Chapter 2
Origin of Macroscopic Approach

With the birth of quantum optics in the 1960s it became clear that it would be easy to describe the interaction between the electromagnetic field and the matter in a cavity even on elimination of matter degrees of freedom. A similar travelling-wave description for the electromagnetic field–matter interaction was considered to be possible in terms of a virtual cavity and a momentum operator of the field.

This approach to quantization was rather distant from the quantum theory of the electromagnetic field. On a fundamental level the theory of the electromagnetic field in the free space does not differ from the theory of this field in the matter. Macroscopic approaches to quantization of the electromagnetic field are not fundamental theories and modify the free-space electromagnetic-field theory. Especially, quantization of the field power has been assumed.

Although the virtual cavity has been beaten, the momentum operator has still enabled one to study quantum aspects of nonlinear optical processes. Quantization restrictions of any kind such as the frequency dispersion of the refractive index were apparent on published work.

Efforts emerged to formulate so simple a quantum theory of the electromagnetic field that it allows one to recognize the role of the momentum operator. Formalisms were presented which, to the contrary, did not consider the momentum operator. With the progress in (classical) optics interest in the quantization of the field power in quantum optics has increased.

Not always is it necessary to utilize the formalism of the electromagnetic field in the matter. For description of experiments with correlated photons it suffices to describe the electromagnetic field between optical devices and to know the input–output relations for the optical elements, both passive and active, with which the radiation is transformed.

2.1 Lossless Nonlinear Dielectric

An approach to the quantum theory of light propagation was considered standard until the critique by Hillery and Mlodinow (1984) and is still. Concerning this approach, let us consider papers by Shen (1967, 1969). Shen (1967) studied

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quantum statistics of nonlinear optics. He contributed to the contemporary research (Glauber 1965). Quantum theory of radiation had long been formulated (Heitler 1954). For investigation of properties of a medium, incoherent scattering has been a useful tool. For nonlinear optics, coherent scattering has been interesting as well or more.

Weak nonlinearity has a significant effect on light only after a longer interaction distance. Light can cover a longer distance easily when contained in a cavity resonator. Quantum statistics has been determined using descriptions suited to the case of a cavity.

In principle, the same treatment can be applied to problems of light propagation in media (Shen 1967). But for coherent scattering, it becomes difficult. This case should be treated by the method of many-body transport theory (Ter Haar 1961). In quantum optics, the cavity treatment of the problems of light propagation in media seems to be valid on the following assumption. Photon fields may be quantized in a box of finite volume, which moves in the z direction with a light velocity c ($\frac{c}{\sqrt{\epsilon}}$ in Shen 1969). One is advised to imagine a box of length cT, where T is the counting time of photodetectors. A partial interaction of the light with the medium can be approximated with no interaction and a complete interaction, which lasts for a time t. The finite medium can be extended to infinity. The resultant change of statistical properties of fields in the box can now be calculated using the cavity treatment (Shen 1967).

In nonlinear optics a number of classical descriptions have been developed both as a cavity problem and as a steady-state propagation problem. Then a cavity problem can be converted to a corresponding steady-state propagation problem by replacing t by $-\frac{z}{c}$ in the field amplitudes and the latter problem can be changed to the former one by replacing z by $-ct$ when $\mathbf{e}_z$ is the direction of propagation. It raises expectations that the same is true in the quantum treatment. Shen (1969) pays attention to replacing t by $-\frac{z\sqrt{\epsilon}}{c}$ and to replacing z by $\frac{ct}{\sqrt{\epsilon}}$.

Here the dependence on the time seems to be more fundamental. It is evident that one is interested in a conversion of a cavity problem to a corresponding steady-state propagation problem. The operators will be space dependent (localized) instead of time dependent. Transformations will be generated with a localized momentum operator instead of the Hamiltonian operator.

On quantizing in a volume L^3 and assuming that the field does not vary appreciably over a distance d large compared with the wavelength, and associating the discrete values of the wave vector k with d (instead of L), the localized annihilation and creation operators $\hat{b}_k(z)$ and $\hat{b}_k^\dagger(z)$ have been proposed. An appropriate component of the vector-potential operator has the expansion of the form

$$\hat{A}(z,t) = c\sum_k \sqrt{\frac{\hbar}{2\omega_k\epsilon_k L^3}}\left\{\hat{b}_k(z)\exp[-\mathrm{i}(\omega_k t - kz)] + \text{H.c.}\right\}, \tag{2.1}$$

where ω_k is the frequency, $\hbar$ is the Planck constant divided by 2π, $\epsilon_k = \epsilon(\omega_k)$ is the value of the dielectric function ϵ at ω_k, and H.c. denotes the term Hermitian conjugate to the previous one. The annihilation and creation operators $\hat{b}_k(z)$ and $\hat{b}_k^\dagger(z)$, respectively, satisfy the equal-space commutation relation

$$\left[\hat{b}_k(z), \hat{b}_{k'}^\dagger(z)\right] = \delta_{kk'}\hat{1}. \tag{2.2}$$

The small variation of the field has been formulated as that of the normally ordered moments $\left\langle \hat{b}_k^{\dagger m}(z)\hat{b}_k^n(z)\right\rangle$. It is also specified that $k = \frac{2\pi n}{d}$, where n is an integer.

There is a difficulty. The above picture of a moving box requires a light velocity c independent of the frequency ω_k. Shen (1969) utilizes the notation $\frac{c}{\sqrt{\epsilon}}$ for this velocity. Here it is replaced by the phase velocity $\frac{c}{\sqrt{\epsilon_k}}$, with $c \equiv c_0$, the free-space speed of light. There is another difficulty in view of this picture that d has been used instead of cT. The localized photon-number operator is realized as a configuration-space photon-number operator (Mandel 1966)

$$\hat{n}(z) = \frac{\mathcal{A}d}{L^3}\sum_k \hat{b}_k^\dagger(z)\hat{b}_k(z), \tag{2.3}$$

where $\mathcal{A}$ is the cross-sectional area of the beam.

A Hamiltonian density $\hat{H}(z,t)$ is considered. The Hamiltonian is

$$\hat{\mathcal{H}}(t) = L^2 \int_L \hat{H}(z', t)\,\mathrm{d}z'. \tag{2.4}$$

A third difficulty is that the localized momentum operator is defined as $\frac{\hat{H}(z,t)}{c}$ essentially, not by using an integration with respect to time. It has been assumed that $k = \frac{2\pi n}{d}$, not that $\omega_k = \frac{2\pi n}{T}$. For free fields, the localized momentum operator is

$$\hat{\mathcal{P}}(z) = \sum_k \hbar k \left[\hat{b}_k^\dagger(z)\hat{b}_k(z) + \tfrac{1}{2}\hat{1}\right]. \tag{2.5}$$

For interacting fields, the localized momentum operator has the form of a Hamiltonian, but with $\hat{b}_k(z)$ and $\hat{b}_k^\dagger(z)$ replacing $\hat{a}_k(t)$ and $\hat{a}_k^\dagger(t)$.

A momentum operator should have the form $\mathbf{e}_z\hat{\mathcal{P}}(z)$, be a vector, but in fact one does not utilize this. The momentum operator generates translations

$$\frac{\mathrm{d}}{\mathrm{d}z}\hat{b}_k(z) = \frac{\mathrm{i}}{\hbar}\left[\hat{b}_k(z), \hat{\mathcal{P}}(z)\right]. \tag{2.6}$$

The electric strength vector is derived from the vector potential according to the relation

$$\hat{E}(z,t) = -\frac{1}{c}\frac{\partial}{\partial t}\hat{A}(z,t). \tag{2.7}$$

We decompose this operator as

$$\hat{E}(z,t) = \hat{E}^{(+)}(z,t) + \hat{E}^{(-)}(z,t), \tag{2.8}$$

where $\hat{E}^{(+)}(z,t)$ $(\hat{E}^{(-)}(z,t))$ contains the functions $\exp(-\mathrm{i}\omega_k t)$ $(\exp(\mathrm{i}\omega_k t))$. In Shen (1967) the opposite convention is used. Then

$$\frac{\mathrm{d}}{\mathrm{d}z}\hat{E}^{(+)}(z,t) = \frac{\mathrm{i}}{\hbar}\left[\hat{E}^{(+)}(z,t), \hat{\mathcal{P}}(z)\right]. \tag{2.9}$$

Something is more suitable for propagation problem: We define all the quantities at a given plane $z = z_0$ for all times and try to obtain the propagation towards $z \geq z_0$. According to equations (2.6) and (2.9), the unitary translation operator is

$$\hat{U}(z,z_0) = \mathcal{S}\exp\left[\frac{\mathrm{i}}{\hbar}\int_{z_0}^{z}\hat{\mathcal{P}}(z')\,\mathrm{d}z'\right], \tag{2.10}$$

where $\mathcal{S}$ is the space-ordering operation. The space-ordered product has a similar definition as the time-ordered product. Field operators at different spatial points z, z_0 are connected by this unitary operator:

$$\hat{E}(z,t) = \hat{U}^{\dagger}(z,z_0)\hat{E}(z_0,t)\hat{U}(z,z_0). \tag{2.11}$$

There are indications that any "alternative" quantum theory is avoided. Such an indication is the fact that the localized momentum operator has been derived from the Hamiltonian density. With this in mind, we pass from the "spatial Heisenberg picture" to a spatial Schrödinger picture. In the latter picture, a localized density matrix (statistical operator) progresses:

$$\hat{\rho}(z) = \hat{U}(z,0)\hat{\rho}(0)\hat{U}^{\dagger}(z,0). \tag{2.12}$$

Here $\hat{\rho}(0)$ is a given statistical operator. Then the correlation function of fields at different times is expressed in two forms:

$$\begin{aligned}
&\left\langle \hat{E}^{(-)}(z,t_1)\ldots\hat{E}^{(-)}(z,t_n)\hat{E}^{(+)}(z,t_n)\ldots\hat{E}^{(+)}(z,t_1)\right\rangle \\
&= \mathrm{Tr}\left\{\hat{\rho}(0)\hat{E}^{(-)}(z,t_1)\ldots\hat{E}^{(-)}(z,t_n)\hat{E}^{(+)}(z,t_n)\ldots\hat{E}^{(+)}(z,t_1)\right\} \\
&= \mathrm{Tr}\left\{\hat{\rho}(z)\hat{E}^{(-)}(0,t_1)\ldots\hat{E}^{(-)}(0,t_n)\hat{E}^{(+)}(0,t_n)\ldots\hat{E}^{(+)}(0,t_1)\right\}.
\end{aligned} \tag{2.13}$$

The equation of motion for a statistical operator $\hat{\rho}(z)$ is

$$\frac{\partial}{\partial z}\hat{\rho}(z) = \frac{\mathrm{i}}{\hbar}\left[\hat{\mathcal{P}}(z), \hat{\rho}(z)\right]. \tag{2.14}$$

With the help of these localized operators, the calculations for steady-state propagation in a medium become the same as the corresponding calculations for a cavity with t replaced by $-\frac{z}{c}$ (Shen 1967) and by $\frac{z\sqrt{\epsilon}}{c}$ (Shen 1969). The problem of beam splitting was mentioned. Essentially, the same proposal has been included in Shen (1969).

2.2 Nondispersive Lossless Linear Dielectric

The study of nonlinear optical phenomena and their inclusion in an effective nonlinear theory of the electromagnetic field has utilized the asymmetry of most optical media, which are nonlinear with respect to the electric-field, but linear relative to the magnetic field. The canonical momentum should be the magnetic induction in place of the more usual electric-field strength. Such a theory may not be capable of describing the Bohm–Aharonov effect. Besides such a theory we expound a simple quantization connected to considerations of the role of the Poynting vector operator and the momentum operator.

A description of the field distribution in space must be completed with a quantum state of the field in quantum physics. A renewed interest in the spatio-temporal description leads to the study of the wave functional of the electromagnetic field despite the doubts of the pioneers of theoretical physics of the photonic wave function.

On neglecting dispersion and nonlinearity, a macroscopic theory of the quantized electromagnetic field in a medium can be very close to the usual theory of this field in free space. In contrast to this, solutions have been disseminated, which include the dispersion and the nonlinearity at least approximately.

2.2.1 Quantization in Terms of a Dual Potential

According to a pioneering paper of Hillery and Mlodinow (1984), the standard macroscopic quantum theory of electrodynamics in a nonlinear medium is due to Shen (1967) and has been elaborated upon by Tucker and Walls (1969). Hillery and Mlodinow (1984) have pointed out some problems with the standard theory, above all that it is not consistent with the macroscopic Maxwell equations.

One approach to the derivation of a macroscopic quantum theory would be to begin from a quantum microscopic theory as explored in the linear case by Hopfield (1958). The other approach is to take the expression for the energy of the radiation in nonlinear medium, which differs from the free-field Hamiltonian in part, and to keep interpreting the electric-field (up to the sign) as the canonically conjugated variable to the vector potential. Then, this macroscopic classical theory is quantized. (Let us note that it differs from Shen (1969)). The Hamiltonian formulation of the theory consists in the noncanonical Hamiltonian

$$\hat{H}_{\text{noncan}} = \hat{H}_{\text{EM}} + \hat{H}_{\text{Inoncan}}, \tag{2.15}$$

where

$$\hat{H}_{\mathrm{EM}} = \frac{1}{2}\int (\hat{\mathbf{E}}^2 + \hat{\mathbf{B}}^2)\,\mathrm{d}^3\mathbf{x}, \tag{2.16}$$

$$\hat{H}_{\mathrm{Inoncan}} = \frac{1}{2}\int \hat{\mathbf{E}}\cdot\hat{\mathbf{P}}\,\mathrm{d}^3\mathbf{x}, \tag{2.17}$$

with $\hat{\mathbf{E}}$ being the electric field strength operator and $\hat{\mathbf{P}}$ being the polarization of the medium, and the Heaviside–Lorentz units having been used. The polarization is a function of the electric field which may be written as a power series. This theory may be called standard. It can easily be seen that, as an undesirable "quantum effect", we obtain an improper expression for the time derivative of the magnetic-induction field $\hat{\mathbf{B}}$.

It is assumed that the medium is lossless, nondispersive, and homogeneous. A Lagrangian is considered which gives proper equations of motion. The electric and magnetic fields are expressed in terms of the vector potential $\mathbf{A}$ and the scalar potential A_0:

$$\mathbf{E} = -\frac{\partial\mathbf{A}}{\partial t} - \nabla A_0, \mathbf{B} = \nabla\times\mathbf{A}. \tag{2.18}$$

The appropriate Lagrangian density depends on the first partial derivatives of the four-vector $A = (A_0, \mathbf{A})$. The momentum canonical to A is $\Pi = (\Pi_0, \mathbf{\Pi})$, where $\Pi_0 = 0$. The vanishing of Π_0 indicates that the system is constrained. It has been shown how to utilize the Dirac quantization procedure for constrained Hamiltonian systems (Dirac 1964). It can be derived that the canonical momentum is $\mathbf{\Pi} = -\mathbf{D}$. The canonical Hamiltonian has the form

$$H = H_{\mathrm{EM}} + H_{\mathrm{I}}, \tag{2.19}$$

where

$$H_{\mathrm{I}} = \int \mathbf{E}\cdot\left[\mathbf{P} - \int_0^1 \mathbf{P}(\lambda\mathbf{E})\,\mathrm{d}\lambda\right]\mathrm{d}^3\mathbf{x}. \tag{2.20}$$

In order to simplify the quantization of the macroscopic Maxwell theory, the dual potential Λ has been introduced along with $\mathbf{\Lambda}$ and Λ_0, which we call the dual vector and scalar potentials. The relation (2.18) is replaced by

$$\mathbf{D} = \nabla\times\mathbf{\Lambda},\quad \mathbf{B} = \frac{\partial\mathbf{\Lambda}}{\partial t} + \nabla\Lambda_0. \tag{2.21}$$

It can be shown that the canonical momentum is $\mathbf{\Pi}^{\times} = \mathbf{B}$. Upon expressing the canonical Hamiltonian functional in terms of the electric-displacement and magnetic-induction fields, the results are the same:

$$H = H^{\times}. \tag{2.22}$$

Then, the usual Hamiltonian theory for the electromagnetic field in a nonlinear dielectric medium and the alternative have been quantized in the ordinary way. We can compare

$$\left[\hat{A}_i(\mathbf{x}, t), \hat{\Pi}_j(\mathbf{x}', t)\right] = \mathrm{i}\delta_{ij}^{\perp}(\mathbf{x} - \mathbf{x}')\hat{1}, \tag{2.23}$$

with

$$\left[\hat{\Lambda}_i(\mathbf{x}, t), \hat{\Pi}_j^{\times}(\mathbf{x}', t)\right] = \mathrm{i}\delta_{ij}^{\perp}(\mathbf{x} - \mathbf{x}')\hat{1}. \tag{2.24}$$

The transverse δ function has been used and made a reference to Bjorken and Drell (1965). Hillery and Mlodinow (1984) do not mention propagation except a paragraph on the interpretation problems, where they recommend to confine the medium to part of the quantization volume and to place the field source and the detector outside of the medium, being aware that they require the consideration of propagation.

It is added that different diagonalizations indicated by the quadratic part of the total Hamiltonian generate different kinds of normal ordering. A doubt is expressed that there is an appropriate kind and the microscopic approach is propounded. Dispersion is also considered a reason for a microscopic theory to be contemplated.

2.2.2 Momentum Operator as Translation Operator

In the late 1980s, the problem of propagation did not seem to be typical of quantum optics. Abram addressed the problem of light propagation through a linear nondispersive lossless medium (Abram 1987). Although this model can be an appropriate limit of the Huttner–Barnett model, we expound the main ideas of Abram (1987). Abram criticized the modal Hamiltonian formalism, especially the inclusion of the linear polarization term in the Hamiltonian:

$$\mathcal{H} \stackrel{?}{=} \frac{1}{8\pi} \int_V (E^2 + H^2 + 4\pi\chi E^2)\,\mathrm{d}V, \tag{2.25}$$

where E (H) is the magnitude of the electrical (magnetic) field strength, χ is the (linear) susceptibility of the material, and V is a quantization volume. This would lead to an incorrect result, mainly to a change of the frequencies of the modes which does not occur. He decided to extend the traditional theory of quantum optics to describe propagation phenomena without invoking the modal Hamiltonian. According to him one of the propagation phenomena, refraction, suggests the momentum as the concept appropriate for the description of these phenomena. Quantum mechanically, space and momentum are canonically conjugate variables. Let us remark that microscopic models demonstrate that a Hamiltonian including light–matter interaction can be considered. These are a good antidote against the idea that "space and momentum are canonically conjugate variables like time and energy".

Propagation of the electromagnetic field is described by the Maxwell equations:

$$\nabla \times \mathbf{H} = \frac{1}{c}\frac{\partial \mathbf{D}}{\partial t}, \tag{2.26}$$

$$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}, \tag{2.27}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{2.28}$$

$$\nabla \cdot \mathbf{D} = 0, \tag{2.29}$$

where $\mathbf{D} = \mathbf{E} + 4\pi\mathbf{P}$ is the electric displacement, $\mathbf{B}$ is the magnetic induction, $\mathbf{E}$ and $\mathbf{H}$ are the electric and magnetic field strengths, respectively, $\mathbf{P}$ is the (linear and nonlinear) polarization induced in the medium, and c is the speed of light. We assume that there are no free charges or currents and that we are dealing with nonmagnetic materials, so that $\mathbf{B} = \mathbf{H}$.

For simplicity, we shall consider only the case of plane waves propagating along the z-axis, with the electric field polarized along the x-axis and the magnetic field along the y-axis. This reduces the Maxwell equations to scalar differential equations, the directions of all vectors being implicit. We shall further assume that light is propagating in a linear dielectric, where the induced polarization is at all times proportional to the incident electric field:

$$P = \chi E, \tag{2.30}$$

where we assume the susceptibility of the material for simplicity to be a scalar (neglecting its tensorial properties), independent of frequency (no dispersion). It is convenient to define also the dielectric function ϵ of the material

$$\epsilon = 1 + 4\pi\chi, \tag{2.31}$$

and the refractive index n,

$$n = \sqrt{\epsilon}. \tag{2.32}$$

The change in the total energy which is given by the integrated energy flux (the Poynting vector) over the surface of a body or volume is proposed in Abram (1987) as the proper quantum-mechanical Hamiltonian. The change in the total momentum is given as the integrated flux of the Maxwell stress tensor. The momentum is treated on the same footing as the Hamiltonian. However, the enigma of the Hamiltonian (2.25) is solved. We may consider a square pulse which enters a dielectric. The total energy is conserved, but the energy density is increased by a factor of n, because the volume V reduces to $V' = \frac{V}{n}$. In volume V' the wavelengths of the modes become $\lambda' = \frac{\lambda}{n}$, but the oscillator frequencies remain unchanged. It is interesting that in the absence of reflection, the electric and magnetic fields of the transmitted (T) waves in the dielectric are related to the corresponding incident (I) fields in free space by

$$E_T = \frac{1}{\sqrt{n}} E_I, \tag{2.33}$$

$$H_T = \sqrt{n} H_I. \tag{2.34}$$

This change in the energy density implies a similar increase for the total momentum of the pulse, the components of which are always proportional to the wave vectors of the excited modes. In propagation along the z-axis the Maxwell stress tensor is replaced by the energy density.

When the propagation along the $\pm z$-axis in free space is considered with the electric field polarized along the x-axis and the magnetic field along the y-axis ($\chi = 0$, $\epsilon = 1$), the electromagnetic vector-potential operator $\hat{A} \equiv \hat{A}(z,t)$ is usually written as ($\hbar = 1$)

$$\hat{A}(z,t) = c \sum_j \left(\frac{2\pi}{V\omega_j} \right)^{\frac{1}{2}} \left(\hat{a}_j^\dagger e^{\mathrm{i}\omega_j t - \mathrm{i}k_j z} + \hat{a}_j e^{-\mathrm{i}\omega_j t + \mathrm{i}k_j z} \right), \tag{2.35}$$

where $\hat{a}_j^\dagger$, $\hat{a}_j$ are the creation, annihilation operators, respectively, for a photon in the jth mode of the wave vector k_j (with $k_{-j} = -k_j$) and the frequency $\omega_j = c|k_j|$ fulfilling the Bose commutation relations. To simplify the notation, we omit unit vectors. It is convenient to rearrange equation (2.35) in a manner that is familiar to solid-state physicists,

$$\hat{A}(z,t) = c \sum_j \left(\frac{2\pi}{V\omega_j} \right)^{\frac{1}{2}} \left(\hat{a}_j^\dagger e^{\mathrm{i}\omega_j t} + \hat{a}_{-j} e^{-\mathrm{i}\omega_j t} \right) e^{-\mathrm{i}k_j z}. \tag{2.36}$$

The electric and magnetic field operators may be obtained as

$$\begin{aligned} \hat{E}(z,t) &= -\frac{1}{c}\frac{\partial}{\partial t}\hat{A}(z,t) = \sum_j \hat{e}_j \\ &= -\mathrm{i} \sum_j \left(\frac{2\pi\omega_j}{V} \right)^{\frac{1}{2}} (\hat{b}_j^\dagger - \hat{b}_{-j}), \end{aligned} \tag{2.37}$$

and

$$\hat{H}(z,t) = \frac{\partial}{\partial z}\hat{A}(z,t) = \sum_j \hat{h}_j$$

$$= -\mathrm{i}\sum_j s_j \left(\frac{2\pi\omega_j}{V}\right)^{\frac{1}{2}} \left(\hat{b}_j^\dagger + \hat{b}_{-j}\right), \tag{2.38}$$

where $s_j \equiv \operatorname{sgn} j$ and

$$\hat{b}_j = \hat{a}_j e^{-\mathrm{i}\omega_j t + \mathrm{i}k_j z}. \tag{2.39}$$

When products of these operators are encountered, we suppose that they are symmetrized. The Hermiticity of the operators $\hat{E} \equiv \hat{E}(z,t)$ and $\hat{H} \equiv \hat{H}(z,t)$ can be verified using the relations

$$\hat{e}_j^\dagger = \hat{e}_{-j}, \tag{2.40}$$

$$\hat{h}_j^\dagger = \hat{h}_{-j}. \tag{2.41}$$

The energy density operator can be written as

$$\hat{u} = \frac{1}{8\pi}\left(\hat{E}^2 + \hat{H}^2\right) \tag{2.42}$$

$$= \sum_j \hat{u}_j \tag{2.43}$$

$$= \frac{1}{8\pi}\sum_j \left(\hat{e}_j\hat{e}_{-j} + \hat{h}_j\hat{h}_{-j}\right)$$

$$= \frac{1}{2V}\sum_j \omega_j \left(\hat{b}_j^\dagger\hat{b}_j + \hat{b}_{-j}^\dagger\hat{b}_{-j} + \hat{1}\right) \tag{2.44}$$

$$= \frac{1}{V}\sum_j \omega_j \left(\hat{b}_j^\dagger\hat{b}_j + \frac{1}{2}\hat{1}\right). \tag{2.45}$$

The energy fluxes due to the forward (backward) waves alone can be expressed uniquely:

$$\hat{u}_+ = \sum_{j(>0)} \frac{\omega_j}{V}\left(\hat{b}_j^\dagger\hat{b}_j + \frac{1}{2}\hat{1}\right), \hat{u}_- = \sum_{j(<0)} \frac{\omega_j}{V}\left(\hat{b}_j^\dagger\hat{b}_j + \frac{1}{2}\hat{1}\right). \tag{2.46}$$

The total momentum operator $\hat{G}$ is then

$$\hat{G} = \frac{V}{c}(\hat{u}_+ - \hat{u}_-) = \sum_j k_j \left(\hat{b}_j^\dagger \hat{b}_j + \frac{1}{2}\hat{1}\right). \tag{2.47}$$

It is important to understand the relations (3.9) and (3.10) in Abram (1987) well. We interpret (3.9) concerning elementary quantum mechanics as

$$\langle z|\hat{p}_z|\psi\rangle = -\mathrm{i}\hbar\frac{\partial}{\partial z}\langle z|\psi\rangle, \tag{2.48}$$

where $|z\rangle$ are position coordinate states and $|\psi\rangle$ is an arbitrary pure state. The similarity with equation (3.10) from Abram (1987)

$$\frac{\partial \hat{Q}}{\partial z} = -\mathrm{i}[\hat{G}, \hat{Q}], \tag{2.49}$$

where $\hat{Q}$ is any operator, fades. We would prefer a definition of the operator $\hat{Q}$. Let us consider

$$\hat{Q} \equiv \hat{Q}(z,t) = Q[\hat{E}(z,t), \hat{H}(z,t)], \tag{2.50}$$

where $Q[\bullet, \bullet]$ is a formal series in $\hat{E}$ and $\hat{H}$. Since the differential operator $\frac{\partial}{\partial z}$ is just as differentiation as the superoperator $-\mathrm{i}[\hat{G}, \bullet]$, it suffices to verify the relation (2.49) for $\hat{Q} = \hat{E}, \hat{H}$. It is true at least in the situations treated in Abram (1987).

Although the operators $\hat{b}_j \equiv \hat{b}_j(z,t)$ are studied using (2.49), the Heisenberg equation of motion, and the initial condition

$$\hat{b}_j(0,0) = \hat{a}_j \tag{2.51}$$

as appropriate for any operator $\hat{Q}(z,t)$, we perceive that the operators do not obey our definition of the operator $\hat{Q}$.

We may calculate the Poynting vector operator as

$$\hat{S} = \frac{c}{4\pi}\hat{E}\hat{H} = \frac{c}{4\pi}\sum_j \hat{e}_j \hat{h}_{-j} \tag{2.52}$$

$$= \sum_j s_j \frac{c\omega_j}{V}\left(\hat{b}_j^\dagger \hat{b}_j + \frac{1}{2}\hat{1}\right). \tag{2.53}$$

The Poynting vector operators due to the forward (backward) waves alone can be expressed uniquely:

$$\hat{S}_+ = \sum_{j(>0)} \frac{c\omega_j}{V}\left(\hat{b}_j^\dagger \hat{b}_j + \frac{1}{2}\hat{1}\right), \quad \hat{S}_- = -\sum_{j(<0)} \frac{c\omega_j}{V}\left(\hat{b}_j^\dagger \hat{b}_j + \frac{1}{2}\hat{1}\right). \tag{2.54}$$

The total energy operator of the free field inside the volume of quantization is thus

$$\hat{\mathcal{H}} = \hat{U} = \frac{V}{c}\left(\hat{S}_+ - \hat{S}_-\right) = \sum_j \omega_j \left(\hat{b}_j^\dagger \hat{b}_j + \frac{1}{2}\hat{1}\right). \tag{2.55}$$

The investigation of the case $\chi \neq 0$, $\epsilon \neq 1$ does not lead to any new expansions of the field operators $\hat{E}$ and $\hat{H}$. The individual components of the rearranged electric and magnetic field operators according to (2.37) and (2.38) satisfy a modified operator algebra with respect to that of the harmonic oscillator:

$$[\hat{e}_j, \hat{e}_l] = [\hat{h}_j, \hat{h}_l] = \hat{0}, \tag{2.56}$$

$$[\hat{e}_j, \hat{h}_l] = s_{-j}\left(\frac{4\pi\omega_j}{V}\right)\delta_{-j,l}\hat{1}, \tag{2.57}$$

where $\delta_{j,l}$ is the Kronecker δ function. The knowledge of these commutators and of the generalized total momentum operator $\hat{G}$, the derivation of (2.42) through (2.47), should have been generalized accordingly, e.g. the relation (2.42) becoming

$$\begin{aligned} \hat{u} &= \frac{1}{8\pi}\left(\epsilon\hat{E}^2 + \hat{H}^2\right) \\ &= \frac{1}{8\pi}\sum_j \left(\epsilon\hat{e}_j\hat{e}_{-j} + \hat{h}_j\hat{h}_{-j}\right), \end{aligned} \tag{2.58}$$

which enables us to derive the Maxwell equations both via the temporal derivatives and via the spatial derivatives.

The energy density operator (2.58) can be generalized. In the expansion (2.43) we can set $\hat{u}_j = \hat{u}_{j\,\mathrm{refr}}$,

$$\hat{u}_{j\,\mathrm{refr}} = \frac{\omega_j}{2V}\left\{\hat{b}_j^\dagger\hat{b}_j + \hat{b}_{-j}\hat{b}_{-j}^\dagger - 2\pi\chi\left(\hat{b}_j^\dagger - \hat{b}_{-j}\right)\left(\hat{b}_{-j}^\dagger - \hat{b}_j\right)\right\}. \tag{2.59}$$

The energy density operator $\hat{u}_{\mathrm{refr}}$ may be diagonalized through a Bogoliubov transformation. To this end we introduce an anti-Hermitian operator $\hat{R}$ of the form

$$\hat{R} = \sum_j \left(\hat{b}_j\hat{b}_{-j} - \hat{b}_j^\dagger\hat{b}_{-j}^\dagger\right) \tag{2.60}$$

and introduce the operators

$$\hat{B}_j = e^{-\gamma\hat{R}}\hat{b}_j e^{\gamma\hat{R}} = (\cosh\gamma)\hat{b}_j - (\sinh\gamma)\hat{b}_{-j}^\dagger, \tag{2.61}$$

where

$$\gamma = \frac{1}{4}\ln\epsilon = \frac{1}{2}\ln n. \tag{2.62}$$

On substitution

$$\hat{b}_j = e^{\gamma \hat{R}} \hat{B}_j e^{-\gamma \hat{R}} = (\cosh \gamma) \hat{B}_j + (\sinh \gamma) \hat{B}^\dagger_{-j}, \tag{2.63}$$

the operator $\hat{R}$ takes the form

$$\hat{R} = \sum_j \left(\hat{B}_j \hat{B}_{-j} - \hat{B}^\dagger_j \hat{B}^\dagger_{-j} \right), \tag{2.64}$$

and the energy density operator has the diagonal form

$$\hat{u}_{j\,\mathrm{refr}} = \frac{n\omega_j}{2V} \left(\hat{B}^\dagger_j \hat{B}_j + \hat{B}_{-j} \hat{B}^\dagger_{-j} \right). \tag{2.65}$$

The momentum operator is then given by

$$\hat{G}_{\mathrm{refr}} = \frac{V}{c} (\hat{u}_+ - \hat{u}_-) = \sum_j K_j \left(\hat{B}^\dagger_j \hat{B}_j + \frac{1}{2} \hat{1} \right), \tag{2.66}$$

with $K_j = nk_j$ and the Hamiltonian can be calculated as

$$\hat{\mathcal{H}}_{\mathrm{refr}} = \sum_j \omega_j \left(\hat{B}^\dagger_j \hat{B}_j + \frac{1}{2} \hat{1} \right). \tag{2.67}$$

By inserting (2.63) into (2.37) and (2.38), respectively, we can obtain the electric and magnetic field operators inside the dielectric:

$$\hat{E}(z,t) = -\mathrm{i} \sum_j \left(\frac{2\pi\omega_j}{nV} \right)^{\frac{1}{2}} (\hat{B}^\dagger_j - \hat{B}_{-j}) \tag{2.68}$$

and

$$\hat{H}(z,t) = -\mathrm{i} \sum_j s_j \left(\frac{2\pi n\omega_j}{V} \right)^{\frac{1}{2}} (\hat{B}^\dagger_j + \hat{B}_{-j}). \tag{2.69}$$

Similarly as above, this relation can be interpreted as a result of the replacement $\hat{b}_j \mapsto \hat{B}_j$ and a consequence of the quantized classical equations (2.33) and (2.34).

For normal incidence on a sharp vacuum–dielectric interface, both reflection and diffraction occur. We will not treat this more general case according to Abram (1987).

2.2.3 Wave Functional Description of Gaussian States

Białynicka-Birula and Białynicki-Birula (1987) have tried first to define the squeezing that is a generalization of the standard definition for one mode of radiation. This definition can be reformulated with respect to Białynicki-Birula (2000). The Riemann–Silberstein–Kramers complex vector has been introduced

$$\mathbf{F}(\mathbf{r}, t) = \frac{1}{\sqrt{2}}\left[\frac{\mathbf{D}(\mathbf{r}, t)}{\sqrt{\epsilon_0}} + \mathrm{i}\frac{\mathbf{B}(\mathbf{r}, t)}{\sqrt{\mu_0}}\right], \tag{2.70}$$

where we have divided by $\sqrt{\epsilon_0}$, $\sqrt{\mu_0}$, as is appropriate with SI units. It has been shown how the Green function method can be used for solving linear equations for the field operator $\hat{\mathbf{F}}(\mathbf{r}, t)$. This approach allows that the medium under investigation is inhomogeneous and time dependent. It is not clear whether the complex vector (2.70) is then useful. It has been suggested that the periodicity of the electric permittivity tensor $\epsilon(\mathbf{r}, t)$ or the magnetic permeability $\mu(\mathbf{r}, t)$ can be important for the generation of squeezed states. Only the dispersion of the medium has not been considered. It has been derived that photon pair production is a necessary condition for squeezing.

It is tempting to generalize the concept of a Gaussian state of the finite-dimensional harmonic oscillator to the case of an infinite oscillator. Białynicka-Birula and Białynicki-Birula (1987) treat the time development of the Gaussian states in the free-field case. There the Schrödinger picture is adopted and an analogue of the Schrödinger representation in quantum mechanics has been introduced. Let us recall the quadrature representation in quantum optics. This representation is a wave functional $\Psi[\mathbf{A}, t]$. Let us observe that contrary to the operator $\hat{\mathbf{A}}(\mathbf{r}, t)$, the argument $\mathbf{A}(\mathbf{r})$ of the wave functional does not depend on t, but the wave functional does depend on t. The Hamiltonian in this representation has the form

$$H = \frac{1}{2}\int \left\{-\frac{\hbar^2}{\epsilon_0}\frac{\delta^2}{\delta\mathbf{A}(\mathbf{r})^2} + \frac{1}{\mu_0}[\nabla \times \mathbf{A}(\mathbf{r})]^2\right\} \mathrm{d}^3\mathbf{r}. \tag{2.71}$$

In Białynicka-Birula and Białynicki-Birula (1987), the wave functional of the vacuum state, i.e. the simplest Gaussian state of the electromagnetic field, can be found, as well as that of the "most general". Thus, the exposition is confined to pure Gaussian states while it is possible to generalize it also to mixed Gaussian states of the electromagnetic field. The pure Gaussian state is determined by a complex matrix kernel, i.e. by two real matrix kernels. It is shown that the expectation values $\langle\hat{\mathbf{B}}\rangle = \mathcal{B}$ and $\langle\hat{\mathbf{D}}\rangle = \mathcal{D}$ (equivalently, $\langle\hat{\mathbf{E}}\rangle = \mathcal{E}$) evolve according to the free-field Maxwell equations and also the equations which the complex matrix kernel obeys can be found there.

The whole electromagnetic field is treated as a huge infinite-dimensional harmonic oscillator. The wave function and the corresponding Wigner function become then functionals of the field variables. Mrówczyński and Müller (1994) have considered only the scalar field. Białynicki-Birula (2000) starts from the wave functional of the vacuum state (Misner et al. 1970)

$$\Psi_0[\mathbf{A}] = C \exp\left[-\frac{1}{4\pi^2\hbar}\sqrt{\frac{\epsilon_0}{\mu_0}}\int\int \mathbf{B}(\mathbf{r})\frac{1}{|\mathbf{r}-\mathbf{r}'|^2}\cdot\mathbf{B}(\mathbf{r}')\,\mathrm{d}^3\mathbf{r}\,\mathrm{d}^3\mathbf{r}'\right] \tag{2.72}$$

and from the wave functional (we change $\tilde{\mathbf{A}} \to -\mathbf{D}$)

$$\tilde{\Psi}_0[-\mathbf{D}] = C \exp\left[-\frac{1}{4\pi^2\hbar}\sqrt{\frac{\mu_0}{\epsilon_0}}\int\int \mathbf{D}(\mathbf{r})\frac{1}{|\mathbf{r}-\mathbf{r}'|^2}\cdot\mathbf{D}(\mathbf{r}')\,\mathrm{d}^3\mathbf{r}\,\mathrm{d}^3\mathbf{r}'\right]. \tag{2.73}$$

The normalization constant C is an issue and it has not been completely solved in Białynicki-Birula (2000). The analogy with the one-dimensional harmonic oscillator leads to other notions. The Wigner functional of the electromagnetic field in the ground state is

$$W_0[\mathbf{A}, -\mathbf{D}] = \exp\{-2N[\mathbf{A}, -\mathbf{D}]\}, \tag{2.74}$$

where

$$\begin{aligned} N[\mathbf{A}, -\mathbf{D}] = \frac{1}{4\pi^2\hbar}\int\int &\left[\sqrt{\frac{\epsilon_0}{\mu_0}}\mathbf{B}(\mathbf{r})\frac{1}{|\mathbf{r}-\mathbf{r}'|^2}\cdot\mathbf{B}(\mathbf{r}')\right. \\ &\left.+\sqrt{\frac{\mu_0}{\epsilon_0}}\mathbf{D}(\mathbf{r})\frac{1}{|\mathbf{r}-\mathbf{r}'|^2}\cdot\mathbf{D}(\mathbf{r}')\right]\mathrm{d}^3\mathbf{r}\,\mathrm{d}^3\mathbf{r}'. \end{aligned} \tag{2.75}$$

The expression (2.75) also plays the role of a norm for the photon wave function (Białynicki-Birula 1996a,b). The Wigner functional for the thermal state of the electromagnetic field has been presented. This state is mixed and it even has infinitely many photons in the whole field. In each of the subsequent cases, the wave functional and the Wigner functional have been introduced. The exception, the mixed state, has no wave functional. Let us remark that for (the statistical operator of) such a state the matrix element can be considered which is a functional of two arguments, $\mathbf{A}$ and $\mathbf{A}'$.

In particular, the Wigner functional for the coherent state of the electromagnetic field $|\mathcal{A}, -\mathcal{D}\rangle$ has been presented, where $\mathcal{A}(\mathbf{r})$, $\mathcal{D}(\mathbf{r})$ are the vector potential and the electric displacement vector, respectively, which characterize the state. The exposition is related to the hot topic of the superpositions of coherent states of the electromagnetic field. The exposition continues with the Wigner functionals for the states of the electromagnetic field that describe a definite number of photons. An example of the functional for the one-photon state with the photon mode function $\mathbf{f}(\mathbf{r})$ has been included.

The norm (2.75) has not been related to any inner product of the photon wave functions, but these notions are connected. In contrast to Białynicka-Birula and

Białynicki-Birula (1987), we introduce quadrature operators as

$$\hat{X}_1[\mathcal{D}] = \int \hat{\mathbf{D}}(\mathbf{r}, 0) \cdot \mathbf{f}(\mathbf{r}) \, \mathrm{d}^3\mathbf{r}, \tag{2.76}$$

$$\hat{X}_2[\mathcal{B}] = \int \hat{\mathbf{B}}(\mathbf{r}, 0) \cdot \mathbf{g}(\mathbf{r}) \, \mathrm{d}^3\mathbf{r}, \tag{2.77}$$

where

$$\mathbf{f}(\mathbf{r}) = \frac{1}{4\pi^2\hbar} \int \sqrt{\frac{\mu_0}{\epsilon_0}} \frac{1}{|\mathbf{r} - \mathbf{r}'|^2} \mathcal{D}(\mathbf{r}') \, \mathrm{d}^3\mathbf{r}', \tag{2.78}$$

$$\mathbf{g}(\mathbf{r}) = \frac{1}{4\pi^2\hbar} \int \sqrt{\frac{\epsilon_0}{\mu_0}} \frac{1}{|\mathbf{r} - \mathbf{r}'|^2} \mathcal{B}(\mathbf{r}') \, \mathrm{d}^3\mathbf{r}'. \tag{2.79}$$

The commutator of the $\hat{X}_1$ and $\hat{X}_2$ operators is

$$[\hat{X}_1[\mathcal{D}], \hat{X}_2[\mathcal{B}]] = \mathrm{i}\hbar \int \mathbf{f}(\mathbf{r}) \cdot [\nabla \times \mathbf{g}(\mathbf{r})] \, \mathrm{d}^3\mathbf{r}. \tag{2.80}$$

Let us note that the right-hand sides of (2.78) and (2.79) comprise the operator $|\nabla|^{-1}$ up to a certain factor (cf. Milburn et al. 1984). Without resorting to this notation, we obtain that

$$[\hat{X}_1[\mathcal{D}], \hat{X}_2[\mathcal{B}]] = \frac{\mathrm{i}}{4\hbar} \int \mathcal{D}(\mathbf{r}_1) \cdot \mathcal{A}(\mathbf{r}_1) \, \mathrm{d}^3\mathbf{r}_1 \, \hat{1}. \tag{2.81}$$

We see easily that the usual commutator $-\frac{1}{2}\mathrm{i}\hat{1}$ is yielded by the field $(\mathcal{D}, \mathcal{B})$ (or $(\mathcal{A}, -\mathcal{D})$) with the property

$$\int [-\mathcal{D}(\mathbf{r}_1)] \cdot \mathcal{A}(\mathbf{r}_1) \, \mathrm{d}^3\mathbf{r}_1 = 2\hbar. \tag{2.82}$$

We have not deepened the contrast by introducing the notation $\hat{X}_1[-\mathcal{D}]$ and $\hat{X}_2[\mathcal{A}]$ on the left-hand sides of (2.76) and (2.77).

Białynicki-Birula (2000) presents the Wigner functional for the squeezed vacuum state:

$$W_{\mathrm{sq}}[\mathbf{A}, -\mathbf{D}] = \exp\left[-\frac{1}{\hbar} \int \int \left(\sqrt{\frac{\epsilon_0}{\mu_0}} \mathbf{B} \cdot \mathbf{K}_{\mathrm{BB}} \cdot \mathbf{B}' + \sqrt{\frac{\mu_0}{\epsilon_0}} \mathbf{D} \cdot \mathbf{K}_{\mathrm{DD}} \cdot \mathbf{D}' + \mathbf{B} \cdot \mathbf{K}_{\mathrm{BD}} \cdot \mathbf{D}' \right) \mathrm{d}^3\mathbf{r} \, \mathrm{d}^3\mathbf{r}' \right], \tag{2.83}$$

where $\mathbf{K}_{\mathrm{BB}}$, $\mathbf{K}_{\mathrm{DD}}$, and $\mathbf{K}_{\mathrm{BD}}$ are real matrix kernels. The kernel $\mathbf{K}_{\mathrm{BD}}$ is not independent of $\mathbf{K}_{\mathrm{BB}}$ and $\mathbf{K}_{\mathrm{DD}}$, but it must obey the condition that is reminiscent of the Schrödinger–Robertson uncertainty relation (Białynicki-Birula 1998). The problem of the time evolution is also discussed. It has been conceded that the Wigner function is not a very powerful tool for making detailed calculations. Just as in the field theory, the symmetric ordering is vexed. Another open question is how the projection of this Wigner functional onto Wigner functions of any orthogonal (complete or incomplete) modal system looks out. It is appropriate to mention here work concerning the photon wave function (Inagaki 1998, Hawton 1999, Kobe 1999), although it is relevant mainly to the electromagnetic field in vacuo.

Using a straightforward procedure, Mendonça et al. (2000) have quantized the linearized equations for an electromagnetic field in a plasma. They have determined an effective mass for the transverse photons. An extension of the quantization procedure leads to the definition of a photon charge operator. Zaleśny (2001) has found that the influence of a medium on a photon can be described by some scalar and vector potentials. He has extended the concept of the vector potential to relativistic velocities of the medium. He has derived formulae for the mass of photon in resting and moving dielectric and the velocity of the photon as a particle.

2.2.4 Source-Field Operator

Knöll et al. (1987) have compared the problem of quantum-mechanical treatment of action of optical devices with the input–output formalism (Collett and Gardiner 1984, Gardiner and Collett 1985, Yamamoto and Imoto 1986, Nilsson et al. 1986, cf. also Gea-Banacloche et al. 1990a,b). Apart from the fact that only a very particular setup is considered in the input–output formalism, the theory does not take into account the full space–time structure of the field.

Knöll et al. (1987) have elaborated on the approach developed on the basis of quantum field theory and applied to the problem of spectral filtering of light (Knöll et al. 1986). The only assumptions are that the interaction between sources and light is linear in the vector potential and the optical system is lossless and that the condition of sufficiently small dispersion is fulfilled. First, the classical Maxwell equations with sources and optical devices are formulated and solved by the procedure of mode expansion and the quantized version is derived. The classical Maxwell equations comprise the relative permittivity

$$\epsilon(\mathbf{r}) = n^2(\mathbf{r}), \tag{2.84}$$

where $n(\mathbf{r})$ is the space-dependent refractive index. The mode functions $\mathbf{A}_\lambda(\mathbf{r})$ are introduced as the solutions of equation

$$\nabla \times (\nabla \times \mathbf{A}_\lambda(\mathbf{r})) - \epsilon(\mathbf{r})\frac{\omega_\lambda^2}{c^2}\mathbf{A}_\lambda(\mathbf{r}) = \mathbf{0}, \tag{2.85}$$

where ω_λ^2 is the separation constant for each λ, from which the gauge condition can be derived

$$\nabla \cdot [\epsilon(\mathbf{r})\mathbf{A}_\lambda(\mathbf{r})] = 0. \tag{2.86}$$

It is assumed that these solutions are normalized and orthogonal in the sense of equation

$$\int \epsilon(\mathbf{r})\mathbf{A}_\lambda(\mathbf{r}) \cdot \mathbf{A}_{\lambda'}(\mathbf{r})\, \mathrm{d}^3\mathbf{r} = \delta_{\lambda\lambda'}\hat{1}. \tag{2.87}$$

In terms of these functions, the vector potential can be decomposed. In the standard manner the destruction and creation operators $\hat{a}_\lambda$ and $\hat{a}_\lambda^\dagger$ are defined, which have the properties

$$\begin{aligned} [\hat{a}_\lambda, \hat{a}_{\lambda'}^\dagger] &= \delta_{\lambda\lambda'}\hat{1}, \\ [\hat{a}_\lambda, \hat{a}_{\lambda'}] &= \hat{0} = [\hat{a}_\lambda^\dagger, \hat{a}_{\lambda'}^\dagger]. \end{aligned} \tag{2.88}$$

On inserting the operators $\hat{a}_\lambda$ and $\hat{a}_\lambda^\dagger$ into the decomposition of the vector potential, the operator of the vector potential $\hat{\mathbf{A}}(\mathbf{r}, t)$ is defined:

$$\hat{\mathbf{A}}(\mathbf{r}, t) = \sum_\lambda \mathbf{A}_\lambda(\mathbf{r}) \left[\hat{a}_\lambda(t) + \hat{a}_\lambda^\dagger(t)\right]. \tag{2.89}$$

The source quantities $\mathbf{r}_a$ and $\mathbf{p}_a$ are considered as the operators $\hat{\mathbf{r}}_a$ and $\hat{\mathbf{p}}_a$, which obey the standard commutation relations

$$\begin{aligned} [\hat{r}_{ka}, \hat{p}_{k'a'}] &= \mathrm{i}\hbar\delta_{aa'}\delta_{kk'}\hat{1}, \\ [\hat{r}_{ka}, \hat{r}_{k'a'}] &= \hat{0} = [\hat{p}_{ka}, \hat{p}_{k'a'}], \end{aligned} \tag{2.90}$$

and the commutation relations

$$\begin{aligned} [\hat{r}_{ka}, \hat{a}_\lambda] &= \hat{0} = [\hat{r}_{ka}, \hat{a}_\lambda^\dagger], \\ [\hat{p}_{ka}, \hat{a}_\lambda] &= \hat{0} = [\hat{p}_{ka}, \hat{a}_\lambda^\dagger]. \end{aligned} \tag{2.91}$$

The operator $\hat{\mathbf{A}}(\mathbf{r}, t)$ can be used for the derivation of the electric-field strength operator which is associated with the radiation field by the relation

$$\hat{\mathbf{E}}(\mathbf{r}, t) = -\frac{\partial}{\partial t}\hat{\mathbf{A}}(\mathbf{r}, t) \tag{2.92}$$

and for the derivation of the magnetic field strength operator

$$\hat{\mathbf{B}}(\mathbf{r}, t) = \nabla \times \hat{\mathbf{A}}(\mathbf{r}, t). \tag{2.93}$$

Nevertheless, the mode functions are redefined so that they obey the normalization condition

$$\int \epsilon(\mathbf{r})\mathbf{A}_\lambda(\mathbf{r}) \cdot \mathbf{A}_{\lambda'}(\mathbf{r})\,\mathrm{d}^3\mathbf{r} = \frac{\hbar}{2\epsilon_0\omega_\lambda}\delta_{\lambda\lambda'}. \tag{2.94}$$

The form of the normalization conditions (2.87) and (2.94) is tailored to real-mode functions and the necessity of modification of some fundamental relations is commented on by Knöll et al. (1987). All of these field operators may be written in the form

$$\hat{\mathbf{F}}(\mathrm{r}, t) = \sum_\lambda \left[\mathbf{F}_\lambda(\mathbf{r})\hat{a}_\lambda(t) + \mathbf{F}^*_\lambda(\mathbf{r})\hat{a}^\dagger_\lambda(t)\right]. \tag{2.95}$$

In dependence on the choice of the operator $\hat{\mathbf{F}}(\mathbf{r}, t)$, the functions $\mathbf{F}_\lambda(\mathbf{r})$ can be derived from the mode functions of the vector potential $\mathbf{A}_\lambda(\mathbf{r})$.

It is often convenient to decompose a given field operator $\hat{\mathbf{F}}(\mathbf{r}, t)$ into two parts by the relation

$$\hat{\mathbf{F}}(\mathbf{r}, t) = \hat{\mathbf{F}}^{(+)}(\mathbf{r}, t) + \hat{\mathbf{F}}^{(-)}(\mathbf{r}, t), \tag{2.96}$$

where

$$\hat{\mathbf{F}}^{(+)}(\mathbf{r}, t) = \sum_\lambda \mathbf{F}_\lambda(\mathbf{r})\hat{a}_\lambda(t), \tag{2.97}$$

$$\hat{\mathbf{F}}^{(-)}(\mathbf{r}, t) = [\hat{\mathbf{F}}^{(+)}(\mathbf{r}, t)]^\dagger. \tag{2.98}$$

Further, the Heisenberg equations of motion for the field operators are derived, so that the field operators can be expressed in terms of free-field and source-field operators. It is typical of the approach of Knöll et al. (1987) that any field operator $\hat{F}^{(+)}_k$ is decomposed into a free-field operator and a source-field operator as follows:

$$\hat{F}^{(+)}_k(\mathbf{r}, t) = \hat{F}^{(+)}_{k\mathrm{free}}(\mathbf{r}, t) + \hat{F}_{k\mathrm{s}}(\mathbf{r}, t), \tag{2.99}$$

where

$$\hat{F}^{(+)}_{k\mathrm{free}}(\mathbf{r}, t) = \sum_\lambda F_{k\lambda}(\mathbf{r})\hat{a}_{\lambda\mathrm{free}}(t), \tag{2.100}$$

$$\hat{F}_{k\mathrm{s}}(\mathbf{r}, t) = \int\int \theta(t - t')K_{kk'}(\mathbf{r}, t; \mathbf{r}', t')\hat{J}_{k'}(\mathbf{r}', t')\,\mathrm{d}^3\mathbf{r}'\,\mathrm{d}t'. \tag{2.101}$$

Here vector components are labelled by the index k and repeated indices k' mean summation.

Unfortunately, the operator $\hat{a}_{\lambda\mathrm{free}}(t)$ was not defined, so that we may only guess that $\hat{a}_{\lambda\mathrm{free}}(t)|_{t=t_0} = \hat{a}_\lambda(t_0)$ for $t = t_0$, and the dynamics for $t \geq t_0$ can be found in Knöll et al. (1987). In equation (2.101), the kernel $K_{kk'}$ is defined by the relation

$$K_{kk'}(\mathbf{r}, t; \mathbf{r}', t') = -\frac{1}{\mathrm{i}\hbar}\sum_\lambda F_{k\lambda}(\mathbf{r})A^*_{k'\lambda}(\mathbf{r}')\exp[-\mathrm{i}\omega_\lambda(t-t')]. \tag{2.102}$$

Inserting equation (2.99) yields the following representation of $\hat{F}_k^{(+)}$:

$$\hat{F}_k^{(+)}(\mathbf{r}, t) = \int\int \theta(t-t')K_{kk'}(\mathbf{r}, t; \mathbf{r}', t')\hat{J}_{k'}(\mathbf{r}', t')\,\mathrm{d}^3\mathbf{r}'\,\mathrm{d}t' + \hat{F}^{(+)}_{k\mathrm{free}}(\mathbf{r}, t). \tag{2.103}$$

In particular, if $\hat{F}_k^{(+)}$ is identified with the vector potential $\hat{A}_k^{(+)}$, it holds that $F_{k\lambda} = A_{k\lambda}$ and the kernel $K_{kk'}$ takes the form

$$K_{kk'}(\mathbf{r}, t; \mathbf{r}', t') = -\frac{1}{\mathrm{i}\hbar}\sum_\lambda A_{k\lambda}(\mathbf{r})A^*_{k'\lambda}(\mathbf{r}')\exp[-\mathrm{i}\omega_\lambda(t-t')]. \tag{2.104}$$

Analogously, if one is interested in the electric-field strength of the radiation $\hat{E}_k^{(+)}$, the appropriate form of the kernel $K_{kk'}$ is

$$K_{kk'}(\mathbf{r}, t; \mathbf{r}', t') = -\frac{1}{\hbar}\sum_\lambda \omega_\lambda A_{k\lambda}(\mathbf{r})A^*_{k'\lambda}(\mathbf{r}')\exp[-\mathrm{i}\omega_\lambda(t-t')]. \tag{2.105}$$

So the symmetry relations

$$K^*_{kk'}(\mathbf{r}, t; \mathbf{r}', t') = \mp K_{k'k}(\mathbf{r}', t'; \mathbf{r}, t) \tag{2.106}$$

are valid for $\hat{A}_k^{(+)}$ and $\hat{E}_k^{(+)}$, respectively. The information on the action of the optical instruments on the source field is contained in the space–time structure of the kernel $K_{kk'}$, which may be regarded as the apparatus function also used in classical optics.

Further, the commutation relations for various combinations of field operators at different times are studied and relationships between field commutators and source-quantity commutators are derived. The following abbreviations of the notation are used:

$$x = \{\mathbf{r}, t\} \tag{2.107}$$

and others, by which the superscripts +, − are introduced also for $\hat{J}_k(x)$ and $K_{kk'}(x, x')$. With these generalizations, it holds that

$$\hat{F}_k^{(j)}(x) = \hat{F}_{k\mathrm{free}}^{(j)}(x) + \hat{F}_{k\mathrm{s}}^{(j)}(x),\ j = +, -, \tag{2.108}$$

$$\hat{F}_{k\mathrm{s}}^{(j)}(x) = \int \theta(t - t')K_{kk'}^{(j)}(x, x')\hat{J}_{k'}^{(j)}(x')\,\mathrm{d}x'. \tag{2.109}$$

When appropriate, the time ordering symbols $\mathcal{T}_+$ and $\mathcal{T}_-$ are used. Let us consider any operator product $\hat{A}_1(t_1)\ \hat{A}_2(t_2)...\hat{A}_n(t_n)$. The symbol $\mathcal{T}_+$ introduces the time ordering of the operators $\hat{A}_i(t_i)$ with the latest time to the far left:

$$\begin{aligned}&\mathcal{T}_+\hat{A}_1(t_1)\hat{A}_2(t_2)...\hat{A}_n(t_n)\\ &= \hat{A}_{i_1}(t_{i_1})\hat{A}_{i_2}(t_{i_2})...\hat{A}_{i_n}(t_{i_n}) \text{ with } t_{i_1} > t_{i_2} > \cdots > t_{i_n},\end{aligned} \tag{2.110}$$

and the symbol $\mathcal{T}_-$ introduces time ordering of the operators $\hat{A}_i(t_i)$ with the latest time to the far right:

$$\begin{aligned}&\mathcal{T}_-\hat{A}_1(t_1)\hat{A}_2(t_2)...\hat{A}_n(t_n)\\ &= \hat{A}_{i_1}(t_{i_1})\hat{A}_{i_2}(t_{i_2})...\hat{A}_{i_n}(t_{i_n}) \text{ with } t_{i_1} < t_{i_2} < \cdots < t_{i_n}.\end{aligned} \tag{2.111}$$

From (2.91) it follows that

$$\begin{aligned}[\hat{F}_{k_1}^{(j_1)}(x_1), \hat{F}_{k_2}^{(j_2)}(x_2)] &= [\hat{F}_{k_1\mathrm{free}}^{(j_1)}(x_1), \hat{F}_{k_2\mathrm{free}}^{(j_2)}(x_2)]\\ &\quad + \hat{D}_{k_1k_2}^{(j_1,j_2)}(x_1, x_2) - \hat{D}_{k_2k_1}^{(j_2,j_1)}(x_2, x_1),\end{aligned} \tag{2.112}$$

where

$$\begin{aligned}\hat{D}_{k_1k_2}^{(j_1,j_2)}(x_1, x_2) &= -\int\int \theta(t_2 - t_2')\theta(t_2' - t_1')\theta(t_1' - t_1)\\ &\otimes K_{k_1k_1'}^{(j_1)}(x_1, x_1')K_{k_2k_2'}^{(j_2)}(x_2, x_2')[\hat{J}_{k_1'}^{(j_1)}(x_1'), \hat{J}_{k_2'}^{(j_2)}(x_2')]\,\mathrm{d}x_1'\,\mathrm{d}x_2'.\end{aligned} \tag{2.113}$$

From an inspection of equation (2.113), we readily learn that

$$\hat{D}_{k_1k_2}^{(j_1,j_2)}(x_1, x_2) = \hat{0} \text{ if } t_1 > t_2. \tag{2.114}$$

The commutators in (2.112) are

$$[\hat{F}_{k_1\mathrm{free}}^{(j)}(x_1), \hat{F}_{k_2\mathrm{free}}^{(j)}(x_2)] = \hat{0},\ \ j = +, -, \tag{2.115}$$

$$[\hat{F}_{k_1\mathrm{free}}^{(+)}(x_1), \hat{F}_{k_2\mathrm{free}}^{(-)}(x_2)] = F_{k_1k_2}(x_1, x_2)\hat{1}, \tag{2.116}$$

where

$$F_{k_1k_2}(x_1, x_2) = \sum_\lambda F_{k_1\lambda}(\mathbf{r}_1)F^*_{k_2\lambda}(\mathbf{r}_2)\exp[-\mathrm{i}\omega_\lambda(t_1 - t_2)]. \tag{2.117}$$

It would be interesting to find the particular forms of the commutators between $\hat{A}$ and $\hat{E}$ or between $\hat{A}^{(+)}$ and $\hat{E}^{(-)}$.

Further, these commutation relations are used to express field correlation functions of free-field operators and source-field operators and to describe the effect of the optical system on the quantum properties of light fields.

The method of transformation of normal and time orderings is demonstrated for the following important class of correlation functions:

$$\begin{aligned} &G^{(m,n)}_{k_1\ldots k_{m+n}}(x_1, \ldots, x_{m+n}) \\ &= \left\langle \left[\mathcal{T}_- \prod_{j=1}^{m} \hat{F}^{(-)}_{k_j}(x_j)\right]\left[\mathcal{T}_+ \prod_{j=m+1}^{m+n} \hat{F}^{(+)}_{k_j}(x_j)\right]\right\rangle. \end{aligned} \tag{2.118}$$

This transformation is understood in the relation

$$\begin{aligned} &\mathcal{T}_+ \hat{F}^{(+)}_{k_1}(x_1)\hat{F}^{(+)}_{k_2}(x_2) \\ &= \mathcal{O}_+ \left[\hat{F}^{(+)}_{k_1\mathrm{free}}(x_1) + \hat{F}^{(+)}_{k_1\mathrm{s}}(x_1)\right]\left[\hat{F}^{(+)}_{k_2\mathrm{free}}(x_2) + \hat{F}^{(+)}_{k_2\mathrm{s}}(x_2)\right]. \end{aligned} \tag{2.119}$$

In equation (2.119) and the following ones the ordering symbols $\mathcal{O}_+$ and $\mathcal{O}_-$ are used. The symbol $\mathcal{O}_+$ introduces the following ordering of operators $\hat{F}^{(+)}_{k_i\mathrm{s}}(x_i)$, $\hat{F}^{(+)}_{k_j\mathrm{free}}(x_j)$:

(i) Ordering of the operators $\hat{F}^{(+)}_{k_i\mathrm{s}}(x_i)$, $\hat{F}^{(+)}_{k_j\mathrm{free}}(x_j)$ with the operators $\hat{F}^{(+)}_{k_j\mathrm{free}}(x_j)$ to the right of the operators $\hat{F}^{(+)}_{k_i\mathrm{s}}(x_i)$.

(ii) Substitution of equation (2.109) for the operators $\hat{F}^{(+)}_{k_i\mathrm{s}}(x_i)$ and $\mathcal{T}_+$ time ordering of the source-quantity operators $\hat{J}_{k'_i}(x'_i)$ in the resulting source-quantity operator products before performing the integrations with respect to t'_i.

The symbol $\mathcal{O}_-$ introduces the following operator ordering in products of operators $\hat{F}^{(-)}_{k_i\mathrm{s}}(x_i)$, $\hat{F}^{(-)}_{k_j\mathrm{free}}(x_j)$:

(i) Ordering of the operators $\hat{F}^{(-)}_{k_i\mathrm{s}}(x_i)$, $\hat{F}^{(-)}_{k_j\mathrm{free}}(x_j)$ with the operators $\hat{F}^{(-)}_{k_j\mathrm{free}}(x_j)$ to the left of the operators $\hat{F}^{(-)}_{k_i\mathrm{s}}(x_i)$.

(ii) Substitution of equation (2.109) for the operators $\hat{F}^{(-)}_{k_i\mathrm{s}}(x_i)$ and $\mathcal{T}_-$ time ordering of the source-quantity operators $\hat{J}^\dagger_{k'_i}(x'_i)$ in the resulting source-quantity operator products before performing the integrations with respect to t'_i.

Equation (2.119) may now be generalized:

$$\mathcal{T}_+ \prod_{j=1}^{n} \hat{F}_{k_j}^{(+)}(x_j) = \mathcal{O}_+ \prod_{j=1}^{n} \left[\hat{F}_{k_j \mathrm{free}}^{(+)}(x_j) + \hat{F}_{k_j \mathrm{s}}^{(+)}(x_j) \right], \tag{2.120}$$

$$\mathcal{T}_- \prod_{j=1}^{n} \hat{F}_{k_j}^{(-)}(x_j) = \mathcal{O}_- \prod_{j=1}^{n} \left[\hat{F}_{k_j \mathrm{free}}^{(-)}(x_j) + \hat{F}_{k_j \mathrm{s}}^{(-)}(x_j) \right]. \tag{2.121}$$

Using relations (2.118), (2.120), and (2.121), we may represent the correlation functions as

$$\begin{aligned} & G_{k_1 \ldots k_{m+n}}^{(m,n)}(x_1, \ldots, x_{m+n}) \\ &= \left\langle \left\{ \mathcal{O}_- \prod_{j=1}^{m} \left[\hat{F}_{k_j \mathrm{free}}^{(-)}(x_j) + \hat{F}_{k_j \mathrm{s}}^{(-)}(x_j) \right] \right\} \right. \\ & \left. \otimes \left\{ \mathcal{O}_+ \prod_{j=m+1}^{m+n} \left[\hat{F}_{k_j \mathrm{free}}^{(+)}(x_j) + \hat{F}_{k_j \mathrm{s}}^{(+)}(x_j) \right] \right\} \right\rangle. \end{aligned} \tag{2.122}$$

When at the points of observation the following conditions are fulfilled

$$\left\langle \ldots \hat{F}_{k\mathrm{free}}^{(+)} \right\rangle = 0 = \left\langle \hat{F}_{k\mathrm{free}}^{(-)} \ldots \right\rangle, \tag{2.123}$$

then the relation (2.122) can be simplified:

$$\begin{aligned} & G_{k_1 \ldots k_{m+n}}^{(m,n)}(x_1, \ldots, x_{m+n}) \\ &= \left\langle \left[\mathcal{O}_- \prod_{j=1}^{m} \hat{F}_{k_j \mathrm{s}}^{(-)}(x_j) \right] \left[\mathcal{O}_+ \prod_{j=m+1}^{m+n} \hat{F}_{k_j \mathrm{s}}^{(+)}(x_j) \right] \right\rangle. \end{aligned} \tag{2.124}$$

When written in more detail, into the relation (2.124), the complex kernels $K_{k_j k_j'}(x_j, x_j')$ are introduced.

It is noted that the effect of the beam splitter that is used for mixing of source light with the reference beam in the case of homodyne detection is described by the assumption that the reference light beam is a free field. In Knöll et al. (1987) the relation (2.122) is specialized to a multimode coherent free field $|\{\alpha_\lambda\}\rangle$,

$$\hat{F}_{k\mathrm{free}}^{(+)}(x)|\{\alpha_\lambda\}\rangle = \mathcal{F}_k(x)|\{\alpha_\lambda\}\rangle, \tag{2.125}$$

that is to say

$$
\begin{aligned}
& G^{(m,n)}_{k_1 \ldots k_{m+n}}(x_1, \ldots, x_{m+n}) \\
&= \left\langle \left\{ \mathcal{O}_- \prod_{j=1}^{m} \left[\mathcal{F}^*_{k_j}(x_j)\hat{1} + \hat{F}^{(-)}_{k_j \mathrm{s}}(x_j) \right] \right\} \right. \\
&\quad \left. \otimes \left\{ \mathcal{O}_+ \prod_{j=m+1}^{m+n} \left[\mathcal{F}_{k_j}(x_j)\hat{1} + \hat{F}^{(+)}_{k_j \mathrm{s}}(x_j) \right] \right\} \right\rangle .
\end{aligned}
\tag{2.126}
$$

Finally, the theory is applied to the photocount statistics. Following Glauber's theory of photon detection (Glauber 1965, Kelley and Kleiner 1964), the probability of observing precisely n events in a counting time interval $[t, t + \Delta t)$ is given by the relation

$$
p_n(t, \Delta t) = \left\langle \mathbf{\Omega} \frac{1}{n!} \left[\hat{\Gamma}(t, \Delta t) \right]^n \exp\left[-\hat{\Gamma}(t, \Delta t) \right] \right\rangle , \tag{2.127}
$$

where

$$
\hat{\Gamma}(t, \Delta t) = \sum_i \int_t^{t+\Delta t} \int_t^{t+\Delta t} S(t_1 - t_2) \hat{E}^{(-)}_k(\mathbf{r}_i, t_1) \hat{E}^{(+)}_k(\mathbf{r}_i, t_2)\, \mathrm{d}t_1\, \mathrm{d}t_2 \tag{2.128}
$$

may be interpreted as the operator of the integrated intensity. Here $\mathbf{r}_i$ are position vectors of the detector atoms and $S(t)$ is a response function. Let us note that one usually assumes that

$$
S(t_1 - t_2) = \eta \delta(t_1 - t_2), \tag{2.129}
$$

with some η. In relation (2.127), the ordering symbol $\mathbf{\Omega}$ introduces the following operator ordering:

(i) The normal ordering of the operators $\hat{E}^{(-)}_k(x)$, $\hat{E}^{(+)}_k(x)$ with the operators $\hat{E}^{(-)}_k(x)$ to the left of the operators $\hat{E}^{(+)}_k(x)$.
(ii) $\mathcal{T}_+$ ordering of the operators $\hat{E}^{(+)}_k(x)$ and $\mathcal{T}_-$ ordering of the operators $\hat{E}^{(-)}_k(x)$.

In analogy with (2.122), relation (2.127) becomes

$$
p_n(t, \Delta t) = \left\langle \mathcal{O} \frac{1}{n!} \left[\hat{\Gamma}(t, \Delta t) \right]^n \exp[-\hat{\Gamma}(t, \Delta t)] \right\rangle , \tag{2.130}
$$

where the $\boldsymbol{\Omega}$ ordering is simply replaced by the $\mathcal{O}$ ordering defined as follows:

(i) The normal ordering of the operators $\hat{E}_{ks}^{(-)}(x)$, $\hat{E}_{k\mathrm{free}}^{(-)}(x)$, $\hat{E}_{ks}^{(+)}(x)$, $\hat{E}_{k\mathrm{free}}^{(+)}(x)$ with the operators $\hat{E}_{ks}^{(-)}(x)$, $\hat{E}_{k\mathrm{free}}^{(-)}(x)$ to the left of the operators $\hat{E}_{ks}^{(+)}(x)$, $\hat{E}_{k\mathrm{free}}^{(+)}(x)$.
(ii) $\mathcal{O}_+$ ordering of the operators $\hat{E}_{ks}^{(+)}(x)$, $\hat{E}_{k\mathrm{free}}^{(+)}(x)$ and $\mathcal{O}_-$ ordering of the operators $\hat{E}_{ks}^{(-)}(x)$, $\hat{E}_{k\mathrm{free}}^{(-)}(x)$.

The fulfilling of the conditions (2.123) causes a modification of relation (2.128) as follows:

$$\hat{\Gamma}(t,\Delta t) = \sum_i \int_t^{t+\Delta t}\int_t^{t+\Delta t} S(t_1-t_2)\hat{E}_{ks}^{(-)}(\mathbf{r}_i,t_1)\hat{E}_{ks}^{(+)}(\mathbf{r}_i,t_2)\,\mathrm{d}t_1\,\mathrm{d}t_2. \tag{2.131}$$

In the case of mixing the source field light with a coherent free-field reference beam, there is an analogy with the relation (2.126):

$$\hat{\Gamma}(t,\Delta t) = \sum_i \int_t^{t+\Delta t}\int_t^{t+\Delta t} S(t_1-t_2)$$
$$\times\left[\mathcal{E}_k^*(\mathbf{r}_i,t_1)\hat{1} + \hat{E}_{ks}^{(-)}(\mathbf{r}_i,t_1)\right]\left[\mathcal{E}_k(\mathbf{r}_i,t_2)\hat{1} + \hat{E}_{ks}^{(+)}(\mathbf{r}_i,t_2)\right]\mathrm{d}t_1\,\mathrm{d}t_2. \tag{2.132}$$

A generalization of the Wick theorem on transforming a time-ordered product onto a sum of normally ordered terms was performed by Agarwal and Wolf (1970).

The quantum theory of the radiation field interacting with atomic sources in the presence of a linear, dispersionless, and absorptionless dielectric with space-dependent refractive index has been applied to the description of the action of a resonator-like cavity with input–output coupling and filled with an active medium (Knöll and Welsch 1992).

2.2.5 Continuum Frequency-Space Description

Blow et al. (1990) have formulated the quantum theory of optical wave propagation without recourse to cavity quantization. This approach avoids the introduction of a box-related mode spacing and enables one to use a continuum frequency-space description. In this chapter and in that by Blow et al. (1991) a continuous-mode quantum theory of electromagnetic field has been developed. As usual in the quantum field theory, the box-related modes are considered whose creation and destruction operators satisfy the usual independent boson commutation relations:

$$[\hat{a}_i,\hat{a}_j^\dagger] = \delta_{ij}\hat{1}. \tag{2.133}$$

Different modes of the cavity, labelled by i and j, have frequencies given by different integer multiples of the mode spacing $\Delta\omega$. The mode spectrum becomes continuous as $\Delta\omega \to 0$ and in this limit the transformation to continuous-mode operators is convenient:

$$\hat{a}_i \to \sqrt{\Delta\omega}\,\hat{a}(\omega). \tag{2.134}$$

A complete orthonormal set of functions was considered which may describe states of finite energy. The set is numerable infinite and to each function in it a destruction operator is assigned. Such operators have all the usual properties of the operators of the monochromatic mode.

Further specific states of the field have been treated such as coherent states, number states, noise and squeezed states. With the use of noncontinuous operators, a generalization of the single-mode normal ordering theorem was proved. Field quantization in a dielectric has been treated including the material dispersion and the theory has been applied to the pulse propagation in an optical fibre. A comparison with results by Drummond (1990, 1994) would be in order.

Let us consider the fields in a lossless dielectric material with the real relative permittivity $\epsilon(\omega)$ and the refractive index $n(\omega)$ related by

$$\epsilon(\omega) = [n(\omega)]^2. \tag{2.135}$$

Let us recall the definition of the phase velocity

$$v_{\mathrm{F}}(\omega) = \frac{\omega}{k} = \frac{c}{n(\omega)} \tag{2.136}$$

and that of the group velocity $v_{\mathrm{G}}(\omega)$

$$\frac{1}{v_{\mathrm{G}}(\omega)} = \frac{\partial k}{\partial\omega} = \frac{1}{c}\frac{\partial}{\partial\omega}[\omega n(\omega)]. \tag{2.137}$$

The normalization of the field operators is fixed by requirement that the normally ordered total energy density operator $\hat{U}(z,t)$ has the diagonal form:

$$\hat{H}_{\text{free}} = \mathcal{A}\int_{-\infty}^{\infty} \hat{U}(z,t)\,\mathrm{d}z = \int \hbar\omega\hat{a}^{\dagger}(\omega)\hat{a}(\omega)\,\mathrm{d}\omega. \tag{2.138}$$

The field operators are obtained in accordance with the relation

$$\hat{E}^{(+)}(z,t) = -\frac{\partial}{\partial t}\hat{A}^{(+)}(z,t),\ \hat{B}^{(+)}(z,t) = \frac{\partial}{\partial z}\hat{A}^{(+)}(z,t) \tag{2.139}$$

and with the expansion of the vector-potential operator

$$\hat{A}^{(+)}(z,t) = \int_{-\infty}^{\infty}\sqrt{\frac{\hbar v_{\mathrm{G}}(\omega)}{4\pi\epsilon_0 c\omega n(\omega)\mathcal{A}}} \times \sum_{\lambda=1,2} \boldsymbol{\epsilon}(k,\lambda)\hat{a}(k,\lambda)\exp[-\mathrm{i}(\omega t - kz)]\,\mathrm{d}k. \tag{2.140}$$

Noting that

$$\mathrm{d}k = \frac{\mathrm{d}\omega}{v_{\mathrm{G}}(\omega)},\ \hat{a}(k,\lambda) = \sqrt{v_{\mathrm{G}}(\omega)}\hat{a}(\omega), \tag{2.141}$$

and taking the polarization to be parallel to the x-axis, it follows from (2.139) that the field operators are

$$\hat{E}^{(+)}(z,t) = \mathrm{i}\int \sqrt{\frac{\hbar\omega}{4\pi\epsilon_0 c\mathcal{A}n(\omega)}} \times \hat{a}(\omega)\exp\left\{-\mathrm{i}\omega\left[t - \frac{n(\omega)z}{c}\right]\right\}\mathrm{d}\omega \tag{2.142}$$

and

$$\hat{B}^{(+)}(z,t) = \mathrm{i}\int \sqrt{\frac{\hbar\omega n(\omega)}{4\pi\epsilon_0 c^3\mathcal{A}}} \times \hat{a}(\omega)\exp\left\{-\mathrm{i}\omega\left[t - \frac{n(\omega)z}{c}\right]\right\}\mathrm{d}\omega. \tag{2.143}$$

Alternatively, the propagation constant can be expanded to the second order in frequency and a partial differential equation can be obtained (cf. Drummond 1990). Assuming a narrow bandwidth, the slowly varying field envelope can be represented by the operator $\hat{a}(z,t)$, which obeys the equation

$$\mathrm{i}\frac{\partial}{\partial z}\hat{a}(z,t) + \frac{k''}{2}\frac{\partial^2}{\partial t^2}\hat{a}(z,t) = \hat{0}, \tag{2.144}$$

where k'' is the second derivative with respect to the frequency of the propagation constant, evaluated at the central frequency. The equation has been simplified using the transformation of envelope into a frame moving with the group velocity. This is necessary for the envelope to be slowly varying. In the classical nonlinear optics the stationary fields have also envelopes, but they seem to be defined otherwise.

The treatment of this problem in the noncontinuous basis proceeds from the replacement

$$\hat{a}(z,t) = \sum_j \phi_j(z,t)\hat{c}_j, \tag{2.145}$$

where $\phi_j(z,t)$ are a complete orthonormal set of functions on z and $\hat{c}_j$ are destruction operators obeying the usual commutation relations. The advantage of this treatment is that the functional dependence on z and t is contained in the c-number functions rather than the operators $\hat{a}(z,t)$ as in the propagation equation (2.144) for example. It is not emphasized by Blow et al. (1990) that the solution of equation (2.144) preserves the equal-space, not equal-time, commutators. Similarly, the set of functions $\phi_j(z,t)$ enjoys the orthonormality and completeness only as the equal-space, but not equal-time, properties. The propagation equation (2.144) now yields the following equations for the noncontinuous basis functions:

$$\mathrm{i}\frac{\partial}{\partial z}\phi_j(z,t) + \frac{k''}{2}\frac{\partial^2}{\partial t^2}\phi_j(z,t) = 0. \tag{2.146}$$

Finally, the process of photodetection in free space is considered and the results applied to homodyne detection with both local oscillator and signal fields pulsed. The results of sets of measurements in which the photocurrent is integrated over periods T can be predicted by the use of an operator

$$\hat{M} = \int_\tau^{\tau+T} \hat{a}^\dagger(t)\hat{a}(t)\,\mathrm{d}t. \tag{2.147}$$

Here τ is the start time of the measurements, the detector is placed at $z = 0$, and

$$\hat{a}(t) = \frac{1}{\sqrt{2\pi}}\int \hat{a}(\omega)\exp(-\mathrm{i}\omega t)\,\mathrm{d}\omega. \tag{2.148}$$

Let us further consider a balanced homodyne detector in which the light beam under study is superposed on a local oscillator by combining them at a 50:50 beam splitter. The measured quantity is the difference in the photocurrents of two detectors placed in the output arms of the beam splitter and it can be represented by the operator (Collett et al. 1987)

$$\hat{O} = \mathrm{i}\int_\tau^{\tau+T} [\hat{a}^\dagger(t)\hat{a}_\mathrm{L}(t) - \hat{a}_\mathrm{L}^\dagger(t)\hat{a}(t)]\,\mathrm{d}t, \tag{2.149}$$

where $\hat{a}_\mathrm{L}^\dagger(t)$ and $\hat{a}_\mathrm{L}(t)$ are the continuum creation and destruction operators of the local oscillator field and $\hat{a}^\dagger(t)$ and $\hat{a}(t)$ correspondingly for the signal field.

For homodyne detection of pulsed signals it is advantageous to use a pulsed local oscillator. The pulsed signal is described by the noncontinuous basis function $\phi_0(t)$ and the local oscillator is described by a normalized function $\phi_\mathrm{L}(t)$, the field of the local oscillator being in the coherent state $|\{\alpha_\mathrm{L}(t)\}\rangle$, where

$$\alpha_\mathrm{L}(t) = \sqrt{N_\mathrm{L}}\exp(\mathrm{i}\theta_\mathrm{L})\phi_\mathrm{L}(t), \tag{2.150}$$

with N_L the mean total number of photons in the pulse and θ_L the externally controlled local oscillator phase. Let us recall the definition of a coherent state:

$$|\{\alpha(t)\}\rangle = \hat{D}(\{\alpha(t)\})|0\rangle, \tag{2.151}$$

with

$$\hat{D}(\{\alpha(t)\}) = \exp\left\{\int [\alpha(t)\hat{a}^\dagger(t) - \alpha^*(t)\hat{a}(t)]\right\}, \tag{2.152}$$

which is close enough to that by Blow et al. (1990) except for the exchange of space for time. It is assumed that the signal field is described by a set of noncontinuous operators $\hat{d}_i$ at the output of a nonlinear system and the signal field at the input to the system is described by a similar set of operators $\hat{c}_i$. The action of the nonlinear system is defined by the relations

$$\hat{d}_0 = \mu\hat{c}_0 + \nu\hat{c}_0^\dagger,$$
$$\hat{d}_i = \hat{c}_i, i > 0. \tag{2.153}$$

In the relation (2.149) it is necessary to substitute

$$\hat{a}(t) = \sum_i \phi_i(t)\hat{d}_i. \tag{2.154}$$

In analogy, we consider

$$\hat{a}_{\mathrm{L}}(t) = \sum_i \phi_{i\mathrm{L}}(t)\hat{c}_{i\mathrm{L}}, \tag{2.155}$$

where the subscript L only modifies the familiar meanings and $\phi_{0\mathrm{L}}(t) = \phi_{\mathrm{L}}(t)$.

It is shown how the formulation of the quantum field theory is modified for the one-dimensional optical system. The fields are defined in an infinite waveguide parallel to the z-axis, but of finite cross-sectional area $\mathcal{A}$ of the rectangular form with sides parallel to the x- and y-axes. The x and y wave-vector components are thus restricted to discrete values and any three-dimensional integral over this spatial region is converted according to

$$\int \mathrm{d}^3k \to \frac{(2\pi)^2}{\mathcal{A}} \sum_{k_x,k_y} \int \mathrm{d}k_z. \tag{2.156}$$

On the assumption that the modes with $k_x \neq 0$ or $k_y \neq 0$ are vacuum ones, a reduced Hilbert (namely Fock) space can be exploited. The summation in (2.156) can, therefore, be removed and putting $k_z = k$, the other conversions are

$$\delta^{(3)}(\mathbf{k} - \mathbf{k}') \to \frac{\mathcal{A}}{(2\pi)^2}\delta(k - k'), \tag{2.157}$$

$$\hat{a}(\mathbf{k}, \lambda) \to \frac{\sqrt{\mathcal{A}}}{2\pi}\hat{a}(k, \lambda). \tag{2.158}$$

The vector-potential operator has been modified for the dispersive lossless medium and compared with Drummond (1990) and Loudon (1963), the positive-frequency part is

$$\hat{A}^{(+)}(\mathbf{r}, t) = \int \sqrt{\frac{\hbar v_{\mathrm{G}}(\omega)}{16\pi^3 \epsilon_0 c\omega n(\omega)}}$$

$$\times \sum_{\lambda=1,2} \boldsymbol{\epsilon}(\mathbf{k}, \lambda)\hat{a}(\mathbf{k}, \lambda) \exp[-\mathrm{i}(\omega t - \mathbf{k} \cdot \mathbf{r})]\, \mathrm{d}^3\mathbf{k}, \tag{2.159}$$

$$\omega = \frac{c}{n(\omega)}|\mathbf{k}|. \tag{2.160}$$

The expression (2.159) can be converted to the one-dimensional form easily (as indicated above, cf. (2.140)).

McDonald (2001) has considered a variation of the physical situation of "slow light" to show that the group velocity can be negative at central frequency. A Gaussian pulse can emerge from the far side of a slab earlier than it hits the near side and the pulse emission at the far side is accompanied by an antipulse emission, the antipulse propagating within the slab so as to annihilate the incident pulse at the near side.

2.3 Quantum Description of Experiments with Stationary Fields

Burnham and Weinberg (1970) found that the measured value of the correlation time between the two optical photons produced in a parametric process was very small, an effect of a practical interest. Laboratory techniques for doing experiments with single photons also have advanced. Since 1985, such photon pairs have become familiar for the study of nonclassical aspects of light (Horne et al. 1990).

The process of optical parametric three-wave mixing in a second-order nonlinear medium consists of the coherent interaction between pump, signal, and idler waves. This process may occur as frequency down conversions, specifically as an optical parametric oscillation and an optical parametric amplification. In a travelling-wave setting, the optical parametric generation is called a spontaneous parametric down-conversion. The photon pairs (biphotons) produced in parametric down-conversion are useful in experiments concerning fundamental questions of quantum theory. The description of experiments has been facilitated by studies of Campos et al. (1990). The autocorrelation and cross-correlation properties of the signal and idler beams produced in the parametric down-conversion have been studied, e.g. in Joobeur et al. (1994). A unified treatment of the experiment on the interference of a "biphoton with itself" and of other three experiments has been provided by Casado et al. (1997a). A fourth-order interference has been obtained in the four cases, and the uniformity has been achieved also by the use of the Wigner (or Weyl) representation of the field operators. A similar treatment of the famous experiment and of another one has been presented by Casado et al. (1997b). A second-order interference has been treated in the two cases and the stochastic properties of the pump beam have been respected.

The studies of the fundamentals of quantum mechanics underlie such interesting applications as quantum cryptography and quantum computing (Bowmeester et al. 2000). The experiments have become very popular (Shih 2003). Design of experiments for undergraduate students has become feasible (Galvez et al. 2005).

Here we return from the Wigner to the Hilbert-space formalism as in Peřinová and Lukš (2003). First we consider the three-dimensional expansion of the operator of a chosen component of the electric vector after Casado et al. (1997a). As in the schematics of the experiments the field is restricted to paths leading to detectors, we introduce one-dimensional expansions of the electric-field operator. We attempt to consider orthogonal modal functions, although we cannot define them everywhere, but only on the paths. We are aware of the dangerous position, where one cannot evaluate the orthogonality property for the lack of a complete definition. In this approach we do not start with the description of the process of parametric down-conversion from a Hamiltonian, but with the response of the output fields of a nonlinear crystal to the input fields (Casado et al. 1997a) when two paths cross such a crystal. Such a response depends also on stochastic properties of the pump beam, which is assumed to be monochromatic however. The experiment on the interference of signal and idler photons (Ghosh et al. 1986) can do with the simple description, when the lack of the second-order interference is derived. In the use of two detectors we consider four paths and modify (double) the description. Nevertheless, we do not reproduce the well-known result.

Similarly we proceed in the case of the experiment of Rarity and Tapster (1990), which was also used to test Bell's inequality using phase and momentum. In contrast, in the case of the experiment of Franson (1989) we are allowed to return to the simple description, as essentially two paths are involved, even though the schematic is more complicated. This experiment was proposed in order to test a Bell inequality for energy and time.

Next we deal with induced coherence and indistinguishability in two-photon interference (Zou et al. 1991). In this case the schematic comprises two nonlinear crystals, the number of paths is greater, but since two paths belong to each crystal, the simple description is appropriate. The lack of induced emission made it a "mind-boggling" experiment (Greenberger et al. 1993), but the indistinguishability of the paths along which the signal photon arrives at the detector (in fact, the biphoton arrives at the two detectors) is still held for the reason of interference. We may refer to Casado et al. (1997b), where stochastic properties of the pump beam are taken into account. Two experiments are analysed: frustrated two-photon creation by interference, and induced coherence and indistinguishability. Coincidences are not studied and a second-order interference has been obtained in the two cases. Last we mention the frustrated two-photon creation via interference (Herzog et al. 1994) restricting ourselves to the second-order interference and the monochromatic pump.

2.3.1 Spatio-temporal Descriptions of Parametric Down-Conversion Experiments

In the Hilbert-space representation of the light field, the electric vector is represented as a sum of two mutually conjugate operators (Casado et al. 1997a)

$$\hat{\mathbf{E}}(\mathbf{r},t) = \hat{\mathbf{E}}^{(+)}(\mathbf{r},t) + \hat{\mathbf{E}}^{(-)}(\mathbf{r},t), \tag{2.161}$$

$$\hat{\mathbf{E}}^{(+)}(\mathbf{r},t) = \mathrm{i}\sum_{\mathbf{k},\lambda}\sqrt{\frac{\hbar\omega_{\mathbf{k}}}{2L^3}}\epsilon_{\mathbf{k},\lambda}\hat{a}_{\mathbf{k},\lambda}(t)e^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}}, \tag{2.162}$$

where L^3 is the normalization volume, $\hat{a}_{\mathbf{k},\lambda}(t)$ is the annihilation operator for a photon whose wave vector is $\mathbf{k}$ and whose polarization vector is $\epsilon_{\mathbf{k},\lambda}$, and $\omega_{\mathbf{k}} = c|\mathbf{k}|$. Equations (2.161) and (2.162) correspond to the Heisenberg picture, where all time dependence of the averages comes from the creation and annihilation operators $\hat{a}^{\dagger}_{\mathbf{k},\lambda}(t)$ and $\hat{a}_{\mathbf{k},\lambda}(t)$. In this picture the state of the field is represented by a time-independent statistical operator $\hat{\rho}$.

As we do not study experiments involving polarizing devices, we find it convenient to use a scalar approximation well known in classical optics. When Casado et al. (1997a) use the subscripts on the (Wigner representations of) field operators they indicate that the light beam contains frequencies within a range and that "transverse" components of wave vectors are limited by small upper values. We believe that such subscripts indicate which part of the field is considered.

The laser theory and, in general, the theory of resonators connect the quantum field with the annihilation operators not via the complex exponentials, but via more general modal functions, which are often related to the device. We suppose that such an approach can be interesting also in our study, after we find modal functions that are connected to the linear devices used and to the mirrors. Obviously, the free evolution of operators $\hat{a}_{\mathbf{k}0}$ (zeroth-order solution) is transformed into a kind of linear dynamics of the "relevant" component $\hat{E}^{(+)}_{\mathrm{ss}\,0}(\mathbf{r},t)$ of the electric vector via the appropriate modal functions, with ss being any subscript. This process can be formalized by a quadratic Hamiltonian, which differs from the free-field Hamiltonian only by the meaning of the creation and annihilation operators. We restrict ourselves to the operator $\hat{\rho}$ that represents a vacuum state.

We suppose that one or two nonlinear crystals involved in the experiment are described in terms of interaction Hamiltonians. The action of the scattering operator on the initial field can be "guessed". The interaction lasting only for a short time and being spatially confined to the medium suggests to us an appropriate modification of the linear dynamics. We modify also the notation for the resulting field by omitting the initial subscript 0.

(i) *The process of parametric down-conversion*

We are going to study the process of parametric down-conversion of light in the Hilbert-space representation. We refer to any of our figures for a sketch of the setup used for parametric down-conversion. A nonlinear crystal is pumped by a laser beam V, producing a continuum of coloured cones around the axis defined by the pump. In experimental practice two narrow correlated beams, called "signal" $\hat{E}_{\mathrm{s}}$ and "idler" $\hat{E}_{\mathrm{i}}$, are selected by means of apertures, filters, or just the detectors. Let us take the origin of the coordinate system, $\mathbf{0} \equiv \mathbf{0}_1$, at the centre of the crystal. We treat the pump beam as an intense monochromatic wave represented, in the scalar approximation, by

$$V(\mathbf{r}, t) = V e^{\mathrm{i}(\mathbf{k}_0 \cdot \mathbf{r} - \omega_0 t)} + \text{c.c.}, \tag{2.163}$$

where V is a complex amplitude of a pump beam, ω_0 is a frequency of the pump beam, $\mathbf{k}_0$ is an appropriate wave vector, and c.c. means the complex conjugate term to the previous one. In a product with the identity operator it may be added to the electric-field operator.

Now, let us consider two narrow correlated beams, called signal and idler, with average frequencies $\omega_{\mathrm{s}}, \omega_{\mathrm{i}}$ and wave vectors $\mathbf{k}_{\mathrm{s}}, \mathbf{k}_{\mathrm{i}}$, respectively, fulfilling the matching conditions

$$\omega_{\mathrm{s}} + \omega_{\mathrm{i}} = \omega_0, \quad \mathbf{k}_{\mathrm{s}} + \mathbf{k}_{\mathrm{i}} = \mathbf{k}_0. \tag{2.164}$$

The response $\hat{E}_{\mathrm{s}}^{(+)}(\mathbf{r}, t)$ and $\hat{E}_{\mathrm{i}}^{(-)}(\mathbf{r}, t)$ of a nonlinear crystal to the input fields $\hat{E}_{0\mathrm{s}}^{(+)}(\mathbf{r}, t)$ and $\hat{E}_{0\mathrm{i}}^{(-)}(\mathbf{r}, t)$ is as

$$\begin{aligned} \hat{E}_{\mathrm{s}}^{(+)}(\mathbf{r}, t) &= (\hat{\hat{1}} + g^2|V|^2\hat{\hat{J}})\hat{E}_{0\mathrm{s}}^{(+)}(\mathbf{r}, t) + e^{-\mathrm{i}\omega_0 t} g V \hat{\hat{G}} \hat{E}_{0\mathrm{i}}^{(-)}(\mathbf{r}, t), \\ \hat{E}_{\mathrm{i}}^{(-)}(\mathbf{r}, t) &= e^{\mathrm{i}\omega_0 t} g V \hat{\hat{G}}^{\dagger} \hat{E}_{0\mathrm{s}}^{(+)}(\mathbf{r}, t) + (\hat{\hat{1}} + g^2|V|^2\hat{\hat{J}})\hat{E}_{0\mathrm{i}}^{(-)}(\mathbf{r}, t), \end{aligned} \tag{2.165}$$

where g is an effective coupling constant, $\hat{\hat{1}}$ is the identity superoperator, $\hat{\hat{G}}$ and $\hat{\hat{J}}$ are antilinear and linear superoperators, respectively, which substitute expansions in annihilation (creation) operators for annihilation (creation) operators ($\hat{\hat{J}}$ yields an expansion in the annihilation operators in the first equation and $\hat{\hat{G}}$ yields an expansion in the creation operators in the same equation), and $\hat{\hat{G}}^{\dagger}\hat{E}_{0\mathrm{s}}^{(+)}(\mathbf{r}, t) \equiv [\hat{\hat{G}}\hat{E}_{0\mathrm{s}}^{(-)}(\mathbf{r}, t)]^{\dagger}$. The relation (2.165) can be modified (doubled) so that it relates output fields $\hat{E}_{\mathrm{s}_j}^{(+)}(\mathbf{r}, t)$, $\hat{E}_{\mathrm{i}_j}^{(-)}(\mathbf{r}, t)$, $j = 1, 2$, to input fields $\hat{E}_{0\mathrm{s}_j}^{(+)}(\mathbf{r}, t)$, $\hat{E}_{0\mathrm{i}_j}^{(-)}(\mathbf{r}, t)$, $j = 1, 2$. Since a pointlike crystal is considered (Casado et al. 1997b), it may be interesting to imagine Equations (2.165) at $\mathbf{r} = \mathbf{0}$ without the subscript 0 on the right-hand side. It can occur, but at the cost of other notation. The interaction does not change the field just in front of the crystal, so we can interpret the initial field as the "in" resulting field. As it is almost at the centre of the crystal, it differs negligibly from the initial field just behind the crystal, which becomes the "out" resulting field.

In order to determine the detection probabilities in the Hilbert-space representation, we adopt the correlation properties (Casado et al. 1997a). In such a work it has proved convenient to substitute slowly varying amplitudes $\hat{F}_J^{(+)}(\mathbf{r}, t)$ $[\hat{F}_J^{(-)}(\mathbf{r}, t)]$ for the amplitudes $\hat{E}_J^{(+)}(\mathbf{r}, t)$ $[\hat{E}_J^{(-)}(\mathbf{r}, t)]$, the relation between them being

$$\hat{F}_J^{(+)}(\mathbf{r}, t) = e^{\mathrm{i}\omega_J t} \hat{E}_J^{(+)}(\mathbf{r}, t),\ J = \mathrm{s},\ \mathrm{i}. \tag{2.166}$$

According to Casado et al. (1997a) it is essential to use the following relation, which is still an approximation:

$$\hat{F}^{(+)}(\mathbf{r}_b, t) = \hat{F}^{(+)}\left(\mathbf{r}_a, t - \frac{r_{ab}}{c}\right) \exp\left(\mathrm{i}\omega_a \frac{r_{ab}}{c}\right), \tag{2.167}$$

where ω_a is some frequency appropriate to a light beam and $r^{ab} = \mathbf{e}_a \cdot (\mathbf{r}_b - \mathbf{r}_a)$, with $\mathbf{e}_a$ being the unit vector in the direction of propagation. Since the vectors whose dot product is taken are usually of the same direction, the magnitude of displacement vector may be evoked.

If we consider the signal beam emerging from the crystal at different times t and t', we can use the autocorrelations (Casado et al. 1997a):

$$\langle \hat{F}_J^{(-)}(\mathbf{r}, t) \hat{F}_J^{(+)}(\mathbf{r}, t') \rangle = g^2 |V|^2 \mu_J(t' - t),\ J = \mathrm{s},\ \mathrm{i}. \tag{2.168}$$

The following autocorrelations, and their complex conjugates, vanish:

$$\langle \hat{F}_J^{(+)}(\mathbf{r}, t) \hat{F}_J^{(+)}(\mathbf{r}, t') \rangle = 0,\ J = \mathrm{s},\ \mathrm{i}. \tag{2.169}$$

The relation holds at any point of the outgoing beam, most interestingly just behind the crystal.

With respect to the cross correlation, we prefer to characterize the signal and idler field operators just behind the crystal at different times (Casado et al. 1997a):

$$\langle \hat{F}_{\mathrm{s\,out}}^{(+)}(\mathbf{0}, t) \hat{F}_{\mathrm{i\,out}}^{(+)}(\mathbf{0}, t') \rangle = gV\nu(t' - t). \tag{2.170}$$

It is useful to know that

$$\langle \hat{F}_{\mathrm{s}}^{(+)}(\mathbf{r}, t) \hat{F}_{\mathrm{i}}^{(-)}(\mathbf{r}, t') \rangle = \langle \hat{F}_{\mathrm{s}}^{(-)}(\mathbf{r}, t) \hat{F}_{\mathrm{i}}^{(+)}(\mathbf{r}, t') \rangle = 0. \tag{2.171}$$

In the Hilbert-space formalism, the usual theory of detection (by photon absorption) is based on the normal ordering. The joint detection rate is given by

$$P_{ab}(\mathbf{r}_a, t; \mathbf{r}_b, t') = K' \langle 0 | \hat{E}^{(-)}(\mathbf{r}_a, t) \hat{E}^{(-)}(\mathbf{r}_b, t') \hat{E}^{(+)}(\mathbf{r}_b, t') \hat{E}^{(+)}(\mathbf{r}_a, t) | 0 \rangle \tag{2.172}$$

in the Heisenberg picture, where $K' = K_a K_b$ and K_a (K_b) is a constant related to the efficiency of the detector and the energy of a single photon. The well-known property of Gaussian random variables A, B, C, and D,

$$\langle ABCD \rangle = \langle AB \rangle \langle CD \rangle + \langle AC \rangle \langle BD \rangle + \langle AD \rangle \langle BC \rangle, \tag{2.173}$$

applies not only in the Weyl (Casado et al. 1997a) but also in the normal ordering and entails that the joint detection rate is written in three terms. The first two terms are fourth order in g, while the last term is second order in g. We may discard the first two terms (Casado et al. 1997a) and finally obtain

$$P_{ab}(\mathbf{r}_a, t; \mathbf{r}_b, t') = K' |\langle \hat{E}^{(+)}(\mathbf{r}_a, t)\hat{E}^{(+)}(\mathbf{r}_b, t')\rangle|^2. \tag{2.174}$$

We will determine the visibility $\mathcal{V}'$ of the intensity interference:

$$\mathcal{V}' = \frac{R_{ab\max} - R_{ab\min}}{R_{ab\max} + R_{ab\min}}, \tag{2.175}$$

where

$$R_{ab\max} = \int_{-\frac{w}{2}}^{\frac{w}{2}} P_{ab\max}(\tau')\,\mathrm{d}\tau',\ R_{ab\min} = \int_{-\frac{w}{2}}^{\frac{w}{2}} P_{ab\min}(\tau')\,\mathrm{d}\tau', \tag{2.176}$$

with w the coincidence window which we choose to be $w = 13 \times 10^{-9}$ s, defined in terms of the integral

$$\begin{aligned} M\left(\frac{d}{c}, \frac{h}{c}\right) &= K' \int_{-\frac{w}{2}}^{\frac{w}{2}} \left|\nu\left(\tau' + \frac{d}{c}\right)\right| \left|\nu\left(\tau' + \frac{h}{c}\right)\right| \mathrm{d}\tau' \\ &= \exp\left[-\frac{\sigma^2}{2}\left(\frac{h-d}{c}\right)^2\right] \\ &\quad \times \frac{1}{2}\left\{\mathrm{erf}\left[\frac{\sigma}{\sqrt{2}}\left(\frac{d+h}{c} + w\right)\right] \mp \mathrm{erf}\left[\mp\frac{\sigma}{\sqrt{2}}\left(w - \frac{d+h}{c}\right)\right]\right\}. \end{aligned} \tag{2.177}$$

Here

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\,\mathrm{d}t, \tag{2.178}$$

d, h are parameters of an experimental setup, $c = 2.998 \times 10^8$ ms^{-1} is the speed of light, $\nu(\tau)$ is a Gaussian,

$$\nu(\tau') = |\nu(\tau')| = \sqrt{\frac{1}{K'}}\left(\frac{2}{\pi}\right)^{\frac{1}{4}} \sqrt{\sigma} e^{-\sigma^2\tau'^2}, \tag{2.179}$$

where $\sigma = 10^{12}$ s^{-1}. Let us remember that for $\sigma^{-1} \ll w$, we have $\mathrm{erf}(\pm\infty) = \pm 1$. Particularly,

$$\begin{aligned} M\left(\frac{d}{c}, \frac{d}{c}\right) &= \frac{1}{2}\left\{\mathrm{erf}\left[\frac{\sigma}{\sqrt{2}}\left(\frac{2d}{c} + w\right)\right] + \mathrm{erf}\left[\frac{\sigma}{\sqrt{2}}\left(w - \frac{2d}{c}\right)\right]\right\}, \\ M(0, 0) &= \mathrm{erf}\left(\frac{\sigma}{\sqrt{2}} w\right). \end{aligned} \tag{2.180}$$

(ii) *Experiment on the interference of signal and idler photons*

Let us start with an experiment demonstrating the coherence properties of the parametric down-conversion photon pairs as proposed in Ghosh et al. (1986). It is assumed that $\omega_\mathrm{s} = \omega_\mathrm{i} = \frac{\omega_0}{2}$ and the signal and idler beams are directed to a screen by means of two mirrors. There is no second-order interference between the two beams. When two detectors are put on the screen one can show a fourth-order, or intensity–intensity, interference. As seen from Fig. 2.1, the tracing of the beams is not so evident, and we modify the well-known result (Casado et al. 1997a, Ghosh et al. 1986). A report on the experiment was brief (Ghosh and Mandel 1987).

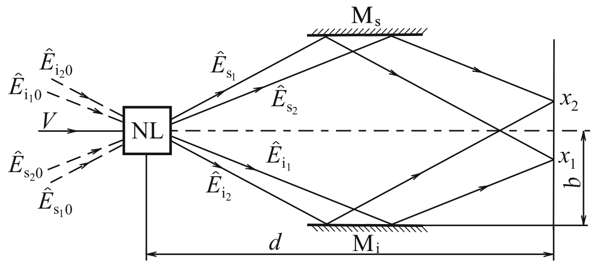

Fig. 2.1 Experimental setup on interference on a screen

In what follows, we will specify modal functions and nonlinear dynamics of field operators. We introduce the notation for the points of reflection $\mathbf{0}_{\mathrm{M}_{\mathrm{s}_j}} = (b, 0, z_{\mathrm{M}_{\mathrm{s}_j}})$, $\mathbf{0}_{\mathrm{M}_{\mathrm{i}_j}} = (-b, 0, z_{\mathrm{M}_{\mathrm{i}_j}})$, $j = 1, 2$, where

$$z_{\mathrm{M}_{\mathrm{s}_j}} = \frac{bd}{2b - x_j}, \quad z_{\mathrm{M}_{\mathrm{i}_j}} = \frac{bd}{2b + x_j}, \quad j = 1, 2, \tag{2.181}$$

with d being the distance from the centre of the crystal to the screen and b being the distance from the axis of the pumping to the mirrors.

We consider the initial electric field in the form

$$\hat{E}_0^{(+)}(\mathbf{r}, t) = V^{(+)}(\mathbf{r}, t)\hat{1} + \sum_{j=1}^{2} \left[\hat{E}_{\mathrm{s}_j 0}^{(+)}(\mathbf{r}, t) + \hat{E}_{\mathrm{i}_j 0}^{(+)}(\mathbf{r}, t) \right], \tag{2.182}$$

where $\mathbf{r} = (x, y, z)$, $\mathbf{k} = (k_x, k_y, k_z)$, and

$$\hat{E}_{\mathrm{s}_j 0}^{(+)}(\mathbf{r}, t) = \sum_{\mathbf{k} \in [\mathbf{k}]_{\mathrm{s}_j}} v_{\mathrm{s}_j \mathbf{k}}(\mathbf{r}) \hat{a}_{\mathrm{s}_j \mathbf{k} 0}(t), \tag{2.183}$$

$$\hat{E}_{\mathrm{i}_j 0}^{(+)}(\mathbf{r}, t) = \sum_{\mathbf{k} \in [\mathbf{k}]_{\mathrm{i}_j}} v_{\mathrm{i}_j \mathbf{k}}(\mathbf{r}) \hat{a}_{\mathrm{i}_j \mathbf{k} 0}(t), \tag{2.184}$$

with the orthonormal systems of functions $v_{\mathrm{s}_j\mathbf{k}}(\mathbf{r})$, $\mathbf{k} \in [\mathbf{k}]_{\mathrm{s}_j}$, $v_{\mathrm{i}_j\mathbf{k}}(\mathbf{r})$, $\mathbf{k} \in [\mathbf{k}]_{\mathrm{i}_j}$, $j = 1, 2$,

$$\|v_{\mathrm{s}_j\mathbf{k}}(\mathbf{r})\|^2 = \int |v_{\mathrm{s}_j\mathbf{k}}(\mathbf{r})|^2 \, \mathrm{d}^3\mathbf{r} = \hbar\omega_{\mathbf{k}}, \quad \mathbf{k} \in [\mathbf{k}]_{\mathrm{s}_j}, \tag{2.185}$$

$$\|v_{\mathrm{i}_j\mathbf{k}}(\mathbf{r})\|^2 = \int |v_{\mathrm{i}_j\mathbf{k}}(\mathbf{r})|^2 \, \mathrm{d}^3\mathbf{r} = \hbar\omega_{\mathbf{k}}, \quad \mathbf{k} \in [\mathbf{k}]_{\mathrm{i}_j}. \tag{2.186}$$

The $[\mathbf{k}]_{\mathrm{s}_j}$ is a set of integer multiples of the vector $\frac{2\pi}{L}\mathbf{e}_{\mathrm{s}_j}$, $\mathbf{e}_{\mathrm{s}_j}$ being a unit vector of the signal beam at the origin. Similarly for $[\mathbf{k}]_{\mathrm{i}_j}$. A formal expression for $v_{J\mathbf{k}}(\mathbf{r})$, $J = \mathrm{s}_j, \mathrm{i}_j$, $j = 1, 2$, is of the form

$$\begin{aligned} v_{J\mathbf{k}}(\mathbf{r}) &= \mathrm{i}\sqrt{\frac{\hbar\omega_{\mathbf{k}}}{\mathcal{A}L}} \exp(\mathrm{i}\mathbf{k}\cdot\mathbf{r}) \;\text{ for }\; \mathbf{r} \parallel \mathbf{k}, \quad z < z_{\mathrm{M}_J}, \quad \mathbf{k} \in [\mathbf{k}]_J, \\ v_{J\mathbf{k}}(\mathbf{r}) &= -\mathrm{i}\sqrt{\frac{\hbar\omega_{\mathbf{k}}}{\mathcal{A}L}} \exp[\mathrm{i}\mathbf{k}'\cdot(\mathbf{r} - \mathbf{0}'_J)] \;\text{ for }\; (\mathbf{r} - \mathbf{0}'_J) \parallel \mathbf{k}', \\ &\qquad z > z_{\mathrm{M}_J}, \quad \mathbf{k} \in [\mathbf{k}]_J, \end{aligned} \tag{2.187}$$

where $J = \mathrm{s}_j, \mathrm{i}_j$, $j = 1, 2$, $\mathcal{A}$ is the effective transverse area of the beam, $\mathbf{0}'_J = (2b, 0, 0)$ for $J = \mathrm{s}_1, \mathrm{s}_2$, $\mathbf{0}'_J = (-2b, 0, 0)$ for $J = \mathrm{i}_1, \mathrm{i}_2$, $\mathbf{k}' = (-k_x, k_y, k_z)$. In a standard fashion, we associate the signal and idler modal functions (2.187) with fields we denote as $\hat{E}^{(+)}_{J0}(\mathbf{r}, t)$, $J = \mathrm{s}_j, \mathrm{i}_j$, $j = 1, 2$.

After switching on the nonlinear interaction, part of the field is not influenced:

$$\hat{E}^{(+)}_{\mathrm{s}_j}(\mathbf{r}, t) = \hat{E}^{(+)}_{\mathrm{s}_j0}(\mathbf{r}, t), \quad \hat{E}^{(+)}_{\mathrm{i}_j}(\mathbf{r}, t) = \hat{E}^{(+)}_{\mathrm{i}_j0}(\mathbf{r}, t) \;\text{ for }\; z < 0, \tag{2.188}$$

whereas for $z > 0$ provided that $g|V| \ll 1$, the perturbative approximation of the solution of the Heisenberg equations of motion that retains terms up to g^2 can be written as

$$\hat{E}^{(+)}_{\mathrm{s}_j}(\mathbf{r}, t) = \hat{E}^{(+)}_{\mathrm{s}_j0}(\mathbf{r}, t) + e^{-\mathrm{i}\omega_0 t} gV\hat{\hat{G}}_j \hat{E}^{(-)}_{\mathrm{i}_j0}(\mathbf{r}, t) + g^2|V|^2\hat{\hat{J}}_j \hat{E}^{(+)}_{\mathrm{s}_j0}(\mathbf{r}, t), \tag{2.189}$$

$$\begin{aligned} \hat{E}^{(+)}_{\mathrm{i}_j}(\mathbf{r}, t) &= \hat{E}^{(+)}_{\mathrm{i}_j0}(\mathbf{r}, t) + e^{-\mathrm{i}\omega_0 t} gV\hat{\hat{G}}_j \hat{E}^{(-)}_{\mathrm{s}_j0}(\mathbf{r}, t) \\ &\quad + g^2|V|^2\hat{\hat{J}}_j \hat{E}^{(+)}_{\mathrm{i}_j0}(\mathbf{r}, t), \quad j = 1, 2, \end{aligned} \tag{2.190}$$

where

$$\hat{E}^{(-)}_{\mathrm{s}_j0}(\mathbf{r}, t) = [\hat{E}^{(+)}_{\mathrm{s}_j0}(\mathbf{r}, t)]^\dagger, \quad \hat{E}^{(-)}_{\mathrm{i}_j0}(\mathbf{r}, t) = [\hat{E}^{(+)}_{\mathrm{i}_j0}(\mathbf{r}, t)]^\dagger, \tag{2.191}$$

and $\hat{\hat{G}}_j$ and $\hat{\hat{J}}_j$ are antilinear and linear superoperators, respectively, which substitute the expansions in annihilation operators ($\hat{\hat{J}}_j$ yields an expansion in the annihilation operators) for annihilation (creation) operators ($\hat{\hat{G}}_j$ yields an expansion in creation operators). Compare Casado et al. (1997a), where G_j and J_j are appropriate linear operators acting on functions of the argument $\mathbf{r}$ for complex-valued functions. The superoperators $\hat{\hat{G}}_j$ and $\hat{\hat{J}}_j$ have the properties

$$\hat{\hat{G}}_j \hat{a}_{j0\mathbf{k}}(t) = \sum_{[\mathbf{k}']_{\mathrm{s}_j}} f(\mathbf{k}, \mathbf{k}') u \left[\frac{\Delta t}{2} (\omega_0 - \omega_{\mathbf{k}} - \omega_{\mathbf{k}'}) \right] \times \exp\left[\mathrm{i} t (\omega_0 - \omega_{\mathbf{k}} - \omega_{\mathbf{k}'}) \right] \hat{a}^{\dagger}_{j0\mathbf{k}'}(t) \text{ for } \mathbf{k} \in [\mathbf{k}]_{\mathrm{i}_j}, \tag{2.192}$$

$$\hat{\hat{J}}_j \hat{a}_{j0\mathbf{k}}(t) = \sum_{[\mathbf{k}']_{\mathrm{i}_j}} \sum_{[\mathbf{k}'']_{\mathrm{s}_j}} f(\mathbf{k}, \mathbf{k}') f^*(\mathbf{k}', \mathbf{k}'') \times u \left[\frac{\Delta t}{2} (\omega_{\mathbf{k}'} + \omega_{\mathbf{k}''} - \omega_0) \right] u \left[\frac{\Delta t}{2} (\omega_{\mathbf{k}''} - \omega_{\mathbf{k}}) \right] \exp\left[\mathrm{i} t (\omega_{\mathbf{k}''} - \omega_{\mathbf{k}}) \right] \times \hat{a}_{j0\mathbf{k}''}(t) \text{ for } \mathbf{k} \in [\mathbf{k}]_{\mathrm{s}_j}, \tag{2.193}$$

respectively, with

$$u(x) = \frac{\sin x}{x} e^{\mathrm{i}x}, \tag{2.194}$$

$$\hat{a}_{j0\mathbf{k}}(t) = \hat{a}_{j0\mathbf{k}}(0) e^{-\mathrm{i}\omega_{\mathbf{k}} t}. \tag{2.195}$$

Supposing that in the sense of classical nonlinear optics, $f(\mathbf{k}, \mathbf{k}')$ is a distribution with a support determined by the condition

$$\omega_0 - \omega_{\mathbf{k}} - \omega_{\mathbf{k}'} = 0, \tag{2.196}$$

we easily obtain that

$$\hat{\hat{G}}_j \hat{a}_{j0\mathbf{k}}(t) = \sum_{[\mathbf{k}']_{\mathrm{s}_j}} f(\mathbf{k}, \mathbf{k}') \hat{a}^{\dagger}_{j0\mathbf{k}'}(t) \text{ for } \mathbf{k} \in [\mathbf{k}]_{\mathrm{i}_j}, \tag{2.197}$$

$$\hat{\hat{J}}_j \hat{a}_{j0\mathbf{k}}(t) = \sum_{[\mathbf{k}'']_{\mathrm{s}_j}} \left[\sum_{[\mathbf{k}']_{\mathrm{i}_j}} f(\mathbf{k}, \mathbf{k}') f^*(\mathbf{k}', \mathbf{k}'') \right] \hat{a}_{j0\mathbf{k}''}(t) \text{ for } \mathbf{k} \in [\mathbf{k}]_{\mathrm{s}_j}, \tag{2.198}$$

which is a great unexpected simplification.

Further we will express the intensity correlations that have been determined in the experiment. Introducing

$$\hat{F}^{(+)}_{\mathrm{s}_j}(\mathbf{r}, t) = \exp(\mathrm{i}\omega_{\mathrm{s}} t) \hat{E}^{(+)}_{\mathrm{s}_j}(\mathbf{r}, t), \tag{2.199}$$

we express the field at a point $\mathbf{r}_j$, $j = 1, 2$, on the screen as

$$\hat{F}^{(+)}(\mathbf{r}_j, t) = \hat{F}^{(+)}_{\mathrm{s}_j\mathrm{out}} \left(\mathbf{0}, t - \frac{r_{\mathrm{s}_j}}{c} \right) \exp\left(\mathrm{i} \frac{\omega_0 r_{\mathrm{s}_j}}{2c} \right) + \hat{F}^{(+)}_{\mathrm{i}_j\mathrm{out}} \left(\mathbf{0}, t - \frac{r_{\mathrm{i}_j}}{c} \right) \exp\left(\mathrm{i} \frac{\omega_0 r_{\mathrm{i}_j}}{2c} \right), \quad j = 1, 2, \tag{2.200}$$

where

$$r_{s_j} = r_s(x_j) = \sqrt{z_{M_{s_j}}^2 + b^2} + \sqrt{(d - z_{M_{s_j}})^2 + (b - x_j)^2}\ , \quad j = 1, 2,$$

$$r_{i_j} = r_i(x_j) = \sqrt{z_{M_{i_j}}^2 + b^2} + \sqrt{(d - z_{M_{i_j}})^2 + (b + x_j)^2}\ , \tag{2.201}$$

and the subscript out indicates that the field behind the nonlinear crystal is considered. Hence, we can obtain the relations (2.209) below.

By taking into account the correlation relations

$$\begin{aligned}\langle \hat{F}^{(+)}_{s_1 \mathrm{out}}(\mathbf{0}, t)\hat{F}^{(+)}_{i_2 \mathrm{out}}(\mathbf{0}, t')\rangle &= \langle \hat{F}^{(+)}_{i_1 \mathrm{out}}(\mathbf{0}, t')\hat{F}^{(+)}_{s_2 \mathrm{out}}(\mathbf{0}, t)\rangle \\ &= gV\nu(t' - t),\end{aligned} \tag{2.202}$$

we get (cf. Casado et al. 1997a)

$$\begin{aligned}&\langle \hat{F}^{(+)}(\mathbf{r}_1, t_1)\hat{F}^{(+)}(\mathbf{r}_2, t_2)\rangle \\ &= gV\left\{\nu\left(t_2 - t_1 + \frac{r_{s_1}}{c} - \frac{r_{i_2}}{c}\right)\exp\left[\mathrm{i}\frac{\omega_0}{2c}(r_{i_2} + r_{s_1})\right]\right. \\ &\quad \left. + \nu\left(t_1 - t_2 + \frac{r_{s_2}}{c} - \frac{r_{i_1}}{c}\right)\exp\left[\mathrm{i}\frac{\omega_0}{2c}(r_{i_1} + r_{s_2})\right]\right\}.\end{aligned} \tag{2.203}$$

Assuming that the beams with different subscripts j are mutually uncorrelated, we finally get (cf. Casado et al. 1997a)

$$\begin{aligned}&P_{12}(\mathbf{r}_1, t + \tau_1; \mathbf{r}_2, t + \tau_2) \approx K' g^2 |V|^2 \\ &\times \left\{ \left|\nu\left(\tau_2 - \tau_1 + \frac{r_{s_1}}{c} - \frac{r_{i_2}}{c}\right)\right|^2 + \left|\nu\left(\tau_1 - \tau_2 + \frac{r_{s_2}}{c} - \frac{r_{i_1}}{c}\right)\right|^2 \right. \\ &+ 2\mathrm{Re}\left[\nu\left(\tau_2 - \tau_1 + \frac{r_{s_1}}{c} - \frac{r_{i_2}}{c}\right)\nu^*\left(\tau_1 - \tau_2 + \frac{r_{s_2}}{c} - \frac{r_{i_1}}{c}\right)\right. \\ &\left.\left. \times \exp\left[\mathrm{i}\frac{\omega_0}{2c}(r_{i_2} + r_{s_1} - r_{i_1} - r_{s_2})\right]\right]\right\},\end{aligned} \tag{2.204}$$

where K' is a constant related to the efficiency of the detectors,

$$K' = K_1 K_2, \quad K_1 = \frac{2\eta_1}{\hbar\omega_0}, \quad K_2 = \frac{2\eta_2}{\hbar\omega_0}, \tag{2.205}$$

η_J, $J = 1, 2$, is the efficiency of the detector D_J.

The visibility is expressed by the formula (2.175), where

$$R_{\mathrm{simax}} + R_{\mathrm{simin}} = 2g^2|V|^2$$
$$\times \left[M\left(\frac{r_{\mathrm{s}_1} - r_{\mathrm{i}_2}}{c}, \frac{r_{\mathrm{s}_1} - r_{\mathrm{i}_2}}{c}\right) + M\left(\frac{r_{\mathrm{i}_1} - r_{\mathrm{s}_2}}{c}, \frac{r_{\mathrm{i}_1} - r_{\mathrm{s}_2}}{c}\right)\right], \tag{2.206}$$

$$R_{\mathrm{simax}} - R_{\mathrm{simin}} = 4g^2|V|^2 M\left(\frac{r_{\mathrm{s}_1} - r_{\mathrm{i}_2}}{c}, \frac{r_{\mathrm{i}_1} - r_{\mathrm{s}_2}}{c}\right). \tag{2.207}$$

We may ask whether the visibility has its proper meaning for all x_1, x_2, whether it is associated only with the extremes of the detection rate, i.e. x_1 and x_2 for which $R_{\mathrm{si}} = R_{\mathrm{simax}}$ or $R_{\mathrm{si}} = R_{\mathrm{simin}}$. By relation (2.204) and the choice (2.179) we are interested in $C = \pm 1$, where

$$C = \cos\left[\frac{\omega_0}{2}\left(\frac{r_{\mathrm{i}_2} + r_{\mathrm{s}_1}}{c} - \frac{r_{\mathrm{i}_1} + r_{\mathrm{s}_2}}{c}\right)\right], \tag{2.208}$$

with $\omega_0 = \frac{2\pi c}{351} \times 10^9$ Hz.

The original formulae for r_{s_j}, r_{i_j} (instead of the relation (2.201)) were as (Ghosh et al. 1986)

$$r_{\mathrm{s}_j} = r_{\mathrm{s}}(x_j) = \sqrt{(2b - x_j)^2 + d^2}\,, \quad j = 1, 2,$$
$$r_{\mathrm{i}_j} = r_{\mathrm{i}}(x_j) = \sqrt{(2b + x_j)^2 + d^2}\,. \tag{2.209}$$

In Fig. 2.2 a short period of the oscillations of the cosine is depicted after Ghosh et al. (1986). An equal phase is assumed on any of the lines $y_2 \equiv x_2 - x_1 =$ constant. The short oscillation period corresponds to the change in the signed distance

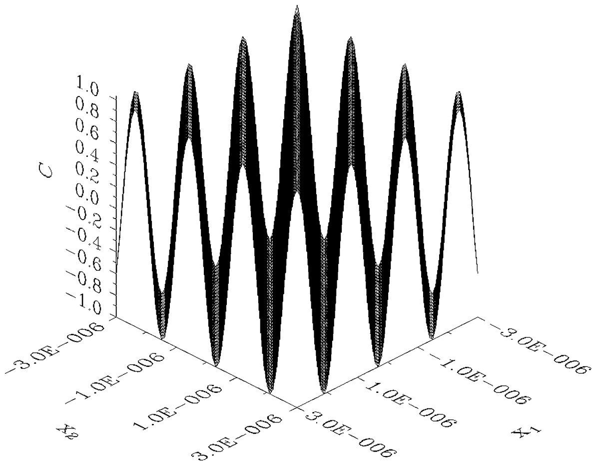

Fig. 2.2 The dependence of the cosine C of the phase on the position coordinates x_1 and x_2 for $d = 1$ and $b = 0.1$. The analysis of the setup in Fig. 2.1 after Ghosh et al. (1986). The cosine C of the phase depends only on the signed distance of the detectors

$y_2 \equiv x_2 - x_1$. As obvious from Fig. 2.3, the complement of the visibility, $1 - \mathcal{V}'$, depends only on the coordinate of the point amid the detectors D1 and D2. An equal visibility $\mathcal{V}'$ is assumed on the lines $y_1 = \frac{1}{2}(x_1 + x_2) =$ constant. For $y_1 = 0.0015$ the visibility almost vanishes.

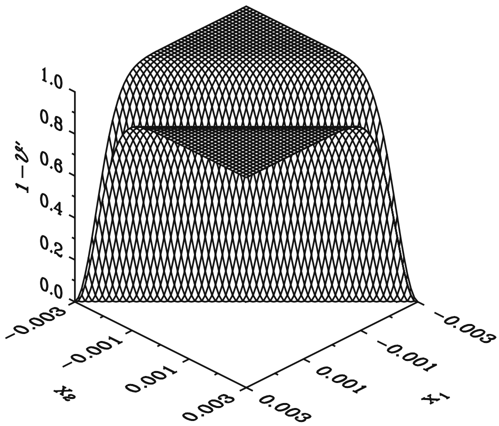

Fig. 2.3 The complement of the visibility, $1 - \mathcal{V}'$, versus the position coordinates x_1 and x_2 for $d = 1$ and $b = 0.1$. The analysis of the setup in Figure 2.1 after Ghosh et al. (1986)

(iii) *The experiment of Rarity and Tapster*

Rarity and Tapster (1990) demonstrated a violation of Bell's inequality using phase and momentum of photon pairs instead of polarization as in previous experiments. They selected two signal beams of the same colour (the frequency ω_s) and two idler beams also of the same colour (frequency $\omega_i \neq \omega_s$). They directed one of the signal beams and one of the idler beams to a mirror M_1 and another mirror M_2 (see Fig. 2.4). On the paths to the mirror M_2 they increased the phase of the signal by φ_s and that of the idler by φ_i. They coherently mixed the two signals and idlers.

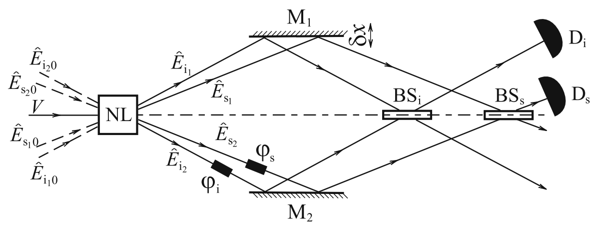

Fig. 2.4 Experimental setup of Rarity and Tapster

Now we will determine nonlinear dynamics of field operators. We assume the electric field in the form (2.161), (2.162), and (2.165), with another orthonormal system of functions $v_{s_j\mathbf{k}}(\mathbf{r})$, $\mathbf{k} \in [\mathbf{k}]_{s_j}$, $v_{i_j\mathbf{k}}(\mathbf{r})$, $\mathbf{k} \in [\mathbf{k}]_{i_j}$, $j = 1, 2$. The distinction is in the sets $[\mathbf{k}]_{s_j}$ and $[\mathbf{k}]_{i_j}$, which are appropriate to the experimental setup.

We will give a specification of $v_{J\mathbf{k}}(\mathbf{r})$, $J = \mathrm{s}_j, \mathrm{i}_j$, $j = 1, 2$. We associate points $\mathbf{0}_{\mathrm{BS_i}}, \mathbf{0}_{\mathrm{BS_s}}$ with the beam splitters. We introduce the notation $z_{\mathrm{BS_i}}, z_{\mathrm{BS_s}}$ such that

$$\mathbf{0}_{\mathrm{BS_i}} = (0, 0, z_{\mathrm{BS_i}}), \mathbf{0}_{\mathrm{BS_s}} = (0, 0, z_{\mathrm{BS_s}}). \tag{2.210}$$

We respect the phase shifters and the spots of reflection on the mirror M_2 with the notation $z_{\mathrm{PS_i}}, z_{\mathrm{M_{i_2}}}, z_{\mathrm{PS_s}}, z_{\mathrm{M_{s_2}}}$. We have located the phase shifter for the idler beam, the spot of reflection of the idler beam, the phase shifter for the signal beam, and the spot of reflection of the signal beam, respectively, at

$$\left(-b\frac{z_{\mathrm{PS_i}}}{z_{\mathrm{M_{i_2}}}}, 0, z_{\mathrm{PS_i}}\right), (-b, 0, z_{\mathrm{M_{i_2}}}), \left(-b\frac{z_{\mathrm{PS_s}}}{z_{\mathrm{M_{s_2}}}}, 0, z_{\mathrm{PS_s}}\right), (-b, 0, z_{\mathrm{M_{s_2}}}). \tag{2.211}$$

The origin $\mathbf{0}$ has its image $\mathbf{0}'_{\mathrm{s_2}} = \mathbf{0}'_{\mathrm{i_2}} = (-2b, 0, 0)$ in the mirror M_2.

Then we assume that the mirror M_1 is simple. We respect the spots of reflection on the mirror M_1 with the notation $z_{\mathrm{M_{i_1}}}, z_{\mathrm{M_{s_1}}}$. We have located the spot of reflection of the idler beam and the spot of reflection of the signal beam, respectively, at

$$(b, 0, z_{\mathrm{M_{i_1}}}), (b, 0, z_{\mathrm{M_{s_1}}}). \tag{2.212}$$

The origin $\mathbf{0}$ has its image $\mathbf{0}'_{\mathrm{s_1}} = \mathbf{0}'_{\mathrm{i_1}} = (2b, 0, 0)$ in the mirror M_1, which is assumed to be simple so far. We specify that

$$v_{J\mathbf{k}}(\mathbf{r}) = \mathrm{i}\sqrt{\frac{\hbar\omega_{\mathbf{k}}}{\mathcal{A}L}}\exp(\mathrm{i}\mathbf{k}\cdot\mathbf{r}) \text{ for } \mathbf{r} \parallel \mathbf{k}, \quad z < z_{\mathrm{M}_J}, \quad \mathbf{k} \in [\mathbf{k}]_J, \tag{2.213}$$

$$v_{J\mathbf{k}}(\mathbf{r}) = -\mathrm{i}\sqrt{\frac{\hbar\omega_{\mathbf{k}}}{\mathcal{A}L}}\exp\left[\mathrm{i}\mathbf{k}'\cdot(\mathbf{r}-\mathbf{0}'_J) + \frac{\omega_{J_1}\delta x}{c}\right] \text{ for } (\mathbf{r}-\mathbf{0}'_J) \parallel \mathbf{k}',$$
$$z_{\mathrm{M}_J} < z < z_{\mathrm{BS}_{J_1}}, \quad \mathbf{k} \in [\mathbf{k}]_J, \tag{2.214}$$

where δx is a path-length difference, $J = \mathrm{s}_1, \mathrm{i}_1$, $J_1 = \mathrm{s}, \mathrm{i}$, $\mathbf{k}' = (-k_x, k_y, k_z)$,

$$v_{J\mathbf{k}}(\mathbf{r}) = \mathrm{i}\sqrt{\frac{\hbar\omega_{\mathbf{k}}}{\mathcal{A}L}}\exp(\mathrm{i}\mathbf{k}\cdot\mathbf{r}) \text{ for } \mathbf{r} \parallel \mathbf{k}, \quad z < z_{\mathrm{PS}_{J_1}}, \quad \mathbf{k} \in [\mathbf{k}]_J, \tag{2.215}$$

$$v_{J\mathbf{k}}(\mathbf{r}) = \mathrm{i}\sqrt{\frac{\hbar\omega_{\mathbf{k}}}{\mathcal{A}L}}\exp[\mathrm{i}(\mathbf{k}\cdot\mathbf{r} + \varphi_{J_1})] \text{ for } \mathbf{r} \parallel \mathbf{k},$$
$$z_{\mathrm{PS}_{J_1}} < z < z_{\mathrm{M}_J}, \quad \mathbf{k} \in [\mathbf{k}]_J, \tag{2.216}$$

$$v_{J\mathbf{k}}(\mathbf{r}) = -\mathrm{i}\sqrt{\frac{\hbar\omega_{\mathbf{k}}}{\mathcal{A}L}}\exp\{\mathrm{i}[\mathbf{k}'\cdot(\mathbf{r}-\mathbf{0}'_J) + \varphi_{J_1}]\} \text{ for } (\mathbf{r}-\mathbf{0}'_J) \parallel \mathbf{k}',$$
$$z_{\mathrm{M}_J} < z < z_{\mathrm{BS}_{J_1}}, \quad \mathbf{k} \in [\mathbf{k}]_J, \tag{2.217}$$

where $J = \mathrm{s}_2, \mathrm{i}_2$, $J_1 = \mathrm{s}, \mathrm{i}$.

Beam splitter $\mathrm{BS_s}$ with the transmissivities $\mathtt{t}_\mathrm{s}$, $\mathtt{t}'_\mathrm{s}$ and reflectivities $\mathtt{r}_\mathrm{s}$, $\mathtt{r}'_\mathrm{s}$: Input values are

$$v_{\mathrm{s}_1\mathbf{k}\mathrm{in}}(\mathbf{0}_{\mathrm{BS_s}}) = -\mathrm{i}\sqrt{\frac{\hbar\omega_\mathbf{k}}{\mathcal{A}L}}\exp\left[\mathrm{i}\mathbf{k}'\cdot(\mathbf{0}_{\mathrm{BS_s}} - \mathbf{0}'_{\mathrm{s}_1}) + \frac{\omega_\mathrm{s}\delta x}{c}\right], \mathbf{k}\in[\mathbf{k}]_{\mathrm{s}_1}, \tag{2.218}$$

$$v_{\mathrm{s}_2\mathbf{k}\mathrm{in}}(\mathbf{0}_{\mathrm{BS_s}}) = -\mathrm{i}\sqrt{\frac{\hbar\omega_\mathbf{k}}{\mathcal{A}L}}\exp\{\mathrm{i}[\mathbf{k}'\cdot(\mathbf{0}_{\mathrm{BS_s}} - \mathbf{0}'_{\mathrm{s}_2}) + \varphi_\mathrm{s}]\}, \mathbf{k}\in[\mathbf{k}]_{\mathrm{s}_2}. \tag{2.219}$$

Under the assumption $\delta x = 0$ we can perform the exchange $\mathbf{k} \leftrightarrow \mathbf{k}'$ and write the output values for $\mathbf{k}\in[\mathbf{k}]_{\mathrm{s}_1}$

$$\begin{aligned} v_{\mathrm{s}_1\mathbf{k}\mathrm{out}}(\mathbf{0}_{\mathrm{BS_s}}) &= -\mathrm{i}\sqrt{\frac{\hbar\omega_\mathbf{k}}{\mathcal{A}L}}\exp\{\mathrm{i}\mathbf{k}'\cdot(\mathbf{0}_{\mathrm{BS_s}} - \mathbf{0}'_{\mathrm{s}_1})\}\mathtt{t}_\mathrm{s} \\ &\quad -\mathrm{i}\sqrt{\frac{\hbar\omega_\mathbf{k}}{\mathcal{A}L}}\exp\{\mathrm{i}[\mathbf{k}\cdot(\mathbf{0}_{\mathrm{BS_s}} - \mathbf{0}'_{\mathrm{s}_2}) + \varphi_\mathrm{s}]\}\mathtt{r}'_\mathrm{s}, \end{aligned} \tag{2.220}$$

$$\begin{aligned} v_{\mathrm{s}_2\mathbf{k}'\mathrm{out}}(\mathbf{0}_{\mathrm{BS_s}}) &= -\mathrm{i}\sqrt{\frac{\hbar\omega_\mathbf{k}}{\mathcal{A}L}}\exp\{\mathrm{i}\mathbf{k}'\cdot(\mathbf{0}_{\mathrm{BS_s}} - \mathbf{0}'_{\mathrm{s}_1})\}\mathtt{r}_\mathrm{s} \\ &\quad -\mathrm{i}\sqrt{\frac{\hbar\omega_\mathbf{k}}{\mathcal{A}L}}\exp\{\mathrm{i}[\mathbf{k}\cdot(\mathbf{0}_{\mathrm{BS_s}} - \mathbf{0}'_{\mathrm{s}_2}) + \varphi_\mathrm{s}]\}\mathtt{t}'_\mathrm{s}. \end{aligned} \tag{2.221}$$

Performing the exchange $\mathbf{k} \leftrightarrow \mathbf{k}'$ in (2.221), we have for $(\mathbf{r} - \mathbf{0}'_{\mathrm{s}_1}) \parallel \mathbf{k}'$, $z > z_{\mathrm{BS_s}}$, $\mathbf{k}\in[\mathbf{k}]_{\mathrm{s}_1}$,

$$\begin{aligned} v_{\mathrm{s}_1\mathbf{k}}(\mathbf{r}) &= -\mathrm{i}\sqrt{\frac{\hbar\omega_\mathbf{k}}{\mathcal{A}L}}\exp[\mathrm{i}\mathbf{k}'\cdot(\mathbf{r} - \mathbf{0}'_{\mathrm{s}_1})]\mathtt{t}_\mathrm{s} \\ &\quad -\mathrm{i}\sqrt{\frac{\hbar\omega_\mathbf{k}}{\mathcal{A}L}}\exp\{\mathrm{i}[\mathbf{k}'\cdot(\mathbf{r} - \mathbf{0}_{\mathrm{BS_s}}) + \mathbf{k}\cdot(\mathbf{0}_{\mathrm{BS_s}} - \mathbf{0}'_{\mathrm{s}_2}) + \varphi_\mathrm{s}]\}\mathtt{r}'_\mathrm{s}, \end{aligned} \tag{2.222}$$

and for $(\mathbf{r} - \mathbf{0}'_{\mathrm{s}_2}) \parallel \mathbf{k}'$, $z > z_{\mathrm{BS_s}}$, $\mathbf{k}\in[\mathbf{k}]_{\mathrm{s}_2}$,

$$\begin{aligned} v_{\mathrm{s}_2\mathbf{k}}(\mathbf{r}) &= -\mathrm{i}\sqrt{\frac{\hbar\omega_\mathbf{k}}{\mathcal{A}L}}\exp\{\mathrm{i}[\mathbf{k}'\cdot(\mathbf{r} - \mathbf{0}_{\mathrm{BS_s}}) + \mathbf{k}\cdot(\mathbf{0}_{\mathrm{BS_s}} - \mathbf{0}'_{\mathrm{s}_1})]\}\mathtt{r}_\mathrm{s} \\ &\quad -\mathrm{i}\sqrt{\frac{\hbar\omega_\mathbf{k}}{\mathcal{A}L}}\exp\{\mathrm{i}[\mathbf{k}'\cdot(\mathbf{r} - \mathbf{0}'_{\mathrm{s}_2}) + \varphi_\mathrm{s}]\}\mathtt{t}'_\mathrm{s}. \end{aligned} \tag{2.223}$$

Beam splitter $\mathrm{BS_i}$ with the transmissivities $\mathtt{t}_\mathrm{i}$, $\mathtt{t}'_\mathrm{i}$ and reflectivities $\mathtt{r}_\mathrm{i}$, $\mathtt{r}'_\mathrm{i}$ can be described analogously: In (2.222) and (2.223) we perform the replacement $\mathrm{s} \leftrightarrow \mathrm{i}$.

In a standard fashion, we associate the modal functions, e.g. (2.213), (2.214), (2.215), (2.216), (2.217), (2.222), and (2.223), with fields we denote $\hat{E}^{(+)}_{j0}(\mathbf{r}, t)$, $J = \mathrm{s}_1, \mathrm{s}_2, \mathrm{i}_1, \mathrm{i}_2$. Nonlinear dynamics is described in the same way as for the experiment on the interference of signal and idler photons by relations (2.188), (2.189), (2.190), (2.192), and (2.193). It allows the fields with $z < 0$ to stay initial and those with $z > 0$ at least to obey the same rules we have used to calculate the modal functions.

We introduce the slowly varying field operators

$$\hat{F}_J^{(+)}(\mathbf{r}, t) = \exp(\mathrm{i}\omega_{J_1} t)\hat{E}_J^{(+)}(\mathbf{r}, t), \tag{2.224}$$

where $J = \mathrm{s}_1, \mathrm{s}_2$, $J_1 = \mathrm{s}$ and $J = \mathrm{i}_1, \mathrm{i}_2$, $J_1 = \mathrm{i}$, for expressing the intensity correlations. The field operators at the signal and idler detectors placed at $\mathbf{r}_\mathrm{s}$, $\mathbf{r}_\mathrm{i}$, respectively, are

$$\begin{aligned}\hat{F}_{\mathrm{s}_2}^{(+)}(\mathbf{r}_\mathrm{s}, t) &= \mathtt{t}'_\mathrm{s}\hat{F}_{\mathrm{s}_2\mathrm{out}}^{(+)}\left(\mathbf{0}, t - \frac{r_\mathrm{s}}{c} + \frac{\varphi_\mathrm{s}}{\omega_\mathrm{s}}\right)\exp\left(\mathrm{i}\frac{\omega_\mathrm{s} r_\mathrm{s}}{c} + \mathrm{i}\varphi_\mathrm{s}\right) \\ &\quad + \mathtt{r}_\mathrm{s}\hat{F}_{\mathrm{s}_1\mathrm{out}}^{(+)}\left(\mathbf{0}, t - \frac{r_\mathrm{s}}{c}\right)\exp\left(\mathrm{i}\frac{\omega_\mathrm{s} r_\mathrm{s}}{c}\right),\end{aligned} \tag{2.225}$$

$$\begin{aligned}\hat{F}_{\mathrm{i}_2}^{(+)}(\mathbf{r}_\mathrm{i}, t') &= \mathtt{t}'_\mathrm{i}\hat{F}_{\mathrm{i}_2\mathrm{out}}^{(+)}\left(\mathbf{0}, t' - \frac{r_\mathrm{i}}{c} + \frac{\varphi_\mathrm{i}}{\omega_\mathrm{i}}\right)\exp\left(\mathrm{i}\frac{\omega_\mathrm{i} r_\mathrm{i}}{c} + \mathrm{i}\varphi_\mathrm{i}\right) \\ &\quad + \mathtt{r}_\mathrm{i}\hat{F}_{\mathrm{i}_1\mathrm{out}}^{(+)}\left(\mathbf{0}, t' - \frac{r_\mathrm{i}}{c}\right)\exp\left(\mathrm{i}\frac{\omega_\mathrm{i} r_\mathrm{i}}{c}\right),\end{aligned} \tag{2.226}$$

with $\mathbf{0}$ the centre of the coordinate system, r_s and r_i the path lengths of the lower signal and idler beams, respectively, to the appropriate detector. In the experiment under consideration both upper paths were modified by δx, since the upper and the lower mirrors were not at exactly the same distance from the pumping beam axis (Casado et al. 1997a) . The mirror above the pumping beam axis is not simple, but a mirror assembly which enables one to change δx (Rarity and Tapster 1990). In the following, we assume $\delta x = 0$.

We will take into account that the signal field $\hat{F}_{\mathrm{s}_1}^{(+)}(\mathbf{r}, t)$ is correlated with the idler field $\hat{F}_{\mathrm{i}_2}^{(+)}(\mathbf{r}, t)$ and also $\hat{F}_{\mathrm{s}_2}^{(+)}(\mathbf{r}, t)$ is correlated with $\hat{F}_{\mathrm{i}_1}^{(+)}(\mathbf{r}, t)$, but $\hat{F}_{\mathrm{s}_j}^{(+)}(\mathbf{r}, t)$ is uncorrelated with $\hat{F}_{\mathrm{i}_j}^{(+)}(\mathbf{r}, t)$, $j = 1, 2$, these pairs not fulfilling matching conditions. If we consider that the time intervals $\frac{r_\mathrm{s}}{c} - \frac{r_\mathrm{i}}{c} - \frac{\varphi_\mathrm{s}}{\omega_\mathrm{s}}$, $\frac{r_\mathrm{s}}{c} - \frac{r_\mathrm{i}}{c} + \frac{\varphi_\mathrm{i}}{\omega_\mathrm{i}}$ are small in comparison with the coherence time of signal and idler given by the function $\nu(\tau)$, we obtain

$$\begin{aligned}\langle\hat{F}_{\mathrm{s}_2}^{(+)}(\mathbf{r}_\mathrm{s}, t)\hat{F}_{\mathrm{i}_2}^{(+)}(\mathbf{r}_\mathrm{i}, t')\rangle \approx &\left\{\mathtt{r}_\mathrm{i}\mathtt{t}'_\mathrm{s}\exp\left[\mathrm{i}\left(\omega_\mathrm{s}\frac{r_\mathrm{s}}{c} + \omega_\mathrm{i}\frac{r_\mathrm{i}}{c} + \varphi_\mathrm{s}\right)\right]\right. \\ &\left. + \mathtt{r}_\mathrm{s}\mathtt{t}'_\mathrm{i}\exp\left[\mathrm{i}\left(\omega_\mathrm{s}\frac{r_\mathrm{s}}{c} + \omega_\mathrm{i}\frac{r_\mathrm{i}}{c} + \varphi_\mathrm{i}\right)\right]\right\}\nu(t' - t).\end{aligned} \tag{2.227}$$

From this we have

$$\begin{aligned}P_\mathrm{si}(\mathbf{r}_\mathrm{s}, t + \tau; \mathbf{r}_\mathrm{i}, t + \tau') &= K' g^2 |V|^2 |\nu(\tau' - \tau)|^2 \\ \times\Big[|\mathtt{r}_\mathrm{i}\mathtt{t}'_\mathrm{s}|^2 &+ |\mathtt{r}_\mathrm{s}\mathtt{t}'_\mathrm{i}|^2 + 2\mathrm{Re}\{\mathtt{r}_\mathrm{s}^*\mathtt{t}'_\mathrm{s}\mathtt{r}_\mathrm{i}\mathtt{t}_\mathrm{i}'^*\exp[\mathrm{i}(\varphi_\mathrm{s} - \varphi_\mathrm{i})]\}\Big].\end{aligned} \tag{2.228}$$

The visibility is expressed by the formula (2.175), where

$$R_{\text{simax}} + R_{\text{simin}} = 2g^2|V|^2 M(0,0)\left(|\mathtt{t}'_{\text{s}}\mathtt{r}_{\text{i}}|^2 + |\mathtt{r}_{\text{s}}\mathtt{t}'_{\text{i}}|^2\right), \tag{2.229}$$

$$R_{\text{simax}} - R_{\text{simin}} = 8g^2|V|^2 M(0,0)\,|\mathtt{t}'_{\text{s}}\mathtt{r}_{\text{i}}||\mathtt{r}_{\text{s}}\mathtt{t}'_{\text{i}}|. \tag{2.230}$$

The dependence of the visibility on the beam splitters with the transmissivities $|\mathtt{t}'_{\text{s}}| \in [0, 1]$, $|\mathtt{t}'_{\text{i}}| \in [0, 1]$ and the reflectivities $|\mathtt{r}_{\text{s}}| = \sqrt{1 - |\mathtt{t}'_{\text{s}}|^2}$, $|\mathtt{r}_{\text{i}}| = \sqrt{1 - |\mathtt{t}'_{\text{i}}|^2}$ is plotted in Fig. 2.5.

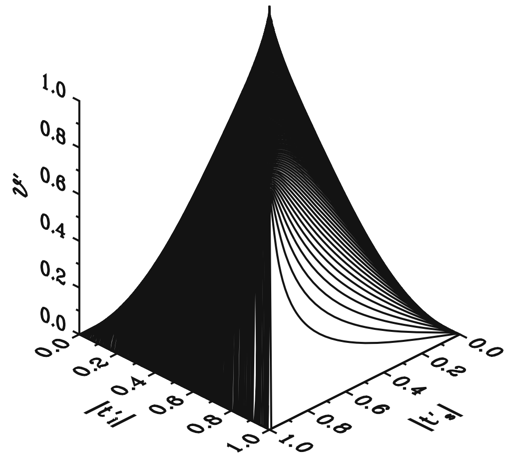

Fig. 2.5 The visibility $\mathcal{V}'$ versus moduli of the amplitude transmissivities $\mathtt{t}'_{\text{s}}$, $\mathtt{t}'_{\text{i}}$ from the "lower side" of the beam splitters for signal and idler beams for $\delta x = 0$

Unfortunately, the quantity under consideration is not dependent on the characteristic of the nonlinear optical process. The surface plotted has a boundary condition zero. It may be equal to unity in the sense of the equality $|\mathtt{t}'_{\text{s}}\mathtt{r}_{\text{i}}| = |\mathtt{t}'_{\text{i}}\mathtt{r}_{\text{s}}|$. The maximum is attained on the line segment connecting the points $|\mathtt{t}'_{\text{s}}| = 0,\quad |\mathtt{t}'_{\text{i}}| = 0$ and $|\mathtt{t}'_{\text{s}}| = 1,\quad |\mathtt{t}'_{\text{i}}| = 1$. The interference manifests itself as a cosine variation of the coincidence rate with $\varphi_{\text{s}} - \varphi_{\text{i}}$.

(iv) *The experiment of Franson*

Franson (1989) proposed a test of "Bell's inequality for energy and time". He arranged two Mach–Zehnder interferometers and let a signal and an idler beam each pass through an interferometer (see Fig. 2.6). The experiment was originally proposed for an atom and free-space propagation.

The coincidence detection shows a fourth-order interference as a cosine dependence on $\frac{1}{c}(\omega_{\text{s}}\Delta L_{\text{s}} + \omega_{\text{i}}\Delta L_{\text{i}})$, where ΔL_{s} (ΔL_{i}) is the length difference between the long (short) route of the signal (idler) beam through the corresponding interferometer. In the past few years several groups have performed experiments of that type. In Tapster et al. (1994) Franson's experiment has been adapted to parametric down-conversion and fibres.

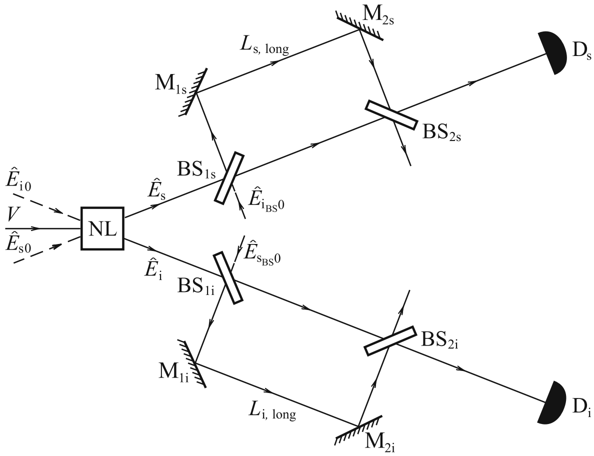

Fig. 2.6 Experimental setup of Franson's type. For simplicity $\hat{E}_{i_{BS}0} \equiv \hat{E}_{i_{BS_{1s}}0}$ and $\hat{E}_{s_{BS}0} \equiv \hat{E}_{s_{BS_{1i}}0}$

For the description of the nonlinear dynamics of field operators, we consider the initial electric-field in the form

$$\hat{E}_0^{(+)}(\mathbf{r},t) = V^{(+)}(\mathbf{r},t)\hat{1} + \hat{E}_{s0}^{(+)}(\mathbf{r},t) + \hat{E}_{i_{BS_{1s}}0}^{(+)}(\mathbf{r},t) + \hat{E}_{i0}^{(+)}(\mathbf{r},t) + \hat{E}_{s_{BS_{1i}}0}^{(+)}(\mathbf{r},t), \tag{2.231}$$

where

$$\hat{E}_{J0}^{(+)}(\mathbf{r},t) = \sum_{\mathbf{k}\in[\mathbf{k}]_J} v_{J\mathbf{k}}(\mathbf{r})\hat{a}_{J\mathbf{k}0}(t), \quad J = \mathrm{s},\ \mathrm{i},\mathrm{i}_{\mathrm{BS}_{1s}}, \mathrm{s}_{\mathrm{BS}_{1i}}. \tag{2.232}$$

The modal functions as restricted to linear segments are

$$v_{\mathrm{s}\mathbf{k}}(\mathbf{r}) = \mathrm{i}\sqrt{\frac{\hbar\omega_{\mathbf{k}}}{\mathcal{A}L}}e^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}} \text{ for } \mathbf{r} \parallel \mathbf{k},\ z < z_{\mathrm{BS}_{1s}},\ \mathbf{k} \in [\mathbf{k}]_{\mathrm{s}}, \tag{2.233}$$

$$v_{\mathrm{i}_{\mathrm{BS}_{1s}}\mathbf{k}}(\mathbf{r}) = \mathrm{i}\sqrt{\frac{\hbar\omega_{\mathbf{k}}}{\mathcal{A}L}}e^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}} \text{ for } (\mathbf{r} - \mathbf{0}_{\mathrm{BS}_{1s}}) \parallel \mathbf{k},\ z > z_{\mathrm{BS}_{1s}},\ \mathbf{k} \in [\mathbf{k}]_{\mathrm{i}_{\mathrm{BS}_{1s}}}. \tag{2.234}$$

Here $\mathbf{0}_{\mathrm{BS}_{1s}}$ is the centre of the beam splitter BS_{1s}, $z_{\mathrm{BS}_{1s}}$ is the corresponding z-coordinate, and $[\mathbf{k}]_{\mathrm{i}_{\mathrm{BS}_{1s}}}$ is the set of wave vectors $\mathbf{k}$ of the beam corresponding to the unused input port of this beam splitter.

The modal functions at the output of this beam splitter are

$$v_{\mathrm{s}\mathbf{k}}(\mathbf{r}) = \mathrm{i}\sqrt{\frac{\hbar\omega_{\mathbf{k}}}{\mathcal{A}L}}e^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}}\mathtt{t}_{\mathrm{s}} + \mathrm{i}\sqrt{\frac{\hbar\omega_{\mathbf{k}}}{\mathcal{A}L}}e^{\mathrm{i}\mathbf{k}\cdot(\mathbf{r}-\mathbf{0}'_{\mathrm{BS}_{1s}\mathrm{i}_{\mathrm{BS}_{1s}}})}\mathtt{r}_{\mathrm{s}}$$
$$\text{for } \mathbf{r} \parallel \mathbf{k},\ z_{\mathrm{BS}_{1s}} < z < z_{\mathrm{BS}_{2s}},\ \mathbf{k} \in [\mathbf{k}]_{\mathrm{s}}, \tag{2.235}$$

$$v_{\mathrm{iBS_{1s}}\mathbf{k}}(\mathbf{r}) = \mathrm{i}\sqrt{\frac{\hbar\omega_{\mathbf{k}}}{\mathcal{A}L}}e^{\mathrm{i}\mathbf{k}\cdot(\mathbf{r}-\mathbf{0}'_{\mathrm{BS_{1s}s}})}\mathfrak{r}_{\mathrm{s}} + \mathrm{i}\sqrt{\frac{\hbar\omega_{\mathbf{k}}}{\mathcal{A}L}}e^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}}\mathfrak{t}_{\mathrm{s}}$$
$$\text{for } (\mathbf{r}-\mathbf{0}_{\mathrm{BS_{1s}}}) \parallel \mathbf{k},\ z_{\mathrm{M_{1s}}} < z < z_{\mathrm{BS_{1s}}},\ \mathbf{k} \in [\mathbf{k}]_{\mathrm{iBS_{1s}}}, \tag{2.236}$$

where $\mathbf{0}'_{\mathrm{BS_{1s}iBS_{1s}}}$, $\mathbf{0}'_{\mathrm{BS_{1s}s}}$ are chosen such that

$$\mathbf{k}\cdot(\mathbf{r}-\mathbf{0}'_{\mathrm{BS_{1s}iBS_{1s}}}) = \mathbf{k}\cdot(\mathbf{r}-\mathbf{0}_{\mathrm{BS_{1s}}}) + \mathbf{k}'\cdot\mathbf{0}_{\mathrm{BS_{1s}}},\quad \mathbf{k}\in[\mathbf{k}]_{\mathrm{s}}, \tag{2.237}$$

$$\mathbf{k}\cdot(\mathbf{r}-\mathbf{0}'_{\mathrm{BS_{1s}s}}) = \mathbf{k}\cdot(\mathbf{r}-\mathbf{0}_{\mathrm{BS_{1s}}}) + \mathbf{k}'\cdot\mathbf{0}_{\mathrm{BS_{1s}}},\quad \mathbf{k}\in[\mathbf{k}]_{\mathrm{iBS_{1s}}}. \tag{2.238}$$

Here $z_{\mathrm{M_{1s}}}$ is the z-coordinate of the centre of the signal mirror and $z_{\mathrm{BS_{2s}}}$ is the z-coordinate of $\mathbf{0}_{\mathrm{BS_{2s}}}$, the centre of the beam splitter $\mathrm{BS_{2s}}$.

After the reflection from the first mirror, the modal function is

$$v_{\mathrm{iBS_{1s}}\mathbf{k}}(\mathbf{r}) = -\mathrm{i}\sqrt{\frac{\hbar\omega_{\mathbf{k}}}{\mathcal{A}L}}e^{\mathrm{i}\mathbf{k}'\cdot(\mathbf{r}-\mathbf{0}'_{\mathrm{M_{1s}BS_{1s}s}})}\mathfrak{r}_{\mathrm{s}}$$
$$-\mathrm{i}\sqrt{\frac{\hbar\omega_{\mathbf{k}}}{\mathcal{A}L}}e^{\mathrm{i}\mathbf{k}'\cdot(\mathbf{r}-\mathbf{0}'_{\mathrm{M_{1s}}})}\mathfrak{t}_{\mathrm{s}} \text{ for } (\mathbf{r}-\mathbf{0}_{\mathrm{M_{1s}}}) \parallel \mathbf{k}',$$
$$z_{\mathrm{M_{1s}}} < z < z_{\mathrm{M_{2s}}},\ \mathbf{k}\in[\mathbf{k}]_{\mathrm{iBS_{1s}}}, \tag{2.239}$$

where $z_{\mathrm{M_{2s}}}$ is the z-coordinate of the centre of the second mirror and $\mathbf{0}'_{\mathrm{M_{1s}BS_{1s}s}}$ and $\mathbf{0}'_{\mathrm{M_{1s}}}$ are chosen such that

$$\mathbf{k}'\cdot(\mathbf{r}-\mathbf{0}'_{\mathrm{M_{1s}BS_{1s}s}}) = \mathbf{k}'\cdot(\mathbf{r}-\mathbf{0}_{\mathrm{M_{1s}}}) + \mathbf{k}\cdot(\mathbf{0}_{\mathrm{M_{1s}}} - \mathbf{0}'_{\mathrm{BS_{1s}s}}),\quad \mathbf{k}\in[\mathbf{k}]_{\mathrm{iBS_{1s}}}, \tag{2.240}$$
$$\mathbf{k}'\cdot(\mathbf{r}-\mathbf{0}'_{\mathrm{M_{1s}}}) = \mathbf{k}'\cdot(\mathbf{r}-\mathbf{0}_{\mathrm{M_{1s}}}) + \mathbf{k}\cdot\mathbf{0}_{\mathrm{M_{1s}}},\quad \mathbf{k}\in[\mathbf{k}]_{\mathrm{iBS_{1s}}}. \tag{2.241}$$

After the reflection from the second mirror, the modal function is

$$v_{\mathrm{iBS_{1s}}\mathbf{k}}(\mathbf{r}) = \mathrm{i}\sqrt{\frac{\hbar\omega_{\mathbf{k}}}{\mathcal{A}L}}e^{-\mathrm{i}\mathbf{k}\cdot(\mathbf{r}-\mathbf{0}'_{\mathrm{M_{2s}M_{1s}BS_{1s}s}})}\mathfrak{r}_{\mathrm{s}}$$
$$+\mathrm{i}\sqrt{\frac{\hbar\omega_{\mathbf{k}}}{\mathcal{A}L}}e^{-\mathrm{i}\mathbf{k}\cdot(\mathbf{r}-\mathbf{0}'_{\mathrm{M_{2s}M_{1s}}})}\mathfrak{t}_{\mathrm{s}} \text{ for } (\mathbf{r}-\mathbf{0}_{\mathrm{M_{2s}}}) \parallel -\mathbf{k},$$
$$z_{\mathrm{M_{2s}}} < z < z_{\mathrm{BS_{2s}}},\ \mathbf{k}\in[\mathbf{k}]_{\mathrm{iBS_{1s}}}, \tag{2.242}$$

where $\mathbf{0}'_{\mathrm{M_{2s}M_{1s}BS_{1s}s}}$ and $\mathbf{0}'_{\mathrm{M_{2s}M_{1s}}}$ are chosen such that

$$-\mathbf{k}\cdot(\mathbf{r}-\mathbf{0}'_{\mathrm{M_{2s}M_{1s}BS_{1s}s}}) = -\mathbf{k}\cdot(\mathbf{r}-\mathbf{0}_{\mathrm{M_{2s}}})$$
$$+\mathbf{k}'\cdot(\mathbf{0}_{\mathrm{M_{2s}}} - \mathbf{0}'_{\mathrm{M_{1s}BS_{1s}s}}),\quad \mathbf{k}\in[\mathbf{k}]_{\mathrm{iBS_{1s}}}, \tag{2.243}$$
$$-\mathbf{k}\cdot(\mathbf{r}-\mathbf{0}'_{\mathrm{M_{2s}M_{1s}}}) = -\mathbf{k}\cdot(\mathbf{r}-\mathbf{0}_{\mathrm{M_{2s}}})$$
$$+\mathbf{k}'\cdot(\mathbf{0}_{\mathrm{M_{2s}}} - \mathbf{0}'_{\mathrm{M_{1s}}}),\quad \mathbf{k}\in[\mathbf{k}]_{\mathrm{iBS_{1s}}}. \tag{2.244}$$

The output modal functions for the beam to be detected are

$$
\begin{aligned}
v_{\mathrm{s}\mathbf{k}}(\mathbf{r}) &= \mathrm{i}\sqrt{\frac{\hbar\omega_{\mathbf{k}}}{\mathcal{A}L}} e^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}} \mathtt{t}_{\mathrm{s}}^{2} \\
&+ \left[\mathrm{i}\sqrt{\frac{\hbar\omega_{\mathbf{k}}}{\mathcal{A}L}} e^{\mathrm{i}\mathbf{k}\cdot(\mathbf{r}-\mathbf{0}'_{\mathrm{BS_{1s}i_{BS_{1s}}}})} + \mathrm{i}\sqrt{\frac{\hbar\omega_{\mathbf{k}}}{\mathcal{A}L}} e^{\mathrm{i}\mathbf{k}\cdot(\mathbf{r}-\mathbf{0}'_{\mathrm{BS_{2s}M_{2s}M_{1s}}})} \right] \mathtt{t}_{\mathrm{s}}\mathtt{r}_{\mathrm{s}} \\
&+ \mathrm{i}\sqrt{\frac{\hbar\omega_{\mathbf{k}}}{\mathcal{A}L}} e^{\mathrm{i}\mathbf{k}\cdot(\mathbf{r}-\mathbf{0}'_{\mathrm{BS_{2s}M_{2s}M_{1s}BS_{1s}s}})} \mathtt{r}_{\mathrm{s}}^{2}, \quad \text{for } \mathbf{r} \parallel \mathbf{k},\ z > z_{\mathrm{BS_{2s}}},\ \mathbf{k} \in [\mathbf{k}]_{\mathrm{s}},
\end{aligned} \tag{2.245}
$$

where $\mathbf{0}'_{\mathrm{BS_{2s}M_{2s}M_{1s}}}$ and $\mathbf{0}'_{\mathrm{BS_{2s}M_{2s}M_{1s}BS_{1s}s}}$ are chosen such that

$$
\mathbf{k}\cdot(\mathbf{r}-\mathbf{0}'_{\mathrm{BS_{2s}M_{2s}M_{1s}}}) = \mathbf{k}\cdot(\mathbf{r}-\mathbf{0}_{\mathrm{BS_{2s}}}) - \mathbf{k}'\cdot(\mathbf{0}_{\mathrm{BS_{2s}}} - \mathbf{0}'_{\mathrm{M_{2s}M_{1s}}}), \quad \mathbf{k} \in [\mathbf{k}]_{\mathrm{s}}, \tag{2.246}
$$

$$
\begin{aligned}
\mathbf{k}\cdot(\mathbf{r}-\mathbf{0}'_{\mathrm{BS_{2s}M_{2s}M_{1s}BS_{1s}s}}) &= \mathbf{k}\cdot(\mathbf{r}-\mathbf{0}_{\mathrm{BS_{2s}}}) \\
&- \mathbf{k}'\cdot(\mathbf{0}_{\mathrm{BS_{2s}}} - \mathbf{0}'_{\mathrm{M_{2s}M_{1s}BS_{1s}s}}), \quad \mathbf{k} \in [\mathbf{k}]_{\mathrm{s}}.
\end{aligned} \tag{2.247}
$$

The output modal functions for the second beam are

$$
\begin{aligned}
v_{\mathrm{i_{BS_{2s}}}\mathbf{k}}(\mathbf{r}) &= \mathrm{i}\sqrt{\frac{\hbar\omega_{\mathbf{k}}}{\mathcal{A}L}} e^{-\mathrm{i}\mathbf{k}\cdot(\mathbf{r}-\mathbf{0}'_{\mathrm{BS_{2s}BS_{1s}i_{BS_{1s}}}})} \mathtt{r}_{\mathrm{s}}^{2} \\
&+ \left[\mathrm{i}\sqrt{\frac{\hbar\omega_{\mathbf{k}}}{\mathcal{A}L}} e^{-\mathrm{i}\mathbf{k}\cdot(\mathbf{r}-\mathbf{0}'_{\mathrm{BS_{2s}s}})} + \mathrm{i}\sqrt{\frac{\hbar\omega_{\mathbf{k}}}{\mathcal{A}L}} e^{-\mathrm{i}\mathbf{k}\cdot(\mathbf{r}-\mathbf{0}'_{\mathrm{M_{2s}M_{1s}BS_{1s}}})} \right] \mathtt{r}_{\mathrm{s}}\mathtt{t}_{\mathrm{s}} \\
&+ \mathrm{i}\sqrt{\frac{\hbar\omega_{\mathbf{k}}}{\mathcal{A}L}} e^{-\mathrm{i}\mathbf{k}\cdot(\mathbf{r}-\mathbf{0}'_{\mathrm{M_{2s}M_{1s}}})} \mathtt{t}_{\mathrm{s}}^{2} \text{ for } (\mathbf{r}-\mathbf{0}_{\mathrm{BS_{1s}}}) \parallel -\mathbf{k}, \\
&\qquad z > z_{\mathrm{BS_{2s}}},\ \mathbf{k} \in [\mathbf{k}]_{\mathrm{i_{BS_{1s}}}},
\end{aligned} \tag{2.248}
$$

where $\mathbf{0}'_{\mathrm{BS_{2s}BS_{1s}i_{BS_{1s}}}}$ and $\mathbf{0}'_{\mathrm{BS_{2s}s}}$ are chosen such that

$$
\begin{aligned}
-\mathbf{k}\cdot(\mathbf{r}-\mathbf{0}'_{\mathrm{BS_{2s}BS_{1s}i_{BS_{1s}}}}) &= -\mathbf{k}\cdot(\mathbf{r}-\mathbf{0}_{\mathrm{BS_{2s}}}) \\
&+ \mathbf{k}'\cdot(\mathbf{0}_{\mathrm{BS_{2s}}} - \mathbf{0}'_{\mathrm{BS_{1s}i_{BS_{1s}}}}), \quad \mathbf{k} \in [\mathbf{k}]_{\mathrm{i_{BS_{1s}}}}.
\end{aligned} \tag{2.249}
$$

$$
-\mathbf{k}\cdot(\mathbf{r}-\mathbf{0}'_{\mathrm{BS_{2s}s}}) = -\mathbf{k}\cdot(\mathbf{r}-\mathbf{0}_{\mathrm{BS_{2s}}}) + \mathbf{k}'\cdot\mathbf{0}_{\mathrm{BS_{2s}}}, \quad \mathbf{k} \in [\mathbf{k}]_{\mathrm{i_{BS_{1s}}}}. \tag{2.250}
$$

In a standard fashion, we associate the modal functions which travel to the above detector, (2.233), (2.234), (2.235), (2.236), (2.239), (2.242), (2.245), and (2.248), with fields we denote as $\hat{E}^{(+)}_{\mathrm{s}0}(\mathbf{r}, t)$, $\hat{E}^{(+)}_{\mathrm{i_{BS_{1s}}}0}(\mathbf{r}, t)$. Exchanging s $\leftrightarrow$ i, we introduce modal functions, which travel to the lower detector. We relate them with fields we denote as $\hat{E}^{(+)}_{\mathrm{i}0}(\mathbf{r}, t)$, $\hat{E}^{(+)}_{\mathrm{s_{BS_{1i}}}0}(\mathbf{r}, t)$. Switching on the nonlinear interaction, we find the field to obey the relations (2.189) and (2.190). Counter to propagation the field stays initial and along with propagation it at least obeys the rules we have used to generate the modal functions. For simplicity, it is assumed that $\mathtt{t}_J = \mathtt{t}'_J = \mathtt{r}_J = \mathtt{r}'_J = \frac{1}{\sqrt{2}}$, $J = \mathrm{s}, \mathrm{i}$.

For the calculation of intensity correlations determined in the experiment, we introduce the slowly varying field operators

$$\hat{F}_{\mathrm{s}}^{(+)}(\mathbf{r},t) = \exp(\mathrm{i}\omega_{\mathrm{s}}t)\hat{E}_{\mathrm{s}}^{(+)}(\mathbf{r},t),$$
$$\hat{F}_{\mathrm{i}}^{(+)}(\mathbf{r},t) = \exp(\mathrm{i}\omega_{\mathrm{i}}t)\hat{E}_{\mathrm{i}}^{(+)}(\mathbf{r},t). \tag{2.251}$$

The field operators at the signal and idler detectors placed at $\mathbf{r}_{\mathrm{s}}$, $\mathbf{r}_{\mathrm{i}}$, respectively, are

$$\begin{aligned}
&\hat{F}_{\mathrm{s}}^{(+)}(\mathbf{r}_{\mathrm{s}},t)\\
&= \frac{1}{2}\Bigg\{\Bigg[\hat{F}_{\mathrm{sout}}^{(+)}\left(\mathbf{0}, t-\frac{|\mathbf{r}_{\mathrm{s}}-\mathbf{r}_{\mathrm{BS}_{2\mathrm{s}}}|}{c}-\frac{L_{\mathrm{s,long}}}{c}-\frac{|\mathbf{r}_{\mathrm{BS}_{1\mathrm{s}}}|}{c}\right)\exp\left(\mathrm{i}\omega_{\mathrm{s}}\frac{|\mathbf{r}_{\mathrm{BS}_{1\mathrm{s}}}|}{c}\right)\\
&\quad -\mathrm{i}\hat{F}_{\mathrm{i}_{\mathrm{BS}_{1\mathrm{s}}}\mathrm{in}}^{(+)}\left(\mathbf{0}_{\mathrm{BS}_{1\mathrm{s}}}, t-\frac{|\mathbf{r}_{\mathrm{s}}-\mathbf{r}_{\mathrm{BS}_{2\mathrm{s}}}|}{c}-\frac{L_{\mathrm{s,long}}}{c}\right)\Bigg]\exp\left(\mathrm{i}\omega_{\mathrm{s}}\frac{L_{\mathrm{s,long}}}{c}\right)\\
&\quad +\Bigg[\hat{F}_{\mathrm{sout}}^{(+)}\left(\mathbf{0}, t-\frac{|\mathbf{r}_{\mathrm{s}}-\mathbf{r}_{\mathrm{BS}_{2\mathrm{s}}}|}{c}-\frac{L_{\mathrm{s,short}}}{c}-\frac{|\mathbf{r}_{\mathrm{BS}_{1\mathrm{s}}}|}{c}\right)\exp\left(\mathrm{i}\omega_{\mathrm{s}}\frac{|\mathbf{r}_{\mathrm{BS}_{1\mathrm{s}}}|}{c}\right)\\
&\quad +\mathrm{i}\hat{F}_{\mathrm{i}_{\mathrm{BS}_{1\mathrm{s}}}\mathrm{in}}^{(+)}\left(\mathbf{0}_{\mathrm{BS}_{1\mathrm{s}}}, t-\frac{|\mathbf{r}_{\mathrm{s}}-\mathbf{r}_{\mathrm{BS}_{2\mathrm{s}}}|}{c}-\frac{L_{\mathrm{s,short}}}{c}\right)\Bigg]\exp\left(\mathrm{i}\omega_{\mathrm{s}}\frac{L_{\mathrm{s,short}}}{c}\right)\Bigg\}\\
&\quad \times\exp\left(\mathrm{i}\omega_{\mathrm{s}}\frac{|\mathbf{r}_{\mathrm{s}}-\mathbf{r}_{\mathrm{BS}_{2\mathrm{s}}}|}{c}\right),
\end{aligned}\tag{2.252}$$

$$\hat{F}_{\mathrm{i}}^{(+)}(\mathbf{r}_{\mathrm{i}},t) = \hat{F}_{\mathrm{s}}^{(+)}(\mathbf{r}_{\mathrm{s}},t)\Big|_{\mathrm{s}\leftrightarrow\mathrm{i}}. \tag{2.253}$$

Let us denote $L_{\mathrm{s,short}}$ ($L_{\mathrm{i,short}}$) a length of the short arm of the interferometer for the signal (idler) beam. Supposing that $\Delta L_{\mathrm{s}} \equiv L_{\mathrm{s,long}} - L_{\mathrm{s,short}}$ ($\Delta L_{\mathrm{i}} \equiv L_{\mathrm{i,long}} - L_{\mathrm{i,short}}$) is much greater than the coherence length of the signal (idler) in order to avoid the second-order interference, we get (Casado et al. 1997a)

$$\begin{aligned}
&\langle\hat{F}_{\mathrm{s}}^{(+)}(\mathbf{r}_{\mathrm{s}},t+\tau)\hat{F}_{\mathrm{i}}^{(+)}(\mathbf{r}_{\mathrm{i}},t+\tau')\rangle = \frac{1}{4}gV\\
&\quad\times\Bigg\{\nu(\tau'-\tau)\exp\Bigg[\frac{\mathrm{i}}{c}\omega_{\mathrm{s}}\left(|\mathbf{r}_{\mathrm{BS}_{1\mathrm{s}}}|+L_{\mathrm{s,long}}+|\mathbf{r}_{\mathrm{s}}-\mathbf{r}_{\mathrm{BS}_{2\mathrm{s}}}|\right)\\
&\quad+\frac{\mathrm{i}}{c}\omega_{\mathrm{i}}\left(|\mathbf{r}_{\mathrm{BS}_{1\mathrm{i}}}|+L_{\mathrm{i,long}}+|\mathbf{r}_{\mathrm{i}}-\mathbf{r}_{\mathrm{BS}_{2\mathrm{i}}}|\right)\Bigg]\\
&\quad+\nu\left(\tau'-\tau+\frac{\Delta L_{\mathrm{i}}-\Delta L_{\mathrm{s}}}{c}\right)\exp\Bigg[\frac{\mathrm{i}}{c}\omega_{\mathrm{s}}\big(|\mathbf{r}_{\mathrm{BS}_{1\mathrm{s}}}|+L_{\mathrm{s,short}}\\
&\quad+|\mathbf{r}_{\mathrm{s}}-\mathbf{r}_{\mathrm{BS}_{2\mathrm{s}}}|\big)+\frac{\mathrm{i}}{c}\omega_{\mathrm{i}}\left(|\mathbf{r}_{\mathrm{BS}_{1\mathrm{i}}}|+L_{\mathrm{i,short}}+|\mathbf{r}_{\mathrm{i}}-\mathbf{r}_{\mathrm{BS}_{2\mathrm{i}}}|\right)\Bigg]\Bigg\},
\end{aligned}\tag{2.254}$$

provided that $|\mathbf{r}_{\mathrm{BS}_{1J}}| + L_{J,\mathrm{short}} + |\mathbf{r}_J - \mathbf{r}_{\mathrm{BS}_{2J}}|$ is the same for $J = \mathrm{s}$ and $J = \mathrm{i}$. We finally obtain

$$\begin{aligned} P_{\mathrm{si}}(\mathbf{r}_{\mathrm{s}}, t+\tau; \mathbf{r}_{\mathrm{i}}, t+\tau') = \frac{1}{16} K' g^2 |V|^2 \Bigg\{ & |\nu(\tau'-\tau)|^2 \\ & + \left| \nu\left(\tau' - \tau + \frac{\Delta L_{\mathrm{i}} - \Delta L_{\mathrm{s}}}{c}\right)\right|^2 \\ & - 2\mathrm{Re}\Bigg\{ \nu(\tau'-\tau)\nu^*\left(\tau' - \tau + \frac{\Delta L_{\mathrm{i}} - \Delta L_{\mathrm{s}}}{c}\right) \\ & \times \exp\left[\frac{\mathrm{i}}{c}(\omega_{\mathrm{s}}\Delta L_{\mathrm{s}} + \omega_{\mathrm{i}}\Delta L_{\mathrm{i}})\right]\Bigg\}\Bigg\}. \end{aligned} \tag{2.255}$$

The visibility is given in (2.175), where

$$\begin{aligned} & R_{\mathrm{simax}} + R_{\mathrm{simin}} \\ & = \frac{1}{8} g^2 |V|^2 \left[M(0,0) + M\left(\frac{\Delta L_{\mathrm{i}} - \Delta L_{\mathrm{s}}}{c}, \frac{\Delta L_{\mathrm{i}} - \Delta L_{\mathrm{s}}}{c}\right)\right], \end{aligned} \tag{2.256}$$

$$R_{\mathrm{simax}} - R_{\mathrm{simin}} = \frac{1}{4} g^2 |V|^2 M\left(0, \frac{\Delta L_{\mathrm{i}} - \Delta L_{\mathrm{s}}}{c}\right). \tag{2.257}$$

The dependence of the visibility on the difference $(\Delta L_{\mathrm{i}} - \Delta L_{\mathrm{s}})$ is plotted in Fig. 2.7. The variation of the visibility is due to the function $M\left(\frac{d}{c}, \frac{h}{c}\right)$, but the function erf does not contribute to it. The distance between the points of inflection is $2\frac{c}{\sigma} = 6 \times 10^{-4}$ m. The interference manifests itself as a cosine variation of the coincidence rate with $\frac{1}{c}(\omega_{\mathrm{s}}\Delta L_{\mathrm{s}} + \omega_{\mathrm{i}}\Delta L_{\mathrm{i}})$.

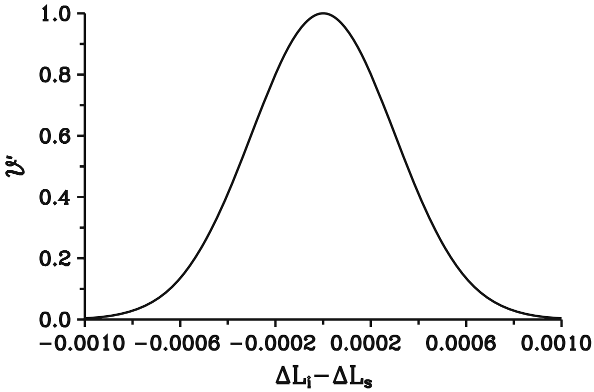

Fig. 2.7 The visibility $\mathcal{V}'$ versus the difference $(\Delta L_{\mathrm{i}} - \Delta L_{\mathrm{s}}) \in [-10^{-3}, 10^{-3}]$ measured in metres

(v) *Induced coherence and indistinguishability in two-photon interference*

Zou et al. (1991) performed an experiment in which fourth-order interference is observed in the superposition of signal photons from two coherently pumped parametric down-conversion crystals, when the paths of the idler photons are aligned. The experimental setup is outlined in Fig. 2.8, in which two nonlinear crystals NL1 and NL2 are optically pumped by two mutually coherent, classical pump waves of complex amplitudes

$$V_j(\mathbf{r}, t) = V_j \exp[\mathrm{i}(\mathbf{k}_0 \cdot \mathbf{r} - \omega_0 t)], \quad j = 1, 2. \tag{2.258}$$

We assume that $V_1 = V_2 \exp(\mathrm{i}\mathbf{k}_0 \cdot \mathbf{0}_2) = V$. On the contrary, there were similar crystals in the experiment, but we consider more general ones. The parametric down-conversion occurs at both crystals, each with the emission of a signal photon and an idler photon. We are interested in the joint detection rate of the detectors $\mathrm{D_s}$ and $\mathrm{D_i}$ when the trajectories of the two idlers i_1, i_2 are aligned and the path difference between the two signals is varied slightly. Fourth-order interference disappears when the idlers are misaligned or separated by a beam stop.

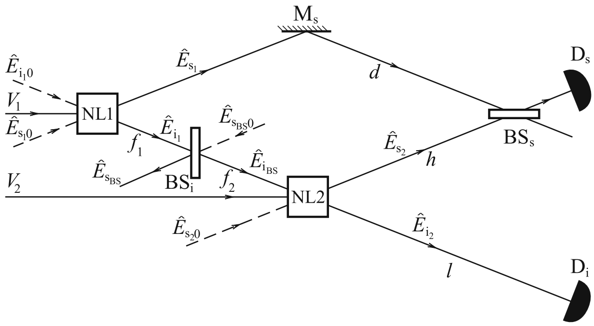

Fig. 2.8 Experimental setup on induced coherence without induced emission. For simplicity, $\hat{E}_{\mathrm{s_{BS}}0} \equiv \hat{E}_{\mathrm{s_{BS_i}}0}$

In what follows, we will specify modal functions and the nonlinear dynamics of field operators. We consider the initial electric field in the form

$$\hat{E}_0^{(+)}(\mathbf{r}, t) = V^{(+)}(\mathbf{r}, t)\hat{1} + \sum_{j=1}^{2} \hat{E}_{\mathrm{s}_j 0}^{(+)}(\mathbf{r}, t) + \hat{E}_{\mathrm{i}0}^{(+)}(\mathbf{r}, t) + \hat{E}_{\mathrm{s_{BS_i}}0}^{(+)}(\mathbf{r}, t), \tag{2.259}$$

where

$$\hat{E}_{J0}^{(+)}(\mathbf{r}, t) = \sum_{\mathbf{k}\in[\mathbf{k}]_J} v_{J\mathbf{k}}(\mathbf{r})\hat{a}_{J\mathbf{k}0}(t), \quad J = \mathrm{s}_1, \mathrm{s}_2, \mathrm{i}, \mathrm{s_{BS_i}}. \tag{2.260}$$

The modal functions as restricted to linear segments are

$$v_{\mathrm{s}_1\mathbf{k}}(\mathbf{r}) = \mathrm{i}\sqrt{\frac{\hbar\omega_{\mathbf{k}}}{\mathcal{A}L}} e^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}} \text{ for } \mathbf{r} \parallel \mathbf{k}, \ z < z_{\mathrm{M_s}}, \ \mathbf{k} \in [\mathbf{k}]_{\mathrm{s}}, \tag{2.261}$$

$$v_{\mathrm{s}_2\mathbf{k}}(\mathbf{r}) = \mathrm{i}\sqrt{\frac{\hbar\omega_{\mathbf{k}}}{\mathcal{A}L}} e^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}} \text{ for } (\mathbf{r} - \mathbf{0}_2) \parallel \mathbf{k}, \ z < z_{\mathrm{BS_s}}, \ \mathbf{k} \in [\mathbf{k}]_{\mathrm{s}}, \tag{2.262}$$

$$v_{\mathrm{s_1}\mathbf{k}}(\mathbf{r}) = -\mathrm{i}\sqrt{\frac{\hbar\omega_{\mathbf{k}}}{\mathcal{A}L}}e^{\mathrm{i}\mathbf{k}'\cdot(\mathbf{r}-\mathbf{0}'_{\mathrm{M_s}})} \text{ for } (\mathbf{r}-\mathbf{0}_{\mathrm{M_s}}) \parallel \mathbf{k}',$$
$$z_{\mathrm{M_s}} < z < z_{\mathrm{BS_s}},\ \mathbf{k} \in [\mathbf{k}]_{\mathrm{s}}. \tag{2.263}$$

Here $\mathbf{0}'_{\mathrm{M_s}}$ and $\mathbf{k}'$ are used as in the definition of modal functions related to Fig. 2.1 and $\mathbf{k}'$ has the same meaning. Here, as in the definitions related to Fig. 2.4, we specify that $\mathbf{0}'_{\mathrm{M_s}}$ has been chosen so that

$$\mathbf{k}'\cdot(\mathbf{r}-\mathbf{0}'_{\mathrm{M_s}}) = \mathbf{k}'\cdot(\mathbf{r}-\mathbf{0}_{\mathrm{M_s}}) + \mathbf{k}\cdot\mathbf{0}_{\mathrm{M_s}},\quad \mathbf{k}\in[\mathbf{k}]_{\mathrm{s}}. \tag{2.264}$$

Similarly as in (2.222) and (2.223) for $\mathtt{t} = \mathtt{t}' = \frac{1}{\sqrt{2}}$, $\mathtt{r} = \mathtt{r}' = \frac{\mathrm{i}}{\sqrt{2}}$, we have

$$v_{\mathrm{s_1}\mathbf{k}}(\mathbf{r}) = -\mathrm{i}\sqrt{\frac{\hbar\omega_{\mathbf{k}}}{\mathcal{A}L}}e^{\mathrm{i}\mathbf{k}'\cdot(\mathbf{r}-\mathbf{0}'_{\mathrm{M_s}})}\frac{1}{\sqrt{2}} + \mathrm{i}\sqrt{\frac{\hbar\omega_{\mathbf{k}}}{\mathcal{A}L}}e^{\mathrm{i}\mathbf{k}'\cdot(\mathbf{r}-\mathbf{0}'_{\mathrm{BS_s}})}\frac{\mathrm{i}}{\sqrt{2}}$$
$$\text{for } (\mathbf{r}-\mathbf{0}_{\mathrm{M_s}}) \parallel \mathbf{k}',\ z > z_{\mathrm{BS_s}},\ \mathbf{k}\in[\mathbf{k}]_{\mathrm{s}}, \tag{2.265}$$
$$v_{\mathrm{s_2}\mathbf{k}}(\mathbf{r}) = -\mathrm{i}\sqrt{\frac{\hbar\omega_{\mathbf{k}}}{\mathcal{A}L}}e^{\mathrm{i}\mathbf{k}\cdot(\mathbf{r}-\mathbf{0}'_{\mathrm{BS_sM_s}})}\frac{\mathrm{i}}{\sqrt{2}} + \mathrm{i}\sqrt{\frac{\hbar\omega_{\mathbf{k}}}{\mathcal{A}L}}e^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}}\frac{1}{\sqrt{2}}$$
$$\text{for } (\mathbf{r}-\mathbf{0}_2) \parallel \mathbf{k},\ z > z_{\mathrm{BS_s}},\ \mathbf{k}\in[\mathbf{k}]_{\mathrm{s}}, \tag{2.266}$$

where $\mathbf{0}'_{\mathrm{BS_s}}$ and $\mathbf{0}'_{\mathrm{BS_sM_s}}$ have been chosen so that, respectively,

$$\mathbf{k}'\cdot(\mathbf{r}-\mathbf{0}'_{\mathrm{BS_s}}) = \mathbf{k}'\cdot(\mathbf{r}-\mathbf{0}_{\mathrm{BS_s}}) + \mathbf{k}\cdot\mathbf{0}_{\mathrm{BS_s}},\quad \mathbf{k}\in[\mathbf{k}]_{\mathrm{s}}, \tag{2.267}$$
$$\mathbf{k}\cdot(\mathbf{r}-\mathbf{0}'_{\mathrm{BS_sM_s}}) = \mathbf{k}\cdot(\mathbf{r}-\mathbf{0}_{\mathrm{BS_s}}) + \mathbf{k}'\cdot\mathbf{0}'_{\mathrm{M_s}},\quad \mathbf{k}\in[\mathbf{k}]_{\mathrm{s}}; \tag{2.268}$$
$$v_{\mathrm{i}\mathbf{k}}(\mathbf{r}) = \mathrm{i}\sqrt{\frac{\hbar\omega_{\mathbf{k}}}{\mathcal{A}L}}e^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}} \text{ for } \mathbf{r}\parallel\mathbf{k},\ z < z_{\mathrm{BS_i}},\ \mathbf{k}\in[\mathbf{k}]_{\mathrm{i}}, \tag{2.269}$$
$$v_{\mathrm{s_{BS_i}}\mathbf{k}}(\mathbf{r}) = \mathrm{i}\sqrt{\frac{\hbar\omega_{\mathbf{k}}}{\mathcal{A}L}}e^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}} \text{ for } (\mathbf{r}-\mathbf{0}_{\mathrm{BS_i}})\parallel\mathbf{k},\ z > z_{\mathrm{BS_i}},\ \mathbf{k}\in[\mathbf{k}]_{\mathrm{s_{BS_i}}}, \tag{2.270}$$
$$v_{\mathrm{i}\mathbf{k}}(\mathbf{r}) = \mathrm{i}\sqrt{\frac{\hbar\omega_{\mathbf{k}}}{\mathcal{A}L}}e^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}}\mathtt{t} + \mathrm{i}\sqrt{\frac{\hbar\omega_{\mathbf{k}}}{\mathcal{A}L}}e^{\mathrm{i}\mathbf{k}\cdot(\mathbf{r}-\mathbf{0}'_{\mathrm{BS_is_{BSi}}})}\mathtt{r}'$$
$$\text{for } \mathbf{r}\parallel\mathbf{k},\ z > z_{\mathrm{BS_i}},\ \mathbf{k}\in[\mathbf{k}]_{\mathrm{i}}, \tag{2.271}$$
$$v_{\mathrm{s_{BS_i}}\mathbf{k}}(\mathbf{r}) = \mathrm{i}\sqrt{\frac{\hbar\omega_{\mathbf{k}}}{\mathcal{A}L}}e^{\mathrm{i}\mathbf{k}\cdot(\mathbf{r}-\mathbf{0}'_{\mathrm{BS_ii}})}\mathtt{r} + \mathrm{i}\sqrt{\frac{\hbar\omega_{\mathbf{k}}}{\mathcal{A}L}}e^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}}\mathtt{t}'$$
$$\text{for } (\mathbf{r}-\mathbf{0}_{\mathrm{BS_i}})\parallel\mathbf{k},\ z < z_{\mathrm{BS_i}},\ \mathbf{k}\in[\mathbf{k}]_{\mathrm{s_{BS_i}}}, \tag{2.272}$$

where $\mathbf{k}'$ is defined relative to the beam splitter $\mathrm{BS_i}$ and $\mathbf{0}'_{\mathrm{BS_ii}}$ and $\mathbf{0}'_{\mathrm{BS_is_{BSi}}}$ have been chosen so that, respectively,

$$\mathbf{k}\cdot(\mathbf{r}-\mathbf{0}'_{\mathrm{BS_is_{BSi}}}) = \mathbf{k}\cdot(\mathbf{r}-\mathbf{0}_{\mathrm{BS_i}}) + \mathbf{k}'\cdot\mathbf{0}_{\mathrm{BS_ii}} \text{ for } \mathbf{k}\in[\mathbf{k}]_{\mathrm{i}}, \tag{2.273}$$
$$\mathbf{k}\cdot(\mathbf{r}-\mathbf{0}'_{\mathrm{BS_ii}}) = \mathbf{k}\cdot(\mathbf{r}-\mathbf{0}_{\mathrm{BS_i}}) + \mathbf{k}'\cdot\mathbf{0}_{\mathrm{BS_i}} \text{ for } \mathbf{k}\in[\mathbf{k}]_{\mathrm{s_{BS_i}}}. \tag{2.274}$$

The modal functions (2.269), (2.270), (2.271), and (2.272) travel to the lower detector. We relate them with fields we denote as $\hat{E}_{\mathrm{i}0}(\mathbf{r}, t)$, $\hat{E}_{\mathrm{s}_{\mathrm{BS}_\mathrm{i}}0}(\mathbf{r}, t)$. Switching on the first nonlinear interaction, we find the field to obey the relations (2.189), (2.190), and (2.191) for $j = 1$ with $\mathrm{i}_1 \to \mathrm{i}$. Counter to propagation the field remains initial and along with propagation it at least obeys the rules we have used to generate the modal functions.

Now we would like to interpret the subscript 0 not as the order of solution but as a number of the initial stage. Since the stage is followed by the first stage, we would like to modify the relations (2.189), (2.190), and (2.191) so that they confess the first-stage operators on the left-hand side, which would lead to the use of the subscript 1.

Switching on the second nonlinear interaction, we find the field to obey the relations (2.189), (2.190), and (2.191) for $j = 2$ with $\mathrm{i}_2 \to \mathrm{i}$, but the first-stage field operators to have been substituted for the operators on the right-hand side. The action of the operators $\hat{\hat{G}}_2$ and $\hat{\hat{J}}_2$ depends on the nonlinear crystal located at $\mathbf{0}_2$. It also depends on the pump beam at the same crystal. Counter to propagation the field stays first stage and along with propagation it still at least obeys the rules to generate the modal functions.

Further we will express the intensity correlations that have been determined in the experiment. Again, we introduce the operators (2.224), where $J = \mathrm{s}_1, \mathrm{s}_2, \mathrm{i}$, $J_1 = \mathrm{s}, \mathrm{s}, \mathrm{i}$. The field operators at the signal and idler detectors placed at $\mathbf{r}_\mathrm{s}$, $\mathbf{r}_\mathrm{i}$, respectively, are

$$\hat{F}_{\mathrm{s}_2}^{(+)}(\mathbf{r}_\mathrm{s}, t) = \frac{1}{\sqrt{2}}\left[-\mathrm{i}\hat{F}_{\mathrm{s}_1}^{(+)}\left(\mathbf{0}_1, t - \frac{d}{c}\right)\exp\left(\mathrm{i}\omega_\mathrm{s}\frac{d}{c}\right) + \hat{F}_{\mathrm{s}_2}^{(+)}\left(\mathbf{0}_2, t - \frac{h}{c}\right)\exp\left(\mathrm{i}\omega_\mathrm{s}\frac{h}{c}\right)\right], \tag{2.275}$$

$$\hat{F}_{\mathrm{i}}^{(+)}(\mathbf{r}_\mathrm{i}, t') = \hat{F}_{\mathrm{i}}^{(+)}\left(\mathbf{0}_2, t' - \frac{l}{c}\right)\exp\left(\mathrm{i}\omega_\mathrm{i}\frac{l}{c}\right). \tag{2.276}$$

We still assume different crystals and derive a slight generalization of the well-known experiment (Casado et al. 1997a). By taking into account the correlation relations

$$\langle \hat{F}_{\mathrm{s}_1}^{(+)}(\mathbf{0}_1, t)\hat{F}_{\mathrm{i}}^{(+)}(\mathbf{0}_2, t')\rangle = \mathtt{t} g V_1 \nu_1\left(t' - t - \frac{f}{c}\right)\exp\left(\mathrm{i}\omega_\mathrm{i}\frac{f}{c}\right), \tag{2.277}$$

$$\langle \hat{F}_{\mathrm{i}}^{(+)}(\mathbf{0}_2, t')\hat{F}_{\mathrm{s}_2}^{(+)}(\mathbf{0}_2, t)\rangle = g V_2 \nu_2(t' - t), \tag{2.278}$$

we get (Casado et al. 1997a)

$$\langle \hat{F}_{\mathrm{s}_2}^{(+)}(\mathbf{r}_{\mathrm{s}}, t)\hat{F}_{\mathrm{i}}^{(+)}(\mathbf{r}_{\mathrm{i}}, t')\rangle = \frac{gV}{\sqrt{2}}$$
$$\times \left[-\mathrm{i}\mathtt{t}\nu_1 \left(\tau' - \tau - \frac{l}{c} - \frac{f}{c} + \frac{d}{c} \right) \exp\left\{ \frac{\mathrm{i}}{c}[\omega_{\mathrm{s}} d + \omega_{\mathrm{i}}(l+f)] \right\} \right.$$
$$\left. + \nu_2 \left(\tau' - \tau - \frac{l}{c} + \frac{h}{c} \right) \exp\left\{ \frac{\mathrm{i}}{c}[\omega_{\mathrm{s}} h + \omega_{\mathrm{i}} l] \right\} \right]. \tag{2.279}$$

We finally obtain

$$P_{\mathrm{si}}(\mathbf{r}_{\mathrm{s}}, t+\tau; \mathbf{r}_{\mathrm{i}}, t+\tau') = \frac{1}{2} K' g^2 |V|^2$$
$$\times \left\{ \left| \mathtt{t}\nu_1 \left(\tau' - \tau - \frac{l}{c} - \frac{f}{c} + \frac{d}{c} \right) \right|^2 + \left| \nu_2 \left(\tau' - \tau - \frac{l}{c} + \frac{h}{c} \right) \right|^2 \right.$$
$$+ 2\mathrm{Im} \left[\mathtt{t}\nu_1 \left(\tau' - \tau - \frac{l}{c} - \frac{f}{c} + \frac{d}{c} \right) \nu_2^* \left(\tau' - \tau - \frac{l}{c} + \frac{h}{c} \right) \right.$$
$$\left. \left. \times \exp\left\{ \frac{\mathrm{i}}{c}[\omega_{\mathrm{s}}(d-h) + \omega_{\mathrm{i}} f] \right\} \right] \right\}. \tag{2.280}$$

We have hopefully corrected the factor, changed a sign with respect to the reflection from the mirror, and changed signs of the argument of $\nu(\tau)$ relying on the identity $\nu(\tau) = \nu(-\tau)$, where $\nu_1(\tau) = \nu_2(\tau) \equiv \nu(\tau)$.

The visibility is expressed by the formula (2.175), where for $d = l + f$, $l = h$

$$R_{\mathrm{simax}} + R_{\mathrm{simin}} = g^2 |V|^2 M(0, 0) \left(|\mathtt{t}|^2 + 1 \right), \tag{2.281}$$
$$R_{\mathrm{simax}} - R_{\mathrm{simin}} = 2g^2 |V|^2 M(0, 0) |\mathtt{t}|. \tag{2.282}$$

The maximum visibility is equal to unity and, in general, it depends on the transmissivity of the beam splitter, as can be seen from Fig. 2.9. The interference manifests itself as a cosine variation of the coincidence rate with $\frac{\omega_0}{c} f$.

(vi) *Frustrated two-photon creation via interference*

Herzog et al. (1994) performed a simple experiment interpreted as showing interference of two processes. They placed three mirrors in the three beams, laser, signal, and idler, that emerge from a nonlinear crystal, NL, and put a detector into the reflected idler beams (see Fig. 2.10). In the standard quantum interpretation a pair of correlated photons can be created either by the laser beam travelling from left to right or when the reflected laser beam travels from right to left. In both cases the idler photon may arrive at the detector. As the two possibilities are indistinguishable

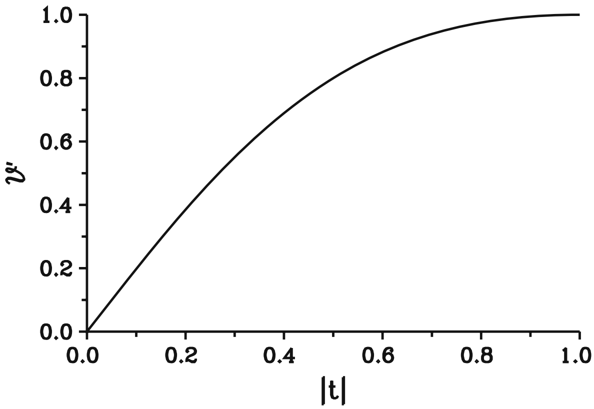

Fig. 2.9 The visibility $\mathcal{V}'$ versus the modulus of the amplitude transmissivity $|\mathrm{t}| \in [0, 1]$ of the beam splitter $\mathrm{BS_i}$. It is assumed that $d = l + f, l = h$

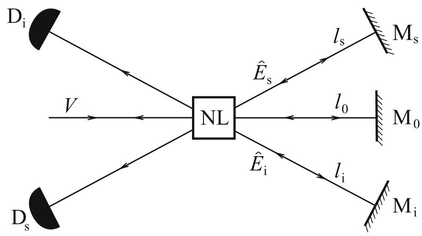

Fig. 2.10 Experimental setup on frustrated two-photon creation via interference

they interfere and the counting rate oscillates depending on the position of a chosen mirror.

Accordingly the description of the pump beam is given by

$$\begin{aligned} V^{(+)}(\mathbf{r}, t) &= V e^{\mathrm{i}(\mathbf{k}_0\cdot\mathbf{r}-\omega_0 t)} - V e^{\mathrm{i}[-\mathbf{k}_0\cdot(\mathbf{r}-2l_0\mathbf{e}_0)-\omega_0 t]} \\ &= V e^{\mathrm{i}(\mathbf{k}_0\cdot\mathbf{r}-\omega_0 t)} - V e^{\mathrm{i}\varphi_0} e^{\mathrm{i}(-\mathbf{k}_0\cdot\mathbf{r}-\omega_0 t)} \\ &\quad \text{for } x = 0,\ y = 0,\ z < l_0, \end{aligned} \tag{2.283}$$

where $\mathbf{e}_0$ is the direction vector of the forward-propagating pump beam, $\varphi_0 = 2|\mathbf{k}_0|l_0 = 2\frac{\omega_0 l_0}{c}$.

The modal functions are

$$\begin{aligned} v_{\mathrm{s}\mathbf{k}}(\mathbf{r}) &= \mathrm{i}\sqrt{\frac{\hbar\omega_\mathbf{k}}{2\mathcal{A}L}} e^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}} - \mathrm{i}\sqrt{\frac{\hbar\omega_\mathbf{k}}{2\mathcal{A}L}} e^{\mathrm{i}\varphi_\mathrm{s}} e^{-\mathrm{i}\mathbf{k}\cdot\mathbf{r}} \\ &\quad \text{for } \mathbf{r} \parallel \mathbf{k},\ z < z_{\mathrm{M_s}},\ \mathbf{k} \in [\mathbf{k}]_\mathrm{s}, \end{aligned} \tag{2.284}$$

where $\varphi_\mathrm{s} = 2\frac{\omega_\mathrm{s} l_\mathrm{s}}{c}$, $z_{\mathrm{M_s}} = \mathbf{e}_0 \cdot \mathbf{0}_{\mathrm{M_s}}$. The modal functions $v_{\mathrm{i}\mathbf{k}}(\mathbf{r})$ are expressed similarly.

Associating the modal functions with quantum fields, we must consider that

$$\hat{E}^{(+)}_{\mathrm{s0}}(\mathbf{r}, t) = \hat{E}^{(+)}_{\mathrm{sF0}}(\mathbf{r}, t) + \hat{E}^{(+)}_{\mathrm{sB0}}(\mathbf{r}, t), \tag{2.285}$$

where

$$\hat{E}^{(+)}_{\mathrm{sF0}}(\mathbf{r},t) = \mathrm{i}\sum_{\mathbf{k}\in[\mathbf{k}]_{\mathrm{s}}}\sqrt{\frac{\hbar\omega_{\mathbf{k}}}{2\mathcal{A}L}}e^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}}\hat{a}_{\mathrm{s}\mathbf{k}0}(t),$$
$$\hat{E}^{(+)}_{\mathrm{sB0}}(\mathbf{r},t) = -\mathrm{i}\sum_{\mathbf{k}\in[\mathbf{k}]_{\mathrm{s}}}\sqrt{\frac{\hbar\omega_{\mathbf{k}}}{2\mathcal{A}L}}e^{\mathrm{i}\varphi_{\mathrm{s}}}e^{-\mathrm{i}\mathbf{k}\cdot\mathbf{r}}\hat{a}_{\mathrm{s}\mathbf{k}0}(t) \tag{2.286}$$

are the forward-propagating component and the backward-propagating component, respectively. The field operators $\hat{E}^{(+)}_{\mathrm{i}0}(\mathbf{r},t)$, $\hat{E}^{(+)}_{\mathrm{iF0}}(\mathbf{r},t)$, and $\hat{E}^{(+)}_{\mathrm{iB0}}(\mathbf{r},t)$ are expressed similarly. Nonlinear dynamics is described by the relations

$$\hat{E}^{(+)}_{\mathrm{sF}}(\mathbf{0}-(r=0)\mathbf{e}_{\mathrm{s}},t) = \hat{E}^{(+)}_{\mathrm{sF0}}(\mathbf{0},t),$$
$$\hat{E}^{(+)}_{\mathrm{iF}}(\mathbf{0}-(r=0)\mathbf{e}_{\mathrm{i}},t) = \hat{E}^{(+)}_{\mathrm{iF0}}(\mathbf{0},t), \tag{2.287}$$

where $\mathbf{e}_j$, $j=\mathrm{s},\mathrm{i}$, is the direction vector of the forward-propagating signal, idler beam, respectively:

$$\hat{E}^{(+)}_{\mathrm{sF}}(\mathbf{0}+(r=0)\mathbf{e}_{\mathrm{s}},t) = (1+g^2|V|^2\hat{\hat{J}})\hat{E}^{(+)}_{\mathrm{sF0}}(\mathbf{0},t) + e^{-\mathrm{i}\omega_0 t}gV\hat{\hat{G}}\hat{E}^{(-)}_{\mathrm{iF0}}(\mathbf{0},t),$$
$$\hat{E}^{(+)}_{\mathrm{iF}}(\mathbf{0}+(r=0)\mathbf{e}_{\mathrm{i}},t) = e^{-\mathrm{i}\omega_0 t}gV\hat{\hat{G}}\hat{E}^{(-)}_{\mathrm{sF0}}(\mathbf{0},t) + (1+g^2|V|^2\hat{\hat{J}})\hat{E}^{(+)}_{\mathrm{iF0}}(\mathbf{0},t), \tag{2.288}$$

$$\begin{aligned}\hat{E}^{(+)}_{\mathrm{sB}}(\mathbf{0}+(r=0)\mathbf{e}_{\mathrm{s}},t) &= -\hat{E}^{(+)}_{\mathrm{sF}}\left(\mathbf{0}+(r=0)\mathbf{e}_{\mathrm{s}},t-\frac{2l_{\mathrm{s}}}{c}\right)\\ &= -(1+g^2|V|^2\hat{\hat{J}})\hat{E}^{(+)}_{\mathrm{sF0}}\left(\mathbf{0},t-\frac{2l_{\mathrm{s}}}{c}\right)\\ &\quad - e^{\mathrm{i}(\varphi_{\mathrm{s}}+\varphi_{\mathrm{i}})}e^{-\mathrm{i}\omega_0 t}gV\hat{\hat{G}}\hat{E}^{(-)}_{\mathrm{iF0}}\left(\mathbf{0},t-\frac{2l_{\mathrm{s}}}{c}\right),\end{aligned} \tag{2.289}$$

$$\begin{aligned}\hat{E}^{(+)}_{\mathrm{iB}}(\mathbf{0}+(r=0)\mathbf{e}_{\mathrm{i}},t) &= -\hat{E}^{(+)}_{\mathrm{iF}}\left(\mathbf{0}+(r=0)\mathbf{e}_{\mathrm{i}},t-\frac{2l_{\mathrm{i}}}{c}\right)\\ &= -e^{\mathrm{i}(\varphi_{\mathrm{s}}+\varphi_{\mathrm{i}})}e^{-i\omega_0 t}gV\hat{\hat{G}}\hat{E}^{(-)}_{\mathrm{sF0}}\left(\mathbf{0},t-\frac{2l_{\mathrm{i}}}{c}\right)\\ &\quad - (1+g^2|V|^2\hat{\hat{J}})\hat{E}^{(+)}_{\mathrm{iF0}}\left(\mathbf{0},t-\frac{2l_{\mathrm{i}}}{c}\right),\end{aligned} \tag{2.290}$$

$$\begin{aligned}\hat{E}^{(+)}_{\mathrm{sBout}}(\mathbf{0},t) &= (1+g^2|V|^2\hat{\hat{J}})\hat{E}^{(+)}_{\mathrm{sB}}(\mathbf{0}+(r=0)\mathbf{e}_{\mathrm{s}},t)\\ &\quad + e^{-\mathrm{i}\omega_0 t}gVe^{\mathrm{i}\varphi_0}\hat{\hat{G}}\hat{E}^{(-)}_{\mathrm{iB}}(\mathbf{0}+(r=0)\mathbf{e}_{\mathrm{i}},t),\\ \hat{E}^{(+)}_{\mathrm{iBout}}(\mathbf{0},t) &= e^{-\mathrm{i}\omega_0 t}gVe^{\mathrm{i}\varphi_0}\hat{\hat{G}}\hat{E}^{(-)}_{\mathrm{sB}}(\mathbf{0}+(r=0)\mathbf{e}_{\mathrm{s}},t)\\ &\quad + (1+g^2|V|^2\hat{\hat{J}})\hat{E}^{(+)}_{\mathrm{iB}}(\mathbf{0}+(r=0)\mathbf{e}_{\mathrm{i}},t).\end{aligned} \tag{2.291}$$

Further we will calculate the quantum mean intensities that have been determined in the experiment. Introducing the slowly varying field operators

$$\begin{aligned}\hat{F}_J^{(+)}(\mathbf{r}, t) &= e^{\mathrm{i}\omega_J t}\hat{E}_J^{(+)}(\mathbf{r}, t),\\ \hat{F}_{J\mathrm{F}}^{(+)}(\mathbf{r}, t) &= e^{\mathrm{i}\omega_J t}\hat{E}_{J\mathrm{F}}^{(+)}(\mathbf{r}, t),\\ \hat{F}_{J\mathrm{B}}^{(+)}(\mathbf{r}, t) &= e^{\mathrm{i}\omega_J t}\hat{E}_{J\mathrm{B}}^{(+)}(\mathbf{r}, t),\ J = \mathrm{s}, \mathrm{i},\end{aligned} \tag{2.292}$$

and $\hat{F}_J^{(-)}(\mathbf{r}, t)$, $\hat{F}_{J\mathrm{F}}^{(-)}(\mathbf{r}, t)$, $\hat{F}_{J\mathrm{B}}^{(-)}(\mathbf{r}, t)$, we express the field operators at the signal and idler detectors placed at $\mathbf{r}_\mathrm{s}$, $\mathbf{r}_\mathrm{i}$, respectively, as

$$\hat{F}_{J\mathrm{B}}^{(+)}(\mathbf{r}_J, t) = \hat{F}_{J\mathrm{Bout}}^{(+)}\left(\mathbf{0}, t - \frac{r_J}{c}\right)\exp\left(\mathrm{i}\frac{\omega_J r_J}{c}\right),\ J = \mathrm{s},\ \mathrm{i}. \tag{2.293}$$

The quantum mean intensity or single photodetection rate in the detector D_s is

$$\begin{aligned}P_\mathrm{s}(\mathbf{r}_\mathrm{s}, t) &= K\left\langle 0 \left| \hat{E}^{(-)}(\mathbf{r}_\mathrm{s}, t)\hat{E}^{(+)}(\mathbf{r}_\mathrm{s}, t) \right| 0\right\rangle && (2.294)\\ &= K\left\langle 0 \left| \hat{F}_{\mathrm{sB}}^{(-)}(\mathbf{r}_\mathrm{s}, t)\hat{F}_{\mathrm{sB}}^{(+)}(\mathbf{r}_\mathrm{s}, t) \right| 0\right\rangle, && (2.295)\end{aligned}$$

where K is a constant related to the efficiency of the detector and the energy of a single photon. Considering the forward propagation, reflections, and the backward propagation, we obtain that

$$\begin{aligned}&\left\langle \hat{F}_{\mathrm{sB}}^{(-)}(\mathbf{r}_\mathrm{s}, t)\hat{F}_{\mathrm{sB}}^{(+)}(\mathbf{r}_\mathrm{s}, t)\right\rangle = \left\langle \hat{F}_{\mathrm{sB}}^{(-)}\left(\mathbf{0}, t - \frac{r_\mathrm{s}}{c}\right)\hat{F}_{\mathrm{sB}}^{(+)}\left(\mathbf{0}, t - \frac{r_\mathrm{s}}{c}\right)\right\rangle\\ &= 2g^2|V|^2\left[\mu_\mathrm{s}(0) + \mu_\mathrm{s}\left(\frac{2l_\mathrm{i}}{c} - \frac{2l_\mathrm{s}}{c}\right)\cos(\varphi_\mathrm{s} + \varphi_\mathrm{i} - \varphi_0)\right],\end{aligned} \tag{2.296}$$

where we have relied on the identity $\mu_\mathrm{s}(\tau) = \mu_\mathrm{s}(-\tau)$. From this,

$$P_\mathrm{s}(\mathbf{r}_\mathrm{s}, t) = 2Kg^2|V|^2\left[\mu_\mathrm{s}(0) + \mu_\mathrm{s}\left(\frac{2l_\mathrm{i}}{c} - \frac{2l_\mathrm{s}}{c}\right)\cos(\varphi_\mathrm{s} + \varphi_\mathrm{i} - \varphi_0)\right]. \tag{2.297}$$

The photodetection rate in the detector D_i is expressed similarly (Casado et al. 1997b).

In conclusion, we have mostly dealt with the fourth-order interference in parametric down-conversion experiments. The 1986, 1990, 1994 (adapted back to free space), and 1991 experiments were chosen according to a review article of other authors. Coincidence measurements in the various setups are essentially (or sufficiently well) described in terms of the cross correlation between the signal and the idler.

We have "promoted" the schemes of the experiments, where only paths through nonlinear and linear optical elements and the free space (with possible reflections from perfect mirrors) to detectors are drawn, to a reason of a certain neglect of the beams' divergence. We have replaced the usual assumption that the electric field is expanded in terms of an incomplete set of plane waves, which is relatively complete with respect to the expected direction of propagation, by the hypothesis that there exists an incomplete or relatively complete system of more complicated modal functions, which have still been specified only on the paths. We have performed conventional quantization by introducing annihilation operators in place of the classical complex amplitudes of the modes. We tried to choose sufficiently realistic values of the parameters for all the four experiments and to find visibilities of the intensity interference.

2.3.2 From Coupled Quantum Harmonic Oscillators Back to Interacting Fields

One of the interference experiments we have described in Section 2.3.1, the experiment of Zou et al. (1991) which has been analysed in Wang et al. (1991a,b), has attracted much attention. The arrangement of two down-converters is pumped by mutually coherent beams and the two down-converters are connected by the idler beam. The spontaneous emission from the first nonlinear crystal in the idler serves as a stimulating idler input to the second nonlinear crystal that acts as an optical amplifier. The interference of signal beams from both the crystals can be observed. A beam splitter placed between the two nonlinear crystals in the idler beam can change the strength of their connection since it attenuates the emerging field.

The parametric down-conversion in the second nonlinear crystal is stimulated by idler photons when the idler field is strong "per frequency unit". In this situation, the polychromatic theory yields results similar to those obtained by the monochromatic treatment, i.e. about multiples of the latter. When the idler field is weak per frequency unit, the second nonlinear crystal is proven to "ignore" the idler photons. The monochromatic description, even though completed by optimal scaling of its results, is far from being persuasive here. The assumption of the strong idler field is implicit in work contributing to the monochromatic theory.

In Řeháček and Peřina (1996) it has been shown that the distribution of photon-number sum in signal modes interpolates between a Bose–Einstein distribution and a convolution of two Bose–Einstein distributions. The latter distribution occurs when the idler beam is blocked. In general, the photon-number sum is distributed as if it corresponded to the number of signal degrees of freedom which varied between 1 and 2. A nonclassical distribution of photon-number sums restricted to even sums of photon numbers cannot occur, because the correlation between the photon numbers of the two signal beams is not complete. It has been found that the distribution of phase difference derived from the Q function narrows when the connection of both the down-converters via the idler mode closes up.

The monochromatic treatment associates each travelling wave with a quantum harmonic oscillator. The simple formalism of several coupled harmonic oscillators is useful for an analysis of the travelling-wave setup of interference experiment due to

Zou et al. (1991) up to suppression of the induced emission. The more complicated approach used originally for the analysis seems to be unsuitable for treating the phenomenon of induced emission. We try to formalize here a comparison between the two approaches.

When the induced emission occurs, it can be utilized. The phenomenon of induced emission makes the phase of an amplified field adopt the same phase as the incident locking field (Wang et al. 1991a, Wiseman and Mølmer 2000). The induced emission can also be used in parametric down-conversion to lock the phase of the idler and, from this, that of the signal (since the phase sum of the signal and idler is locked to the pump phase). If the field used to lock the idler of one down-converter (crystal NL2 in Fig. 2.11) is itself the idler output of another down-converter (crystal NL1 in Fig. 2.11), the two signal fields will also be locked in phase. Thus, they will have, in principle, perfect first-order coherence and so will interfere at a final beam splitter not included in Fig. 2.11. If there is no connection between the two down-converters, and hence no induced emission, the two signals will be incoherent and there will be no interference.

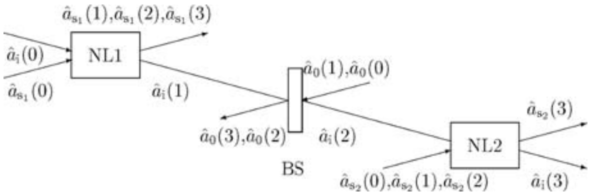

Fig. 2.11 Scheme of two parametric processes with aligned idler beams with the spatial Heisenberg picture made explicit

Zou et al. (1991) and Wang et al. (1991a) had a negligible probability of both crystals producing a down-converted photon pair and used a quantum-mechanical explanation based on indistinguishability of paths to explain the interference they observed in the experiment. To the contrary, the interference was lost when one could tell which crystal had emitted each signal photon. Using multimode analysis of the experiment, they derived that there could be no induced emission in their experiment. Nonetheless, they found that for perfect matching of idler modes, the signal fields from NL1 and NL2 show perfect interference.

The multimode approach to the analysis used by Wang et al. (1991a) yields an explanation involving a sufficient number of realistic parameters. Even though in the foregoing section the formalism yielded results similar enough to those of Wang et al. (1991a), here we try to come near their analysis. However, there exists a simple quantal description that may claim that it conforms to results of the multimode analysis. Such simple models have been published. Concerning this, we may refer to Řeháček and Peřina (1996), Wiseman and Mølmer (2000), and Peřinová et al. (2000) and provide what is a continuation of Peřinová et al. (2000).

(i) *Formalism of several modes*

We turn to the quantum analysis of the Zou–Wang–Mandel experiment. The experimental arrangement consists of two parametric down-conversion crystals with aligned idler beams, which are partially connected due to the presence of a beam splitter in between, and is illustrated in Fig. 2.11.

Restricting ourselves to the quasimonochromatic light beams (or quasimonochromatic components of these), we can describe the system by four modes, s_1 (the signal mode for crystal NL1), i (the idler modes, which are identified), s_2 (the signal mode for crystal NL2), and 0 (the escape mode for the beam splitter) (Peřinová et al. 2000). We consider the input annihilation operators $\hat{a}_{s_1}(0)$, $\hat{a}_0(1)$, $\hat{a}_{s_2}(2)$, $\hat{a}_i(0)$, the output annihilation operators $\hat{a}_{s_1}(1)$, $\hat{a}_0(2)$, $\hat{a}_{s_2}(3)$, $\hat{a}_i(3)$, and the intermediate annihilation operators $\hat{a}_i(1)$, $\hat{a}_i(2)$. Here s_1, s_2 stand for the signal mode of crystal 1 and that of crystal 2, respectively, i for the idler mode, and 0 for the "escape" mode of the beam splitter. To obtain four-mode unitary transformations between the stages 0, 1, 2, 3, we consider also the appendage input annihilation operators $\hat{a}_0(0)$, $\hat{a}_{s_2}(0)$, $\hat{a}_{s_2}(1)$ and the appendage output annihilation operators $\hat{a}_{s_1}(2)$, $\hat{a}_{s_1}(3)$, $\hat{a}_0(3)$. Of course, in the description of the dynamics below, we will have to be consistent with the identities $\hat{a}_0(0) = \hat{a}_0(1)$, $\hat{a}_{s_2}(0) = \hat{a}_{s_2}(1) = \hat{a}_{s_2}(2)$, $\hat{a}_{s_1}(1) = \hat{a}_{s_1}(2) = \hat{a}_{s_1}(3)$, $\hat{a}_0(2) = \hat{a}_0(3)$. Let us consider, in the Hilbert space of these four modes, an arbitrary operator $\hat{M}(j)$, $j = 0, 1, 2, 3$, a jth-stage operator. We will write the equation giving the transformation of $\hat{M}(j)$ from its value $\hat{M}(0)$ before the interaction to its value $\hat{M}(1)$ after the action of the first down converter, to its value $\hat{M}(2)$ after the action of the beam splitter, and to its value $\hat{M}(3)$ after the action of the second down converter.

The appropriate relations read

$$\hat{M}(j+1) = \hat{U}^{\dagger}_{j+1}(j)\hat{M}(j)\hat{U}_{j+1}(j), \quad j = 0, 1, 2. \tag{2.298}$$

Having prescribed equations of motion of operators, we have adopted a spatial modified Heisenberg picture of the dynamics. In the Heisenberg picture the input state does not change, while in the modified Heisenberg picture it changes like that of the free field. In our case of the "discrete" space (cf. $j = 0, 1, 2, 3$), the change of the state cannot be specified satisfactorily, but fortunately, we will not need it. In (2.298) $\hat{U}_{j+1}(j)$ for $j = 0, 2$ describe the down conversion in the undepleted pump approximation. The crystals are assumed to be identical or distinct and pumps are assumed to be identical so that

$$\hat{U}_{j+1}(j) = \exp\left\{ i\kappa_{\frac{j}{2}+1}\left[\hat{a}_{s_{\frac{j}{2}+1}}(j)\hat{a}_i(j) + \text{H.c.}\right]\right\}, \quad j = 0, 2, \tag{2.299}$$

where $\kappa_{\frac{j}{2}+1} = \frac{\chi}{c}v_{p_{\frac{j}{2}+1}}l_{\frac{j}{2}+1}$, χ is the quadratic susceptibility of the matter of which both the nonlinear crystals are made, $v_{p_{\frac{j}{2}+1}}$ are classical complex amplitudes of pumping beams, c is the speed of light, l_1 and l_2 are the lengths of the first crystal and the second crystal, respectively. In between the down converters the idler from crystal NL1 is put through a beam splitter BS and becomes the idler for crystal NL2. This process is described by

$$\hat{U}_2(1) = \exp\left\{ i[\bar{\omega}_0\hat{a}_0^{\dagger}(1)\hat{a}_0(1) + \bar{\omega}_i\hat{a}_i^{\dagger}(1)\hat{a}_i(1) + (\gamma^*\hat{a}_0^{\dagger}(1)\hat{a}_i(1) + \text{H.c.})]\right\}, \tag{2.300}$$

where

$$\left.\begin{array}{c}\bar{\omega}_0\\ \bar{\omega}_{\mathrm{i}}\end{array}\right\} = \arg\left(\frac{\mathtt{t}+\mathtt{t}'}{2}\right) \mp \mathrm{i}\frac{(\mathtt{t}-\mathtt{t}')}{2} f(\mathtt{t},\mathtt{t}'), \tag{2.301}$$

$$\gamma = -\mathrm{i}\mathtt{r} f(\mathtt{t},\mathtt{t}'), \tag{2.302}$$

with

$$f(\mathtt{t},\mathtt{t}') = \frac{\mathrm{Arccos}\left|\frac{\mathtt{t}+\mathtt{t}'}{2}\right|}{\sqrt{1-\left|\frac{\mathtt{t}+\mathtt{t}'}{2}\right|^2}} \frac{(\mathtt{t}+\mathtt{t}')^*}{|\mathtt{t}+\mathtt{t}'|}. \tag{2.303}$$

Here $\mathtt{t}$ and $\mathtt{r}$ are the transmission and reflection amplitude coefficients, respectively, for the idler mode and $\mathtt{t}'$ and $\mathtt{r}'$ are those for the "escape" mode. The modulus of the transmission amplitude coefficient $|\mathtt{t}|$ can vary between zero (where the second emission is spontaneous) and unity (where the second down conversion is stimulated in the highest degree).

Hence,

$$\begin{aligned}\hat{a}_{\mathrm{i}}(2) &= \hat{U}_2^\dagger(1)\hat{a}_{\mathrm{i}}(1)\hat{U}_2(1) = \mathtt{t}\hat{a}_{\mathrm{i}}(1) + \mathtt{r}'\hat{a}_0(1),\\ \hat{a}_0(2) &= \hat{U}_2^\dagger(1)\hat{a}_0(1)\hat{U}_2(1) = \mathtt{r}\hat{a}_{\mathrm{i}}(1) + \mathtt{t}'\hat{a}_0(1),\end{aligned} \tag{2.304}$$

and the unitarity of the transformation matrix implies that

$$|\mathtt{t}|^2 + |\mathtt{r}|^2 = 1, \quad |\mathtt{r}'|^2 + |\mathtt{t}'|^2 = 1, \quad \mathtt{t}\mathtt{r}'^* + \mathtt{r}\mathtt{t}'^* = 0. \tag{2.305}$$

It is advantageous to assume that $\mathtt{t}' = \mathtt{t}^*$, and from this $\mathtt{r} = -\mathtt{r}'^*$ (Řeháček and Peřina 1996), and that $\mathrm{Re}\,\mathtt{t} > 0$. Then

$$f(\mathtt{t},\mathtt{t}') = \frac{\mathrm{Arccos}(\mathrm{Re}\,\mathtt{t})}{\sqrt{1-(\mathrm{Re}\,\mathtt{t})^2}}, \tag{2.306}$$

$$\left.\begin{array}{c}\bar{\omega}_0\\ \bar{\omega}_{\mathrm{i}}\end{array}\right\} = \pm(\mathrm{Im}\,\mathtt{t}) f(\mathtt{t},\mathtt{t}'). \tag{2.307}$$

On applying the relation (2.298) at the input ($j = 0$) and at the stage 2 ($j = 2$), we obtain that

$$\begin{aligned}\hat{a}_{\mathrm{s}_1}(1) &= \hat{a}_{\mathrm{s}_1}(0)\cosh(\kappa_1) + \mathrm{i}\hat{a}_{\mathrm{i}}^\dagger(0)\sinh(\kappa_1),\\ \hat{a}_{\mathrm{i}}(1) &= \mathrm{i}\hat{a}_{\mathrm{s}_1}^\dagger(0)\sinh(\kappa_1) + \hat{a}_{\mathrm{i}}(0)\cosh(\kappa_1),\end{aligned} \tag{2.308}$$

and

$$\begin{aligned}\hat{a}_{\mathrm{s}_2}(3) &= \hat{a}_{\mathrm{s}_2}(2)\cosh(\kappa_2) + \mathrm{i}\hat{a}_{\mathrm{i}}^\dagger(2)\sinh(\kappa_2),\\ \hat{a}_{\mathrm{i}}(3) &= \mathrm{i}\hat{a}_{\mathrm{s}_2}(2)\sinh(\kappa_2) + \hat{a}_{\mathrm{i}}(2)\cosh(\kappa_2).\end{aligned} \tag{2.309}$$

Using Equations (2.304), we easily obtain the following relations:

$$\hat{a}_{s_1}(1) = \cosh(\kappa_1)\hat{a}_{s_1}(0) + \mathrm{i}\sinh(\kappa_1)\hat{a}_{\mathrm{i}}^{\dagger}(0), \tag{2.310}$$

$$\hat{a}_{s_2}(3) = \mathtt{t}^* \sinh(\kappa_1)\sinh(\kappa_2)\hat{a}_{s_1}(0) + \mathrm{i}\mathtt{t}^* \cosh(\kappa_1)\sinh(\kappa_2)\hat{a}_{\mathrm{i}}^{\dagger}(0) + \mathrm{i}\mathtt{t}^* \cosh(\kappa_2)\hat{a}_{s_2}(2) + \mathrm{i}\mathtt{r}'^* \sinh(\kappa_2)\hat{a}_0^{\dagger}(1). \tag{2.311}$$

The statistical properties of the system in the Heisenberg picture can be obtained when we take into account that the initial, in fact, "permanent" statistical operator of the system is given as $\hat{\rho} \equiv \hat{\rho}(0)$ and when we average

$$\langle \hat{M}(j) \rangle = \mathrm{Tr}\{\hat{\rho}\hat{M}(j)\}. \tag{2.312}$$

Here, concretely, the statistical operator is a tensor product of separate vacuum statistical operators

$$\hat{\rho} = \prod_{j=s_1,\mathrm{i},s_2,0} |0\rangle_j\, {}_j\langle 0|. \tag{2.313}$$

We may introduce also the abbreviations $\hat{M} \equiv \hat{M}(0)$ and we consider the Schrödinger picture, where the relation (2.298) is replaced by the evolution relations

$$\hat{\rho}(j+1) = \hat{U}_{j+1}\hat{\rho}(j)\hat{U}_{j+1}^{\dagger}, \quad j = 0, 1, 2, \tag{2.314}$$

with $\hat{U}_j \equiv \hat{U}_j(0)$ given in (2.299) and (2.300). The equivalence of both the pictures can be proved and the statistical properties can be expressed in similar terms as in (2.312)

$$\langle \hat{M} \rangle(j) = \mathrm{Tr}\{\hat{\rho}(j)\hat{M}\} = \langle \hat{M}(j) \rangle. \tag{2.315}$$

Since all the initial fields are in the vacuum states, it is easy to obtain the expectation values

$$\langle \hat{a}_{s_1}^{\dagger}(1)\hat{a}_{s_1}(1) \rangle = \sinh^2(\kappa_1), \tag{2.316}$$

$$\langle \hat{a}_{s_2}^{\dagger}(3)\hat{a}_{s_2}(3) \rangle = \sinh^2(\kappa_2)[1 + |\mathtt{t}|^2 \sinh^2(\kappa_1)], \tag{2.317}$$

$$\langle \hat{a}_{s_1}^{\dagger}(1)\hat{a}_{s_2}(3) \rangle = \mathtt{t}^* \sinh(\kappa_1)\cosh(\kappa_1)\sinh(\kappa_2). \tag{2.318}$$

We will show in the Heisenberg picture that the input–output relation is connected to the SU(2,2) group. In fact,

$$\begin{pmatrix} \hat{a}_{s_1}(3) \\ \hat{a}_{\mathrm{i}}^{\dagger}(3) \\ \hat{a}_{s_2}(3) \\ \hat{a}_0^{\dagger}(3) \end{pmatrix} = \begin{pmatrix} m_{s_1s_1} & m_{s_1\mathrm{i}} & m_{s_1s_2} & m_{s_10} \\ m_{\mathrm{i}s_1} & m_{\mathrm{ii}} & m_{\mathrm{i}s_2} & m_{\mathrm{i}0} \\ m_{s_2s_1} & m_{s_2\mathrm{i}} & m_{s_2s_2} & m_{s_20} \\ m_{0s_1} & m_{0\mathrm{i}} & m_{0s_2} & m_{00} \end{pmatrix} \begin{pmatrix} \hat{a}_{s_1}(0) \\ \hat{a}_{\mathrm{i}}^{\dagger}(0) \\ \hat{a}_{s_2}(0) \\ \hat{a}_0^{\dagger}(0) \end{pmatrix}, \tag{2.319}$$

where

$$\begin{aligned} m_{s_1s_1} &= \cosh(\kappa_1), \quad m_{s_1i} = \mathrm{i}\sinh(\kappa_1), \quad m_{s_1s_2} = m_{s_10} = 0; \\ m_{is_1} &= -\mathrm{i}\mathtt{t}^*\sinh(\kappa_1)\cosh(\kappa_2), \quad m_{ii} = \mathtt{t}^*\cosh(\kappa_1)\cosh(\kappa_2), \\ m_{is_2} &= -\mathrm{i}\sinh(\kappa_2), \quad m_{i0} = \mathtt{r}'^*\cosh(\kappa_2); \\ m_{s_2s_1} &= \mathtt{t}^*\sinh(\kappa_1)\sinh(\kappa_2), \quad m_{s_2i} = \mathrm{i}\mathtt{t}^*\cosh(\kappa_1)\sinh(\kappa_2), \\ m_{s_2s_2} &= \cosh(\kappa_2), \quad m_{s_20} = \mathrm{i}\mathtt{r}'^*\sinh(\kappa_2); \\ m_{0s_1} &= \mathrm{i}\mathtt{r}'\sinh(\kappa_1), \quad m_{0i} = -\mathtt{r}'\cosh(\kappa_1), \quad m_{0s_2} = 0, \quad m_{00} = \mathtt{t}. \end{aligned} \tag{2.320}$$

From the form of the relation (2.319) it is evident that the operator

$$\hat{N}(j) = \hat{n}_{s_1}(j) + \hat{n}_{s_2}(j) - \hat{n}_i(j) - \hat{n}_0(j), \quad j = 0, 3, \tag{2.321}$$

is independent of j. This conservation law suggests the SU(2,2) group. The coefficients of the transformation (2.319) verify the pseudoorthogonality relations

$$m_{js_1}m^*_{ks_1} + m_{js_2}m^*_{ks_2} - m_{ji}m^*_{ki} - m_{j0}m^*_{k0} = g_{jk}, \quad j,k = s_1, i, s_2, 0, \tag{2.322}$$

where

$$g_{jk} = g_{jj}\delta_{jk}, \quad g_{s_1s_1} = g_{s_2s_2} = 1, \quad g_{ii} = g_{00} = -1. \tag{2.323}$$

We observe that the antinormally ordered moments have the expression

$$\langle \hat{a}_j(3)\hat{a}_j^\dagger(3)\rangle = |m_{js_1}|^2 + |m_{js_2}|^2, \quad j = s_1, s_2, \tag{2.324}$$

and the normally ordered moments

$$\langle \hat{a}_j^\dagger(3)\hat{a}_j(3)\rangle = |m_{js_1}|^2 + |m_{js_2}|^2, \quad j = i, 0. \tag{2.325}$$

More generally,

$$\begin{aligned} \langle \hat{a}_{s_1}(3)\hat{a}^\dagger_{s_2}(3)\rangle &= m_{s_1s_1}m^*_{s_2s_1} + m_{s_1s_2}m^*_{s_2s_2}, \\ \langle \hat{a}_{s_2}(3)\hat{a}^\dagger_{s_1}(3)\rangle &= \langle \hat{a}_{s_1}(3)\hat{a}^\dagger_{s_2}(3)\rangle^*, \end{aligned} \tag{2.326}$$

$$\begin{aligned} \langle \hat{a}^\dagger_i(3)\hat{a}_0(3)\rangle &= m_{is_1}m^*_{0s_1} + m_{is_2}m^*_{0s_2}, \\ \langle \hat{a}^\dagger_0(3)\hat{a}_i(3)\rangle &= \langle \hat{a}^\dagger_i(3)\hat{a}_0(3)\rangle^*. \end{aligned} \tag{2.327}$$

Further nonvanishing moments are

$$\langle \hat{a}_j(3)\hat{a}_k(3)\rangle = m_{js_1}m^*_{ks_1} + m_{js_2}m^*_{ks_2}, \quad j = s_1, s_2, \quad k = i, 0, \tag{2.328}$$

and

$$\langle \hat{a}^\dagger_j(3)\hat{a}^\dagger_k(3)\rangle = \langle \hat{a}_j(3)\hat{a}_k(3)\rangle^*. \tag{2.329}$$

The rest second-order moments vanish:

$$\langle \hat{a}_j(3)\hat{a}_k^\dagger(3)\rangle = \langle \hat{a}_j^\dagger(3)\hat{a}_k(3)\rangle = 0, \quad j = \mathrm{s}_1, \mathrm{s}_2, \quad k = \mathrm{i}, 0, \tag{2.330}$$

$$\begin{aligned}\langle \hat{a}_j(3)\hat{a}_k(3)\rangle = \langle \hat{a}_j^\dagger(3)\hat{a}_k^\dagger(3)\rangle = 0, \quad j = \mathrm{s}_1, \mathrm{s}_2, \quad k = \mathrm{s}_1, \mathrm{s}_2,\\ \text{and} \quad j = \mathrm{i}, 0, \quad k = \mathrm{i}, 0.\end{aligned} \tag{2.331}$$

Quantum statistics of radiation in the process under study is that of a four-mode Gaussian state, starting with the quantum characteristic function:

$$\begin{aligned}&C_{\mathcal{S}}(\beta_{\mathrm{s}_1}, \beta_{\mathrm{s}_2}, \beta_{\mathrm{i}}, \beta_0, 3)\\ &\quad = \mathrm{Tr}\{\hat{\rho}(3)\hat{D}_{\mathrm{s}_1}(\beta_{\mathrm{s}_1}, 0)\hat{D}_{\mathrm{s}_2}(\beta_{\mathrm{s}_2}, 0)\hat{D}_{\mathrm{i}}(\beta_{\mathrm{i}}, 0)\hat{D}_0(\beta_0, 0)\}\\ &\quad = \mathrm{Tr}\{\hat{\rho}\hat{D}_{\mathrm{s}_1}(\beta_{\mathrm{s}_1}, 3)\hat{D}_{\mathrm{s}_2}(\beta_{\mathrm{s}_2}, 3)\hat{D}_{\mathrm{i}}(\beta_{\mathrm{i}}, 3)\hat{D}_0(\beta_0, 3)\},\end{aligned} \tag{2.332}$$

where the displacement operators are given by

$$\hat{D}_j(\beta_j, k) = \exp[\beta_j\hat{a}_j^\dagger(k) - \beta_j^*\hat{a}_j(k)], \quad j = \mathrm{s}_1, \mathrm{s}_2, \mathrm{i}, 0, \quad k = 0, 3. \tag{2.333}$$

By the remark above, $\hat{D}_j(\beta_j) \equiv \hat{D}_j(\beta_j, 0)$. On substituting into the relation (2.332) according to (2.319), we obtain that

$$\begin{aligned}&C_{\mathcal{S}}(\beta_{\mathrm{s}_1}, \beta_{\mathrm{s}_2}, \beta_{\mathrm{i}}, \beta_0, 3)\\ &= \mathrm{Tr}\{\hat{\rho}(0)\hat{D}_{\mathrm{s}_1}(\beta_{\mathrm{s}_1}(3))\hat{D}_{\mathrm{s}_2}(\beta_{\mathrm{s}_2}(3))\hat{D}_{\mathrm{i}}(\beta_{\mathrm{i}}(3))\hat{D}_0(\beta_0(3))\},\end{aligned} \tag{2.334}$$

where

$$\begin{aligned}-\beta_{\mathrm{s}_1}^*(3) &= -\beta_{\mathrm{s}_1}^* m_{\mathrm{s}_1\mathrm{s}_1} - \beta_{\mathrm{s}_2}^* m_{\mathrm{s}_2\mathrm{s}_1} + \beta_{\mathrm{i}} m_{\mathrm{i}\mathrm{s}_1} + \beta_0 m_{0\mathrm{s}_1},\\ -\beta_{\mathrm{s}_2}^*(3) &= -\beta_{\mathrm{s}_1}^* m_{\mathrm{s}_1\mathrm{s}_2} - \beta_{\mathrm{s}_2}^* m_{\mathrm{s}_2\mathrm{s}_2} + \beta_{\mathrm{i}} m_{\mathrm{i}\mathrm{s}_2} + \beta_0 m_{0\mathrm{s}_2},\\ \beta_{\mathrm{i}}(3) &= -\beta_{\mathrm{s}_1}^* m_{\mathrm{s}_1\mathrm{i}} - \beta_{\mathrm{s}_2}^* m_{\mathrm{s}_2\mathrm{i}} + \beta_{\mathrm{i}} m_{\mathrm{ii}} + \beta_0 m_{0\mathrm{i}},\\ \beta_0(3) &= -\beta_{\mathrm{s}_1}^* m_{\mathrm{s}_1 0} - \beta_{\mathrm{s}_2}^* m_{\mathrm{s}_2 0} + \beta_{\mathrm{i}} m_{\mathrm{i}0} + \beta_0 m_{00}.\end{aligned} \tag{2.335}$$

From the known quantum characteristic function for the initial vacuum state

$$C_{\mathcal{S}}(\beta_{\mathrm{s}_1}, \beta_{\mathrm{s}_2}, \beta_{\mathrm{i}}, \beta_0, 0) = \exp\left\{-\frac{1}{2}\sum_{j=\mathrm{s}_1,\mathrm{s}_2,\mathrm{i},0} |\beta_j|^2\right\}, \tag{2.336}$$

we derive that

$$\begin{aligned}C_{\mathcal{S}}(\beta_{\mathrm{s}_1}, \beta_{\mathrm{s}_2}, \beta_{\mathrm{i}}, \beta_0, 3) = \exp\Bigg\{ &- \sum_{j=\mathrm{s}_1,\mathrm{s}_2,\mathrm{i},0} |\beta_j|^2 B_{j\mathcal{S}}\\ + \Bigg[-\beta_{\mathrm{s}_1}\beta_{\mathrm{s}_2}^* B_{\mathrm{s}_1\mathrm{s}_2}^* - \beta_{\mathrm{i}}\beta_0^* B_{\mathrm{i}0}^* &+ \sum_{j=\mathrm{s}_1,\mathrm{s}_2}\sum_{k=\mathrm{i},0} \beta_j\beta_k C_{jk}^* + \mathrm{c.c.}\Bigg]\Bigg\}.\end{aligned} \tag{2.337}$$

Here the coefficients $B_{j\mathcal{S}}$, B_{jk}, C_{jk} can be expressed in the form

$$
\begin{aligned}
B_{j\mathcal{S}} &= \langle \hat{a}_j(3)\hat{a}_j^\dagger(3)\rangle - \frac{1}{2}, \quad j = \mathrm{s}_1, \mathrm{s}_2,\\
B_{j\mathcal{S}} &= \langle \hat{a}_j^\dagger(3)\hat{a}_j(3)\rangle + \frac{1}{2}, \quad j = \mathrm{i}, 0,\\
B_{\mathrm{s}_1\mathrm{s}_2} &= \langle \hat{a}_{\mathrm{s}_1}(3)\hat{a}_{\mathrm{s}_2}^\dagger(3)\rangle,\\
B_{\mathrm{i}0} &= \langle \hat{a}_{\mathrm{i}}(3)\hat{a}_0^\dagger(3)\rangle,\\
C_{jk} &= \langle \hat{a}_j(3)\hat{a}_k(3)\rangle, \quad j = \mathrm{s}_1, \mathrm{s}_2, \quad k = \mathrm{i}, 0.
\end{aligned} \tag{2.338}
$$

Taking into account that $\langle \hat{a}_j(3)\rangle = 0$, $j = \mathrm{s}_1, \mathrm{s}_2, \mathrm{i}, 0$, we see that we are consistent with the more general notation

$$
\begin{aligned}
B_{j\mathcal{A}} &= \langle \Delta\hat{a}_j(3)\Delta\hat{a}_j^\dagger(3)\rangle, \quad j = \mathrm{s}_1, \mathrm{s}_2,\\
B_{j\mathcal{N}} &= \langle \Delta\hat{a}_j^\dagger(3)\Delta\hat{a}_j(3)\rangle, \quad j = \mathrm{i}, 0,
\end{aligned} \tag{2.339}
$$

where $\Delta\hat{a} = \hat{a} - \langle\hat{a}\rangle$, and with the coefficients $B_{\mathrm{s}_1\mathrm{s}_2}$, $B_{\mathrm{i}0}$, C_{jk}, $\quad j = \mathrm{s}_1, \mathrm{s}_2, \quad k = \mathrm{i}, 0$, after similar replacement.

We confine ourselves to the study of the signal beams in what follows, which are described by the reduced statistical operator

$$
\hat{\rho}_{\mathrm{signal}}(3) = \mathrm{Tr}_{\mathrm{i}}\mathrm{Tr}_0\{\hat{\rho}(3)\}, \tag{2.340}
$$

where Tr_{i} and Tr_0 are partial traces over the idler and escape modes, respectively. Quantum characteristic function in the state described by the statistical operator (2.340) can easily be obtained:

$$
C_{\mathcal{S}}(\beta_{\mathrm{s}_1}, \beta_{\mathrm{s}_2}) \equiv C_{\mathcal{S}}(\beta_{\mathrm{s}_1}, \beta_{\mathrm{s}_2}, 3) = C_{\mathcal{S}}(\beta_{\mathrm{s}_1}, \beta_{\mathrm{s}_2}, 0, 0, 3). \tag{2.341}
$$

In Peřinová et al. (2003), the same function has been introduced as

$$
\begin{aligned}
&C_{\mathcal{S}}(\beta_{\mathrm{s}_1}, \beta_{\mathrm{s}_2}; 1, 3) = \mathrm{Tr}\left\{\hat{\rho}(0)\hat{D}_{\mathrm{s}_1}(\beta_{\mathrm{s}_1}, 1)\hat{D}_{\mathrm{s}_2}(\beta_{\mathrm{s}_2}, 3)\right\}\\
&= \exp\left[-\sum_{j=\mathrm{s}_1,\mathrm{s}_2} |\beta_j|^2 B_{j\mathcal{S}} + \left(-\beta_{\mathrm{s}_1}\beta_{\mathrm{s}_2}^* B_{\mathrm{s}_1\mathrm{s}_2}^* + \mathrm{c.c.}\right)\right].
\end{aligned} \tag{2.342}
$$

In the following we simplify the notation s_1, s_2 for the signal modes to 1, 2, respectively. From the characteristic function

$$
C_s(\beta_1, \beta_2) = \exp\left[\frac{s}{2}(|\beta_1|^2 + |\beta_2|^2)\right] C_{\mathcal{S}}(\beta_1, \beta_2), \tag{2.343}
$$

where $s = 1, 0, -1$ in the subscript and also $s = \mathcal{N}, \mathcal{S}, \mathcal{A}$ denote the normal, symmetrical, and antinormal orderings of field operators, we can establish the Φ_s

quasidistribution related to the respective ordering of field operators

$$\Phi_s(\alpha_1,\alpha_2) = \frac{1}{\pi^4} \times \int C_s(\beta_1,\beta_2) \exp\left(\alpha_1\beta_1^* - \alpha_1^*\beta_1 + \alpha_2\beta_2^* - \alpha_2^*\beta_2\right) \, \mathrm{d}^2\beta_1 \, \mathrm{d}^2\beta_2. \tag{2.344}$$

After integrating, we obtain

$$\Phi_s(\alpha_1,\alpha_2) = \frac{1}{\pi^2 K_{12s}} \times \exp\left\{\frac{1}{K_{12s}}[-B_{2s}|\alpha_1|^2 - B_{1s}|\alpha_2|^2 + (B_{12}^*\alpha_1\alpha_2^* + \text{c.c.})]\right\}, \tag{2.345}$$

where

$$\begin{aligned} B_{1\mathcal{A}} &= \cosh^2(\kappa_1),\ B_{1\mathcal{S}} = B_{1\mathcal{A}} - \frac{1}{2},\ B_{1\mathcal{N}} = B_{1\mathcal{A}} - 1, \\ B_{2\mathcal{A}} &= \cosh^2(\kappa_2) + |\mathtt{t}| \sinh^2(\kappa_2)\sinh^2(\kappa_1), \\ B_{2\mathcal{S}} &= B_{2\mathcal{A}} - \frac{1}{2},\ B_{2\mathcal{N}} = B_{2\mathcal{A}} - 1, \\ B_{12}^* &= \mathtt{t}^* \sinh(\kappa_1)\cosh(\kappa_1)\sinh(\kappa_2), \end{aligned} \tag{2.346}$$

and

$$K_{12s} = B_{1s}B_{2s} - |B_{12}|^2. \tag{2.347}$$

Especially, for $s = -1$ it holds that

$$\Phi_{\mathcal{A}}(\alpha_1,\alpha_2) = \frac{1}{\pi^2}\langle\alpha_1,\alpha_2|\hat{\rho}_{\text{signal}}(3)|\alpha_1,\alpha_2\rangle, \tag{2.348}$$

where $|\alpha_1,\alpha_2\rangle$ is the two-mode coherent state, which yields the expansion

$$\Phi_{\mathcal{A}}(\alpha_1,\alpha_2) = \frac{1}{\pi^2}\exp(-|\alpha_1|^2 - |\alpha_2|^2) \sum_{q=-\infty}^{\infty} \sum_{m_1=\max(0,-q)}^{\infty} \times \sum_{n_2=\max(0,-q)}^{\infty} \rho(m_1+q, m_1, n_2, n_2+q) \frac{\alpha_1^{*(m_1+q)}\alpha_1^{m_1}\alpha_2^{*n_2}\alpha_2^{n_2+q}}{\sqrt{(m_1+q)!m_1!n_2!(n_2+q)!}} \tag{2.349}$$

for any $\Phi_{\mathcal{A}}$ quasidistribution that does not depend on $\alpha_1\alpha_2$, $\alpha_1^*\alpha_2^*$. Here

$$\rho(n_1,m_1,n_2,m_2) = \langle n_1,n_2|\hat{\rho}_{\text{signal}}(3)|m_1,m_2\rangle \tag{2.350}$$

are the usual matrix elements. Equating the expansion coefficients for (2.345) with $s = -1$ and those of (2.349), we arrive at the expression

$$\rho(m_1+q, m_1, n_2, n_2+q) = \sum_{p=\max(0,-q)}^{\min(m_1,n_2)} \frac{\sqrt{(m_1+q)!m_1!n_2!(n_2+q)!}}{(m_1-p)!(n_2-p)!p!(p+q)!} \times \frac{(K_{12\mathcal{A}}-B_{2\mathcal{A}})^{m_1-p}(K_{12\mathcal{A}}-B_{1\mathcal{A}})^{n_2-p}B_{12}^{*p}B_{12}^{p+q}}{K_{12\mathcal{A}}^{m_1+n_2+q+1}}, \tag{2.351}$$

while obviously

$$\rho(m_1+q_1, m_1, n_2, n_2-q_2) = 0 \ \text{ for } \ q_1 \neq -q_2. \tag{2.352}$$

(ii) *Photon-number statistics*

Numbers of photons in signal modes complete the picture of the quantum correlation between these beams. The joint photon-number distribution $p(n_1, n_2)$ can be expressed in the concise form (Řeháček and Peřina 1996). A substitution into (2.351) leads to slightly more complicated expression:

$$p(n_1, n_2) = \rho(n_1, n_1, n_2, n_2). \tag{2.353}$$

This distribution can be seen in Fig. 2.12 for $|\mathtt{t}| = 1$, $B_{1\mathcal{N}} = B_{2\mathcal{N}} = 3$, $|B_{12}| = 3$. It differs from the product of pertinent marginal distributions by larger "diagonal" probabilities. To the contrary for $|\mathtt{t}|$ small, the joint photon-number distribution is approximately the product of its marginal photon-number distributions.

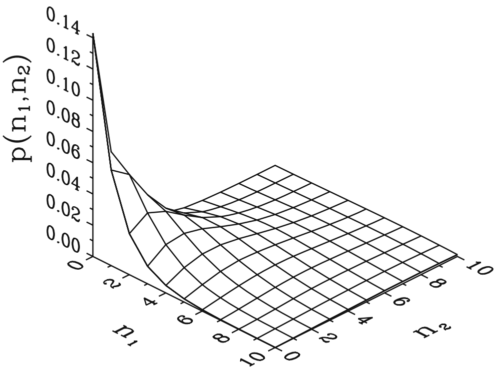

Fig. 2.12 Joint photon-number distribution for $|\mathtt{t}| = 1$; $B_{1\mathcal{N}} = B_{2\mathcal{N}} = 3$, $n_j \in [0, 10]$, $j = 1, 2$

As for the experimental arrangement under study, it depends on l_1, $\mathtt{t}$, l_2, whereas the numerical demonstration is restricted to the case when the length of the first crystal is kept fixed. Consequently, the mean photon number $B_{1\mathcal{N}}$ in the first signal mode is constant and this convenient behaviour is, for the sake of illustrations, extended also to the second one as a relationship between $|\mathtt{t}|$ and κ_2,

$$\sinh^2(\kappa_2) = \frac{B_{2\mathcal{N}}}{1 + |\mathtt{t}|^2 B_{1\mathcal{N}}}. \tag{2.354}$$

The Glauber degree of coherence (Peřina 1991) $\gamma_{12}^{(2)}$ is the complex-valued quantum correlation measure related to the normal ordering:

$$\gamma_{12}^{(2)} = \frac{B_{12}^*}{\sqrt{B_{1\mathcal{N}} B_{2\mathcal{N}}}}. \tag{2.355}$$

In the numerical illustration of the quantum correlation measures, we assume $\kappa_1 = \kappa_2 = \kappa$ and find κ_1 by the inversion of the formula

$$B_{1\mathcal{N}} = \sinh^2(\kappa_1) = \langle \hat{a}_{\mathrm{s}_1}^\dagger(1)\hat{a}_{\mathrm{s}_1}(1)\rangle = \overline{n}_1(1). \tag{2.356}$$

The limit case $|\mathtt{t}| = 1$ is interesting, $|\gamma_{12}^{(2)}| = R_{\mathcal{N}} = 1$, see Fig. 2.13. This maximum correlation does not correspond to a weaker correlation between the signal photon numbers. In the multimode analysis (Wang et al. 1991a) of the experiment (Zou et al. 1991), the visibility of the interference between the signal fields has been expressed, the interference manifests itself as oscillations in the counting rate I_{s} (see (2.412) below) when propagation times of the idler beam from NL1 to NL2, of the first signal beam from NL1 to $\mathrm{D_s}$, and of the second signal beam from NL2 to $\mathrm{D_s}$ are incremented by $\delta\tau_0$, $\delta\tau_1$, $\delta\tau_2$, respectively. Deriving a simplified visibility $\mathcal{V}_{\mathrm{simple}}$ for the formalism of several modes provides

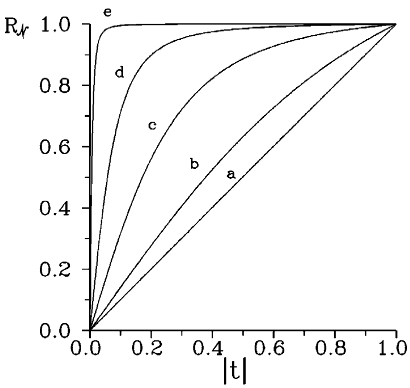

Fig. 2.13 Quantum correlation measure $R_{\mathcal{N}}$ versus the modulus of the transmission amplitude coefficient $|\mathtt{t}| \in [0, 1]$; it is assumed that $\overline{n}_1(1) = 10^{-2}, 1, 10, 100, 10^4$ (the curves a, b, c, d, e, respectively)

$$\mathcal{V}_{\mathrm{simple}} = \frac{2R_{\mathcal{N}}\sqrt{B_{1\mathcal{N}} B_{2\mathcal{N}}}}{B_{1\mathcal{N}} + B_{2\mathcal{N}}}. \tag{2.357}$$

As $\sqrt{B_{1\mathcal{N}} B_{2\mathcal{N}}} \le \frac{1}{2}(B_{1\mathcal{N}} + B_{2\mathcal{N}})$, the visibility cannot exceed the correlation measure $R_{\mathcal{N}}$ and the equality is attained for $B_{1\mathcal{N}} = B_{2\mathcal{N}}$. The maximum obtainable visibility between two fields in an experiment is given by the correlation measure $R_{\mathcal{N}}$, cf. Wiseman and Mølmer (2000).

On substituting (2.338) with (2.316), (2.317), and (2.318) into (2.355), we find that

$$R_{\mathcal{N}} = \frac{|\mathtt{t}|\cosh(\kappa_1)}{\sqrt{1+|\mathtt{t}|^2\sinh^2(\kappa_1)}}. \tag{2.358}$$

Noting that the idler beam, before it enters the beam splitter, has the same statistics as the output signal 1, we can rewrite (2.358) in terms of the mean photon number $\bar{n}_1(1) = \sinh^2(\kappa_1)$ as

$$R_{\mathcal{N}} = |\mathtt{t}|\sqrt{\frac{1+\overline{n}_1(1)}{1+|\mathtt{t}|^2\overline{n}_1(1)}}. \tag{2.359}$$

Wiseman and Mølmer (2000) considered the relevant limits in this form. The single-photon regime which is the regime of experiment and theory in Zou et al. (1991) and Wang et al. (1991a,b) occurs for $\overline{n}_1(1) \ll 1$. Up to the first order in the rescaled lengths κ_1, κ_2 of the crystals, we simply obtain $R_{\mathcal{N}} = |\mathtt{t}|$. The probability of a down conversion at crystal NL1 over interaction time is less than or equal to $\overline{n}_1(1)$, or it is small. The probability to have down conversions at both crystals over interaction time is less than or equal to $\left\langle \hat{a}_{s_1}^{\dagger}(1)\hat{a}_{s_1}(1)\hat{a}_{s_2}^{\dagger}(3)\hat{a}_{s_2}(3)\right\rangle = B_{1\mathcal{N}}B_{2\mathcal{N}} + |B_{12}|^2$, or it is negligible. The single-photon regime applies in the multimode analysis of Wang et al. (1991a), because each of the narrow-bandwidth signal modes $(\mathbf{k}_{s_1}, \omega_{s_1})$, $(\mathbf{k}_{s_2}, \omega_{s_2})$, with directions characterized by $\mathbf{k}_{s_j}$ and with the frequencies ω_{s_j}, $j = 1, 2$, that form broad-band signal fields, receives only a small part of the pumping photons over interaction time. The same applies to idler modes and idler fields. The signal fields s_1 and s_2 from the two down converters are allowed to come together and interfere at the detector D_s.

In the spatial interaction picture, the state of the field produced by the crystals is given by

$$|\psi(3)\rangle \equiv \hat{U}_3(0)\hat{U}_2(0)\hat{U}_1(0)|0\rangle_{s_1,i,s_2,0}, \tag{2.360}$$

where $\hat{U}_{j+1}(0)$ for $j = 1, 2, 3$ are given by relations (2.299) and (2.300), where the annihilation operators $\hat{a}_{s_{\frac{j}{2}+1}}(j) \to \hat{a}_{s_{\frac{j}{2}+1}}(0)$, $\hat{a}_i(j) \to \hat{a}_i(0)$, $\hat{a}_0(j) \to \hat{a}_0(0)$. We will drop the argument (0) at the annihilation operators in what follows. Here $|0\rangle_{s_1,i,s_2,0} \equiv |0\rangle_{s_1}|0\rangle_i|0\rangle_{s_2}|0\rangle_0$ and, in general,

$$|n_{s_1}, n_i, n_{s_2}, n_0\rangle \equiv |n_{s_1}\rangle_{s_1}|n_i\rangle_i|n_{s_2}\rangle_{s_2}|n_0\rangle_0. \tag{2.361}$$

In the Schrödinger picture, the operators do not change and in the interaction picture, which is the modified Schrödinger picture, they change like the Heisenberg picture free-field operators. An analogue of relation (2.298) for a discrete space is not used in quantum optics (the free-field propagation is absorbed in the interaction). Fortunately, we will use just the interaction-picture annihilation operators. Expanding the operators $\hat{U}_{j+1}(0)$ according to $\kappa_{\frac{j}{2}+1}$, $j=0,2$, we obtain that

$$|\psi(3)\rangle \simeq |0\rangle_{\mathrm{s}_1,\mathrm{i},\mathrm{s}_2,0} + \mathrm{i}\kappa_2|0,1,1,0\rangle + \mathrm{i}\kappa_1(\mathtt{t}|1,1,0,0\rangle + \mathtt{r}|1,0,0,1\rangle), \quad (2.362)$$

when $\kappa_{\frac{j}{2}+1}$ are small. For $|\mathtt{t}|=1$ we have a single-photon state in the idler mode and in the collection of the signal modes. In general, one can infer a conversion at crystal NL1 after a photocount in the escape mode. We introduce the probability of the detection

$$p_{1,0,0,1}(3) = |\kappa_1|^2|\mathtt{r}|^2. \quad (2.363)$$

Let us assume that one infers a conversion at crystal NL2 after no photocounts in the escape mode. We introduce the probability of a correct inference of the conversion at crystal NL2

$$p_{0,1,1,0}(3) = |\kappa_2|^2 \quad (2.364)$$

and that of such a wrong inference

$$p_{1,1,0,0}(3) = |\kappa_1|^2|\mathtt{t}|^2. \quad (2.365)$$

On a photocount in the escape mode it is certain that the conversion has happened at NL1. On no photocounts in this mode, the posterior probabilities are

$$\mathrm{Prob}(\underline{n}_{\mathrm{s}_1}=0 \cap \underline{n}_{\mathrm{s}_2}=1|\underline{n}_{\mathrm{i}}=1 \cap \underline{n}_0=0) = \frac{|\kappa_2|^2}{|\kappa_1|^2|\mathtt{t}|^2+|\kappa_2|^2}, \quad (2.366)$$

$$\mathrm{Prob}(\underline{n}_{\mathrm{s}_1}=1 \cap \underline{n}_{\mathrm{s}_2}=0|\underline{n}_{\mathrm{i}}=1 \cap \underline{n}_0=0) = \frac{|\kappa_1|^2|\mathtt{t}|^2}{|\kappa_1|^2|\mathtt{t}|^2+|\kappa_2|^2}. \quad (2.367)$$

Here the underlining means a random variable. The counting rate registered by D_{s} exhibits perfect interference when the idler fields are perfectly aligned. This may be regarded as reflecting the intrinsic impossibility of knowing whether the detected photon comes from NL1 or NL2 (Wang et al. 1991a). The multiphoton conditional states can be found in Luis and Peřina (1996a).

Let us consider the annihilation operators $\hat{a}_{\mathrm{s}_{\frac{j}{2}+1}}$, $j=0,2$. The action of the two operators on the state $|\psi(3)\rangle$ is asymptotically for small $\kappa_{\frac{j}{2}+1}$ expressed as

$$\hat{a}_{\mathrm{s}_1}|\psi(3)\rangle \simeq \mathrm{i}\kappa_1(\mathtt{t}|0,1,0,0\rangle + \mathtt{r}|0,0,0,1\rangle), \quad (2.368)$$

$$\hat{a}_{\mathrm{s}_2}|\psi(3)\rangle \simeq \mathrm{i}\kappa_2|0,1,0,0\rangle. \quad (2.369)$$

Hence

$$\langle \hat{a}^\dagger_{\mathrm{s}_1}\hat{a}_{\mathrm{s}_2}\rangle \simeq \kappa_1\kappa_2\mathtt{t}^*, \quad (2.370)$$

$$\langle \hat{a}^\dagger_{\mathrm{s}_1}\hat{a}_{\mathrm{s}_1}\rangle \simeq \kappa_1^2, \quad \langle \hat{a}^\dagger_{\mathrm{s}_2}\hat{a}_{\mathrm{s}_2}\rangle \simeq \kappa_2^2. \quad (2.371)$$

It can be verified that these relations hold up to the second order in κ_1, κ_2. We can find that $\gamma_{12}^{(2)} = \mathtt{t}^*$ in the limit of small $\kappa_{\frac{j}{2}+1}$, $j = 0, 2$. Obviously, the equation holds up to the zeroth order, but it can be verified that it is valid up to the first order in κ_1, κ_2.

The opposite regime is that where $\overline{n}_1(1) \gg 1$. Here there are many photons on average in all of the down-converted beams. That is, the phase of the stage-1 or stage-2 idler mode should lock the phase of the output signal-2 mode for any nonzero transmission amplitude coefficient $\mathtt{t}$.

(iii) *Multimode formalism*

In Wang et al. (1991a) the pump beams at each crystal are represented by complex analytic signals $V_1(t)$ and $V_2(t)$ such that $|V_j(t)|^2$ is in units of photons per second ($j = 1, 2$). The multimode formalism enables one to respect that the two crystals are centred at $\mathbf{0}_1$ and $\mathbf{0}_2$. The multimode formalism views the electric fields as temporal-interaction-picture operators

$$\hat{E}_m^{(+)}(\mathbf{r}, t) = \sqrt{\frac{\delta\omega}{2\pi}} \sum_{\omega_m} \hat{a}_m(\omega_m) \exp[\mathrm{i}(\mathbf{k}_m \cdot \mathbf{r} - \omega_m t)], \quad m = \mathrm{s}_1, \mathrm{i}, \mathrm{s}_2, 0, \tag{2.372}$$

where $\delta\omega$ is the mode spacing and $\hat{a}_m(\omega_m)$ is the photon annihilation operator for narrow-bandwidth signal ($m = \mathrm{s}_j$, $j = 1, 2$), idler ($m = \mathrm{i}$), and "escape" ($m = 0$) modes $(\mathbf{k}_m, \omega_m)$ at the frequency ω_m. The Hilbert space for the multimode analysis is a tensor product of those Hilbert spaces of separate modes, whose vacuum states may be designated as $|0\rangle_m(\omega_m)$, $m = \mathrm{s}_1, \mathrm{i}, \mathrm{s}_2, 0$. We consider photon-flux amplitude operators $\hat{E}_{\mathrm{s}_j}^{(+)}(t)$, $j = 1, 2$, at the appropriate detector, $\hat{E}_{\mathrm{s}_j}^{(+)}(t) \equiv \hat{E}_{\mathrm{s}_j}(\mathbf{r}_\mathrm{s}, t) = \hat{E}_{\mathrm{s}_j}^{(+)}(\mathbf{0}_j, t - \tau_j)$, where τ_j, $j = 1, 2$, is the propagation time of s_j from NLj to D_s.

In order to compare the sophisticated multimode formalism with the simple formalism of several modes, we must present another appropriate description of the dynamics of the same down-conversion experiment. We adopt the temporal interaction picture and combine it with the spatial interaction picture. In this case, the state produced by the crystal is given by

$$|\psi(3, t)\rangle \equiv \hat{U}_3(0, t)\hat{U}_2(0, t)\hat{U}_1(0, t)|0\rangle_{\mathrm{s}_1,\mathrm{i},\mathrm{s}_2,0}. \tag{2.373}$$

Here $|0\rangle_{\mathrm{s}_1,\mathrm{i},\mathrm{s}_2,0}$ is the vacuum state of all the narrow-bandwidth signal, idler, and "escape" modes $(\mathbf{k}_{\mathrm{s}_j}, \omega_{\mathrm{s}_j})$, $j = 1, 2$, $(\mathbf{k}_\mathrm{i}, \omega_\mathrm{i})$, $(\mathbf{k}_0, \omega_0)$, respectively. $\hat{U}_{j+1}(0, t)$, $j = 0, 2$, are unitary operators:

$$\hat{U}_{j+1}(0, t) = \lim_{t_0 \to -\infty} \hat{U}_{j+1}(0, t, t_0), \tag{2.374}$$

where $\hat{U}_{j+1}(0, t, t_0)$ are unitary operators that obey the initial condition

$$\hat{U}_{j+1}(0, t, t_0)\Big|_{t=t_0} = \hat{1}, \tag{2.375}$$

$\hat{U}_{j+1}(0,t,t_0)$ for $j = 0, 2$ describe the parametric down conversion and are expressed, indirectly, in terms of $\hat{E}^{(+)}_{\mathrm{s}_{\frac{j}{2}+1}}(\mathbf{r},t)$, $\hat{E}^{(+)}_{\mathrm{i}}(\mathbf{r},t)$; $\hat{U}_2(0,t)$ describes the beam splitter and is expressed as

$$\hat{U}_2(0,t) = \exp\left\{\mathrm{i}\sum_{\omega''}\left[\overline{\omega}_0\hat{a}_0^\dagger(\omega'')\hat{a}_0(\omega'') + \overline{\omega}_\mathrm{i}\hat{a}_\mathrm{i}^\dagger(\omega'')\hat{a}_\mathrm{i}(\omega'') + \left(\gamma^*\hat{a}_0^\dagger(\omega'')\hat{a}_\mathrm{i}(\omega'') + \mathrm{H.c.}\right)\right]\right\}. \tag{2.376}$$

In this point we differ from the paper by Wang et al. (1991a), who used the initial condition at $t' = t - t_1$ and they did not write down the decomposition into stages. Let $\mathcal{T}$ denote the time ordering. We will introduce the unitary operator

$$\hat{U}_3(0,t,t-t_1) = \mathcal{T}\exp\left\{\int_{t-t_1}^{t}\left[\nu_2 V_2(0)\delta\omega\sum_{\omega'}\sum_{\omega''}\phi(\omega_0-\omega'',\omega'') \times e^{-\mathrm{i}(\mathbf{k}'_{\mathrm{s}_2}+\mathbf{k}'')\cdot\mathbf{0}_2}e^{-\mathrm{i}(\omega_0-\omega'-\omega'')t'}\hat{a}^\dagger_{\mathrm{s}_2}(\omega')\hat{a}^\dagger_\mathrm{i}(\omega'') - \mathrm{H.c.}\right]\mathrm{d}t'\right\}, \tag{2.377}$$

where ν_j, $j = 1, 2$, is a constant such that $|\nu_j|^2$ gives the fraction of incident pump photons that is spontaneously down converted in the steady state, ω_0 is the frequency of the monochromatic pump beam, $\mathbf{k}'_{\mathrm{s}_2}$ ($\mathbf{k}''$) is a wave vector that is determined by the frequency ω'_{s_2} (ω'') and the direction of the second signal beam (the idler beam). To the first order in the processes the unitary operator may be expressed as

$$\hat{U}_3(0,t,t-t_1) = \hat{1} + \left\{\nu_2 V_2(0)\delta\omega\sum_{\omega'}\sum_{\omega''}\phi(\omega_0-\omega'',\omega'') \times e^{-\mathrm{i}(\mathbf{k}'_{\mathrm{s}_2}+\mathbf{k}'')\cdot\mathbf{0}_2}\frac{\sin\left[\frac{1}{2}(\omega_0-\omega'-\omega'')t_1\right]}{\frac{1}{2}(\omega_0-\omega'-\omega'')} \times \exp\left[-\mathrm{i}(\omega_0-\omega'-\omega'')\left(t-\frac{t_1}{2}\right)\right]\hat{a}^\dagger_{\mathrm{s}_2}(\omega')\hat{a}^\dagger_\mathrm{i}(\omega'') - \mathrm{H.c.}\right\}. \tag{2.378}$$

From this we obtain the vector

$$\begin{aligned}|\psi(3,t,t-t_1)\rangle &= \hat{U}_3(0,t,t-t_1)|\psi(2,t,t-t_1)\rangle \\ &= |\psi(2,t,t-t_1)\rangle + \nu_2 V_2(0)\delta\omega\sum_{\omega'}\sum_{\omega''}\phi(\omega_0-\omega'',\omega'') \\ &\quad\times e^{-\mathrm{i}(\mathbf{k}'_{\mathrm{s}_2}+\mathbf{k}'')\cdot\mathbf{0}_2}\frac{\sin\left[\frac{1}{2}(\omega_0-\omega'-\omega'')t_1\right]}{\frac{1}{2}(\omega_0-\omega'-\omega'')} \\ &\quad\times\exp\left[-\mathrm{i}(\omega_0-\omega'-\omega'')\left(t-\frac{t_1}{2}\right)\right]|\omega'\rangle_{\mathrm{s}_2}|\omega''\rangle_\mathrm{i}|0\rangle_{\mathrm{s}_1,0},\end{aligned} \tag{2.379}$$

where $|\omega'\rangle_{s_2}$ and $|\omega''\rangle_{\rm i}$ are frequency eigenstates of the second signal and the idler beam, respectively, $|0\rangle_{s_1,0}$ is the vacuum state of the first signal and escape modes, and

$$\begin{aligned}|\psi(2,t,t-t_1)\rangle &= \hat{U}_2(0,t)\hat{U}_1(0,t,t-t_1)|0\rangle_{s_1,i,s_2,0}\\ &= \hat{U}_2(0,t)\hat{U}_1(0,t,t-t_1)\hat{U}_2^{\dagger}(0,t)|0\rangle_{s_1,i,s_2,0}. \end{aligned}\tag{2.380}$$

We transform the vector $|\psi(3,t,t-t_1)\rangle$ to a vector

$$\begin{aligned}\hat{E}_{s_2}(t)|\psi(3,t,t-t_1)\rangle &= \nu_2 V_2(0)\sqrt{\frac{\delta\omega}{2\pi}}\delta\omega\sum_{\omega'}\sum_{\omega''}\phi(\omega_0-\omega'',\omega'')\\ &\times e^{-\mathrm{i}\mathbf{k}''\cdot\mathbf{0}_2}\frac{\sin\left[\frac{1}{2}(\omega_0-\omega'-\omega'')t_1\right]}{\frac{1}{2}(\omega_0-\omega'-\omega'')}\\ &\times\exp\left[-\mathrm{i}(\omega_0-\omega'-\omega'')\left(\tau_2-\frac{t_1}{2}\right)\right]\exp[-\mathrm{i}(\omega_0-\omega'')(t-\tau_2)]\\ &\times|\omega''\rangle_{\rm i}|0\rangle_{s_1,s_2,0}\end{aligned}\tag{2.381}$$

$$\begin{aligned}&\simeq \nu_2 V_2(0)\sqrt{\frac{\delta\omega}{2\pi}}\sum_{\omega''}\phi(\omega_0-\omega'',\omega'')\\ &\times e^{-\mathrm{i}\mathbf{k}''\cdot\mathbf{0}_2}\int_{-\infty}^{\infty}\frac{\sin\left[\frac{1}{2}(\omega_0-\omega'-\omega'')t_1\right]}{\frac{1}{2}(\omega_0-\omega'-\omega'')}\\ &\times\exp\left[-\mathrm{i}(\omega_0-\omega'-\omega'')\left(\tau_2-\frac{t_1}{2}\right)\right]\mathrm{d}\omega'\exp[-\mathrm{i}(\omega_0-\omega'')(t-\tau_2)]\\ &\times|\omega''\rangle_{\rm i}|0\rangle_{s_1,s_2,0}\end{aligned}\tag{2.382}$$

$$= \nu_2 V_2(t-\tau_2)|1(\mathbf{0}_2,t-\tau_2)\rangle_{\rm i}|0\rangle_{s_1,s_2,0},\tag{2.383}$$

where the single-photon state of the idler beam

$$\begin{aligned}|1(\mathbf{r},t)\rangle_{\rm i} = \sqrt{2\pi\delta\omega}&\sum_{\omega''}\phi(\omega_0-\omega'',\omega'')\\ &\times e^{-\mathrm{i}(\mathbf{k}''\cdot\mathbf{r}-\omega''t)}|\omega''\rangle_{\rm i},\end{aligned}\tag{2.384}$$

$\phi(\tilde{\omega}'',\omega'')$ is connected with spectral functions $\phi_j(\omega',\omega'';\omega)$ characterizing the signal and idler fields at any crystal NLj,

$$\begin{aligned}\phi(\tilde{\omega},\omega) &= \phi_1(\tilde{\omega},\omega)=\phi_2(\tilde{\omega},\omega),\\ \phi_j(\tilde{\omega}'',\omega'') &= \phi_j(\tilde{\omega}'',\omega'';\omega_0),\quad j=1,2.\end{aligned}\tag{2.385}$$

The frequency eigenstates are single-photon states:

$$|\omega''\rangle_m = \hat{a}_m^{\dagger}(\omega'')|0\rangle_m,\tag{2.386}$$

where

$$|0\rangle_m = \bigotimes_{\omega''} |0\rangle_m(\omega''), \quad m = \mathrm{i}, 0. \tag{2.387}$$

Unfortunately, the nonvanishing result is obtained only for

$$0 < \tau_2 < t_1. \tag{2.388}$$

To resolve this, Wang et al. (1991a) let $t_1 \to \infty$. Should t_1 mean the interaction time, it is better to change the integration limits, namely not to consider the integration interval $[t - t_1, t]$, but, for instance,

$$[t - (K_2 + 1)t_1, t - K_2 t_1], \tag{2.389}$$

for

$$K_2 t_1 < \tau_2 < (K_2 + 1)t_1 \tag{2.390}$$

to hold.

We further calculate

$$\hat{E}_{\mathrm{s}_1}(t)|\psi(3, t, t - t_1)\rangle = \hat{E}_{\mathrm{s}_1}(t)|\psi(2, t, t - t_1)\rangle. \tag{2.391}$$

We obtain the appropriate component of the vector $|\psi(2, t, t - t_1)\rangle$ by the action of the unitary operator

$$\begin{aligned}\hat{U}_2(0, t)\hat{U}_1(0, t, t - t_1)\hat{U}_2^\dagger(0, t) = \mathcal{T} \exp\Bigg\{\int_{t-t_1}^{t} \Big[\nu_1 V_1(0)\delta\omega \\ \times \sum_{\omega'}\sum_{\omega''} \phi(\omega_0 - \omega'', \omega'')e^{-\mathrm{i}(\mathbf{k}'_{\mathrm{s}_1}+\mathbf{k}'')\cdot\mathbf{0}_1} e^{-\mathrm{i}(\omega_0-\omega'-\omega'')t'} \\ \times \hat{a}_{\mathrm{s}_1}^\dagger(\omega')[\mathtt{t}^*\hat{a}_{\mathrm{i}}(\omega'') + \mathtt{r}^*\hat{a}_0(\omega'')]^\dagger - \mathrm{H.c.}\Big] \mathrm{d}t'\Bigg\}.\end{aligned} \tag{2.392}$$

The calculation proceeds similarly as in the case of NL2, but we replace $\hat{a}_{\mathrm{i}}^\dagger(\omega'')$ by $\mathtt{t}\hat{a}_{\mathrm{i}}^\dagger(\omega'') + \mathtt{r}\hat{a}_0^\dagger(\omega'')$, $|\omega''\rangle_{\mathrm{i}}$ by $\mathtt{t}|\omega''\rangle_{\mathrm{i}} + \mathtt{r}|\omega''\rangle_0$ and we change all the other subscripts that underlie to a change, so that

$$\begin{aligned}\hat{E}_{\mathrm{s}_1}(t)|\psi(3, t, t - t_1)\rangle = \nu_1 V_1(\mathbf{0}_1, t - \tau_1)\big[\mathtt{t}|1(\mathbf{0}_1, t - \tau_1)\rangle_{\mathrm{i}}|0\rangle_{\mathrm{s}_1,\mathrm{s}_2,0} \\ + \mathtt{r}|1(\mathbf{0}_1, t - \tau_1)\rangle_0|0\rangle_{\mathrm{s}_1,\mathrm{i},\mathrm{s}_2}\big],\end{aligned} \tag{2.393}$$

where $|0\rangle_{\mathrm{s}_1,\mathrm{s}_2,0} \equiv |\psi_{\mathrm{vac}}\rangle_{\mathrm{s}_1,\mathrm{s}_2,0}$, $|0\rangle_{\mathrm{s}_1,\mathrm{i},\mathrm{s}_2} \equiv |\psi_{\mathrm{vac}}\rangle_{\mathrm{s}_1,\mathrm{i},\mathrm{s}_2}$ stand for vacuum states, $|1(\mathbf{r}, t)\rangle_{\mathrm{i}}$ is defined in (2.384), and $|1(\mathbf{r}, t)\rangle_0$ stands for single-photon state of the "escape" beam

$$|1(\mathbf{r}, t)\rangle_0 \equiv \sqrt{2\pi\delta\omega}\sum_{\omega''} \phi(\omega_0 - \omega'', \omega'') \exp[-\mathrm{i}(\mathbf{k}'' \cdot \mathbf{r} - \omega'' t)]|\omega''\rangle_0. \tag{2.394}$$

Here a nonvanishing result is obtained only for

$$0 < \tau_1 < t_1. \tag{2.395}$$

Considering a change of the integration limits as above, we see that, to the first order of the processes, no difficulties arise if we change the limits independently of NL2. We do not consider the integration interval $[t - t_1, t]$, but, for instance,

$$[t - (K_1 + 1)t_1, t - K_1 t_1], \tag{2.396}$$

for

$$K_1 t_1 < \tau_1 < (K_1 + 1)t_1 \tag{2.397}$$

to hold.

In other words, the relations (2.383) and (2.393) can be generalized to provide

$$\hat{A}_{c\tau_1}[\hat{E}_{\mathrm{s}_1}^{(+)}(t)|\psi(3,t)\rangle] = \hat{E}_{\mathrm{s}_1}(t)|\psi(3, t - K_1 t_1, t - (K_1 + 1)t_1)\rangle, \tag{2.398}$$

$$\hat{A}_{c(\tau_0+\tau_2)}[\hat{E}_{\mathrm{s}_2}^{(+)}(t)|\psi(3,t)\rangle] = \hat{E}_{\mathrm{s}_2}(t)|\psi(3, t - K_2 t_1, t - (K_2 + 1)t_1)\rangle, \tag{2.399}$$

where τ_0 is the propagation time of the idler from NL1 to NL2. $\hat{A}_f$ denotes an attenuation of the field down to the vacuum state outside an interaction length centred at the distance f from NL1 in the direction of propagation of the beam. The operator $\hat{A}_f$ compensates for the difference we have caused with the initial condition at $t' = t_0 \to -\infty$ instead of the Wang–Zou–Mandel shortening of the integration interval. The operator $\hat{A}_f$ is not unitary and is even "slightly" nonlinear. Its consideration depends on a neglect of the coherence length in comparison with the interaction length. Using such an operator we can describe, where (within which interaction length) the single-photon states are localized at the time t,

$$\hat{A}_{c\tau}[|1(\mathbf{0}_1, t - \tau)\rangle_{\mathrm{i}}|0\rangle_{\mathrm{s}_1,\mathrm{s}_2,0}] = |1(\mathbf{0}_1, t - \tau)\rangle_{\mathrm{i}}|0\rangle_{\mathrm{s}_1,\mathrm{s}_2,0}, \tag{2.400}$$

$$\hat{A}_{c\tau}[|1(\mathbf{0}_1, t - \tau)\rangle_0|0\rangle_{\mathrm{s}_1,\mathrm{i},\mathrm{s}_2}] = |1(\mathbf{0}_1, t - \tau)\rangle_0|0\rangle_{\mathrm{s}_1,\mathrm{i},\mathrm{s}_2}. \tag{2.401}$$

The angular brackets will mean averages and, when operators are involved, the brackets are supposed to average in the state $|\psi(3,t)\rangle$,

$$\langle \hat{M} \rangle = \langle \psi(3,t)|\hat{M}|\psi(3,t)\rangle, \tag{2.402}$$

with $\hat{M}$ being an operator. When the operator is situated inside an interaction length centred in the propagation distance f from NL1, it also holds that

$$\langle \hat{M} \rangle = \hat{A}_f[\langle \psi(3,t)|]\hat{M}\hat{A}_f[|\psi(3,t)\rangle]. \tag{2.403}$$

Hence, one may omit the unusual notation when no ambiguities arise.

Letting ω_s and ω_i denote the centre frequency of the signal beam and the idler beam, respectively, we have $\omega_\mathrm{s} + \omega_\mathrm{i} = \omega_0$. Introducing the normalized correlation function $\mu(\tau)$ of the down-converted light

$$_\mathrm{i}\langle 1(\mathbf{r}_1, t-\tau_1)|1(\mathbf{r}_2, t-\tau_2)\rangle_\mathrm{i} = \mu(\tau_0+\tau_2-\tau_1)\exp[-\mathrm{i}\omega_\mathrm{i}(\tau_0+\tau_2-\tau_1)], \quad (2.404)$$

where

$$e^{-\mathrm{i}\omega_\mathrm{i}\tau}\mu(\tau) = 2\pi\int_0^{\omega_0} |\phi(\tilde{\omega},\omega)|^2 e^{-\mathrm{i}\omega\tau}\,\mathrm{d}\omega, \quad (2.405)$$

we obtain that the relations (2.370) and (2.371) ought to read

$$\begin{aligned}\langle \hat{E}_{\mathrm{s}_1}^{(-)}(t)\hat{E}_{\mathrm{s}_2}^{(+)}(t)\rangle &\simeq \nu_1^*\nu_2\mathtt{t}^*\langle V_1^*(t-\tau_1)V_2(t-\tau_2)\rangle \\ &\times\mu(\tau_0+\tau_2-\tau_1)\exp[-\mathrm{i}\omega_\mathrm{i}(\tau_0+\tau_2-\tau_1)], \quad (2.406)\\ \langle \hat{E}_{\mathrm{s}_1}^{(-)}(t)\hat{E}_{\mathrm{s}_1}^{(+)}(t)\rangle &\simeq |\nu_1|^2\langle|V_1(t-\tau_1)|^2\rangle,\\ \langle \hat{E}_{\mathrm{s}_2}^{(-)}(t)\hat{E}_{\mathrm{s}_2}^{(+)}(t)\rangle &\simeq |\nu_2|^2\langle|V_2(t-\tau_2)|^2\rangle, \quad (2.407)\end{aligned}$$

where we introduce $\hat{E}_{\mathrm{s}_j}^{(-)}(t) \equiv \left[\hat{E}_{\mathrm{s}_j}^{(+)}(t)\right]^\dagger$. Hence the modulus of the normalized correlation function is

$$\begin{aligned}&\frac{|\langle \hat{E}_{\mathrm{s}_1}^{(-)}(t)\hat{E}_{\mathrm{s}_2}^{(+)}(t)\rangle|}{\sqrt{\langle \hat{E}_{\mathrm{s}_1}^{(-)}(t)\hat{E}_{\mathrm{s}_1}^{(+)}(t)\rangle\langle \hat{E}_{\mathrm{s}_2}^{(-)}(t)\hat{E}_{\mathrm{s}_2}^{(+)}(t)\rangle}}\\ &= \frac{\langle V_1^*(t-\tau_1)V_2(t-\tau_2)\rangle}{\sqrt{\langle|V_1(t-\tau_1)|^2\rangle\langle|V_2(t-\tau_2)|^2\rangle}}|\mu(\tau_0+\tau_2-\tau_1)||\mathtt{t}|. \quad (2.408)\end{aligned}$$

The maximum value is equal to $|\mathtt{t}|$, which is predicted also by equation (2.359). A linear dependence of visibility on $|\mathtt{t}|$, as seen convincingly in the original work (Zou et al. 1991, Wang et al. 1991a), is the true signature of induced coherence without induced emission.

As concerns $\hat{E}_{\mathrm{s}_j}^{(+)}(t)|\psi(t)\rangle$, $j = 1, 2$, they are not explicitly presented in Wang et al. (1991a), but they may be derived. It emerges that the parameters of the beam splitter do not enter the relation for $\hat{E}_{\mathrm{s}_1}^{(+)}(t)|\psi(t)\rangle$. On the contrary, $\hat{E}_{\mathrm{s}_1}^{(+)}(t)|\psi(t)\rangle$ in Wang et al. (1991a) comprises the parameters $\mathtt{t}^*$, $\mathtt{r}'^*$. Nevertheless, the statistical properties in Peřinová et al. (2003) coincide with those in Wang et al. (1991a), because the differences under discussion resemble distinct, yet equivalent pictures. Especially, considering the photon-flux amplitude operators $\hat{E}_\mathrm{s}^{(+)}(t)$ at the detector D_s with a quantum efficiency η_s (Wang et al. 1991a),

$$\hat{E}_\mathrm{s}^{(+)}(t) = \frac{1}{\sqrt{2}}\left[\mathrm{i}\hat{E}_{\mathrm{s}_1}^{(+)}(t) + \hat{E}_{\mathrm{s}_2}^{(+)}(t)\right], \quad (2.409)$$

substituting into the formula for the average rate of photon counting

$$I_\mathrm{s} = \eta_\mathrm{s}\langle\psi(t)|\hat{E}_\mathrm{s}^{(-)}(t)\hat{E}_\mathrm{s}^{(+)}(t)|\psi(t)\rangle, \quad (2.410)$$

where

$$\hat{E}_{\rm s}^{(-)}(t) = [\hat{E}_{\rm s}^{(+)}(t)]^{\dagger}, \tag{2.411}$$

and taking into account the orthogonality of single-photon states $|1(\mathbf{r}_1, t)\rangle_{\rm i}$ and $|1(\mathbf{r}_1, t)\rangle_0$ uniquely results in the relation

$$I_{\rm s} = \frac{1}{2}\eta_{\rm s}\{|\nu_1|^2\langle|V_1(t-\tau_1)|^2\rangle + |\nu_2|^2\langle|V_2(t-\tau_2)|^2\rangle$$
$$+[-{\rm i}\nu_1^*\nu_2\langle V_1^*(t-\tau_1)V_2(t-\tau_2)\rangle{\rm t}^*\mu(\tau_0+\tau_2-\tau_1)e^{-{\rm i}\omega_{\rm i}(\tau_0+\tau_2-\tau_1)} + {\rm c.c.}]\}. \tag{2.412}$$

Peřinová et al. (2000) have studied quantum statistics of radiation in signal modes of the two-mode parametric processes with aligned idler beams. They have found that the signal beams are in the correlated chaotic state. The strength of correlation depends on the degree to which the paths of the idler beams are superposed and aligned. They have compared different measures of correlation, especially the entropic or information-based measure with the modulus of the usual degree of coherence in the dependence on absolute value of the transmission amplitude coefficient of the beam splitter inserted as an attenuator of the perfect alignment. Some other measures have been introduced taking into account the symmetrical and antinormal orderings of field operators. In contrast to the normal ordering, these orderings do not indicate the maximum correlation for the perfect alignment. The situation with the photon numbers in the signal modes, whose correlation is not maximum for the perfect alignment, serves as motivation for such a more general consideration. The theory of canonical correlation has been applied to the quasidistribution of complex amplitudes related to the symmetrical ordering of field operators. They have taken into account that the quantum correlation has a significant effect on the photon-number sum, photon-number difference, and quantum phase-difference statistics. Essentially, it concerned the variances of number sum and number difference and the dispersions of quantum phase differences according to various definitions. A comparison of distributions of quantum phase difference derived from the phase-space distributions has shown that the phase-difference uncertainty increases from the normal ordering, through the symmetrical and antinormal orderings, whereas the system of canonical phase related to the antinormal ordering of exponential phase operators ranges between the symmetrical and antinormal orderings, but by no means exactly. The paper (Peřinová et al. 2000) reveals that the correlated chaotic state is the mixed partial phase-difference state. In addition to the marginal distributions, the joint number-sum and phase-difference distribution has been considered, but for the canonical quantum phase difference and the Luis–Sánchez-Soto phase difference only. The quasidistribution of number difference and phase difference has been defined with the properties that the marginal distribution of the phase difference is the canonical one. They have addressed the number sum and the quantum phase difference as simultaneously measurable observables and the number difference and the quantum phase difference as canonically conjugate observables.

Peřinová et al. (2003) have compared the simple formalism of several coupled harmonic oscillators with multimode formalism in the analysis of an interference experiment. On focusing on several modes they have been able to study phase properties of "correlated chaotic beams". Then they have assumed the single-photon regime as also previous authors did. They have indicated that, assuming several coupled harmonic oscillators, the previous authors did not try to include time delays between optical elements into the analysis. Peřinová et al. (2003) have also formally expressed, for instance, that one works with a single-photon state of some signal modes in the several-mode formalism whenever one describes the experiment with a superposition of single-photon states of modes that form the signal beam.

The utility of a simple single-mode theory has been clarified in the case where single spatial mode filters and narrow-band optical filters are used to filter the output state of parametric down-conversion Li et al. 2005).

Peřina and Křepelka (2005) have derived joint photon-number distributions in signal and idler modes and have illustrated related concepts taking into account experimental data. Peřina and Křepelka (2006) have provided the generalization of this description to stimulated parametric down-conversion. Peřina et al. (2007) have reported on a measurement of the joint signal–idler photoelectron distribution of twin beams. Parameters of the previously published model (Peřina and Křepelka 2005) have been estimated. The specific result that the joint signal–idler quasidistribution of integrated intensities can be approximated by a well-behaved function even in the case where the quasidistribution is not an ordinary function has been comprised. Peřina (2008) has shown that a nonlinear planar waveguide pumped by a beam orthogonal to its surface may serve as a versatile source of photon pairs. He considers the pump-pulse duration, pump-beam transverse width, and angular decomposition of the pump-beam frequency and their effect on characteristics of a photon pair, such as the spectral widths of signal and idler fields, the pair time duration, and the degree of entanglement between the two fields.

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