

Preface

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This volume contains various results on partial differential equations where Sobolev spaces are used. Their selection is motivated by the research interests of the editor and the geographical links to the places where S.L. Sobolev worked and lived: St. Petersburg, Moscow, and Novosibirsk. Most of the papers are written by leading experts in control theory and inverse problems. Another reason for the selection is a strong link to applied areas. In my opinion, control theory and inverse problems are main areas of differential equations of importance for some branches of contemporary science and engineering. S.L. Sobolev, as many great mathematicians, was very much motivated by applications. He did not distinguish between pure and applied mathematics, but, in his own words, between “good mathematics and bad mathematics.” While he possessed a brilliant analytical technique, he most valued innovative ideas, solutions of deep conceptual problems, and not mathematical decorations, perfecting exposition, and “generalizations.”

S.L. Sobolev himself never published papers on inverse problems or control theory, but he was very much aware of the state of art and he monitored research on inverse problems. In particular, in his lecture at a Conference on Differential Equations in 1954 (found in Sobolev’s archive and made available to me by Alexander Bukhgeim), he outlined main inverse problems in geophysics: the inverse seismic problem, the electromagnetic prospecting, and the inverse problem of gravimetry. While at that time one of the main achievements was the solution of the one-dimensional Sturm–Liouville and spectral problems, he emphasized importance and possibilities of multi-dimensional inverse problems.

I was a part of Sobolev Seminar at the Institute of Mathematics, Novosibirsk, the USSR, from 1971 to 1980, until he left for Moscow. When presenting

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new results on inverse problems I was always aware that S.L. Sobolev followed presentations very closely and liked new ideas and not technicalities.

Now it is hard to imagine papers on the theory of partial differential equations written without use of Sobolev spaces. The papers in this collection are not an exception: almost all of them to some extent make use of these spaces. Their collection shows a variety of situations where Sobolev spaces are used. One of the main contributions of S.L. Sobolev is the introduction and use of Sobolev spaces and generalized (weak) solutions to boundary value problems to establish existence and uniqueness of generalized solutions to some basic boundary value problems (in particular, for hyperbolic and higher order elliptic equations) and their relations to classical solutions. Together with K.O. Friedrichs he can be viewed as a founding father of the contemporary theory of partial differential equations.

Now we briefly review the contents of the volume.

M. Belishev exposes an approach to the problem of reconstruction of a Riemannian Ω from the elliptic or hyperbolic Dirichlet-to-Neumann maps based on representation of algebras of functions. The basic idea is that these maps uniquely determine certain sufficiently large algebra of functions on Ω , and hence the spectrum of this algebra which is isometric to Ω . This approach applies to elliptic operators at least in the two-dimensional case, where there are strong connections between (generalized) analytic functions and elliptic partial differential equations. For second order hyperbolic equations the needed algebra is generated by the family of projection operators. A projection operator in this case projects onto the $L^2(\Omega)$ -closure of waves sent from the boundary of Ω at times from 0 to $\theta < T$. This approach shows unusual aspects of the powerful and innovative Boundary Control Method initiated by M. Belishev and has a strong promise for new applications.

The memoir of A. Fursikov, M. Gunzburger, and J. Peterson is devoted to the theory of boundary value problems for the stochastic Ginzburg–Landau equation. This semilinear equation models the famous phenomenon of superconductivity in materials under low temperatures. The authors introduce into the Ginzburg–Landau model (complex-valued) white noise. This change has some regularizing effect. In addition, the system enjoys some interesting ergodic properties. They demonstrate existence and uniqueness of weak solutions. The exposition contains a convenient definition and useful properties of the Wiener measures and of the Wiener process, discusses the Ito integral and the Ito formula. The existence of a weak solution is proved by obtaining a priori estimates of solutions, including the mean modulus of continuity, some special compactness theorems, and passing to limits when the size of spatial grid goes to zero. Finally, the existence of the strong statistical solution is established.

The paper of V. Isakov and N. Kim contains a first proof of Carleman type estimate with two large parameters for a general second order partial

differential operator with real-valued coefficients. Carleman estimates are estimates in an L^2 -space with the exponential weight containing a large parameter. They are the basic tool in proving uniqueness and stability of the continuation of solutions to partial differential equations. The continuation of solutions is of fundamental importance for boundary control theory and inverse problems. While their theory is well developed for scalar operators, systems of partial differential equations present currently a formidable challenge. Some classical isotropic systems can be principally diagonalized and handled like scalar equations, but more general and important anisotropic in some cases can be only transformed into principally triangular ones. Then two large parameters are essential to achieve global results. As an example, the classical Lamé system with residual stress is handled.

The paper of V. Ivrii, one of the best known students of Sobolev, discloses more detail about his deep research on spectral asymptotics. He considers the weighted integral of the spectral kernel of the magnetic Schrödinger operator. The (singular) weight is the inverse of the distance to the diagonal, so this integral can be interpreted as the Dirac correction term related to the ground state energy of atoms or molecules. The goal is to justify the Weyl type asymptotic behavior of this integral, i.e., to obtain best possible estimate for the remainder. The basic technique is the microlocal analysis and the leading idea is to make use of (micro)hyperbolicity. An essential tool of proofs is splitting integration domains into special zones, somehow in the spirit of the classical potential theory. Conditions for estimates of remainders include some assumptions on the distribution of eigenvalues and on microhyperbolicity of the Schrödinger operator.

In their contribution, I. Lasiecka and R. Triggiani develop a complete theory of boundary controllability for important partial differential equations modeling the Kirchhoff plate and the Euler–Bernoulli plate with various physically meaningful (linear and nonlinear) boundary controls. In addition, they consider the Schrödinger equation with the Dirichlet and Neumann boundary controls which is of significance for plates and shells equations due to their factorization into products of Schrödinger operators. One of the main goals is to achieve exact controllability (i.e., driving the system into zero final state by boundary control in a finite time) and the feedback stabilization of solutions for large times. The basic tool is the semigroup theory. Very delicate related questions concern continuity of boundary operators in Sobolev spaces. Several proofs are given and there are references to an extensive bibliography on the subject.

V. Maz'ya and A. Movchan obtain and justify asymptotic expansions of Green's functions of the transmission and Neumann problems for the Laplace equation in a domain with several holes. The important feature of their result is the uniformity of the asymptotic expansions with respect to the arguments of Green's functions. The small parameter ε in these expansions is the average size of the holes. The authors employ the potential theory and the theory of

Sobolev spaces to obtain uniform bounds of the remainder in the asymptotic expansion of quadratic order with respect to ε .

M. Taylor is discussing the use of the Finsler metric to study the wave propagation. The Riemannian metric is a particular case of the Finsler metric and its importance for the theory of elliptic and hyperbolic equations is widely recognized. The author introduces the Finsler symbol, connects it to pseudodifferential operators and hyperbolic partial differential equations, and interprets a construction of the fundamental solution to the hyperbolic Cauchy problem by using the Finsler symbol. He gives an example of a strictly hyperbolic equation of the fourth order that gives rise to a non-Riemannian Finsler metric. Finally, he describes some applications to the ergodic theory and harmonic analysis. The main technique of this paper is the theory of pseudodifferential operators.

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