My Love Affair with the Sobolev Inequality

David R. Adams

Abstract Reminiscence about different versions of the Sobolev inequality obtained by the author and others.

Due to the fact that the Sobolev Inequality is so central to much of mathematical analysis, especially to partial differential equations, it is not surprising that there are by now, 70 years after Sobolev's original paper, many different versions of the Sobolev Inequality and by many different authors. This paper is a tribute to S.L. Sobolev.

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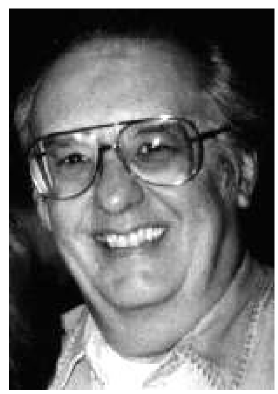

On a cold December morning shortly after Christmas 1969, a small group of people were huddled against the cold near a limousine type bus parked in the middle of the main street of a very small South Dakota farming community in the central plains of the USA – my ancestral hometown. The bus, run by the Greyhound Company, was the "east-west connector bus" and the middle of Main Street was the usual passenger pick-up and drop-off spot in town. Here Main Street consisted of just one block of store front businesses and it

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was wide enough for angular parking on each side and still with plenty of room to accommodate car traffic on either side of the parked bus and the waiting people. Most of these people were my relatives, gathered in town for Christmas and now taking the opportunity to see me off. This bus will meet a larger north-south bus at some highway crosswords out on the nearby prairie. I am expecting to tranfer to that bus for the next stage of my journey.

When I boarded the small bus, there was only one other passenger, an elderly woman. And as I sat down and waved a final farewell to my family, the woman turned to me and asked, "where are you going, young man?" I could have been very dramatic and responded, "into History!", but I did not, nor did it even cross my mind to say such a thing. I just said, "to Rome, Italy", which I am sure was dramatic enough. "My land", she responded, "I am only going to see my sister in Minnesota, you are going a long ways."

And I guess it was, both physically and psychologically for all concerned. And now I can confess that I was not very well prepared for my Italian sojourn. Though I did eventually adjust and adapt to life in Rome (January–August 1970), and even began to thrive there toward the end of my stay. However my budding Italian speech never broke away from my American-midwestern accent. Though if I kept my mouth shut, I eventually could pass for Italian at least in dress and demeanor. Once an American tourist stopped me on the street during my last days to ask, in English, where some place-street was located. I responded by telling her where it was and how to get there. "Wow, you speak good English!" she said. "Thank you" I replied and walked away, leaving her with the illusion.

Thus with my initial bus trip, I began my mathematical odyssey – first to the CNR in Rome as a Post Doc under the direction of Guido Stampacchia, later as an instructor at Rice University, an acting Assistant Professor at the University of California, San Diego, a visiting professor at Indiana University, and finally a Professor at the University of Kentucky – for the past 30+ years. When I left for Italy, I had just days earlier received my Ph.D. degree from the University of Minnesota under the direction of N.G. Meyers. And I began my Post Doc studies by tackling a question posed earlier by him and then at the CNR under the watchful eye of Stampacchia. As it turned out, it was this question of Meyers' that essentially started me down the road of looking at variations of the now classical Sobolev Inequality. For I did not consciously look to that direction, but as it all transpired, I have over the years returned again and again to this theme, eventually giving five or six versions of the Sobolev Inequality during my career. And due to the fact that the Sobolev Inequality is so central to much of mathematical analysis, especially to partial differential equations, it is not surprising that there are by now, 70 years after Sobolev's original paper, many different versions of the Sobolev Inequality and by many different authors. This paper is a tribute to S.L. Sobolev. I do not pretend to review all of this literature, only at best, part of my role in it. This is after all the story of my romance with the Sobolev Inequality.

The classical Sobolev Inequality that I refer to can take on one or two equivalent forms. For example, if $u(x)$ is a smooth function of compact support, then for $1 < p < n/m$, there is a constant c depending only on n, m and p such that

$$(*) \qquad ||u||_{L^q(I\mathbb{R}^n)} \leqslant c||D^m u||_{L^p(\mathbb{R}^n)},$$

where $1/q = 1/p - m/n$[1] Here $D^m u$ denotes the vector of all mth order derivatives of u. Or, if we use Riesz potentials, an equivalent form of $(*)$ is

$$(**) \qquad ||I_\alpha f||_{L^q(\mathbb{R}^n)} \leqslant c'||f||_{L^p(\mathbb{R}^n)}$$

with again $q = np/(n - \alpha p)$, where $\alpha = m$. Here I have written

$$I_\alpha f(x) = \int_{\mathbb{R}^n} |x-y|^{\alpha-n} f(y)\, dy,$$

$0 < \alpha < n$, $1 < p < n/\alpha$.

On the connection between the two, simply note $|u(x)| \leqslant cI_m\,(|D^m u|)(x)$ and singular integrals obtained by differentiating $I_m\, f$ m-times.

So now I begin the story of my involvement with the Sobolev Inequality – from my first struggle at the CNR to prove a trace inequality to my more recent work on a vanishing mean exponential integrability condition with R. Hurri-Syrjänen.

Some time lines are:

1971 – The trace inequality (Sect. 1)

1973 – An exponential trace inequality (Sect. 6)

1974 – A mixed norm inequality; with R. Bagby (Sect. 2)

1975 – A Morrey–Sobolev Inequality (Sect. 3)

1976 – A trace inequality with CSI, $q = p$ (Sect. 1)

1982 – A Morrey–Besov inequality; with J. Lewis (Sect. 4)

1988 – Exponential integrability (Sect. 5)

1998 – Estimates for $M_\alpha f$ (Sect. 7, (4))

2003 – Vanishing exponential integrability; with R. Hurri-Syrjänen (Sect. 6)

2004 – Trace estimates for Morrey–Sobolev functions; with J. Xiao (Sect. 3)

[1] Inequality $(*)$ also holds for $p = 1$ by the Gagliardo–Nirenberg estimates (see [55]). However, $(**)$ does not hold for $p = 1$.

1 The Trace Inequality

The struggle alluded to above was the question of finding necessary and sufficient conditions on a Borel measures μ defined on subsets of $\mathbb{R}^n$ that insures, in the language of N.G. Meyers, that μ has positive capacity. In the late 1960's, Meyers wrote a paper (unpublished, to this date) titled: *Capacities, extremal length and traces of strongly differentiable functions.* Here, to unify the ideas of capacity and extremal length, among other ideas, he defined a capacity of a set of measures on $\mathbb{R}^n$. The usual capacity of a standard subset $K \subset \mathbb{R}^n$ then reduced to taking the sets of Dirac measures $\{\delta_x\}_{x\in K}$. A simplified version of this might be

$$\mathbb{C}_{\alpha,p,q}(\mathcal{K}) = \inf\{||f||^p_{L^p(\mathbb{R}^n)} : ||I_\alpha f||_{L^q(\nu)} \geqslant 1 \ \forall \nu \in \mathcal{K} \text{ and } f \geqslant 0\},$$

where $\mathcal{K} \subset \mathcal{M}^+ =$ all Borel measures on $\mathbb{R}^n$. The question posed by Meyers was to characterize all measures μ with positive capacity $\mathbb{C}_{\alpha,p,q}(\{\mu\}) > 0$. This is clearly equivalent to the trace estimate

$$||I_\alpha f||_{L^q(\mu)} \leqslant c_1 ||f||_{L^p(\mathbb{R}^n)}. \tag{1.1}$$

What I eventually proved, in [1], was the simple and elegant necessary and sufficient condition

$$\mu(B(x,r)) \leqslant Ar^d \tag{1.2}$$

for all $r > 0$ and all $x \in \mathbb{R}^n$. Here $B(x,r)$ is an open ball centered at x and of radius $r > 0$. The conditions of equivalency are: $0 < d \leqslant n$, $q = dp/(n-\alpha p)$, $1 < p < q < \infty$. The coefficient A is a constant independent of x and r. I worked this out in the Spring of 1970. At the time the only result I was aware of along these lines was the trace estimate of Il'in [37] which is (1.1) with $\mu =$ Lebesgue measure on a hyperplane. Later in [2], I found a much simpler proof of the equivalency of (1.1) and (1.2). There, the result followed easily from the weak type estimate

$$\mu([I_\alpha f > t]) \leqslant \left(\frac{c}{t}||f||_{L^p(\mathbb{R}^n)}\right)^q \tag{1.3}$$

for $f \geqslant 0$, followed by an application of the Marcinkiewicz Interpolation Theorem, since here $1/q < 1/p$ (see [11, Theorem 7.2.2] or [41, Theorem 1, p. 52] or even [55, Theorem 4.7.2]).

Of course, it easily becomes clear that (1.1) and (1.2) are no longer equivalent when $q = p$. Indeed, if $d = n - \alpha p$ in (1.2), then there is an $f \in L^p_+(\mathbb{R}^n)$ such that $I_\alpha f = +\infty$ on a set of positive finite μ measure, or to say it another way, $C_{\alpha,p}(K) \equiv \mathbb{C}_{\alpha,p}(\{\delta_x : x \in K\})$ can be zero for a compact set $K \subset \mathbb{R}^n$ with positive Hausdorff capacity (content) $H_\infty^{n-\alpha p}(K) > 0$. Here

$$H_\infty^d(K) = \inf \left\{ \sum_i r_i^d : \begin{array}{l} K \text{ is covered by a countable} \\ \text{number of balls of radius } r_i > 0 \end{array} \right\}.$$

Left to right: V.P. Havin, V.G. Maz'ya, and D.R. Adams. Maz'ya lecturing on the finer points of Potential Theory on the wall of a building in Leningrad. Summer 1974.

All of my attempts to find a simple substitute for condition (1.2) when $q = p$ failed – until I made a pilgrimage to Leningrad (with L. Hedberg and J. Brennan) to meet with V. Maz'ya (and V. Havin) in the summer of 1974. Discussions with Maz'ya profoundly changed my view of the trace question. The limiting case $q = p$ needs the Maz'ya type capacity inequality

$$\int_0^\infty C_{\alpha,p}([I_\alpha f > t])\, dt^p \leqslant c||f||^p_{L^p(\mathbb{R}^n)} \tag{1.4}$$

for $f \geqslant 0$ and $1 < p < n/\alpha$. In [4], I dubbed (1.4) a *capacity strong type inequality* (or CSI) in analogy to the weak and strong type estimates for singular integrals in Harmonic Analysis; note that the weak type estimate

$$C_{\alpha,p}([I_\alpha f > t]) \leqslant t^{-p}||f||^p_{L^p(\mathbb{R}^n)}$$

is trivial. Maz'ya was the first to establish (1.4); the paper [39] treats the cases $\alpha = 1$ and 2. And then after my Leningrad meeting, I managed to prove (1.4) for α a positive integer (see [4]). Later, Dahlberg [26] noted how to get (1.4) in the remaining fractional α cases, and finally Hansson [34] gave a general argument that extended (1.4) considerably (see also [11, p. 187f]). Maz'ya [40] proved a CSI for Besov spaces also around this time. The connection of (1.4) to (1.1) is of course

$$\mu(K) \leqslant c\, C_{\alpha,p}(K) \tag{1.5}$$

for all compact sets $K \subset \mathbb{R}^n$. A similar condition: $\mu(K) \leqslant c\, C_{\alpha,p}(K)^{q/p}$ for all compact K works to give (1.1) when $1 < p < q < \infty$, but the advantage of checking (1.2) for only balls is enormous. But as we have observed above (1.1) does not hold simply by having (1.5) for $K = B(x,r)$ only. There is a ball condition in [38], but it is no longer so simple and a famous ball condition of Fefferman–Phong in [27], but that one is only sufficient.

Another interesting sufficient condition for (1.5) is the boundedness of the nonlinear potential $U^{\mu}_{\alpha,p}(x) = I_\alpha(I_\alpha\mu)^{p'-1}(x)$ or equivalently, for the Wolff potential, where $\alpha p < n$

$$W^{\mu}_{\alpha,p}(x) \equiv \int\limits_0^\infty [r^{\alpha p - n}\mu(B(x,r))]^{p'-1}\,\frac{dt}{t}.$$

This last statement is a consequence of the Wolff inequality

$$||I_\alpha \mu||^{p'}_{L^{p'}(\mathbb{R}^n)} \leqslant c \int\limits_{\mathbb{R}^n} W^{\mu}_{\alpha,p}(x)d\mu(x) \tag{1.8}$$

and

$$\begin{aligned}\mu(K) &\leqslant \int I_\alpha f d\mu^K \leqslant ||f||_{L^p(\mathbb{R}^n)}||I_\alpha\mu^K||_{L_{p'}(\mathbb{R}^n)}\\ &\leqslant ||f||_{L^p(\mathbb{R}^n)}\; c||W^{\mu}_{\alpha,p}||^{1/p'}_{L^\infty}\,\mu(K)^{1/p'}\end{aligned}$$

or

$$\mu(K) \leqslant c||W^{\mu}_{\alpha,p}||^{p-1}_{L^\infty}\, C_{\alpha,p}(K).$$

Here I have written $\mu^K = \mu\angle K$. Notice also that

$$||I_\alpha\mu||^{p'}_{L^{p'}(\mathbb{R}^n)} = \int U^{\mu}_{\alpha,p}\, d\mu$$

and so a lower bound matching (1.8) for p' norm of the potential $I_\alpha\mu$ holds – it is a simple estimate, whereas the Wolff inequality requires much heavier work (see [11, p. 109] or [6]).

Finally, I wish to say a few words about the case $q < p$, though I have had little to do with it. The result that I find most interesting here is due to Verbitsky et al [25]. Again, it involves the Wolff potential; the inequality (1.1) holds, $0 < q < p$, $1 < p < n/\alpha$ if and only if

$$W^{\mu}_{\alpha,p} \in L^{q(p-1)/(p-q)}(\mu). \tag{1.9}$$

My small contribution to this case is for $0 < p \leqslant 1$, where the $L^p(\mathbb{R}^n)$ spaces are now replaced by the real Hardy spaces $H^p(\mathbb{R}^n)$. Here is what I can prove for $p = 1$:

(1) (1.1) holds with $p = 1$ if and only if (1.2) holds,

$$q = d/(n-\alpha), \quad q > 1;$$

(2) (1.1) holds with $p = 1$ and $q = 1$ if and only if

$$\mu(K) \leqslant cH^{n-\alpha}_{\infty}(K) \tag{1.10}$$

holds for all compact set (K);

(3) (1.1) holds with $p = 1$ if and only if

$$\int (M_\alpha \mu)^{q/(1-q)}\, d\mu < \infty \tag{1.11}$$

for $0 < q < 1$.

The first result follows from the Semmes inequality

$$|I_\alpha f(x)| \leqslant c(s)[I_{\alpha s}(f^*)^s]^{1/s} \tag{1.12}$$

for $0 < s < 1$. Here f^* is the "grand maximal function" as used in H^p-theory (see [51] or [20, p. 217]). The estimate (1.11) follows from a covering argument; here

$$M_\alpha \mu(x) = \sup_{r > 0} r^{\alpha - n} \int\limits_{|x-y|<r} d\mu(y)$$

the fractional maximal function of the measure μ. There are also analogues of (1)-(3) for the Hardy spaces H^p, $0 < p < 1$, replacing H^1, by similar methods.

2 A Mixed Norm Inequality

In the early 1970's, I had the idea to try to extend the L^p-capacity theory to the case of parabolic Riesz potentials. One distinctive feature of these

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