

Volume Growth and Escape Rate of Brownian Motion on a Cartan–Hadamard Manifold

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Abstract We prove an upper bound for the escape rate of Brownian motion on a Cartan–Hadamard manifold in terms of the volume growth function. One of the ingredients of the proof is the Sobolev inequality on such manifolds.

1 Introduction

Let M be a geodesically complete noncompact Riemannian manifold. We denote by $d(x, y)$ the geodesic distance between x and y and by μ the Riemannian volume measure. We use \mathbb{P}_x to denote the diffusion measure generated by the Laplace–Beltrami operator Δ . Let $X = \{X_t, t \in \mathbb{R}_+\}$ be the coordinate process on the path space $W(M) = C(\mathbb{R}_+, M)$. By definition, \mathbb{P}_x is a probability measure on $W(M)$ under which X is a Brownian motion starting from x .

Fix a reference point $z \in M$, and let $\rho(x) = d(x, z)$. We say that a function $R(t)$ is an *upper rate function* for Brownian motion on M if

$$\mathbb{P}_z\{\rho(X_t) \leq R(t) \text{ for all sufficiently large } t\} = 1.$$

The purpose of this paper is to study the rate of escape of Brownian motion on M in terms of the volume growth function. Let us first point out that the notion of an upper rate function makes sense only if the lifetime of Brownian motion is infinite. In this case, the manifold M is called stochastically

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complete. The stochastic completeness is equivalent to the identity

$$\int_M p(t, x, y) d\mu(y) = 1,$$

where $p(t, x, y)$ is the minimal heat kernel on M , which is also the transition density function of Brownian motion on M .

Let $B(z, R)$ be the geodesic ball of radius R centered at z . It was proved [3] that M is stochastically complete if

$$\int_0^\infty \frac{r dr}{\log \mu(B(z, R))} = \infty. \quad (1.1)$$

The integral in (1.1) will be used in this paper to construct an upper rate function. Before we state the result, let us briefly survey the existing estimates of escape rate.

- The classical Khinchin law of the iterated logarithm says that for a Brownian motion in \mathbb{R}^n with probability 1

$$\limsup_{t \rightarrow \infty} \frac{\rho(X_t)}{\sqrt{4t \log \log t}} = 1$$

(the factor 4 instead of the classical 2 appears because, in our setting, a Brownian motion is generated by Δ rather than $\frac{1}{2}\Delta$). It follows that for any $\varepsilon > 0$

$$R(t) = \sqrt{(4 + \varepsilon)t \log \log t} \quad (1.2)$$

is an upper rate function.

- If M has nonnegative Ricci curvature, then (1.2) is again an upper rate function on M (see [9, Theorem 1.3] and [7, Theorem 4.2]).
- If the volume growth function is at most polynomial, i.e.,

$$\mu(B(z, r)) \leq Cr^D$$

for large enough r and some positive constants C and D , then the function

$$R(t) = \text{const } \sqrt{t \log t} \quad (1.3)$$

is an upper rate function (see [15, Theorem 5.1], [7, Theorem 1.1], and [9, Theorem 1.1]). Note that the logarithm in (1.3) is single in contrast to (1.2) and, in general, cannot be replaced by the iterated logarithm (see [1, 10]).

- If the volume growth function admits a sub-Gaussian exponential estimate

$$\mu(B(z, r)) \leq \exp(Cr^\alpha)$$

where $0 < \alpha < 2$, then the function

$$R(t) = \text{const } t^{\frac{1}{2-\alpha}}$$

is an upper rate function (see [7, Theorem 4.1]).

Note that (1.1) is satisfied if the volume growth function admits the Gaussian exponential estimate

$$\mu(B(z, r)) \leq \exp(Cr^2) \quad (1.4)$$

(under the condition (1.4), the stochastic completeness was also proved by different methods in [15], [12], and [2]). However, none of the existing results provided any estimates of escape rate under the condition (1.4), let alone under the volume growth function $\exp(Cr^2 \log r)$ and the like.

We construct an upper rate function under the most general condition (1.1). However, we assume, in addition, that M is a Cartan–Hadamard manifold, i.e., a geodesically complete simply connected Riemannian manifold of nonnegative sectional curvature. The property of Cartan–Hadamard manifolds that we use is the Sobolev inequality: if $N = \dim M$, then for any function $f \in C_0^\infty(M)$

$$\left(\int_M |f|^{\frac{N}{N-1}} d\mu \right)^{\frac{N-1}{N}} \leq C_N \int_M |\nabla f| d\mu, \quad (1.5)$$

where C_N is a constant depending only on N (see [11]). The Sobolev inequality allows us to carry through the Moser iteration argument in [14] and prove a mean value estimate for solutions of the heat equation on M , which is one of the ingredients of our proof.

Now, we state our main result.

Theorem 1.1. *Let M be a Cartan–Hadamard manifold. Assume that the following volume estimate holds for a fixed point $z \in M$ and all sufficiently large R :*

$$\mu(B(z, R)) \leq \exp(f(R)), \quad (1.6)$$

where $f(R)$ is a positive, strictly increasing, and continuous function on $[0, +\infty)$ such that

$$\int_0^\infty \frac{r dr}{f(r)} = \infty. \quad (1.7)$$

Let $\varphi(t)$ be the function on \mathbb{R}_+ defined by

$$t = \int_0^{\varphi(t)} \frac{r dr}{f(r)}. \quad (1.8)$$

Then $R(t) = \varphi(Ct)$ is an upper rate function for Brownian motion on M for some absolute constant C (for example, for any $C > 128$).

If we set $f(R) = \log \mu(B(z, R))$ for large R , then the condition (1.7) becomes identical to (1.1). Under this condition, Theorem 1.1 guarantees the existence of an upper rate function $R(t)$. This, in particular, means that in a finite time Brownian motion stays with probability one in a bounded set, which implies that the life time of Brownian motion is infinite almost surely. Hence the manifold M is stochastically complete. This recovers the above cited result that (1.1) on geodesically complete manifolds implies the stochastic completeness, although under the additional assumption that M is Cartan–Hadamard.

Let us show some examples.

- If

$$\mu(B(z, R)) \leq CR^D \quad (1.9)$$

for some constants C and D , then (1.6) holds with

$$f(R) = D \log R + \text{const}$$

and (1.8) yields

$$t \simeq \frac{\varphi^2}{2D \log \varphi}.$$

It follows that $\log t \simeq \log \varphi^2$ and

$$\varphi(t) \simeq \sqrt{Dt \log t}.$$

Hence the function

$$R(t) = \sqrt{CDt \log t}$$

is an upper rate function which matches the above cited results of [7, 9, 15].

- If $\mu(B(z, R)) \leq \exp(Cr^\alpha)$ for some $0 < \alpha < 2$, then (1.6) holds with $f(R) = Cr^\alpha$ and (1.8) yields $t \simeq \varphi(t)^{2-\alpha}$. Hence we obtain the upper rate function

$$R(t) = Ct^{\frac{1}{2-\alpha}}$$

which matches the above cited result [7].

- If

$$\mu(B(z, R)) \leq \exp(CR^2),$$

then $f(R) = CR^2$. Then (1.8) yields $t \simeq \ln \varphi(t)$. Hence we obtain the upper rate function

$$R(t) = \exp(Ct).$$

This result is new. Similarly, if

$$\mu(B(z, R)) \leq \exp(CR^2 \log R),$$

then (1.8) yields $t \simeq \log \log \varphi$. Hence

$$R(t) = \exp(\exp Ct).$$

The hypothesis that M is Cartan–Hadamard can be replaced by the requirement that the Sobolev inequality (1.5) holds on M . Furthermore, the method goes through also in the setting of weighted manifolds, when measure μ is not necessarily the Riemannian measure, but has a smooth positive density, say $\sigma(x)$, with respect to the Riemannian measure. Then, instead of the Laplace–Beltrami operator, one should consider the weighted Laplace operator

$$\Delta_\mu = \frac{1}{\sigma} \operatorname{div}(\sigma \nabla)$$

which is symmetric with respect to μ . Theorem 1.1 extends to the weighted manifolds that are geodesically complete and satisfy the Sobolev inequality (1.5).

This paper is organized as follows. Section 2 contains the proof of an upper bound for certain positive solutions of the heat equation. In Sect. 3, we prove the main result stated above. In Sect. 4, we compute the sharp upper rate function on model manifold and show that, for a certain range of volume growth functions, the upper rate function of Theorem 1.1 is sharp up to a constant factor in front of t .

2 Heat Equation Solution Estimates

In this section, we prove a pointwise upper bound of certain solutions of the heat equation on a Cartan–Hadamard manifold M (Theorem 2.3). It is an easy consequence of an L^2 -bound for a general complete manifold and a mean value type inequality for a Cartan–Hadamard manifold. These two upper bounds are known, and we state them as lemmas.

For any set $A \subset M$ let A_r be the open r -neighborhood of A in M .

Lemma 2.1. *Let M be a geodesically complete Riemannian manifold. Suppose that $u(x, t)$ is a smooth subsolution to the heat equation in the cylinder $A_r \times [0, T]$, where $A \subset M$ is a compact set and $r, T > 0$ (see Fig. 1). Assume also that $0 \leq u(x, t) \leq 1$ and $u(x, 0) = 0$ on A_r . Then for any $t \in (0, T]$*

$$\int_A u^2(x, t) d\mu(x) \leq \mu(A_r) \max\left(1, \frac{r^2}{2t}\right) \exp\left(-\frac{r^2}{2t} + 1\right).$$

For the proof see [6, Theorem 3] (see also [7, Proposition 3.6]). Note that no geometric assumption about M is made except for the geodesic completeness.

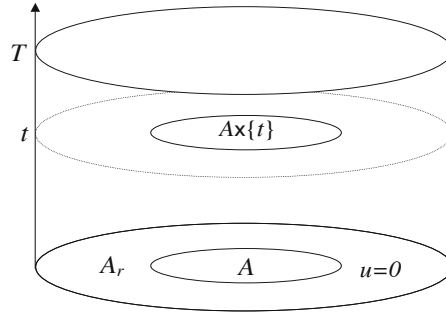


Fig. 1 Illustration to Lemma 2.1.

The proof exploits essentially the property of the geodesic distance function that $|\nabla d| \leq 1$.

Takeda [15] proved a similar estimate for $\int_A u(x, t) d\mu(x)$ by using a different probabilistic argument. However, it is more convenient for us to work with the L^2 rather than with L^1 estimate in view of the following lemma.

Lemma 2.2. *Let M be a Cartan–Hadamard manifold of dimension N . Suppose that $u(x, t)$ is a smooth nonnegative subsolution to the heat equation in a cylinder $B(y, r) \times [0, T]$, where $r, T > 0$ (see Fig. 2). Then*

$$u(y, T)^2 \leq \frac{C_N}{\min(\sqrt{T}, r)^{N+2}} \int_0^T \int_{B(y, r)} u^2(x, t) d\mu(x) dt, \quad (2.1)$$

where C_N is a constant depending only on N .

Proof. As was already mentioned above, a Cartan–Hadamard manifold admits the Sobolev inequality (1.5). By a standard argument, (1.5) implies the Sobolev–Moser inequality

$$\int_M |f|^{2+\frac{4}{N}} d\mu \leq C_N \left(\int_M |f|^2 d\mu \right)^{2/N} \int_M |\nabla f|^2 d\mu,$$

which leads, by the Moser iteration argument [14], to the mean value inequality (2.1). Note that the value of C_N may be different in all the above inequalities.

An alternative proof of the implication (1.5) \Rightarrow (2.1) can be found in [4] (see also [5, Theorem 3.1 and formula (3.4)]). \square

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