Chapter 2
Function Spaces

We introduce in this chapter the function spaces that will be relevant to the subsequent developments in this monograph. The function spaces to be discussed include spaces of continuous and continuously differentiable functions, Lebegue and Sobolev spaces, associated with an open bounded domain $\Omega \subset \mathbb{R}^d$. In order to treat time-dependent problems, we also introduce spaces of vector-valued functions, i.e., spaces of mappings defined on a time interval $[0,T] \subset \mathbb{R}$ with values into a Banach or Hilbert space X.

2.1 The Spaces $C^m(\overline{\Omega})$ and $L^p(\Omega)$

Let Ω be an open bounded subset of $\mathbb{R}^d$, where d is a positive integer. We denote by Γ the boundary of Ω, and $\overline{\Omega} = \Omega \cup \Gamma$ the closure of Ω. A typical point in $\mathbb{R}^d$ is denoted by $\boldsymbol{x} = (x_1, \ldots, x_d)$ or $\boldsymbol{x} = (x_1, \ldots, x_d)^T$. It is convenient to use the multi-index notation for partial derivatives. An ordered collection of d non-negative integers, $\alpha = (\alpha_1, \ldots, \alpha_d)$, is called a *multi-index*. The quantity $|\alpha| = \sum_{i=1}^d \alpha_i$ is said to be the *length* of α. If v is an m-times differentiable function, then for each α with $|\alpha| \le m$,

$$D^\alpha v(\boldsymbol{x}) = \frac{\partial^{|\alpha|} v(\boldsymbol{x})}{\partial x_1^{\alpha_1} \cdots \partial x_d^{\alpha_d}}$$

is the α^{th} partial derivative. This is a handy notation for partial derivatives. Some examples are

$$\partial_1 v = \frac{\partial v}{\partial x_1} = D^\alpha v, \quad \alpha = (1, 0, \ldots, 0),$$
$$\frac{\partial^d v}{\partial x_1 \cdots \partial x_d} = D^\alpha v, \quad \alpha = (1, 1, \ldots, 1).$$

The set of all the partial derivatives of order m of a function v can be expressed as $\{ D^\alpha v : |\alpha| = m \}$. There are other notations commonly used for partial derivatives; e.g., the partial derivative $\partial v/\partial x_i$ is also written as $\partial_{x_i} v$, or $\partial_i v$, or v_{x_i}, or $v_{,i}$.

The spaces $C^m(\overline{\Omega})$. Let $C(\overline{\Omega})$ be the space of functions that are uniformly continuous on Ω. Each function in $C(\overline{\Omega})$ is bounded. The notation $C(\overline{\Omega})$ is consistent with the fact that a uniformly continuous function on Ω has a unique continuous extension to $\overline{\Omega}$. For $v \in C(\overline{\Omega})$, we always understand its boundary values on Γ to be the continuous extension of the values of v in Ω. The space $C(\overline{\Omega})$ is a Banach space with the norm

$$\|v\|_{C(\overline{\Omega})} = \sup\{ |v(\boldsymbol{x})| : \boldsymbol{x} \in \Omega \} \equiv \max\{ |v(\boldsymbol{x})| : \boldsymbol{x} \in \overline{\Omega} \}.$$

Denote $\mathbb{Z}_+$ the set of non-negative integers. For every $m \in \mathbb{Z}_+$, $C^m(\overline{\Omega})$ is the space of functions that, together with their derivatives of order less than or equal to m, are continuous up to the boundary,

$$C^m(\overline{\Omega}) = \{ v \in C(\overline{\Omega}) : D^\alpha v \in C(\overline{\Omega}) \text{ for } |\alpha| \le m \}.$$

When $m = 0$, we usually write $C(\overline{\Omega})$ instead of $C^0(\overline{\Omega})$. The space $C^m(\overline{\Omega})$ is a Banach space with the norm

$$\|v\|_{C^m(\overline{\Omega})} = \sum_{|\alpha| \le m} \|D^\alpha v\|_{C(\overline{\Omega})}.$$

We also set

$$C^\infty(\overline{\Omega}) = \bigcap_{m=0}^{\infty} C^m(\overline{\Omega}) \equiv \{ v \in C(\overline{\Omega}) : v \in C^m(\overline{\Omega}) \quad \forall\, m \in \mathbb{Z}_+ \},$$

the space of infinitely differentiable functions.

Given a function $v : \Omega \to \mathbb{R}$, its *support* is defined to be

$$\operatorname{supp} v = \overline{\{ \boldsymbol{x} \in \Omega : v(\boldsymbol{x}) \ne 0 \}}.$$

We say that v has a *compact support* if $\operatorname{supp} v$ is a proper subset of Ω: $\operatorname{supp} v \subset \Omega$. Thus, if v has a compact support, then there is a neighboring open strip about the boundary Γ such that v is zero on the part of the strip that lies inside Ω. Later on, we will need the space

$$C_0^\infty(\Omega) = \{ v \in C^\infty(\Omega) : \operatorname{supp} v \subset \Omega \}.$$

Obviously, $C_0^\infty(\Omega) \subset C^\infty(\overline{\Omega})$.

The spaces $L^p(\Omega)$. For every number $p \in [1, \infty)$ we denote by $L^p(\Omega)$ the linear space of (equivalence classes of) measurable functions $v : \Omega \to \mathbb{R}$

for which

$$\|v\|_{L^p(\Omega)} = \left(\int_\Omega |v(\boldsymbol{x})|^p dx \right)^{1/p} < \infty, \tag{2.1}$$

where the integration is understood to be in the sense of Lebesgue. The mapping $v \mapsto \|v\|_{L^p(\Omega)}$ is a norm on $L^p(\Omega)$. Here and below, it is understood that v represents an equivalence class of functions, two functions being equivalent if they are equal almost everywhere (a.e.), that is, equal except on a subset of Ω of Lebesgue measure zero.

The definition of the spaces $L^p(\Omega)$ can be extended to include the case $p = \infty$ in the following manner. For every measurable function $v : \Omega \to \mathbb{R}$ denote

$$\begin{aligned} \|v\|_{L^\infty(\Omega)} &= \operatorname*{ess\,sup}_{\boldsymbol{x}\in\Omega} |v(\boldsymbol{x})| \\ &= \inf \{ M \in [0,\infty] \, : \, |v(\boldsymbol{x})| \le M \quad \text{a.e. } \boldsymbol{x} \in \Omega \}. \end{aligned} \tag{2.2}$$

The quantity (2.2) is called the *essential supremum* of $|v|$, and we say that v is *essentially bounded* if $\|v\|_{L^\infty(\Omega)} < \infty$. We denote by $L^\infty(\Omega)$ the linear space of (equivalence classes of) measurable functions $v : \Omega \to \mathbb{R}$ that are essentially bounded. The mapping $v \mapsto \|v\|_{L^\infty(\Omega)}$ is a norm on the space $L^\infty(\Omega)$.

For $p \in [1,\infty]$, its conjugate exponent $q \in [1,\infty]$ is defined by the relations

$$\begin{aligned} \frac{1}{p} + \frac{1}{q} &= 1 \quad \text{if } p \neq 1, \\ q &= \infty \quad \text{if } p = 1. \end{aligned}$$

Here, we adopt the convention $1/\infty = 0$ and therefore $q = 1$ if $p = \infty$.

Some basic properties of the L^p spaces are summarized below.

Theorem 2.1. *Let Ω be an open bounded set in $\mathbb{R}^d$ and let $p \in [1,\infty]$. Then:*

(a) *$L^p(\Omega)$ is a Banach space.*
(b) *Every Cauchy sequence in $L^p(\Omega)$ has a subsequence that converges pointwise a.e. on Ω.*
(c) (Hölder's inequality) *Let $u \in L^p(\Omega)$, $v \in L^q(\Omega)$. Then*

$$\int_\Omega |u(\boldsymbol{x})\, v(\boldsymbol{x})| \, dx \le \|u\|_{L^p(\Omega)} \|v\|_{L^q(\Omega)}.$$

(d) *For $1 \le p < \infty$, the dual $(L^p(\Omega))'$ of $L^p(\Omega)$ is the space $L^q(\Omega)$. Moreover, if $p \in (1,\infty)$, the space $L^p(\Omega)$ is reflexive.*
(e) *For $1 \le p < \infty$, $L^p(\Omega)$ is a separable space.*

The case $p = 2$ is special since in this case $p = q = 2$. A simple consequence of the last theorem is the following result.

Corollary 2.2. *The space $L^2(\Omega)$ is a Hilbert space with the inner product*

$$(u, v)_{L^2(\Omega)} = \int_\Omega u(\boldsymbol{x})\, v(\boldsymbol{x})\, dx \quad \forall\, u, v \in L^2(\Omega).$$

Moreover, $L^2(\Omega)$ is separable and the following Cauchy-Schwarz inequality holds:

$$\left| \int_\Omega u(\boldsymbol{x})\, v(\boldsymbol{x})\, dx \right| \leq \|u\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} \quad \forall\, u, v \in L^2(\Omega).$$

We now introduce the spaces of locally integrable functions that will be used in the next section in order to introduce the weak derivatives of real-valued functions defined on Ω.

Definition 2.3. Let $1 \leq p < \infty$. A function $v : \Omega \subset \mathbb{R}^d \to \mathbb{R}$ is said to be *locally p-integrable*, $v \in L^p_{\text{loc}}(\Omega)$, if for every $\boldsymbol{x} \in \Omega$, there is an open neighborhood Ω' of $\boldsymbol{x}$ such that $\Omega' \subset\subset \Omega$ (i.e., $\overline{\Omega'} \subset \Omega$) and $v \in L^p(\Omega')$.

We have the following useful result ([153, p. 18]).

Lemma 2.4. (Generalized Variational Lemma) *Let $v \in L^1_{\text{loc}}(\Omega)$ with Ω a nonempty open set in $\mathbb{R}^d$. If*

$$\int_\Omega v(\boldsymbol{x})\, \phi(\boldsymbol{x})\, dx = 0 \quad \forall\, \phi \in C_0^\infty(\Omega),$$

then $v = 0$ a.e. on Ω.

In Section 2.3, we shall need the space

$$L^2(\Omega)^d = \{\, \boldsymbol{u} = (u_i) \,:\, u_i \in L^2(\Omega),\, i = 1, \ldots, d \,\}. \tag{2.3}$$

This is a Hilbert space with the canonical inner product

$$(\boldsymbol{u}, \boldsymbol{v})_{L^2(\Omega)^d} = \sum_{i=1}^{d} \int_\Omega u_i(\boldsymbol{x}) v_i(\boldsymbol{x})\, dx$$

and the associated norm

$$\|\boldsymbol{u}\|_{L^2(\Omega)^d} = \left(\sum_{i=1}^{d} \int_\Omega u_i^2(\boldsymbol{x})\, dx \right)^{\frac{1}{2}}.$$

2.2 Sobolev Spaces

Sobolev spaces are indispensable tools in the study of boundary value problems. To introduce Sobolev spaces, we first need to extend the definition of derivatives. The starting point is the classical "integration by parts" formula

$$\int_{\Omega} v(\boldsymbol{x})\, D^{\alpha}\phi(\boldsymbol{x})\, dx = (-1)^{|\alpha|} \int_{\Omega} D^{\alpha} v(\boldsymbol{x})\, \phi(\boldsymbol{x})\, dx, \tag{2.4}$$

which holds for $v \in C^k(\Omega)$, $\phi \in C_0^{\infty}(\Omega)$ and $|\alpha| \leq k$. This formula, relating differentiation and integration, is the most important formula in calculus. The *weak derivative* is an extension of the classical derivative, i.e., if the classical derivative exists, then the two derivatives coincide. To ensure that the weak derivative is useful, we require that the integration by parts formula (2.4) holds. A more general approach for the extension of the classical derivatives is to first introduce the derivatives in the distributional sense. A detailed discussion of distributions and the derivatives in the distributional sense can be found in several monographs, e.g., [138]. Here we choose to introduce the concept of the weak derivatives directly, which is sufficient for this work.

Definition 2.5. Let Ω be a nonempty open set in $\mathbb{R}^d$, $v, w \in L^1_{\mathrm{loc}}(\Omega)$. Then w is called an α^{th} *weak derivative* of v if

$$\int_{\Omega} v(\boldsymbol{x})\, D^{\alpha}\phi(\boldsymbol{x})\, dx = (-1)^{|\alpha|} \int_{\Omega} w(\boldsymbol{x})\, \phi(\boldsymbol{x})\, dx \quad \forall\, \phi \in C_0^{\infty}(\Omega). \tag{2.5}$$

Applying Lemma 2.4, we see that when a weak derivative exists, it is uniquely defined up to a set of measure zero. Also, from the definition of the weak derivative, we see that if v is k-times continuously differentiable on Ω, then, for each α with $|\alpha| \leq k$, the classical partial derivative $D^{\alpha} v$ is also the α^{th} weak derivative of v. For this reason, we use the notation $D^{\alpha} v$ also for the α^{th} weak derivative of v and we note that $D^{\alpha} v = v$ if $|\alpha| = 0$.

We continue with the following definition.

Definition 2.6. Let k be a non-negative integer, $k \in \mathbb{Z}_+$, and let $p \in [1, \infty]$. The *Sobolev space* $W^{k,p}(\Omega)$ is the set of all the functions $v \in L^1_{\mathrm{loc}}(\Omega)$ such that for each multi-index α with $|\alpha| \leq k$, the α^{th} weak derivative $D^{\alpha} v$ exists and belongs to $L^p(\Omega)$. The norm in the space $W^{k,p}(\Omega)$ is defined as

$$\|v\|_{W^{k,p}(\Omega)} = \begin{cases} \left(\sum_{|\alpha| \leq k} \|D^{\alpha} v\|^p_{L^p(\Omega)} \right)^{1/p} & \text{if} \quad 1 \leq p < \infty, \\ \max_{|\alpha| \leq k} \|D^{\alpha} v\|_{L^{\infty}(\Omega)} & \text{if} \quad p = \infty. \end{cases}$$

We note that $W^{0,p}(\Omega) = L^p(\Omega)$, and when $p = 2$, we write $W^{k,2}(\Omega) \equiv H^k(\Omega)$.

A seminorm over the space $W^{k,p}(\Omega)$ is

$$|v|_{W^{k,p}(\Omega)} = \begin{cases} \left(\sum_{|\alpha| = k} \|D^{\alpha} v\|^p_{L^p(\Omega)} \right)^{1/p} & \text{if} \quad 1 \leq p < \infty, \\ \max_{|\alpha| = k} \|D^{\alpha} v\|_{L^{\infty}(\Omega)} & \text{if} \quad p = \infty. \end{cases}$$

It is not difficult to see that $W^{k,p}(\Omega)$ is a normed space. Moreover, we have the following results, which summarize the basic properties of Sobolev spaces.

Theorem 2.7. *Let Ω be an open bounded set in $\mathbb{R}^d$, $k \in \mathbb{Z}_+$ and $p \in [1,\infty]$. Then:*

(a) $W^{k,p}(\Omega)$ *is a Banach space.*
(b) $W^{k,p}(\Omega)$ *is reflexive if* $1 < p < \infty$.
(c) $W^{k,p}(\Omega)$ *is separable if* $1 \leq p < \infty$.

A simple consequence of the Theorem 2.7 combined with Corollary 2.2 is the following result.

Corollary 2.8. *The Sobolev space $H^k(\Omega)$ is a separable Hilbert space with the inner product*

$$(u,v)_{H^k(\Omega)} = \int_\Omega \sum_{|\alpha|\leq k} D^\alpha u(\boldsymbol{x})\, D^\alpha v(\boldsymbol{x})\, dx \quad \forall\, u,v \in H^k(\Omega).$$

The closure of the space $C_0^\infty(\Omega)$ with respect to the norm $\|\cdot\|_{W^{k,p}(\Omega)}$ gives a closed subspace of $W^{k,p}(\Omega)$, denoted $W_0^{k,p}(\Omega)$. When $p = 2$, we use the notation $H_0^k(\Omega) \equiv W_0^{k,2}(\Omega)$. It follows from Theorem 2.7 and Corollary 2.8 that $W_0^{k,p}(\Omega)$ is a Banach space and $H_0^k(\Omega)$ is a Hilbert space. It can be shown that the seminorm $|\cdot|_{W^{k,p}(\Omega)}$ is a norm on $W_0^{k,p}(\Omega)$ and there exists a constant $c > 0$ such that

$$|v|_{W^{k,p}(\Omega)} \leq \|v\|_{W^{k,p}(\Omega)} \leq c\,|v|_{W^{k,p}(\Omega)} \quad \forall\, v \in W_0^{k,p}(\Omega).$$

We now collect some important properties of Sobolev spaces. Some of them require a certain degree of regularity on the boundary Γ of the domain Ω and, for this reason, we introduce the following definition.

Definition 2.9. Let Ω be open and bounded in $\mathbb{R}^d$. We say that Ω has a *Lipschitz continuous boundary* Γ if for each point $\boldsymbol{x}_0 \in \Gamma$ there exists $r > 0$ and a Lipschitz continuous function $g : \mathbb{R}^{d-1} \to \mathbb{R}$ such that, upon relabeling the coordinate axes if necessary, we have

$$\Omega \cap B(\boldsymbol{x}_0, r) = \{\, \boldsymbol{x} \in B(\boldsymbol{x}_0, r) : \, x_d > g(x_1, \ldots, x_{d-1}) \,\}.$$

Here, $B(\boldsymbol{x}_0, r)$ denotes the ball of $\mathbb{R}^d$ centered at $\boldsymbol{x}_0$ with radius r. Also, we recall that a real-valued function $g : \mathbb{R}^{d-1} \to \mathbb{R}$ is said to be *Lipschitz continuous* if for some constant L_g, there holds the inequality

$$|g(\boldsymbol{x}) - g(\boldsymbol{y})| \leq L_g\, \|\boldsymbol{x} - \boldsymbol{y}\| \quad \forall\, \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^{d-1},$$

where $\|\boldsymbol{x} - \boldsymbol{y}\|$ denotes the standard Euclidean distance between $\boldsymbol{x}$ and $\boldsymbol{y}$,

$$\|\boldsymbol{x} - \boldsymbol{y}\| = \left(\sum_{i=1}^{d-1} |x_i - y_i|^2 \right)^{1/2}.$$

We note that, since Γ is a compact set in $\mathbb{R}^d$, in Definition 2.9 we can actually find a finite number of points $\{\boldsymbol{x}_i\}_{i=1}^N$ on the boundary, positive numbers $\{r_i\}_{i=1}^N$ and Lipschitz continuous functions $\{g_i\}_{i=1}^N$, such that Γ is covered by the union of the balls $B(\boldsymbol{x}_i, r_i)$, $1 \le i \le N$, and

$$\Omega \cap B(\boldsymbol{x}_i, r_i) = \{ \boldsymbol{x} \in B(\boldsymbol{x}_i, r_i) : x_d > g_i(x_1, \ldots, x_{d-1}) \}$$

upon relabeling the coordinate axes, if necessary.

With a slight abuse of terminology, we say that a domain Ω is a *Lipschitz domain* if it is an open bounded domain with Lipschitz continuous boundary. In the following, we always assume that Ω is a Lipschitz domain, and we note that this assumption is, in fact, not needed for some of the results stated below.

We have the following result on the approximation of Sobolev functions by smooth functions.

Theorem 2.10. *Assume that $\Omega \subset \mathbb{R}^d$ is a Lipschitz domain, $k \in \mathbb{Z}_+$ and $p \in [1, \infty)$. Then for each $v \in W^{k,p}(\Omega)$, there exists a sequence $\{v_n\} \subset C^\infty(\overline{\Omega})$ such that*

$$\|v_n - v\|_{W^{k,p}(\Omega)} \to 0 \quad \text{as } n \to \infty.$$

From the definition of the space $W_0^{k,p}(\Omega)$, we immediately obtain the following density result.

Theorem 2.11. *Under the assumptions of Theorem* 2.10, *for every $v \in W_0^{k,p}(\Omega)$, there exists a sequence $\{v_n\} \subset C_0^\infty(\Omega)$ such that*

$$\|v_n - v\|_{W^{k,p}(\Omega)} \to 0 \quad \text{as } n \to \infty.$$

To compare Sobolev spaces with different indices, we need the following definitions.

Definition 2.12. Let X and Y be two normed spaces with $X \subset Y$. We say the space X is *continuously embedded* in Y and write $X \hookrightarrow Y$, if the identity operator $I : X \to Y$ is continuous. We say the space X is *compactly embedded* in Y and write $X \hookrightarrow\hookrightarrow Y$, if the identity operator $I : X \to Y$ is compact.

It is easy to see that $X \hookrightarrow Y$ if and only if there exists $c > 0$ such that

$$\|v\|_Y \le c \, \|v\|_X \quad \forall \, v \in X \tag{2.6}$$

or, equivalently,

$$\forall \{v_n\} \subset X, \quad v_n \to v \text{ in } X \Longrightarrow v_n \to v \text{ in } Y.$$

Also, it can be proved that $X \hookrightarrow\hookrightarrow Y$ iff

$$\forall \{v_n\} \subset X, \quad v_n \rightharpoonup v \text{ in } X \Longrightarrow v_n \to v \text{ in } Y.$$

Some properties regarding embeddings and compact embeddings involving Sobolev spaces, which are important in analyzing the regularity of weak solutions of boundary value problems, are summarized in the following theorem.

Theorem 2.13. *Let $\Omega \subset \mathbb{R}^d$ be a Lipschitz domain, $k \in \mathbb{Z}_+$ and $p \in [1, \infty)$. Then the following statements are valid.*

(a) *If $k < \frac{d}{p}$, then $W^{k,p}(\Omega) \hookrightarrow L^q(\Omega)$ for every $q \le p^*$ and $W^{k,p}(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega)$ for every $q < p^*$, where $\frac{1}{p^*} = \frac{1}{p} - \frac{k}{d}$.*

(b) *If $k = \frac{d}{p}$, then $W^{k,p}(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega)$ for every $q < \infty$.*

(c) *If $k > \frac{d}{p}$, then $W^{k,p}(\Omega) \hookrightarrow\hookrightarrow C^m(\overline{\Omega})$ for every integer m that satisfies $0 \le m < k - \frac{d}{p}$.*

A direct consequence of Theorem 2.13 is the following result.

Corollary 2.14. *Let $\Omega \subset \mathbb{R}^d$ be a Lipschitz domain, $p \in [1, \infty)$ and let k be a positive integer. Then $W^{k,p}(\Omega) \hookrightarrow\hookrightarrow L^p(\Omega)$.*

In this work, we shall use the embedding result

$$H^1(\Omega) \hookrightarrow\hookrightarrow L^2(\Omega) \tag{2.7}$$

which is obtained from Corollary 2.14 by taking $k = 1$ and $p = 2$.

Sobolev spaces are defined through $L^p(\Omega)$ spaces. Hence Sobolev functions are uniquely defined only a.e. in Ω. Since the boundary Γ has measure zero in $\mathbb{R}^d$, the boundary values of a Sobolev function seem to be not well defined. Nevertheless, it is possible to define the trace of a Sobolev function on the boundary in such a way that for a Sobolev function that is continuous up to the boundary, its trace coincides with its boundary value.

Theorem 2.15. *Assume that Ω is a Lipschitz domain in $\mathbb{R}^d$ with boundary Γ and $1 \le p < \infty$. Then there exists a linear continuous operator $\gamma : W^{1,p}(\Omega) \to L^p(\Gamma)$ such that $\gamma v = v|_\Gamma$ if $v \in W^{1,p}(\Omega) \cap C(\overline{\Omega})$. Moreover, the mapping $\gamma : W^{1,p}(\Omega) \to L^p(\Gamma)$ is compact, i.e., for every bounded sequence $\{v_n\}$ in $W^{1,p}(\Omega)$, there is a subsequence $\{v_{n_k}\} \subset \{v_n\}$ such that $\{\gamma v_{n_k}\}$ is convergent in $L^p(\Gamma)$.*

The operator γ is called the *trace operator*, and γv can be termed the *trace* of $v \in W^{1,p}(\Omega)$. For the sake of simplicity, when no ambiguity occurs, we usually write v instead of γv. Note that the continuity of γ implies that there exists a constant $c > 0$, which depends on Ω, such that

$$\|\gamma v\|_{L^p(\Gamma)} \leq c\, \|v\|_{W^{1,p}(\Omega)} \quad \forall\, v \in W^{1,p}(\Omega). \tag{2.8}$$

Generally, the trace operator is neither an injection nor a surjection from $W^{1,p}(\Omega)$ to $L^p(\Gamma)$ (the only exception is when $p = 1$, since then the trace operator is surjective from $W^{1,1}(\Omega)$ to $L^1(\Gamma)$). When $p > 1$, the range $\gamma(W^{1,p}(\Omega))$ is a space smaller than $L^p(\Gamma)$.

2.3 Equivalent Norms on the Space $H^1(\Omega)$

In studying second-order boundary value problems, the Sobolev space $H^1(\Omega)$ plays a crucial role. For this reason, since the antiplane contact problems we study in Part IV of this monograph lead to second-order boundary value problems, in this section we pay a particular attention to the Sobolev space $H^1(\Omega)$.

Let Ω be a Lipschitz domain in $\mathbb{R}^d$. Recall that

$$H^1(\Omega) = \{u \in L^2(\Omega) \,:\, \nabla u \in L^2(\Omega)^d\,\}$$

where $\nabla u = (\partial_1 u, \ldots, \partial_d u)$ represents the *gradient operator*. The space $H^1(\Omega)$ is a Hilbert space with the inner product

$$(u, v)_{H^1(\Omega)} = (u, v)_{L^2(\Omega)} + (\nabla u, \nabla v)_{L^2(\Omega)^d} \tag{2.9}$$

and the associated norm

$$\|u\|_{H^1(\Omega)} = \big(\|u\|^2_{L^2(\Omega)} + \|\nabla u\|^2_{L^2(\Omega)^d}\big)^{\frac{1}{2}}. \tag{2.10}$$

A seminorm over the space $H^1(\Omega)$ is

$$|u|_{H^1(\Omega)} = \|\nabla u\|_{L^2(\Omega)^d}.$$

It also follows from Theorem 2.15 that the trace map $\gamma : H^1(\Omega) \to L^2(\Gamma)$ is a linear continuous and compact operator. In what follows, we simply write v for the trace γv of an element $v \in H^1(\Omega)$ and we note that inequality (2.8) shows that there exists $c > 0$, which depends on Ω, such that

$$\|v\|_{L^2(\Gamma)} \leq c\, \|v\|_{H^1(\Omega)} \quad \forall\, v \in H^1(\Omega). \tag{2.11}$$

The following result can be used to generate various equivalent norms on $H^1(\Omega)$.

Theorem 2.16. *Let Ω be an open, bounded, and connected set in $\mathbb{R}^d$ with a Lipschitz boundary Γ. Assume that $p_1, p_2, \dots, p_N$ are continuous seminorms on $H^1(\Omega)$ with the property that the functional p defined by*

$$p(v) = \|\nabla v\|_{L^2(\Omega)^d} + \sum_{j=1}^{N} p_j(v) \quad \forall\, v \in H^1(\Omega) \tag{2.12}$$

is a norm on $H^1(\Omega)$. Then p and $\|\cdot\|_{H^1(\Omega)}$ are equivalent norms on $H^1(\Omega)$.

Proof. Since $p_1, p_2, \dots, p_N$ are continuous seminorms on $H^1(\Omega)$ there exist positive constants $M_1, M_2 \dots, M_N$ such that

$$p_j(v) \le M_j \|v\|_{H^1(\Omega)} \quad \forall\, v \in H^1(\Omega),\ j = 1, \dots N$$

and therefore (2.12) implies that

$$p(v) \le c_1 \, \|v\|_{H^1(\Omega)} \quad \forall\, v \in H^1(\Omega), \tag{2.13}$$

where $c_1 = \max\{1, M_1, \dots, M_N\} > 0$.

We shall prove that there exists $c_2 > 0$ such that

$$p(v) \ge c_2 \, \|v\|_{H^1(\Omega)} \quad \forall\, v \in H^1(\Omega). \tag{2.14}$$

To this end, we argue by contradiction. Suppose that this inequality is false; then, for all $n \in \mathbb{N}$ there exists $v_n \in H^1(\Omega)$ such that

$$\|v_n\|_{H^1(\Omega)} = 1, \tag{2.15}$$

$$p(v_n) \le \frac{1}{n}. \tag{2.16}$$

We combine (2.12) and (2.16) to see that

$$\|\nabla v_n\|_{L^2(\Omega)^d} + \sum_{j=1}^{N} p_j(v_n) \le \frac{1}{n} \quad \forall\, n \in \mathbb{N},$$

which shows that $\nabla v_n \to \mathbf{0}$ in $L^2(\Omega)^d$ and $p_j(v_n) \to 0$ for all $j = 1, \dots, N$, as $n \to \infty$. Equality (2.15) shows that $\{v_n\}$ is a bounded sequence in $H^1(\Omega)$ and, using the compact embedding (2.7), it follows that there exists a function $v \in L^2(\Omega)$ and a subsequence of the sequence $\{v_n\}$, still denoted by $\{v_n\}$, such that $v_n \to v$ in $L^2(\Omega)$ as $n \to \infty$. Since

$$\int_\Omega v_n(\boldsymbol{x})\, \partial_i \phi(\boldsymbol{x})\, dx = -\int \partial_i v_n(\boldsymbol{x})\, \phi(\boldsymbol{x})\, dx \quad \forall\, \phi \in C_0^\infty(\Omega),$$

$v_n \to v$ in $L^2(\Omega)$ and $\partial_i v_n \to 0$ in $L^2(\Omega)$ $\forall\, i = 1, \ldots, d$, passing to the limit in the previous equality we obtain that

$$\int_\Omega v(\boldsymbol{x})\, \partial_i \phi(\boldsymbol{x})\, dx = 0 \quad \forall\, \phi \in C_0^\infty(\Omega) \quad \forall\, i = 1, \ldots, d.$$

This last equality shows that v has weak derivatives of first order and these derivatives vanish a.e. in Ω. We conclude that $v \in H^1(\Omega)$ and $\nabla v = \mathbf{0}$. It follows from (2.10) that

$$\|v_n - v\|_{H^1(\Omega)} = \left(\|v_n - v\|^2_{L^2(\Omega)} + \|\nabla v_n\|^2_{L^2(\Omega)^d}\right)^{\frac{1}{2}}$$

and since $v_n \to v \in L^2(\Omega)$, $\nabla v_n \to \mathbf{0}$ in $L^2(\Omega)^d$, we find that

$$v_n \to v \qquad \text{in} \quad H^1(\Omega) \quad \text{as} \quad n \to \infty. \tag{2.17}$$

Next, since the seminorms $p_1, \ldots, p_N$ are continuous, (2.12) and (2.17) imply that

$$p(v_n) \to p(v) \qquad \text{as } n \to \infty$$

and, therefore, by (2.16) we find that $p(v) = 0$. Using now the fact that p is a norm on the space $H^1(\Omega)$, it follows that $v = 0$, which contradicts the equality $\|v\|_{H^1(\Omega)} = 1$ obtained by combining (2.15) and (2.17). Therefore, inequality (2.14) holds.

It follows that the norm p satisfies (2.13) and (2.14), which concludes the proof. □

Many useful inequalities can be derived as consequence of Theorem 2.16, see [57] and [126] for details. Here, we present only one consequence of the theorem, which will be used in Section 8.2 of this manuscript.

Corollary 2.17. *Let Ω be an open, bounded, and connected set in $\mathbb{R}^d$ with a Lipschitz boundary Γ and let Γ_1 be a measurable part of Γ such that meas $\Gamma_1 > 0$. Then the functional p on $H^1(\Omega)$ defined by*

$$p(v) = \|\nabla v\|_{L^2(\Omega)^d} + \int_{\Gamma_1} |v|\, da \quad \forall\, v \in H^1(\Omega) \tag{2.18}$$

is a norm on $H^1(\Omega)$, equivalent to the norm $\|\cdot\|_{H^1(\Omega)}$.

Proof. We apply Theorem 2.16 to the special case $N = 1$ and

$$p_1(v) = \int_{\Gamma_1} |v|\, da.$$

Clearly, p_1 is a seminorm on the space $H^1(\Omega)$; moreover, by using the Cauchy-Schwarz inequality and (2.11), we have

$$p_1(v) = \int_{\Gamma_1} |v|\, da \le \left(\int_{\Gamma_1} da\right)^{\frac{1}{2}} \left(\int_{\Gamma_1} v^2\, da\right)^{\frac{1}{2}}$$
$$\le (meas\,\Gamma_1)^{\frac{1}{2}} \|v\|_{L^2(\Gamma_1)} \le c\,(meas\,\Gamma_1)^{\frac{1}{2}} \|v\|_{H^1(\Omega)} \quad \forall\, v \in H^1(\Omega),$$

which shows that p_1 is continuous. Assume now that $p(v) = 0$; then $\nabla v = \mathbf{0}$ and $\int_{\Gamma_1} v\, da = 0$; the first equality shows that v is a constant in Ω, $v(\boldsymbol{x}) = k$ a.e. $\boldsymbol{x} \in \Omega$, and the second one shows that

$$\int_{\Gamma_1} k\, da = k\,(meas\,\Gamma_1) = 0,$$

which implies that $k = 0$. Thus, $v = 0$ and therefore p is a norm on $H^1(\Omega)$. Corollary 2.17 is now a consequence of Theorem 2.16. □

2.4 Spaces of Vector-valued Functions

We shall need the spaces of vector-valued functions in studying time-dependent variational problems. In the following, if it is not specified otherwise, $(X, \|\cdot\|_X)$ will denote a real Banach space and $[0, T]$ will denote the time interval of interest, for $T > 0$.

$C^m([0,T];X)$ **spaces.** We define $C([0,T];X)$ to be the space of functions $v : [0,T] \to X$ that are continuous on the closed interval $[0,T]$. With the norm

$$\|v\|_{C([0,T];X)} = \max_{t \in [0,T]} \|v(t)\|_X,$$

the space $C([0,T];X)$ is a Banach space.

Definition 2.18. A function $v : [0,T] \to X$ is said to be *(strongly) differentiable* at $t_0 \in [0,T]$ if there exists an element in X, denoted as $v'(t_0)$ and called the *(strong) derivative* of v at t_0, such that

$$\lim_{h \to 0} \left\| \frac{1}{h}(v(t_0 + h) - v(t_0)) - v'(t_0) \right\|_X = 0,$$

where the limit is taken with respect to h with $t_0 + h \in [0,T]$. The derivative at $t_0 = 0$ is defined as a right-sided limit and that at $t_0 = T$ as a left-sided limit. The function v is said to be *differentiable on* $[0,T]$ if it is differentiable at every $t_0 \in [0,T]$. It is *differentiable a.e.* if it is differentiable a.e. on $(0,T)$. In this case, the function v' is called the (strong) derivative of v. Higher derivatives $v^{(j)}$, $j \geq 2$, are defined recursively by $v^{(j)} = (v^{(j-1)})'$. Usually we use the notation $\dot{v} = v'$ and we understand $v^{(0)}$ to be v.

For an integer $m \geq 0$, we define the space

$$C^m([0,T];X) = \{\, v \in C([0,T];X) : \ v^{(j)} \in C([0,T];X), \ j = 1, \ldots, m \,\}.$$

This is a Banach space with the norm

$$\|v\|_{C^m([0,T];X)} = \sum_{j=0}^{m} \max_{t\in[0,T]} \|v^{(j)}(t)\|_X.$$

In particular, $C^1([0,T];X)$ denotes the space of continuously differentiable functions on $[0,T]$ with values in X. This is a Banach space with the norm

$$\|v\|_{C^1([0,T];X)} = \max_{t\in[0,T]} \|v(t)\|_X + \max_{t\in[0,T]} \|\dot{v}(t)\|_X.$$

We also set

$$\begin{aligned} C^\infty([0,T];X) &= \bigcap_{m=0}^{\infty} C^m([0,T];X) \\ &\equiv \{\, v \in C([0,T];X) : \ v \in C^m([0,T];X) \quad \forall\, m \in \mathbb{Z}_+ \},\end{aligned}$$

the space of infinitely differentiable functions defined on $[0,T]$ with values in X.

The spaces $L^p(0,T;X)$. For $p \in [1,\infty)$, we define $L^p(0,T;X)$ to be the space of all measurable functions $v : [0,T] \to X$ such that $\int_0^T \|v(t)\|_X^p dt < \infty$. With the norm

$$\|v\|_{L^p(0,T;X)} = \left(\int_0^T \|v(t)\|_X^p dt \right)^{1/p},$$

the space $L^p(0,T;X)$ becomes a Banach space. We define $L^\infty(0,T;X)$ to be the space of all measurable functions $v : [0,T] \to X$ such that $t \mapsto \|v(t)\|_X$ is essentially bounded on $[0,T]$. The space $L^\infty(0,T;X)$ is a Banach space with the norm

$$\|v\|_{L^\infty(0,T;X)} = \operatorname*{ess\,sup}_{t\in[0,T]} \|v(t)\|_X.$$

When $(X, (\cdot,\cdot)_X)$ is a Hilbert space, $L^2(0,T;X)$ is also a Hilbert space with the inner product given by

$$(u,v)_{L^2(0,T;X)} = \int_0^T (u(t), v(t))_X \, dt.$$

Also, below we shall use the notation $L^p(0,T)$ for the space $L^p(0,T;\mathbb{R})$.

The following theorem summarizes some basic properties of the spaces $L^p(0,T;X)$.

Theorem 2.19. *Let $p \in [1,\infty]$. Then:*

(a) *$C([0,T];X)$ is dense in $L^p(0,T;X)$ and the embedding is continuous.*
(b) *If $X \hookrightarrow Y$, then $L^p(0,T;X) \hookrightarrow L^q(0,T;Y)$ for $1 \le q \le p \le \infty$.*

The following result of Lebesgue is useful when we localize a global relation.

Theorem 2.20. *Assume that $v \in L^1(0,T;X)$. Then*

$$\lim_{h\to 0} \frac{1}{h}\int_{t_0}^{t_0+h} \|v(t)-v(t_0)\|_X \, dt = 0 \quad \text{for almost all } t_0 \in (0,T).$$

We see that Theorem 2.20 implies that if $v \in L^1(0,T;X)$, then

$$\lim_{h\to 0} \frac{1}{h}\int_{t_0}^{t_0+h} v(t)\, dt = v(t_0) \quad \text{for almost all } t_0 \in (0,T).$$

Here the limit is understood in the sense of the norm of X, that is,

$$\lim_{h\to 0} \left\| \frac{1}{h}\int_{t_0}^{t_0+h} v(t)\, dt - v(t_0)\right\|_X = 0 \quad \text{for almost all } t_0 \in (0,T).$$

The next result describes the dual of the space $L^p(0,T;X)$, in the Hilbertian case.

Theorem 2.21. *Let $p \in [1,\infty)$ and let $q \in (1,\infty]$ be its conjugate. Assume X is a Hilbert space. Then the dual of the space $L^p(0,T;X)$ is $L^q(0,T;X)$.*

The duality pairing between the space $L^p(0,T;X)$ and its dual is given by

$$\langle u', u\rangle = \int_0^T (u'(t), u(t))_X \, dt.$$

The spaces $W^{k,p}(0,T;X)$. To define the Sobolev spaces of vector-valued functions, we need to define in an appropriate way derivatives with respect to the time variable for functions defined on $[0,T]$ with values in X. The approach is similar to that taken in the case of weak derivatives of real-valued functions, see Definition 2.5; that is, we take as a starting point the elementary integration by parts formula

$$\int_0^T v(t)\phi^{(m)}(t)\, dt = (-1)^m \int_0^T v^{(m)}(t)\phi(t)\, dt$$

which holds for all functions $v \in C^m([0,T];X)$ and $\phi \in C_0^\infty(0,T)$. Here m is a positive integer and $(\cdot)^{(m)} = d^m(\cdot)/dt^m$.

Definition 2.22. A function $v : [0,T] \to X$ is said to be *locally integrable*, $v \in L^1_{\text{loc}}(0,T;X)$, if for every closed interval $B \subset [0,T]$, we have

$$\int_B \|v(t)\|_X dt < \infty.$$

Definition 2.23. Let $v,\, w \in L^1_{\text{loc}}(0,T;X)$. Then w is called a m^{th} *weak derivative* of v if

$$\int_0^T v(t)\,\phi^{(m)}(t)\,dt = (-1)^m \int_0^T w(t)\,\phi(t)\,dt \quad \forall\,\phi \in C_0^\infty(0,T). \tag{2.19}$$

It can be proved that when a weak derivative exists, it is unique. For this reason, if v and w satisfy (2.19) for some positive integer m we write simply $w = v^{(m)}$. The first two weak derivatives are also denoted by $\dot{v}$, $\ddot{v}$ and we note that $v^{(0)} = v$.

For $k \in \mathbb{Z}_+$ and $1 \le p \le \infty$, we introduce the space

$$W^{k,p}(0,T;X) = \{\, v \in L^p(0,T;X) :\, \|v^{(m)}\|_{L^p(0,T;X)} < \infty \quad \forall\, m \le k\,\}.$$

When $p < \infty$, we define the norm in the space $W^{k,p}(0,T;X)$ by

$$\|v\|_{W^{k,p}(0,T;X)} = \left(\int_0^T \sum_{0 \le m \le k} \|v^{(m)}(t)\|_X^p dt\right)^{1/p}.$$

When $p = \infty$, the norm is defined as

$$\|v\|_{W^{k,\infty}(0,T;X)} = \max_{0 \le m \le k} \operatorname*{ess\,sup}_{t \in [0,T]} \|v^{(m)}(t)\|_X.$$

If X is a Hilbert space and $p = 2$, then the space

$$H^k(0,T;X) \equiv W^{k,2}(0,T;X)$$

is a Hilbert space with the inner product

$$(u,v)_{H^k(0,T;X)} = \sum_{0 \le m \le k} \int_0^T (u^{(m)}(t), v^{(m)}(t))_X\, dt.$$

We have the following result.

Theorem 2.24. *Assume that X is a Banach space and $p \in [1,\infty]$. Then $W^{1,p}(0,T;X) \hookrightarrow C([0,T];X)$.*

The previous theorem shows that every element $v \in W^{1,p}(0,T;X)$ can be identified with an element, still denoted by v, in the space $C([0,T];X)$, possibly after a modification on a subset of $[0,T]$ with zero measure. Moreover, there is a positive constant c such that

$$\|v\|_{C([0,T];X)} \leq c\,\|v\|_{W^{1,p}(0,T;X)} \quad \forall\, v \in W^{1,p}(0,T;X).$$

The next result can help us to understand the nature of the functions in the spaces $W^{k,p}(0,T;X)$. For this, we recall the definition of absolute continuity.

Definition 2.25. A function $v : [0,T] \to X$ is *absolutely continuous* if for any $\varepsilon > 0$, there exists a $\delta = \delta(\varepsilon) > 0$ such that $\sum_n \|v(b_n) - v(a_n)\|_X < \varepsilon$ whenever $\{(a_n, b_n)\}_n \subset [0,T]$ is a countable family of pairwisely disjoint intervals with total length $\sum_n (b_n - a_n) < \delta$.

We have the following characterization for the space $W^{k,p}(0,T;X)$.

Theorem 2.26. *Let $k > 0$ be an integer, $p \in [1,\infty]$, and $v \in L^p(0,T;X)$. Then $v \in W^{k,p}(0,T;X)$ if and only if $v(t) = w(t)$ a.e. on $(0,T)$ for some $w \in C^{k-1}([0,T];X)$ such that w, w', …, $w^{(k-1)}$ are absolutely continuous on $[0,T]$, the k-th strong derivative $w^{(k)}$ exists a.e., and $w^{(k)} \in L^p(0,T;X)$.*

Because of Theorem 2.26, we usually simply say $v \in W^{k,p}(0,T;X)$ iff v, v', …, $v^{(k-1)}$ are absolutely continuous on $[0,T]$ and $v^{(k)} \in L^p(0,T;X)$.

It is well known that every real-valued absolutely continuous function $v : [0,T] \to \mathbb{R}$ is differentiable a.e. on $(0,T)$ and it can be expressed as the indefinite integral of its derivative. Some simple examples (see for instance [11, p. 15]) show that this fails when v is absolutely continuous from $[0,T]$ to a general Banach space X. However, the following theorem, obtained in [92], holds.

Theorem 2.27. *Let X be a reflexive Banach space and let $v : [0,T] \to X$ be an absolutely continuous function. Then v is a.e. differentiable on $(0,T)$, its derivative $\dot{v}$ belongs to $L^1(0,T;X)$, and*

$$v(t) = v(0) + \int_0^t \dot{v}(s)\,ds \quad \forall\, t \in [0,T]. \tag{2.20}$$

Using Theorems 2.26 and 2.27, we can show the following results, which will be used on several occasions in the rest of the manuscript.

Proposition 2.28. *Let X be a reflexive Banach space and let $v : [0,T] \to X$. Then:*

(a) *v belongs to $W^{1,1}(0,T;X)$ iff v is an absolutely continuous function on $[0,T]$ to X;*
(b) *v belongs to $W^{1,\infty}(0,T;X)$ iff v is a Lipschitz continuous function on $[0,T]$ to X.*

Proposition 2.29. *Let X be a reflexive Banach space and $u \in W^{1,p}(0,T;X)$ for some $p \in [1,\infty]$. Then*

$$\|u(t)-u(s)\|_X \leq \int_s^t \|\dot{u}(\tau)\|_X d\tau, \quad 0 \leq s \leq t \leq T.$$

We also have

$$\|u(t)-u(s)\|_X^p \leq (t-s)^{p-1} \int_s^t \|\dot{u}(\tau)\|_X^p d\tau, \quad 0 \leq s \leq t \leq T,$$

if $p<\infty$, and

$$\|u(t)-u(s)\|_X \leq (t-s)\|\dot{u}\|_{L^\infty(0,T;X)}, \quad 0 \leq s \leq t \leq T,$$

if $p=\infty$.

Proposition 2.30. *Let X be a Hilbert space and let $a : X \times X \to \mathbb{R}$ be a bilinear symmetric continuous and X-elliptic form. Then, for all $u \in W^{1,2}(0,T;X)$, the real-valued function $t \mapsto \frac{1}{2}\, a(u(t),u(t))$ is absolutely continuous on $[0,T]$,*

$$\frac{d}{dt}\left(\frac{1}{2}\, a(u(t),u(t))\right) = a(u(t),\dot{u}(t)) \quad \text{a.e. on } (0,T),$$

and the following equality holds:

$$\frac{1}{2}\, a(u(t),u(t)) = \frac{1}{2}\, a(u(0),u(0)) + \int_0^t a(u(s),\dot{u}(s))\, ds,$$

for all $t \in [0,T]$.

We end this section with some integrability results that will be useful in the study of evolutionary variational inequalities. The following results represent direct consequences of more general results that can be found in [16, p. 160].

Theorem 2.31. *Let X be a Banach space and let $j : X \to (-\infty,\infty]$ be a proper convex l.s.c. function. Then:*

(1) For all $v \in L^1(0,T;X)$ the function $t \mapsto j(v(t))$ is measurable on $[0,T]$. Moreover, it is integrable if and only if there exists $g \in L^1(0,T)$ such that $j(v(t)) \leq g(t)$ a.e. $t \in (0,T)$.

(2) The function $\phi : L^1(0,T;X) \to (-\infty,\infty]$ defined by

$$\phi(v) = \begin{cases} \displaystyle\int_0^T j(v(t))\, dt & \text{if} \quad j(v) \in L^1(0,T;X), \\ \infty & \text{otherwise}, \end{cases}$$

is proper, convex, and lower semicontinuous.

Direct consequences of Theorem 2.31 are provided by the following results.

Corollary 2.32. *Let X be a Banach space and let $j : X \to \mathbb{R}$ be a convex l.s.c. function that satisfies*

$$j(v) \le c\left(\|v\|_X^2 + 1\right) \quad \forall\, v \in X, \tag{2.21}$$

for some $c > 0$. Then for all $v \in L^2(0,T;X)$, the function $t \mapsto j(v(t))$ is integrable on $[0,T]$. Moreover,

$$v \mapsto \phi(v) = \int_0^T j(v(t))\,dt \tag{2.22}$$

is a convex l.s.c. function on the space $L^2(0,T;X)$.

Proof. Let $v \in L^2(0,T;X)$ and take $g(t) = c\left(\|v(t)\|_X^2 + 1\right)$ a.e. $t \in (0,T)$. It follows that $g \in L^1(0,T)$. We use now (2.21) and Theorem 2.31(1) to conclude the first part of the corollary. The second part is a consequence of Theorem 2.31(2), combined with the inclusion $L^2(0,T;X) \hookrightarrow L^1(0,T;X)$ provided by Theorem 2.19. □

Corollary 2.33. *Let $(X, (\cdot,\cdot)_X)$ be a Hilbert space and let $j : X \to \mathbb{R}$ be a convex Gâteaux differentiable function that satisfies*

$$\|\nabla j(v)\|_X \le c\left(\|v\|_X + 1\right) \quad \forall\, v \in X, \tag{2.23}$$

for some $c > 0$. Then for all $v \in L^2(0,T;X)$ the function $t \mapsto j(v(t))$ is integrable on $[0,T]$. Moreover, the function $\phi : L^2(0,T;X) \to \mathbb{R}$ defined by (2.22) *is convex and l.s.c.*

Proof. We use Corollary 1.37 to see that j is lower semicontinuous. Moreover, it follows from Proposition 1.36 that

$$j(w) - j(v) \ge (\nabla j(v), w - v)_X \quad \forall\, v,\, w \in X.$$

We choose $w = 0_X$ in this inequality and use the Cauchy-Schwarz inequality to find that

$$j(v) \le \|\nabla j(v)\|_X \|v\|_X + j(0_X) \quad \forall\, v \in X. \tag{2.24}$$

We combine inequalities (2.24) and (2.23) to see that j satisfies an inequality of the form (2.21), then we use Corollary 2.32 to conclude the proof. □

Bibliographical Notes

The material presented in Chapter 1 is standard and can be found in many books on functional analysis. For more information in the field, we refer the reader to the books [7], [18], [150]–[156]. A complete treatment of the general theory of convex functions as well as proofs of the results on convex analysis presented in Section 1.4 can be found in the works [13, 38, 44, 66, 151], for instance.

For a comprehensive treatment of basic aspects of spaces introduced in Sections 2.1 and 2.2, we refer the reader to [1, 18, 95, 108, 121]. The proof of Theorem 2.16 was written following [126, p. 201]. A more general result that can be used to generate various equivalent norms on the Sobolev space $W^{k,p}(\Omega)$ can be found in [57, p. 117]. More details on the spaces of vector-valued functions presented in Section 2.4 can be found in [11, 13, 17, 22].

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