Preface

How to solve partial differential systems by completing the square. This could well have been the title of this monograph as it grew into a project to develop a systematic approach for associating suitable nonnegative energy functionals to a large class of partial differential equations (PDEs) and evolutionary systems. The minima of these functionals are to be the solutions we seek, not because they are critical points (i.e., from the corresponding Euler-Lagrange equations) but from also being zeros of these functionals. The approach can be traced back to Bogomolnyi's trick of "completing squares" in the basic equations of quantum field theory (e.g., Yang-Mills, Seiberg-Witten, Ginzburg-Landau, etc.,), which allows for the derivation of the so-called self (or antiself) dual version of these equations. In reality, the *"self-dual Lagrangians"* we consider here were inspired by a variational approach proposed – over 30 years ago – by Brézis and Ekeland for the heat equation and other gradient flows of convex energies. It is based on Fenchel-Legendre duality and can be used on any convex functional – not just quadratic ones – making them applicable in a wide range of problems. In retrospect, we realized that the "energy identities" satisfied by Leray's solutions for the Navier-Stokes equations are also another manifestation of the concept of self-duality in the context of evolution equations.

The book could have also been entitled *How to solve nonlinear PDEs via convex analysis on phase space.* Indeed, the *self-dual vector fields* we introduce and study here are natural extensions of gradients of convex energies – and hence of self-adjoint positive operators – which usually drive dissipative systems but also provide representations for the superposition of such gradients with skew-symmetric operators, which normally generate conservative flows. Most remarkable is the fact that self-dual vector fields turned out to coincide with *maximal monotone operators*, themselves being far-reaching extensions of subdifferentials of convex potentials. This means that we have a one-to-one correspondence between three fundamental notions of modern nonlinear analysis: maximal monotone operators, semigroups of contractions, and self-dual Lagrangians. As such, a large part of nonlinear analysis can now be reduced to classical convex analysis on phase space, with self-dual Lagrangians playing the role of potentials for monotone vector fields according to a suitable calculus that we develop herein. This then leads to variational formulations and resolutions of a large class of differential systems that cannot otherwise be Euler-Lagrange equations of action functionals.

A note of caution, however, is in order about our chosen terminology. Unlike its use in quantum field theory, our concept of self-duality refers to an invariance under the Legendre transform – up to an automorphism of phase space – of the Lagrangians we consider. It also reflects the fact that many of the functionals we consider here are self-dual in the sense of convex optimization, meaning that the value of the infimum in the primal minimization problem is exactly the negative of the value of the supremum in the corresponding dual problem and therefore must be zero whenever there is no duality gap.

Another note, of a more speculative nature, is also in order, as our notion of self-duality turned out to be also remarkably omnipresent outside the framework of quantum field theory. Indeed, the class of self-dual partial differential systems – as presented here – becomes quite encompassing, as it now also contains many of the classical PDEs, albeit stationary or evolutionary, from gradient flows of convex potentials (such as the heat and porous media equations), Hamiltonian systems, and nonlinear transport equations to Cauchy-Riemann systems, Navier-Stokes evolutions, Schrödinger equations, and many others. As such, many of these basic PDEs can now be perceived as the "self-dual representatives" of families of equations that are still missing from current physical models. They are the absolute minima of newly devised self-dual energy functionals that may have other critical points that correspond – via Euler-Lagrange theory – to a more complex and still uncharted hierarchy of equations.

The prospect of exhibiting a unifying framework for the existence theory of such a disparate set of equations was the main motivating factor for writing this book. The approach is surprising because it suggests that basic convex analysis – properly formulated on phase space – can handle a large variety of PDEs that are normally perceived to be inherently nonlinear. It is also surprisingly simple because it essentially builds on a single variational principle that applies to a deceivingly restrictive-looking class of self-dual energy functionals. The challenges then shift from the analytical issues connected with the classical calculus of variations towards more algebraic/functional analytic methods for identifying and constructing self-dual functionals as well as ways to combine them without destroying their self-dual features.

With this in mind, the book is meant to offer material for an advanced graduate course on convexity methods for PDEs. The generality we chose for our statements definitely puts it under the "functional analysis" classification. The examples – deliberately chosen to be among the simplest of those that illustrate the proposed general principles – require, however, a fair knowledge of classical analysis and PDEs, which is needed to make – among other things – judicious choices of function spaces where the self-dual variational principles need to be applied. These choices necessarily require an apriori knowledge of the expected regularity of the (weak) solutions. We are therefore well aware that this project runs the risk of being perceived as "too much PDEs for functional analysts, and too much functional analysis for PDErs." This is a price that may need to be paid whenever one ventures into any attempt at a unification or classification scheme within PDE theory.

At this stage, I would like to thank Ivar Ekeland for pointing me toward his 1976 conjecture with Haïm Brézis, that triggered my initial interest and eventually led to the development of this program. Most of the results in this book have been obtained in close collaboration with my postdoctoral fellow Abbas Moameni and my former MSc student Leo Tzou. I can certainly say that without their defining contributions – both conceptual and technical – this material would never have reached its present state of readiness.

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Big Bar Lake, British Colombia, Canada N. A. Ghoussoub
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