*(r,z) + T^*(-r+q,-z+y)\}\\
&= \inf_{z\in X}\left\{L^*(q,z) + \psi(V^{-1}(-z+y))\right\}.
\end{aligned}
$$

Finally,

$$
\begin{aligned}
(L\star M)^*(q,y) &= (L+T)^*(q,y) + \varphi^*((U^{-1})^*q)\\
&= \inf_{z\in X}\left\{L^*(q,z) + \psi(V^{-1}(-z+y))\right\} + \varphi^*((U^{-1})^*q)\\
&= \inf_{z\in X}\{L^*(q,z) + (\varphi\circ U)^*(q) + (\psi^*\circ V^*)^*(-z+y)\}\\
&= (L^*\oplus M^*)(q,y).
\end{aligned}
$$

For (4), again fix $(q,y)\in X^*\times X$, and write

$$
\begin{aligned}
(L\star M)^*(q,y) &= \sup_{(z,x,p)\in X\times X\times X^*}\{\langle q,x\rangle + \langle y,p\rangle - L(z,p) - M(x-z,p)\}\\
&= \sup_{(z,v,p)\in X\times X\times X^*}\{\langle q,v+z\rangle + \langle y,p\rangle - L(z,p) - M(v,p)\}\\
&= \sup_{(z,v,p)\in X\times X\times X^*}\{-\varphi^*(-z,-v,-p) - \psi^*(z,v,p)\}
\end{aligned}
$$

with $\varphi^*(z,v,p) = \langle q,z\rangle + L(-z,-p)$ and $\psi^*(z,v,p) = -\langle y,p\rangle - \langle q,v\rangle + M(v,p)$. Note that now

$$
\begin{aligned}
\varphi(r,s,x) &= \sup_{(z,v,p)\in X\times X\times X^*}\{\langle r,z\rangle + \langle v,s\rangle + \langle x,p\rangle - \langle q,z\rangle - L(-z,-p)\}\\
&= \sup_{(z,v,p)\in X\times X\times X^*}\{\langle r-q,z\rangle + \langle v,s\rangle + \langle x,p\rangle - L(-z,-p)\}\\
&= \sup_{v\in X}\{\langle v,s\rangle + L^*(q-r,-x)\},
\end{aligned}
$$

which is equal to $+\infty$ whenever $s\neq 0$. Similarly, we have

$$\begin{aligned}
\psi(r,s,x) &= \sup_{(z,v,p)\in X\times X\times X^*} \{\langle r,z\rangle + \langle v,s\rangle + \langle x,p\rangle + \langle y,p\rangle + \langle v,q\rangle - M(v,p)\} \\
&= \sup_{(z,v,p)\in X\times X\times X^*} \{\langle r,z\rangle + \langle v,q+s\rangle + \langle x+y,p\rangle - M(v,p)\} \\
&= \sup_{z\in X}\{\langle z,r\rangle + M^*(q+s,x+y)\},
\end{aligned}$$

which is equal to $+\infty$ whenever $r\neq 0$. If now $\mathrm{Dom}_2(L^*) - \mathrm{Dom}_2(M^*)$ contains a neighborhood of the origin, then we apply the theorem of Fenchel and Rockafellar to get

$$\begin{aligned}
(L\star M)^*(q,y) &= \sup\{-\varphi^*(-z,-v,-p) - \psi^*(z,v,p);\,(z,v,p)\in X\times X\times X^*\} \\
&= \inf\{\varphi(r,s,x)+\psi(r,s,x);\,(r,s,x)\in X^*\times X^*\times X\} \\
&= \inf_{(r,s,x)\in X^*\times X^*\times X} \Big\{ \sup_{v\in X}\{\langle v,s\rangle + L^*(q-r,-x)\} \\
&\qquad\qquad + \sup_{z\in X}\{\langle z,r\rangle + M^*(q+s,x+y)\}\Big\} \\
&= \inf\{L^*(q,-x)+M^*(q,x+y); x\in X\} \\
&= (L^*\oplus M^*)(q,y).
\end{aligned}$$

Assertion (5) can be proved in a similar fashion.

For (6), fix $(q,y)\in X^*\times X$, set $r=\Gamma x+p$ and write

$$\begin{aligned}
(L_\Gamma)^*(q,y) &= \sup\{\langle q,x\rangle + \langle y,p\rangle - L(x,-\Gamma x+p);\,(x,p)\in X\times X^*\} \\
&= \sup\{\langle q,x\rangle + \langle y,r+\Gamma x\rangle - L(x,r);\,(x,r)\in X\times X^*\} \\
&= \sup\{\langle q+\Gamma^* y,x\rangle + \langle y,r\rangle - L(x,r);\,(x,r)\in X\times X^*\} \\
&= L^*(q+\Gamma^* y,y).
\end{aligned}$$

For (7), let $r = x-\Gamma^{-1}p$ and $s=\Gamma x$ and write

$$\begin{aligned}
({}_\Gamma L)^*(q,y) &= \sup\{\langle q,x\rangle + \langle y,p\rangle - L(x-\Gamma^{-1}p,\Gamma x);\,(x,p)\in X\times X^*\} \\
&= \sup\{\langle q,\Gamma^{-1}s\rangle + \langle y,s-\Gamma r\rangle - L(r,s);\,(r,s)\in X\times X^*\} \\
&= \sup\{\langle (\Gamma^{-1})^* q+y,s\rangle - \langle \Gamma^* y,r\rangle - L(r,s);\,(r,s)\in X\times X^*\} \\
&= L^*(-\Gamma^* y,(\Gamma^{-1})^* q+y).
\end{aligned}$$

The proof of (8) is obvious, while for (9) notice that if $(\tilde z,\tilde r)\in (X\times Y)\times(X^*\times Y^*)$, where $\tilde z=(x,y)$ and $\tilde r=(p,q)$, we can write

$$L\oplus_A M(\tilde z,\tilde r) = (L+M)(\tilde z,\tilde A\tilde z+\tilde r),$$

where $\tilde A: X\times Y\to X^*\times Y^*$ is the skew-adjoint operator defined by $\tilde A(\tilde z)=\tilde A((x,y)) = (-A^*y,Ax)$. Now apply (6) and (8) to $L+M$ and $\tilde A$ to obtain

$$(L\oplus_A M)^*((p,q),(x,y)) = (L+M)^*(\tilde r+\tilde A^*\tilde z,\tilde z)$$

$$= (L^* + M^*)(\tilde{r} - \tilde{A}\tilde{z}, \tilde{z})$$
$$= L^*(A^*y + p, x) + M^*(-Ax + q, y).$$

Assertion (10) follows again from (6) since the Lagrangian $M((x,y),(p,q)) = \varphi(x,y) + \psi^*(-A^*y - p, Ax - q)$ is of the form $L((x,y), \tilde{A}(x,y) + (p,q))$, where $L((x,y),(p,q)) = \varphi(x,y) + \psi^*(p,q)$ and $\tilde{A} : X \times Y \to X^* \times Y^*$ is again the skew-adjoint operator defined by $\tilde{A}((x,y)) = (-A^*y, Ax)$. The Legendre transform is then equal to $L^*((p,q),(x,y)) = \psi(x,y) + \varphi^*(A^*y + p, -Ax + q)$.

2.6 Legendre transforms on various path spaces

Legendre transform on the path space $L^\alpha([0,T],X)$

For $1 < \alpha < +\infty$, we consider the space $L^\alpha_X[0,T]$ of Bochner integrable functions from $[0,T]$ into X with norm

$$\|u\|_{L_\alpha(X)} = \left(\int_0^T \|u(t)\|_X^\alpha dt \right)^{\frac{1}{\alpha}}.$$

Definition 2.4. Let $[0,T]$ be a time interval and let X be a reflexive Banach space. A *time-dependent convex function on* $[0,T] \times X$ (*resp., a time-dependent convex Lagrangian on* $[0,T] \times X \times X^*$) is a function $\varphi : [0,T] \times X \to \mathbf{R} \cup \{+\infty\}$ (resp., $L : [0,T] \times X \times X^* \to \mathbf{R} \cup \{+\infty\}$) such that :

1. φ (resp., L) is measurable with respect to the σ-field generated by the products of Lebesgue sets in $[0,T]$ and Borel sets in X (resp., in $X \times X^*$).
2. For each $t \in [0,T]$, the function $\varphi(t,\cdot)$ (resp., $L(t,\cdot,\cdot)$) is convex and lower semi-continuous on X (resp., $X \times X^*$).

The Hamiltonian H_L of L is the function defined on $[0,T] \times X \times X^*$ by

$$H_L(t,x,y) = \sup\{\langle y,p\rangle - L(t,x,p); p \in X^*\}.$$

To each time-dependent Lagrangian L on $[0,T] \times X \times X^*$, one can associate the corresponding Lagrangian $\mathscr{L}$ on the path space $L^\alpha_X \times L^\beta_{X^*}$, where $\frac{1}{\alpha} + \frac{1}{\beta} = 1$ to be

$$\mathscr{L}(u,p) := \int_0^T L(t,u(t),p(t))dt,$$

as well as the associated Hamiltonian on $L^\alpha_X \times L^\alpha_X$,

$$H_{\mathscr{L}}(u,v) = \sup\left\{ \int_0^T (\langle p(t),v(t)\rangle - L(t,u(t),p(t)))dt \ ; p \in L^\beta_{X^*} \right\}$$

The Fenchel-Legendre dual of $\mathscr{L}$ is defined for any $(q,v) \in L^{\beta}_{X^*} \times L^{\alpha}_{X}$ as

$$\mathscr{L}^*(q,v) = \sup_{(u,p)\in L^{\alpha}_{X}\times L^{\beta}_{X^*}} \int_0^T \{\langle q(t),u(t)\rangle + \langle p(t),v(t)\rangle - L(t,u(t),p(t))\}\,dt.$$

Proposition 2.7 immediately yields the following.

Proposition 2.9. *Suppose that L is a Lagrangian on $[0,T]\times X\times X^*$, and let $\mathscr{L}$ be the corresponding Lagrangian on the path space $L^{\alpha}_{X}\times L^{\beta}_{X^*}$. Then,*

1. $\mathscr{L}^*(p,u) = \int_0^T L^*(t,p(t),u(t))dt$.
2. $H_{\mathscr{L}}(u,v) = \int_0^T H_L(t,u(t),v(t))dt$.

Suppose now that H is a Hilbert space, and consider the space A^2_H of all functions in L^2_H such that $\dot{u}\in L^2_H$ equipped with the norm

$$\|u\|_{A^2_H} = (\|u\|^2_{L^2_H} + \|\dot{u}\|^2_{L^2_H})^{1/2}.$$

Theorem 2.3. *Suppose ℓ is a convex lower semicontinuous function on $H\times H$, and let L be a time-dependent Lagrangian on $[0,T]\times H\times H$ such that*

$$\text{For each } p\in L^2_H, \text{ the map } u\to \int_0^T L(t,u(t),p(t))dt \text{ is continuous on } L^2_H. \tag{2.16}$$

$$\text{The map } u\to \int_0^T L(t,u(t),0)dt \text{ is bounded on the unit ball of } L^2_H. \tag{2.17}$$

$$-C\le \ell(a,b)\le \tfrac{1}{2}(1+\|a\|^2_H+\|b\|^2_H) \text{ for all } (a,b)\in H\times H. \tag{2.18}$$

Consider the following Lagrangian on $L^2_H\times L^2_H$:

$$\mathscr{L}(u,p) = \begin{cases} \int_0^T L(t,u(t),p(t)-\dot{u}(t))dt + \ell(u(0),u(T)) & \text{if } u\in A^2_H \\ +\infty & \text{otherwise.}\end{cases}$$

The Legendre transform of $\mathscr{L}$ is then

$$\mathscr{L}^*(p,u) = \begin{cases} \int_0^T L^*(t,p(t)-\dot{u}(t),u(t))dt + \ell^*(-u(0),u(T)) & \text{if } u\in A^2_H \\ +\infty & \text{otherwise.}\end{cases}$$

Proof. For $(q,v)\in L^2_H\times A^2_H$, write

$$\begin{aligned}\mathscr{L}^*(q,v) &= \sup_{u\in L^2_H}\sup_{p\in L^2_H}\Big\{\int_0^T (\langle u(t),q(t)\rangle + \langle v(t),p(t)\rangle - L(t,u(t),p(t)-\dot{u}(t)))dt \\ &\qquad -\ell(u(0),u(T))\Big\} \\ &= \sup_{u\in A^2_H}\sup_{p\in L^2_H}\Big\{\int_0^T (\langle u(t),q(t)\rangle + \langle v(t),p(t)\rangle - L(t,u(t),p(t)-\dot{u}(t)))dt \\ &\qquad -\ell(u(0),u(T))\Big\}.\end{aligned}$$

Make a substitution $p(t) - \dot{u}(t) = r(t) \in L^2_H$. Since u and v are both in A^2_H, we have

$$\int_0^T \langle v, \dot{u} \rangle = -\int_0^T \langle \dot{v}, u \rangle + \langle v(T), u(T) \rangle - \langle v(0), u(0) \rangle.$$

Since the subspace $A^{2,0}_H = \{u \in A^2_H; u(0) = u(T) = 0\}$ is dense in L^2_H, and since $u \to \int_0^T L(t, u(t), p(t))dt$ is continuous on L^2_H for each p, we obtain

$$\begin{aligned}
\mathscr{L}^*(q,v) &= \sup_{u \in A^2_H} \sup_{r \in L^2_H} \Big\{ \int_0^T \{\langle u(t), q(t) \rangle + \langle v(t), r(t) + \dot{u}(t) \rangle - L(t, u(t), r(t))\} dt \\
&\qquad\qquad -\ell(u(0), u(T))\} \\
&= \sup_{u \in A^2_H} \sup_{r \in L^2_H} \Big\{ \int_0^T \{\langle u(t), q(t) - \dot{v}(t) \rangle + \langle v(t), r(t) \rangle - L(t, u(t), r(t))\} dt \\
&\qquad\qquad + \langle v(T), u(T) \rangle - \langle v(0), u(0) \rangle - \ell(u(0), u(T)) \Big\} \\
&= \sup_{u \in A^2_H} \sup_{r \in L^2_H} \sup_{u_0 \in A^{2,0}_H} \Big\{ \int_0^T \{\langle u, q - \dot{v} \rangle + \langle v, r \rangle - L(t, u(t), r(t))\} dt \\
&\qquad\qquad + \langle v(T), (u + u_0)(T) \rangle - \langle v(0), (u + u_0)(0) \rangle \Big\} \\
&\qquad\qquad - \ell((u + u_0)(0), (u + u_0)(T)) \Big\} \\
&= \sup_{w \in A^2_H} \sup_{r \in L^2_H} \sup_{u_0 \in A^{2,0}_H} \Big\{ \int_0^T \langle w(t) - u_0(t), q(t) + \dot{v}(t) \rangle + \langle v(t), r(t) \rangle \, dt \\
&\qquad\qquad - \int_0^T L(t, w(t) - u_0(t), r(t)) \, dt \Big\} \\
&\qquad\qquad + \langle v(T), w(T) \rangle - \langle v(0), w(0) \rangle - \ell(w(0), w(T)) \Big\} \\
&= \sup_{w \in A^2_H} \sup_{r \in L^2_H} \sup_{x \in L^2_H} \Big\{ \int_0^T \{\langle x, q - \dot{v} \rangle + \langle v(t), r(t) \rangle - L(t, x(t), r(t))\} dt \\
&\qquad\qquad + \langle v(T), w(T) \rangle - \langle v(0), w(0) \rangle - \ell(w(0), w(T)) \Big\}.
\end{aligned}$$

Now, for each $(a,b) \in H \times H$, there is $w \in A^2_H$ such that $w(0) = a$ and $w(T) = b$, namely the linear path $w(t) = \frac{(T-t)}{T} a + \frac{t}{T} b$. Since ℓ is continuous on H, we finally obtain that

$$\begin{aligned}
\mathscr{L}^*(q,v) &= \sup_{(a,b) \in H \times H} \sup_{(r,x) \in L^2_H \times L^2_H} \Big\{ \int_0^T \{\langle x, q - \dot{v} \rangle + \langle v, r \rangle - L(t, x(t), r(t))\} dt \\
&\qquad\qquad + \langle v(T), b \rangle - \langle v(0), a \rangle - \ell(a, b) \Big\} \\
&= \sup_{x \in L^2_H} \sup_{r \in L^2_H} \Big\{ \int_0^T \{\langle x(t), q(t) - \dot{v}(t) \rangle + \langle v(t), r(t) \rangle - L(t, x(t), r(t))\} dt \Big\}
\end{aligned}$$

$$+\sup_{a\in H}\sup_{b\in H}\left\{\langle v(T),b\rangle-\langle v(0),a\rangle-\ell(a,b)\right\}$$
$$=\int_0^T L^*(t,q(t)-\dot v(t),v(t))dt+\ell^*(-v(0),v(T)).$$

If now $(q,v)\in L^2_H\times L^2_H\setminus A^2_H$, then we use the fact that $u\to\int_0^T L(t,u(t),0)dt$ is bounded on the unit ball of A^2_H and the growth condition on ℓ to deduce that

$$\begin{aligned}\mathscr{L}^*(q,v)\geq\sup_{u\in A^2_H}\sup_{r\in A^2_H}&\left\{\int_0^T\langle u(t),q(t)\rangle+\langle v(t),r(t)\rangle+\langle v(t),\dot u(t)\rangle-L(t,u(t),r(t))dt\right.\\&\left.-\ell(u(0),u(T))\right\}\\ \geq\sup_{u\in A^2_H}\sup_{r\in A^2_H}&\left\{-\|u\|_{L^2_H}\|q\|_{L^2_H}-\|v\|_{L^2_H}\|r\|_{L^2_H}+\int_0^T\langle v,\dot u\rangle-L(t,u(t),r(t))dt\right.\\&\left.-\ell(u(0),u(T))\right\}\\ \geq\sup_{\|u\|_{A^2_H}\leq 1}&\left\{-\|q\|_2+\int_0^T\{\langle v(t),\dot u(t)\rangle-L(t,u(t),0)\}dt-\ell(u(0),u(T))\right\}\\ \geq\sup_{\|u\|_{A^2_H}\leq 1}&\left\{C+\int_0^T\langle v(t),\dot u(t)\rangle-L(t,u,0)dt-\frac{1}{2}(\|u(0)\|^2+\|u(T)\|^2)\right\}\\ \geq\sup_{\|u\|_{A^2_H}\leq 1}&\left\{D+\int_0^T\langle v(t),\dot u(t)\rangle dt-\frac{1}{2}(\|u(0)\|_H^2+\|u(T)\|_H^2)\right\}.\end{aligned}$$

Since now v does not belong to A^2_H, we have that

$$\sup_{\|u\|_{A^2_H}\leq 1}\left\{\int_0^T\langle v(t),\dot u(t)\rangle dt-\frac{1}{2}(\|u(0)\|_H^2+\|u(T)\|_H^2)\right\}=+\infty,$$

which means that $\mathscr{L}^*(q,v)=+\infty$.

Legendre transform on spaces of absolutely continuous functions

Consider now the path space $A^2_H=\{u:[0,T]\to H;\,\dot u\in L^2_H\}$ equipped with the norm

$$\|u\|_{A^2_H}=\left(\|u(0)\|_H^2+\int_0^T\|\dot u\|^2dt\right)^{\frac{1}{2}}.$$

One way to represent the space A^2_H is to identify it with the product space $H\times L^2_H$ in such a way that its dual $(A^2_H)^*$ can also be identified with $H\times L^2_H$ via the formula

$$\langle u,(p_1,p_0)\rangle_{A^2_H, H\times L^2_H} = \langle u(0),p_1\rangle_H + \int_0^T \langle \dot u(t), p_0(t)\rangle dt, \tag{2.19}$$

where $u\in A^2_H$ and $(p_1,p_0)\in H\times L^2_H$. With this duality, we have the following theorem.

Theorem 2.4. *Let L be a time-dependent convex Lagrangian on $[0,T]\times H\times H$ and let ℓ be a proper convex lower semicontinuous function on $H\times H$. Consider the Lagrangian on $A^2_H\times(A^2_H)^* = A^2_H\times(H\times L^2_H)$ defined by*

$$\mathscr{N}(u,p) = \int_0^T L(t,u(t)-p_0(t),-\dot u(t))dt + \ell(u(0)-a,u(T)), \tag{2.20}$$

where $u\in A^2_H$ and $(p_0(t),a)\in L^2_H\times H$ represents an element p in the dual of A^2_H. Then, for any $(v,q)\in A^2_H\times(A^2_H)^*$ with q of the form $(q_0(t),0)$, we have

$$\mathscr{N}^*(q,v) = \int_0^T L^*(t,-\dot v(t),v(t)-q_0(t),)dt + \ell^*(-v(0),v(T)). \tag{2.21}$$

Proof. For $(v,q)\in A^2_H\times(A^2_H)^*$ with q represented by $(q_0(t),0)$, write

$$\begin{aligned}\mathscr{N}^*(q,v) = \sup_{p_1\in H}\sup_{p_0\in L^2_H}\sup_{u\in A^2_H}\Big\{&\langle p_1,v(0)\rangle + \int_0^T \langle p_0(t),\dot v(t)\rangle + \langle q_0(t),\dot u(t)\rangle\, dt\\ &- \int_0^T L(t,u(t)-p_0(t),-\dot u(t))\,dt - \ell(u(0)-p_1,u(T))\Big\}.\end{aligned}$$

Making a substitution $u(0)-p_1 = a\in H$ and $u(t)-p_0(t)=y(t)\in L^2_H$, we obtain

$$\begin{aligned}\mathscr{N}^*(q,v) = \sup_{a\in H}\sup_{y\in L^2_H}\sup_{u\in A^2_H}\Big\{&\langle u(0)-a,v(0)\rangle - \ell(a,u(T))\\ &+ \int_0^T \{\langle u(t)-y(t),\dot v(t)\rangle + \langle q_0(t),\dot u(t)\rangle - L(t,y(t),-\dot u(t))\}dt\Big\}.\end{aligned}$$

Since $\dot u$ and $\dot v\in L^2_H$, we have

$$\int_0^T \langle u,\dot v\rangle = -\int_0^T \langle \dot u, v\rangle + \langle v(T),u(T)\rangle - \langle v(0),u(0)\rangle,$$

which implies

$$\begin{aligned}\mathscr{N}^*(q,v) = \sup_{a\in H}\sup_{y\in L^2_H}\sup_{u\in A^2_H}\Big\{&\langle -a,v(0)\rangle + \langle v(T),u(T)\rangle - \ell(a,u(T))\\ &\int_0^T [-\langle y(t),\dot v(t)\rangle + \langle v(t)-q_0(t),-\dot u(t)\rangle - L(t,y(t),-\dot u(t))]\,dt\Big\}.\end{aligned}$$

Now identify A^2_H with $H\times L^2_H$ via the correspondence

$$(b,r) \in H \times L^2_H \mapsto b + \int_t^T r(s)\,ds \in A^2_H,$$
$$u \in A^2_H \mapsto \big(u(T), -\dot u(t)\big) \in H \times L^2_H.$$

We finally obtain

$$\begin{aligned}
\mathscr{N}^*(q,v) &= \sup_{a\in H}\sup_{b\in H}\Big\{\langle a, -v(0)\rangle + \langle v(T), b\rangle - \ell(a,b)\Big\} \\
&+ \sup_{y\in L^2_H}\sup_{r\in L^2_H}\Big\{\int_0^T -\langle y(t), \dot v(t)\rangle + \langle v(t) - q_0(t), r(t)\rangle - L(t,y(t),r(t))dt\Big\} \\
&= \int_0^T L^*(t, -\dot v(t), v(t) - q_0(t))dt + \ell^*(-v(0), v(T)).
\end{aligned}$$

Legendre transform for a symmetrized duality on spaces of absolutely continuous functions

Consider again $A^2_H := \{u : [0,T] \to H; \dot u \in L^2_H\}$ equipped with the norm

$$\|u\|_{A^2_H} = \left\{\left\|\frac{u(0)+u(T)}{2}\right\|_H^2 + \int_0^T \|\dot u\|_H^2\,dt\right\}^{\frac{1}{2}}.$$

We can again identify the space A^2_H with the product space $H \times L^2_H$ in such a way that its dual $(A^2_H)^*$ can also be identified with $H \times L^2_H$ via the formula

$$\Big\langle u, (p_1, p_0)\Big\rangle_{A^2_H, H\times L^2_H} = \Big\langle \frac{u(0)+u(T)}{2}, p_1\Big\rangle + \int_0^T \langle \dot u(t), p_0(t)\rangle\,dt,$$

where $u \in A^2_H$ and $(p_1, p_0(t)) \in H \times L^2_H$.

Theorem 2.5. *Suppose L is a time-dependent Lagrangian on $[0,T]\times H \times H$ and ℓ is a Lagrangian on $H \times H$. Consider the following Lagrangian defined on the space $A^2_H \times (A^2_H)^* = A^2_H \times (H \times L^2_H)$ by*

$$\mathscr{M}(u,p) = \int_0^T L\big(t, u(t) + p_0(t), -\dot u(t)\big)\,dt + \ell\Big(u(T) - u(0) + p_1, \frac{u(0)+u(T)}{2}\Big).$$

The Legendre transform of $\mathscr{M}$ on $A^2_H \times (L^2_H \times H)$ is given by

$$\mathscr{M}^*(p,u) = \int_0^T L^*\big(t, -\dot u(t), u(t) + p_0(t)\big)\,dt + \ell^*\Big(\frac{u(0)+u(T)}{2}, u(T) - u(0) + p_1\Big).$$

Proof. For $(q,v) \in A^2_H \times (A^2_H)^*$ with q represented by $(q_0(t), q_1)$, we have

$$\begin{aligned}\mathscr{M}^*(q,v) = \sup_{p_1\in H} \sup_{p_0\in L^2_H} \sup_{u\in A^2_H} \Big\langle p_1, \frac{v(0)+v(T)}{2}\Big\rangle + \Big\langle q_1, \frac{u(0)+u(T)}{2}\Big\rangle \\ - \int_0^T \left[\langle p_0(t),\dot v(t)\rangle + \langle q_0(t),\dot u\rangle - L\big(t,u(t)+p_0(t),-\dot u(t)\big)\right] dt \\ -\ell\big(u(T)-u(0)+p_1, \frac{u(0)+u(T)}{2}\big)\Big\}.\end{aligned}$$

Making a substitution $u(T)-u(0)+p_1 = a \in H$ and $u(t)+p_0(t) = y(t) \in L^2_H$, we obtain

$$\begin{aligned}\mathscr{M}^*(q,v) = \sup_{a\in H} \sup_{y\in L^2_H} \sup_{u\in A^2_H} \Big\langle a-u(T)+u(0), \frac{v(0)+v(T)}{2}\Big\rangle + \Big\langle q_1, \frac{u(0)+u(T)}{2}\Big\rangle \\ - \int_0^T \left[\langle y(t)-u(t),\dot v\rangle + \langle q_0(t),\dot u(t)\rangle - L\big(t,y(t),-\dot u(t)\big)\right] dt \\ -\ell\Big(a, \frac{u(0)+u(T)}{2}\Big)\Big\}.\end{aligned}$$

Again, since $\dot u$ and $\dot v \in L^2_H$, we have

$$\int_0^T \langle u(t),\dot v(t)\rangle\, dt = -\int_0^T \langle \dot u(t), v(t)\rangle\, dt + \langle u(T),v(T)\rangle - \langle v(0),u(0)\rangle,$$

which implies

$$\begin{aligned}\mathscr{M}^*(q,v) = \sup_{a\in H} \sup_{y\in L^2_H} \sup_{u\in A^2_H} \Big\{ \Big\langle a, \frac{v(0)+v(T)}{2}\Big\rangle - \Big\langle u(T), \frac{v(0)+v(T)}{2} - v(T)\Big\rangle \\ -\Big\langle u(0), v(0) - \frac{v(0)+v(T)}{2}\Big\rangle + \Big\langle q_1, \frac{u(0)+u(T)}{2}\Big\rangle \\ - \int_0^T \left[\langle y(t),\dot v\rangle + \langle \dot u(t), v(t)+q_0(t)\rangle - L\big(t,y(t),-\dot u(t)\big)\right] dt \\ -\ell\Big(a, \frac{u(0)+u(T)}{2}\Big)\Big\}.\end{aligned}$$

Hence,

$$\begin{aligned}\mathscr{M}^*(q,v) = \sup_{a\in H, y\in L^2_H, u\in A^2_H} \Big\langle a, \frac{v(0)+v(T)}{2}\Big\rangle + \Big\langle q_1 + v(T) - v(0), \frac{u(0)+u(T)}{2}\Big\rangle \\ -\ell\Big(a, \frac{u(0)+u(T)}{2}\Big) \\ - \int_0^T \left[\langle y(t),\dot v(t)\rangle + \langle \dot u(t), v(t)+q_0(t)\rangle - L\big(t,y(t),-\dot u(t)\big)\right] dt \Big\}.\end{aligned}$$

Now identify A_H^2 with $H \times L_H^2$ via the correspondence:

$$\big(b, f(t)\big) \in H \times L_H^2 \longmapsto b + \frac{1}{2}\left(\int_t^T f(s)\,ds - \int_0^t f(s)\,ds\right) \in A_H^2,$$
$$u \in A_H^2 \longmapsto \big(\frac{u(0)+u(T)}{2}, -\dot{u}(t)\big) \in H \times L_H^2.$$

We finally obtain

$$\begin{aligned}\mathscr{M}^*(q,v) &= \sup_{a\in H}\sup_{b\in H}\Big\{\langle a, \frac{v(0)+v(T)}{2}\rangle + \langle q_1 + v(T) - v(0), b\rangle - \ell(a,b)\Big\} \\ &\quad + \sup_{y\in L_H^2 r\in L_H^2}\Big\{\int_0^T -\langle y(t), \dot{v}(t)\rangle + \langle v(t) + q_0(t), r(t)\rangle - L\big(t, y(t), r(t)\big)dt\Big\} \\ &= \ell^*\big(\frac{v(0)+v(T)}{2}, q_1 + v(T) - v(0)\big) + \int_0^T L^*\big(t, -\dot{v}(t), v(t) + q_0(t)\big)\,dt.\end{aligned}$$

Exercises 2.B. Legendre transforms on path spaces

1. Prove Proposition 2.9.
2. Establish the identification between the Hilbert spaces $A_H^2[0,T]$ and $H \times L_H^2$ via the isomorphism $u \in A_H^2 \mapsto \big(u(T), -\dot{u}(t)\big) \in H \times L_H^2$.
3. Establish the identification between the Hilbert spaces $A_H^2[0,T]$ and $H \times L_H^2$ via the isomorphism $u \in A_H^2 \mapsto \big(\frac{u(0)+u(T)}{2}, -\dot{u}(t)\big) \in H \times L_H^2$.
4. Show that the Legendre transform of the Lagrangian on $L_H^2 \times L_H^2$

$$\mathscr{L}(u,p) = \begin{cases} \int_0^T L(t, u(t), p(t) - \dot{u}(t))dt + \ell\big(u(T) - u(0), \frac{u(0)+u(T)}{2}\big) & \text{if } u \in A_H^2 \\ +\infty & \text{otherwise}\end{cases}$$

is

$$\mathscr{L}^*(p,u) = \begin{cases} \int_0^T L^*(t, p(t) - \dot{u}(t), u(t))dt + \ell^*\big(\frac{u(0)+u(T)}{2}, u(T) - u(0)\big) & \text{if } u \in A_H^2 \\ +\infty & \text{otherwise,}\end{cases}$$

provided the conditions of Theorem 2.3 are satisfied.

2.7 Primal and dual problems in convex optimization

Consider the problem of minimizing a convex lower semicontinuous function I that is bounded below on a Banach space X. This is usually called the *primal problem*:

$$(\mathscr{P}) \qquad \inf_{x\in X} I(x). \tag{2.22}$$

One can sometimes associate to I a family of perturbed problems in the following way. Let Y be another Banach space, and consider a convex lower semicontinuous

Lagrangian $L : X \times Y \to \mathbf{R} \cup \{+\infty\}$ such that the following holds:

$$I(x) = L(x,0) \quad \text{for all } x \in X. \tag{2.23}$$

For any $p \in Y$, one can consider the perturbed minimization problem

$$(\mathscr{P}_p) \qquad \inf_{x \in X} L(x,p) \tag{2.24}$$

in such a way that $(\mathscr{P}_0)$ is clearly the initial primal problem. By considering the Legendre transform L^* of L on the dual space $X^* \times Y^*$, one can consider the so-called *dual problem*

$$(\mathscr{P}^*) \qquad \sup_{p^* \in Y^*} -L^*(0,p^*). \tag{2.25}$$

Consider the function $h : Y \to \mathbf{R} \cup \{+\infty\}$ on the space of perturbations Y defined by

$$h(p) = \inf_{x \in X} L(x,p) \quad \text{for every } p \in Y. \tag{2.26}$$

The following proposition summarizes the relationship between the primal problem and the behavior of the value function h.

Theorem 2.6. *Assume L is a proper convex lower semicontinuous Lagrangian that is bounded below on $X \times Y$. Then, the following assertions hold:*

1. (Weak duality) $\quad -\infty < \sup_{p^* \in Y^*} \{-L^*(0,p^*)\} \le \inf_{x \in X} L(x,0) < +\infty.$

2. h is a convex function on Y such that $h^*(p^*) = L^*(0,p^*)$ for every $p^* \in Y^*$, and

$$h^{**}(0) = \sup_{p^* \in Y^*} \{-L^*(0,p^*)\}.$$

3. h is lower semicontinuous at 0 (i.e., $(\mathscr{P})$ is normal) if and only if there is no duality gap, i.e., if

$$\sup_{p^* \in Y^*} \{-L^*(0,p^*)\} = \inf_{x \in X} L(x,0).$$

4. h is subdifferentiable at 0 (i.e., $(\mathscr{P})$ is stable) if and only if $(\mathscr{P})$ is normal and $(\mathscr{P}^*)$ has at least one solution. Moreover, the set of solutions for $(\mathscr{P}^*)$ is equal to $\partial h^{**}(0)$.

5. If for some $x_0 \in X$ the function $p \to L(x_0,p)$ is bounded on a ball centered at 0 in Y, then $(\mathscr{P})$ is stable and $(\mathscr{P}^*)$ has at least one solution.

Proof. (1) For each $p^* \in Y^*$, we have

$$\begin{aligned} L^*(0,p^*) &= \sup\{\langle p^*,p\rangle - L(x,p); x \in X, p \in Y\} \\ &\ge \sup\{\langle p^*,0\rangle - L(x,0); x \in X\} \\ &= -\inf\{L(x,0); x \in X\}. \end{aligned}$$

(2) To prove the convexity of h, consider $\lambda \in (0,1)$ and elements $p,q \in Y$ such that $h(p)$ and $h(q)$ are finite. For every $a > h(p)$ (resp., $b > h(q)$), find $u \in X$ (resp., $v \in X$) such that

$$h(p) \leq L(x,p) \leq a \quad \text{and} \quad h(q) \leq L(v,q) \leq b.$$

Now use the convexity of L in both variables to write

$$\begin{aligned} h(\lambda p + (1-\lambda)q) &= \inf\{L(x,\lambda p + (1-\lambda)q); x \in X\} \\ &\leq L(\lambda u + (1-\lambda)v, \lambda p + (1-\lambda)q) \\ &\leq \lambda L(u,p) + (1-\lambda)L(v,q) \\ &\leq \lambda a + (1-\lambda)b, \end{aligned}$$

from which the convexity of h follows.

(3) Note first that the Legendre dual of h can be written for $p^* \in Y^*$ as

$$\begin{aligned} h^*(p^*) &= \sup\{\langle p^*, p\rangle - h(p); p \in Y\} \\ &= \sup\left\{\langle p^*, p\rangle - \inf_{x\in X}\{L(x,p); p \in Y\}\right\} \\ &= \sup\{\langle p^*, p\rangle - L(x,p); p \in Y, x \in X\} \\ &= L^*(0,p^*). \end{aligned}$$

It follows that

$$\sup_{p^*\in Y^*}\{-L^*(0,p^*)\} = \sup_{p^*\in Y^*} -h^*(p^*) = h^{**}(0) \leq h(0) = \inf_{x\in X} L(x,0). \tag{2.27}$$

Our claim follows from the fact that h is lower semicontinuous at 0 if and only if $h(0) = h^{**}(0)$.

For claim 4), we start by establishing that the set of solutions for $(\mathscr{P}^*)$ is equal to $\partial h^{**}(0)$. Indeed, if $p^* \in Y^*$ is a solution of $(\mathscr{P}^*)$, then

$$\begin{aligned} -h^*(p^*) &= -L^*(0,p^*) \\ &= \sup\{-L^*(0,q^*); q^* \in Y^*\} \\ &= \sup\{-h^*(q^*); q^* \in Y^*\} \\ &= \sup\{\langle 0, q^*\rangle - h^*(q^*); q^* \in Y^*\} \\ &= h^{**}(0), \end{aligned}$$

which is equivalent to $p^* \in \partial h^{**}(0)$.

Suppose now that $\partial h(0) \neq \emptyset$. Then, $h(0) = h^{**}(0)$ (i.e., $(\mathscr{P})$ is normal) and $\partial h(0) = \partial h^{**}(0) \neq \emptyset$, and hence $(\mathscr{P}^*)$ has at least one solution. Conversely, if h is lower semicontinuous at 0, then $h(0) = h^{**}(0)$, and if $\partial h^{**}(0) \neq \emptyset$, then $\partial h(0) = \partial h^{**}(0) \neq \emptyset$.

The condition in (5) readily implies that h is bounded above on a neighborhood of zero in Y^*, which implies that h is subdifferentiable at 0.

Further comments

The first four sections summarize the most basic concepts and relevant tools of convex analysis that will be used throughout this text. Proofs are not included, as they can be found in a multitude of books on convex analysis. We refer to the books of Aubin and Ekeland [8], Brézis [26], Ekeland and Temam [47], Ekeland [46], and Phelps [130].

The particularities of convex calculus on phase space were developed in Ghoussoub [55]. Legendre transforms on path space for the basic action functionals of the calculus of variations have already been dealt with by Rockafellar [137]. Theorem 2.4 is due to Ghoussoub and Tzou [68], while the new symmetrized duality for A_H^2 and the corresponding Legendre transform were first discussed in Ghoussoub and Moameni [63].

Springer

http://www.springer.com/978-0-387-84896-9

Self-dual Partial Differential Systems and Their
Variational Principles
Ghoussoub, N.
2009, XIV, 354 p., Hardcover
ISBN: 978-0-387-84896-9