Chapter 2
Green's Functions

Ernian Pan

2.1 Introduction

Coupling between mechanical and electric fields has stimulated interesting research related to the microelectromechanical system [1, 2]. The major applications are in sensor and actuator devices by which an electric voltage can induce an elastic deformation and vice versa. Because many novel materials, such as the nitride group semiconductors, are piezoelectric, study on quantum nanostructures is currently a cutting-edge topic with the strain energy band engineering in the center [3, 4]. Novel laminated composites (with adaptive and smart components) are continuously attracting great attention from mechanical, aerospace, and civil engineering branches [5]. In materials property study, the Eshelby-based micromechanics theory has been very popular [6]. In most of these exciting research topics, the fundamental solution of a given system under a unit concentrated force/charge or simply the Green's function solution is required. This motivates the writing of this chapter. In this chapter, however, only the static case with general anisotropic piezoelectricity is considered, even though a couple of closely related references on vibration and/or dynamics (time-harmonic) wave propagation are briefly reviewed. Furthermore, although emphasis is given to the generalized point and line forces, the Green's functions to the corresponding point and line dislocations, as well as point and line eigenstrain are also discussed or presented based on Betti's reciprocal theorem.

Ernian Pan
Department of Civil Engineering, Dept. of Applied Mathematics, The University of Akron, Akron, OH 44325-3905, USA, e-mail: pan2@uakron.edu

J. Yang (ed.), *Special Topics in the Theory of Piezoelectricity*, DOI: 10.1007/978-0-387-89498-0_2, 13

2.2 Governing Equations

Consider a linear, anisotropic piezoelectric and heterogeneous solid occupying the domain V bounded by the boundary S. In discussing the Green's functions, the problem domain and the corresponding boundary conditions are clearly described later. We also assume that the deformation is static, and thus the field equations for such a solid consist of [7]:
(a) *Equilibrium equations* (*including Gauss equation*)*:*

$$\sigma_{ji,j} + f_i = 0 \qquad D_{i,i} - q = 0, \tag{2.1}$$

where $\sigma_{\rm ij}$ and $D_{\rm i}$ are the stress and electric displacement, respectively; $f_{\rm i}$ and q are the body force and electric charge, respectively. In this and the following sections, summation from 1 to 3 (1 to 4) over repeated lowercase (uppercase) subscripts is implied. A subscript comma denotes the partial differentiation.

In the Cartesian coordinate system, the equilibrium equations are

$$\begin{aligned}
&\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} + f_x = 0 \\
&\frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} + f_y = 0 \\
&\frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{zy}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + f_z = 0 \qquad (2.2a) \\
&\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} - q = 0\,. \qquad (2.2b)
\end{aligned}$$

In the cylindrical coordinate system, the equilibrium equations are

$$\begin{aligned}
&\frac{\partial \sigma_{rr}}{\partial r} + \frac{\partial \sigma_{r\theta}}{r\partial \theta} + \frac{\partial \sigma_{rz}}{\partial z} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} + f_r = 0 \\
&\frac{\partial \sigma_{r\theta}}{\partial r} + \frac{\partial \sigma_{\theta\theta}}{r\partial \theta} + \frac{\partial \sigma_{\theta z}}{\partial z} + \frac{2\sigma_{r\theta}}{r} + f_\theta = 0 \qquad (2.3a) \\
&\frac{\partial \sigma_{rz}}{\partial r} + \frac{\partial \sigma_{\theta z}}{r\partial \theta} + \frac{\partial \sigma_{zz}}{\partial z} + \frac{\sigma_{rz}}{r} + f_z = 0 \\
&\frac{\partial D_r}{\partial r} + \frac{\partial D_\theta}{r\partial \theta} + \frac{\partial D_z}{\partial_z} - q = 0\,. \qquad (2.3b)
\end{aligned}$$

(b) *Constitutive relations:*

$$\sigma_{ij} = C_{ijlm}\gamma_{lm} - e_{kji}E_k \qquad D_i = e_{ijk}\gamma_{jk} + \varepsilon_{ij}E_j, \tag{2.4}$$

where $\gamma_{\rm ij}$ is the strain and $E_{\rm i}$ the electric field; $C_{\rm ijlm}$, $e_{\rm ijk}$, and $\varepsilon_{\rm ij}$ are the elastic moduli, piezoelectric coefficients, and dielectric constants, respectively. The uncoupled state (purely elastic and purely electric deformation) can be obtained by simply setting $e_{\rm ijk} = 0$. For transversely isotropic piezoelectric materials with the z-axis being the material symmetric (or the poling) axis,

the constitutive relation in the Cartesian coordinate system is (using the reduced indices for C_{ijkl} and e_{ijk}, with the following correspondence between the one and two indices: 1 = 11, 2 = 22, 3 = 33, 4 = 23, 5 = 13, 6 = 12)

$$\begin{aligned}
\sigma_{xx} &= C_{11}\gamma_{xx} + C_{12}\gamma_{yy} + C_{13}\gamma_{zz} - e_{31}E_z \\
\sigma_{yy} &= C_{12}\gamma_{xx} + C_{11}\gamma_{yy} + C_{13}\gamma_{zz} - e_{31}E_z \\
\sigma_{zz} &= C_{13}\gamma_{xx} + C_{13}\gamma_{yy} + C_{33}\gamma_{zz} - e_{33}E_z \\
\sigma_{yz} &= 2C_{44}\gamma_{yz} - e_{15}E_y \\
\sigma_{xz} &= 2C_{44}\gamma_{xz} - e_{15}E_x \\
\sigma_{xy} &= 2C_{66}\gamma_{xy} \qquad (2.5a) \\
D_x &= 2e_{15}\gamma_{xz} + \varepsilon_{11}E_x \\
D_y &= 2e_{15}\gamma_{yz} + \varepsilon_{11}E_y \\
D_z &= e_{31}(\gamma_{xx} + \gamma_{yy}) + e_{33}\gamma_{zz} + \varepsilon_{33}E_z\,, \qquad (2.5b)
\end{aligned}$$

where $C_{66} = (C_{11} - C_{12})/2$.

Similarly, in the cylindrical coordinate system, the constitutive relation is

$$\begin{aligned}
\sigma_{rr} &= C_{11}\gamma_{rr} + C_{12}\gamma_{\theta\theta} + C_{13}\gamma_{zz} - e_{31}E_z \\
\sigma_{\theta\theta} &= C_{12}\gamma_{rr} + C_{11}\gamma_{\theta\theta} + C_{13}\gamma_{zz} - e_{31}E_z \\
\sigma_{zz} &= C_{13}\gamma_{rr} + C_{13}\gamma_{\theta\theta} + C_{33}\gamma_{zz} - e_{33}E_z \\
\sigma_{\theta z} &= 2C_{44}\gamma_{\theta z} - e_{15}E_\theta \\
\sigma_{rz} &= 2C_{44}\gamma_{rz} - e_{15}E_r \\
\sigma_{r\theta} &= 2C_{66}\gamma_{r\theta} \qquad (2.6a) \\
D_r &= 2e_{15}\gamma_{rz} + \varepsilon_{11}E_r \\
D_\theta &= 2e_{15}\gamma_{\theta z} + \varepsilon_{11}E_\theta \\
D_z &= e_{31}(\gamma_{rr} + \gamma_{\theta\theta}) + e_{33}\gamma_{zz} + \varepsilon_{33}E_z\,. \qquad (2.6b)
\end{aligned}$$

(c) *Elastic strain-displacement and electric field-potential relations:*

$$\gamma_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \qquad E_i = -\phi_{,i}\,, \qquad (2.7)$$

where u_i and ϕ are the elastic displacement and electric potential, respectively.

In the Cartesian coordinate system, we have

$$\gamma_{xx} = \frac{\partial u_x}{\partial x}, \qquad \gamma_{yy} = \frac{\partial u_y}{\partial y}, \qquad \gamma_{zz} = \frac{\partial u_z}{\partial z} \qquad \gamma_{yz} = 0.5\left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right)$$

$$\gamma_{xz} = 0.5\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right) \qquad \gamma_{xy} = 0.5\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) \qquad (2.8a)$$

$$E_x = -\frac{\partial \phi}{\partial x}, \qquad E_y = -\frac{\partial \phi}{\partial y}, \qquad E_z = -\frac{\partial \phi}{\partial z}\,, \qquad (2.8b)$$

and in cylindrical coordinate system, we obtain

$$\gamma_{rr} = \frac{\partial u_r}{\partial r}, \qquad \gamma_{\theta\theta} = \frac{\partial u_\theta}{r\partial\theta} + \frac{u_r}{r}, \qquad \gamma_{zz} = \frac{\partial u_z}{\partial z}$$
$$\gamma_{\theta z} = 0.5\left(\frac{\partial u_\theta}{\partial z} + \frac{\partial u_z}{r\partial\theta}\right) \qquad \gamma_{rz} = 0.5\left(\frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z}\right)$$
$$\gamma_{r_\theta} = 0.5\left(\frac{\partial u_r}{r\partial\theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right) \tag{2.9a}$$
$$E_r = -\frac{\partial\phi}{\partial r}, \qquad E_\theta = -\frac{\partial\phi}{r\partial\theta}, \qquad E_z = -\frac{\partial\phi}{\partial z}. \tag{2.9b}$$

The notation introduced by Barnett and Lothe [8] has been shown to be very convenient for the analysis of piezoelectric problems. With this notation, the elastic displacement and electric potential, the elastic strain and electric field, the stress and electric displacement, and the elastic and electric moduli (or coefficients) can be grouped together as [9]

$$u_I = \begin{cases} u_i & I = i = 1,2,3 \\ \phi & I = 4 \end{cases} \tag{2.10}$$

$$\gamma_{Ij} = \begin{cases} \gamma_{ij} & I = i = 1,2,3 \\ -E_j & I = 4 \end{cases} \tag{2.11}$$

$$\sigma_{iJ} = \begin{cases} \sigma_{ij} & J = j = 1,2,3 \\ D_i & J = 4 \end{cases} \tag{2.12a}$$

$$T_J = \sigma_{iJ}n_i = \begin{cases} \sigma_{ij}n_i & J = j = 1,2,3 \\ D_i n_i & J = 4 \end{cases} \tag{2.12b}$$

$$C_{iJKl} = \begin{cases} C_{ijkl} & J,K = j,k = 1,2,3 \\ e_{lij} & J = j = 1,2,3; K = 4 \\ e_{ikl} & J = 4; K = k = 1,2,3 \\ -\varepsilon_{il} & J,K = 4 \end{cases}. \tag{2.13}$$

It is noted that we have kept the original symbols instead of introducing new ones because they can be easily distinguished by the range of their subscript. In terms of this shorthand notation, the constitutive relations can be unified into a single equation as

$$\sigma_{iJ} = C_{iJKl}\gamma_{Kl}, \tag{2.14}$$

where the material property coefficients C_{iJKl} can be location-dependent in the region.

Similarly, the equilibrium equations in terms of the extended stresses can be recast into

$$\sigma_{iJ,i} + f_J = 0 \tag{2.15}$$

with the extended force f_{J} being defined as

$$f_J = \begin{cases} f_j & J = j = 1, 2, 3 \\ -q & J = 4 \end{cases}. \tag{2.16}$$

For the Green's function solutions, the body force and electric charge density are replaced by the following concentrated unit sources ($k = 1, 2, 3$),

$$f_I = \begin{cases} \delta_{ik}\delta(\boldsymbol{x} - \boldsymbol{x}_0), & I = i = 1, 2, 3 \\ \delta(\boldsymbol{x} - \boldsymbol{x}_0), & I = 4. \end{cases}. \tag{2.17}$$

It is observed that Equations (2.14) and (2.15) are exactly the same as their purely elastic counterparts. The only difference is the dimension of the index of the involved quantities. Therefore, the solution method developed for anisotropic elasticity can be directly applied to the piezoelectric case. For ease of reference, in this chapter, we still use *displacement* to stand for the elastic displacement and electric potential as defined in Equation (2.10), use *stress* for the stress and electric displacement as defined in Equation (2.12a), and use *traction* for the elastic traction and normal electric displacement as defined in Equation (2.12b).

2.3 Relations Among Different Sources and Their Responses

Relations among different concentrated sources and their responses can be studied via Betti's reciprocal theorem, which states that for two systems (1) and (2) belonging to the same material space, the following relation holds (i.e., [9])

$$\sigma_{iJ}^{(1)} u_{J,i}^{(2)} = \sigma_{iJ}^{(2)} u_{J,i}^{(1)}. \tag{2.18}$$

From (2.18), one can easily derive the following integral equation for these two systems

$$\int_S \sigma_{iJ}^{(1)} u_J^{(2)} n_i dS - \int_V \sigma_{iJ,i}^{(1)} u_J^{(2)} dV = \int_S \sigma_{iJ}^{(2)} u_J^{(1)} n_i dS - \int_V \sigma_{iJ,i}^{(2)} u_J^{(1)} dV. \tag{2.19}$$

We let system (1) be the real boundary value problem and (2) be the corresponding "point-force" Green's function problem; that is,

$$\sigma_{iJ,i} = -\delta_{JK}\delta(x_p^f - x_p^s), \tag{2.20}$$

where the field point is at x_p^f and the extended point force is applied at x_p^s in the K-direction (with $K = 4$ corresponding to a negative electric charge). Then (2.19) can be reduced to a well-known integral representation of the displacement field:

$$u_K(x_p^s) = \int_S [\sigma_{iJ}(x_p^f) u_J^{fK}(x_p^s; x_p^f) = \sigma_{iJ}^{fK}(x_p^s; x_p^f) u_J(x_p^f)] n_i(x_p^f) dS(x_p^f) + \int_V u_J^{fK}(x_p^s; x_p^f) f_J(x_p^f) dV(x_p^f), \tag{2.21}$$

where in the Green's function expressions, the first superscript f denotes that the Green's function corresponds to an extended point force, and the second superscript K is the direction of the point force.

Now, we wish to find the displacement response due to a prescribed dislocation (displacement discontinuity) across a surface Σ embedded in V (or the dislocation Green's function). Let $n_{\rm i} (= n_{\rm i}^- = -n_{\rm i}^+)$ be the unit normal to Σ, $b_{\rm I} = u_{\rm I}^+ - u_{\rm I}^-$ being the (extended) dislocation. This dislocation along Σ may have any form provided that the following (extended) traction continuity condition holds.

$$\sigma_{iJ}^+ n_i^+ + \sigma_{iJ}^- n_i^- = 0. \tag{2.22}$$

This type of displacement discontinuity is also called a Somigliana dislocation with the Volterra dislocation (or the dislocation of Volterra–Weingarten) being its special case [10]. In the latter case,

$$\triangle u_I \equiv u_I^+ - u_I^- = \begin{cases} U_i + \Omega_{ij} x_j^s; & I \le 3, \\ U_4; & I = 4 \end{cases} \tag{2.23}$$

where $U_{\rm I}$ and $\Omega_{\rm ij}$ are constants. If, furthermore, $\Omega_{ij} = 0$, the dislocation then reduces to the (extended) Burger's vector. Assume that the displacement and stress fields satisfy the same homogeneous boundary condition on the outer boundary S, and apply Equation (2.21) to the region bounded internally by Σ and externally by S; we then come to (also omit the volumetric integral term associated with the body force)

$$u_K(x_p^s) = \int_\Sigma \sigma_{iJ}^{fK}(x_p^s; x_p^f) b_J(x_p^f) n_i(x_p^f) d\Sigma(x_p^f). \tag{2.24}$$

This is the integral expression of the displacement due to a dislocation along Σ with its density tensor being defined as

$$D(i, J) \equiv n_i(x_p^f) b_J(x_p^f) d\Sigma(x_p^f). \tag{2.25}$$

In (elastic) seismology, they are defined as fault elements $d\Sigma$ with an outward normal $n_{\rm i}$ (defined with respect to the positive side of the fault) having a displacement discontinuity $\triangle_{\rm i} = b_{\rm i}$. It is noted that the kernel function in Equation (2.24) is the Green's stress with component (iJ) at the

field point x_p^f due to a point force at x_p^s in the Kth direction. Alternatively, the displacement response due to the dislocation density tensor can also be expressed by the kernel displacement function due to a point dislocation, namely,

$$u_K(x_p^f) = \int_\Sigma u_K^{diJ}(x_p^s; x_p^f) b_J(x_p^s) n_i(x_p^s) d\Sigma(x_p^s), \tag{2.26}$$

where the superscript d denotes dislocation and (i, J) is the normal direction of the dislocation plane (i) and the Burger's vector direction (J). Comparing Equation (2.26) to (2.24), we immediately obtain the following important equivalence between the stress due to a point force and the displacement due to a point dislocation

$$u_K^{diJ}(x_p^s; x_p^f) = \sigma_{iJ}^{fK}(x_p^f; x_p^s). \tag{2.27}$$

That is, the position of the source and field points in the point-force Green's stresses need to be exchanged in order to obtain the point-dislocation Green's displacements. This is a most simple and yet very important relation. Similar results for poroelastic media were derived by Pan [11]. Several important observations of (2.27) are listed below:

(a) In general, once the point-force Green's functions are solved, the corresponding point-dislocation Green's functions can be obtained through the relation (2.27). In deriving relation (2.27), we have assumed that the system is linear piezoelectric, but can be of general *anisotropy* and *heterogeneity*. In particular, this relation can be used to derive the point-dislocation Green's functions in horizontally layered systems, including half-space and bimaterial domains as special cases. For example, the point-dislocation Green's functions in horizontally layered media can be derived in both the Fourier transform or the physical domains using Equation (2.27) and the point force solutions [12–14].

(b) Direct solution of the point-dislocation Green's functions is also possible but the procedure may be very complicated. The way to achieve this is to derive the equivalent body force of the point dislocation, find the related discontinuity of the physical quantities, and solve for the unknowns, using the method as previously employed by Pan [12] for the transversely isotropic and layered half-space.

(c) For the elastic isotropic or transversely isotropic bimaterial, half-space, or full-space, each term on the right-hand side of Equation (2.27) is proportional to various eigenstrains, such as the misfit lattice strain, the nucleus of strain (or a nucleus of strain multiplied by the elastic constants), and so on. With Equation (2.27), however, it is unnecessary to add all the related nuclei of strain together and enforce the boundary or interface condition to solve the coefficients involved.

(d) In using Equation (2.27), one must be very careful that on the left-hand side, x_p^s and x_p^f are the source and field points, respectively; and on the right-hand side, x_p^f and x_p^s are the field and source points, respectively.

Therefore, the Green's displacements due to a point dislocation can be obtained from the Green's stresses due to a point force by exchanging the position of the field and source points and by assigning the suitable meanings to the associated indexes.

(e) For a homogeneous and infinite domain, expressing the point-force Green's stresses by the strain and substituting the result back to Equation (2.26), we then have the Volterra relation. It is noted that for this specific case, the point-force Green's stresses are functions of the relative vector from the source to field points $\left(\text{i.e., } x_p^f - x_p^s\right)$, and they satisfy the following relation,

$$\sigma_{iJ}^{fK}(x_P^f; x_p^s) = \sigma_{iJ}^{fK}(x_p^s - x_p^f) = -\sigma_{iJ}^{fK}(x_p^f - x_p^s) = -\sigma_{iJ}^{fK}(x_p^s; x_p^f). \quad (2.28)$$

We therefore have

$$u_K^{diJ}(x_p^s; x_p^f) = -\sigma_{iJ}^{fK}(x_p^s; x_p^f). \quad (2.29)$$

It should be emphasized that only for the homogeneous infinite domain, can the Green's displacements due to a point dislocation be obtained directly from the Green's stresses due to a point force, without exchanging the field and source positions! For all other situations, the dislocation-induced Green displacements should be obtained strictly using Equation (2.27).

For a homogeneous and infinite solid of purely elastic isotropy, (2.29) is reduced to

$$u_k^{dij}(x_p^s; x_p^f) = -\sigma_{ij}^{fk}(x_p^s; x_p^f) = -\lambda u_{l,l}^k \delta_{ij} - \mu(u_{i,j}^k + u_{j,i}^k), \quad (2.30)$$

where λ and μ are the two Lame's elastic constants. The derivatives of the Green's elastic displacements due to a point force at x_p^s in the k-direction are taken with respect to the field point x_p^f.

On the other hand, if we take the derivatives of the point-force Green displacements with respect to the source point x_p^s, we then have

$$\varepsilon^{ij}(x_p^s; x_p^f) = \lambda u_{l,l}^k \delta_{ij} + \mu(u_{i,j}^k + u_{j,i}^k), \quad (2.31)$$

which has an opposite sign to that given by (2.30).

The components in (2.31) are called nuclei of strain by Mindlin [15]. The first term corresponds to the center of compression or dilatation and the other terms are double forces. Therefore, from the physical point of view, the point-dislocation Green's functions can be constructed through super position of various nuclei of strain (or the derivatives of the point-force Green's displacements), with their coefficients being solely related to the elastic constants. This is a physical explanation for the mathematical and arbitrary equivalent relation (2.29). It is obvious that if the point-force Green's functions can be derived in an exact closed-form (or explicit form), the

corresponding point dislocation solutions will also have the same features because they are obtained by the superposition of various nuclei of strain. Detailed analyses can be found in [10, 16, 17] for the isotropic elastic case and in [18] for the transversely isotropic elastic case. For materials of either isotropy or transverse isotropy, these exact closed-form nuclei of strain can also be employed to derive the point-force Green's functions in a half-space or a bimaterial space, and to derive the solutions corresponding to various inclusions.

It is noted that the Green's function relations between those due to a point dislocation and those due to a point force are applicable to 3D only. For the 2D case, the Green's functions due to the line force and those due to the line dislocations (open all the way to the half-infinite line) have the same singularity order. These are clearly observed in the following section on 2D Green's functions, and one should pay particular attention to this difference.

2.4 Green's Functions in Anisotropic Two-Dimensional Infinite, Half, and Bimaterial Planes

Our two-dimensional (2D) problem is in the x–z plane, and it is under the assumption that all the field and source quantities are independent of the y-variable (i.e., $\partial()/\partial y = 0$). Therefore, the Green's functions presented are rigorously for the generalized 2D plane strain case. Furthermore, presented below are only the displacements and tractions (on the $z =$ constant plane) based on the Stroh formalism in terms of the complex variables. Summation of the repeated subscript R from 1 to 4 is implied.

2.4.1 Green's Functions in Anisotropic 2D Infinite Planes Due to a Line Force and Line Dislocation

The Green's functions for the displacements and tractions at field point $\boldsymbol{x}(x, z)$ due to a line force at $\boldsymbol{X}(X, Z)$ can be expressed as

$$U_{KJ}(\boldsymbol{x}, \boldsymbol{X}) = \frac{1}{\pi} \operatorname{Im}\{A_{JR} \ln(z_R - s_R) A_{KR}\} \tag{2.32a}$$

$$T_{KJ}(\boldsymbol{x}, \boldsymbol{X}) = -\frac{1}{\pi} \operatorname{Im}\{B_{JR} \frac{p_R n_1 - n_3}{z_R - s_R} A_{KR}\}. \tag{2.32b}$$

The first subscript K is the component of the line force of unit value $\boldsymbol{f} = (f_1, f_2, f_3, -Q)$ and the second subscript J is the component of the displacement (2.32a) and the traction (2.32b). Also in these equations, "Im" stands for the imaginary part of the complex value; A_{IJ} and B_{IJ} are two constant matrices related to the piezoelectric material property; n_1 and n_3 (functions of $\boldsymbol{x}$) are the unit outward normal components along the x- and

z-directions; p_R $(R = 1, 2, 3, 4)$ are the Stroh eigenvalues; and $z_R = x + p_R z$ and $s_R = X + p_R Z$ are related to the field $\boldsymbol{x}(x, z)$ and source $\boldsymbol{X}(X, Z)$ points, respectively. These displacement and traction Green's functions are required in the conventional boundary integral equation formulation to solve the general boundary value problems in piezoelectric solids. In order to find the Green's strain and electric fields, one only needs to take the derivation of the Green's displacement (2.32a) with respect to the field point $\boldsymbol{x}$ (refer to (2.7)). The corresponding stress and electric displacements can be obtained thorough the piezoelectric constitutive relations (2.14).

The Stroh eigenvalues and eigenmatrices involved in (2.32) are obtained by solving the following eigensystem of equations [19]. First, the eigenvalue p and the corresponding eigenvector $\boldsymbol{a}$ are solved from the eigenrelation:

$$[\boldsymbol{Q} + p(\boldsymbol{R} + \boldsymbol{R}^T) + p^2\boldsymbol{T}]\boldsymbol{a} = 0, \tag{2.33}$$

where the superscript T denotes matrix transpose, and

$$Q_{IK} = C_{1IK1}, \qquad R_{IK} = C_{1IK3}, \qquad T_{IK} = C_{3IK3}, \tag{2.34}$$

where C_{iJKl} are the elastic and electric moduli (or coefficients) defined in (2.13).

Then, the eigenvector $\boldsymbol{b}$ is obtained from

$$\boldsymbol{b} = (\boldsymbol{R}^T + p\boldsymbol{T})\boldsymbol{a} = -\frac{1}{p}(\boldsymbol{Q} + p\boldsymbol{R})\boldsymbol{a}. \tag{2.35}$$

Denoting by p_m, $\boldsymbol{a}_m$, and $\boldsymbol{b}_m$ $(m = 1, 2, \ldots, 8)$ the eigenvalues and the associated eigenvectors, we then order them in such a way so that

$$\begin{aligned} &\text{Im}\, p_J > 0, \qquad p_{J+4} = \overline{p}_J, \qquad \boldsymbol{a}_{J+4} = \overline{\boldsymbol{a}}_J, \qquad \boldsymbol{b}_{J+4} = \overline{\boldsymbol{b}}_J \qquad (J = 1, 2, 3, 4) \\ &\boldsymbol{A} = [\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3, \boldsymbol{a}_4], \qquad \boldsymbol{B} = [\boldsymbol{b}_1, \boldsymbol{b}_2, \boldsymbol{b}_3, \boldsymbol{b}_4], \end{aligned} \tag{2.36}$$

where an overbar denotes the complex conjugate. We have also assumed that p_J are distinct and the eigenvectors $\boldsymbol{a}_J$ and $\boldsymbol{b}_J$ satisfy the normalization relation [8, 19]

$$\boldsymbol{b}_I^T\boldsymbol{a}_J + \boldsymbol{a}_I^T\boldsymbol{b}_J = \delta_{IJ} \tag{2.37}$$

with δ_{iJ} being the 4×4 Kronecker delta (i.e., the 4×4 identity matrix). We also remark that repeated eigenvalues p_J can be avoided by using slightly perturbed material coefficients with negligible errors [20]. In so doing, the simple structure of the Green's function solutions (2.32) can always be employed.

Similarly, the Green's functions at $\boldsymbol{x}$ due to the generalized line dislocations (Burger's vector and electric potential discontinuity) $\boldsymbol{b} = (\triangle u_1, \triangle u_2, \triangle u_3, \triangle\phi)$ of unit value at $\boldsymbol{X}$ can be found as

$$U_{KJ}(\boldsymbol{x}, \boldsymbol{X}) = \frac{1}{\pi}\text{Im}\,\{A_{JR}\ln(z_R - s_R)B_{KR}\} \tag{2.38a}$$

$$T_{KJ}(\boldsymbol{x},\boldsymbol{X}) = -\frac{1}{\pi}\mathrm{Im}\left\{B_{JR}\frac{p_R n_1 - n_3}{z_R - s_R}B_{KR}\right\}. \tag{2.38b}$$

Comparing (2.32) and (2.38), we notice that the line force and line dislocation Green's functions are very similar to each other and they both have the same order of singularity.

2.4.2 Green's Functions in Anisotropic 2D Half-Planes Due to a Line Force and Line Dislocation

The half-plane Green's functions for the displacements and tractions (the Jth component) with outward normal n_1 and n_3 (at $\boldsymbol{x}$) due to a line force at $\boldsymbol{X}$ with component K can be expressed as

$$U_{KJ}(\boldsymbol{x},\boldsymbol{X}) = \frac{1}{\pi}\mathrm{Im}\left\{A_{JR}\ln(z_R - s_R)A_{KR} + \sum_{v=1}^{4}\left[A_{JR}\ln(z_R - \overline{s}_v)Q^v_{RK}\right]\right\} \tag{2.39a}$$

$$T_{KJ}(\boldsymbol{x},\boldsymbol{X}) = -\frac{1}{\pi}\mathrm{Im}\left\{B_{JR}\frac{p_R n_1 - n_3}{z_R - s_R}A_{KR} + \sum_{v=1}^{4}\left[B_{JR}\frac{p_R n_1 - n_3}{z_R - \overline{s}_v}Q^v_{RK}\right]\right\}, \tag{2.39b}$$

where

$$Q^v_{RN} = B^{-1}_{RS}\overline{B}_{SP}(I_v)_P\overline{A}_{NP} \tag{2.40}$$

with

$$\begin{aligned} \boldsymbol{I}_1 &= \mathrm{diag}[1,0,0,0]; \qquad \boldsymbol{I}_2 = \mathrm{diag}[0,1,0,0] \\ \boldsymbol{I}_3 &= \mathrm{diag}[0,0,1,0]; \qquad \boldsymbol{I}_4 = \mathrm{diag}[0,0,0,1]. \end{aligned} \tag{2.41}$$

Similarly, the half-plane Green's functions for the displacements and tractions (the Jth component) with outward normal n_1 and n_3 (at $\boldsymbol{x}$) due to the line dislocations at $\boldsymbol{X}$ with component K can be expressed as

$$U_{KJ}(\boldsymbol{x},\boldsymbol{X}) = \frac{1}{\pi}\mathrm{Im}\left\{A_{JR}\ln(z_R - s_R)B_{KR} + \sum_{v=1}^{4}\left[A_{JR}\ln(z_R - \overline{s}_v)Q^v_{RK}\right]\right\} \tag{2.42a}$$

$$T_{KJ}(\boldsymbol{x},\boldsymbol{X}) = -\frac{1}{\pi}\mathrm{Im}\left\{B_{JR}\frac{p_R n_1 - n_3}{z_R - s_R}B_{KR} + \sum_{v=1}^{4}\left[B_{JR}\frac{p_R n_1 - n_3}{z_R - \overline{s}_v}Q^v_{RK}\right]\right\}, \tag{2.42b}$$

where

$$Q^v_{RN} = B^{-1}_{RS}\overline{B}_{SP}(I_v)_P\overline{B}_{NP}. \tag{2.43}$$

Similar Green's function expressions can be obtained for the general boundary conditions on the surface of the anisotropic elastic and anisotropic piezoelectric half-plane. The detailed discussions can be found in [21, 22].

2.4.3 Green's Functions in 2D Anisotropic Bimaterial Plane Due to a Line Force and Line Dislocation

Depending upon the relative locations of the source and field points, there are four sets of Green's functions for the bimaterial case. We assume that materials 1 and 2 occupy the half-planes $z > 0$ and $z < 0$, respectively. Let us again assume that a line force $\boldsymbol{f} = (f_1, f_2, f_3, -Q)$ or line dislocation $\boldsymbol{b} = (\triangle u_1, \triangle u_2, \triangle u_3, \triangle\phi)$ is applied at the source point (X, Z) in one of the half-planes. To derive the Green's functions, it is sufficient to find the displacement vector $\boldsymbol{u}$ and traction vector $\boldsymbol{t}$ due to the line force or dislocation [19], which are presented below for different combinations of the source and field points.

Assume that the source point (X, Z) is in the half-plane of material $\lambda(\lambda = 1 \text{ or } 2)$. Then if the field point $\boldsymbol{x} = (x, z)$ is in the source plane (i.e., the half-plane of material λ), the displacement and traction vectors can be expressed as [23]

$$\begin{aligned}
\boldsymbol{u}^{(\lambda)} &= \frac{1}{\pi}\mathrm{Im}\left\{\boldsymbol{A}^{(\lambda)}\langle\ln(z^{(\lambda)}_* - s^{(\lambda)}_*)\rangle\,\boldsymbol{q}^{\infty,\lambda}\right\} + \frac{1}{\pi}\mathrm{Im}\sum_{J=1}^{4}\left\{\boldsymbol{A}^{(\lambda)}\langle\ln(z^{(\lambda)}_* - \overline{s}^{(\lambda)}_J)\rangle\,\boldsymbol{q}^{(\lambda)}_J\right\}\\
\boldsymbol{t}^{(\lambda)} &= -\frac{1}{\pi}\mathrm{Im}\left\{\boldsymbol{B}^{(\lambda)}\left\langle\frac{p^{(\lambda)}_* n_1 - n_3}{z^{(\lambda)}_* - s^{(\lambda)}{}_*}\right\rangle\boldsymbol{q}^{\infty,\lambda}\right\} - \frac{1}{\pi}\mathrm{Im}\sum_{J=1}^{4}\left\{\boldsymbol{B}^{(\lambda)}\left\langle\frac{p^{(\lambda)}_* n_1 - n_3}{z^{(\lambda)}_* - \overline{s}^{(\lambda)}_*}\right\rangle\boldsymbol{q}^{(\lambda)}_J\right\}.
\end{aligned} \tag{2.44}$$

If the field point (x, z) is in the other half-plane of material $\mu(\mu \neq \lambda)$ $(\lambda, \mu = 1 \text{ or } 2)$, then the displacement and traction vectors can be expressed as

$$\begin{aligned}
\boldsymbol{u}^{(\mu)} &= \frac{1}{\pi}\mathrm{Im}\sum_{J=1}^{4}\left\{\boldsymbol{A}^{(\mu)}\langle\ln(z^{(\mu)}_* - s^{(\lambda)}_J)\rangle\boldsymbol{q}^{(\mu)}_J\right\}\\
\boldsymbol{t}^{(\mu)} &= -\frac{1}{\pi}\mathrm{Im}\sum_{J=1}^{4}\left\{\boldsymbol{B}^{(\mu)}\left\langle\frac{p^{(\mu)}_* n_1 - n_3}{z^{(\mu)}_* - s^{(\lambda)}_J}\right\rangle\boldsymbol{q}^{(\mu)}_J\right\}.
\end{aligned} \tag{2.45}$$

In (2.44) and (2.45), the superscripts (λ) and (μ) denote the quantities associated with the material domains 1 and 2; $p^{(\lambda)}_J$, $\boldsymbol{A}^{\lambda}$, and $\boldsymbol{B}^{(\lambda)}$ ($\lambda = 1$ and 2 for the two half-planes) are the Stroh eigenvalues and the corresponding

eigenmatrices as given before. Also in (2.44) and (2.45), we defined:

$$\begin{aligned}\langle \ln(z_*^{(\lambda)} - s_*^{(\lambda)})\rangle &= \mathrm{diag}[\ln(z_1^{(\lambda)} - s_1^{(\lambda)}), \ln(z_2^{(\lambda)} - s_2^{(\lambda)}),\\ &\qquad \ln(z_3^{(\lambda)} - s_3^{(\lambda)}), \ln(z_4^{(\lambda)} - s_4^{(\lambda)})]\\ \langle \ln(z_*^{(\lambda)} - \overline{s}_*^{(\lambda)})\rangle &= \mathrm{diag}[\ln(z_1^{(\lambda)} - \overline{s}_J^{(\lambda)}), \ln(z_2^{(\lambda)} - \overline{s}_J^{(\lambda)}),\\ &\qquad \ln(z_3^{(\lambda)} - \overline{s}_J^{(\lambda)}), \ln(z_4^{(\lambda)} - \lambda\overline{s}_J^{(\lambda)})],\end{aligned} \tag{2.46}$$

where z_J^{α} and $s_J^{(\alpha)}$ $(\alpha = 1, 2)$ are complex variables associated with the field and source points, respectively. They are defined as

$$\begin{aligned} z_J^{(\alpha)} &= x + p_J^{(\alpha)} z,\\ s_J^{(\alpha)} &= X + p_J^{(\alpha)} Z. \end{aligned} \tag{2.47}$$

We further observe that the first term in (2.44) corresponds to the full-plane Green's functions in material λ with:

$$\boldsymbol{q}^{\infty,\lambda} = (\boldsymbol{A}^{(\lambda)})^T \boldsymbol{f} \tag{2.48}$$

for the line force, and

$$\boldsymbol{q}^{\infty,\ \lambda} = (\boldsymbol{B}^{(\lambda)})^T \boldsymbol{b} \tag{2.49}$$

for the line dislocation.

The second term in (2.44) and the term in (2.45) are the complementary parts of the Green's function solutions. The complex vectors $\boldsymbol{q}_J^{(\lambda)}$ $(\lambda = 1, 2;\ J = 1, 2, 3, 4)$ in (2.44) and $q_J^{(\mu)}(\mu = 1, 2;\ J = 1, 2, 3, 4)$ in (2.45) are determined using the continuity conditions along the interface of the two half-planes. Assume that the interface is perfect and after certain algebraic calculations, these vectors can be obtained as ($\lambda, \mu = 1$ or 2, but $\mu \neq \lambda$):

$$\boldsymbol{q}_J^{(\lambda)} = (\boldsymbol{A}^{(\lambda)})^{-1}(\boldsymbol{M}^{(\lambda)} + \overline{\boldsymbol{M}}^{(\mu)})^{-1}(\overline{\boldsymbol{M}}^{(\mu)} - \overline{\boldsymbol{M}}^{(\lambda)})\overline{\boldsymbol{A}}^{(\lambda)} \boldsymbol{I}_J \overline{\boldsymbol{q}}^{\infty,\lambda} \tag{2.50}$$

for (2.44), and

$$\boldsymbol{q}_J^{(\mu)} = (\boldsymbol{A}^{(\mu)})^{-1}(\overline{\boldsymbol{M}}^{(\lambda)} + \boldsymbol{M}^{(\mu)})^{-1}(\boldsymbol{M}^{(\lambda)} + \overline{\boldsymbol{M}}^{(\lambda)})\boldsymbol{A}^{(\lambda)} \boldsymbol{I}_J \boldsymbol{q}^{\infty,\lambda} \tag{2.51}$$

for (2.45).

In (2.50) and (2.51), matrix $\boldsymbol{M}^{(\alpha)}$ is the impedance tensor defined as

$$\boldsymbol{M}^{(\alpha)} = -i\boldsymbol{B}^{(\alpha)}(\boldsymbol{A}^{(\alpha)})^{-1} \qquad (\alpha = 1, 2) \tag{2.52}$$

and the diagonal matrix $\boldsymbol{I}_J$ is the same as that defined by (2.41).

We point out that similar Green's function expressions can be derived for the bimaterial with general (or imperfect) interface conditions. Detailed discussion can be found in [24, 25].

2.5 Green's Functions in Three-Dimensional Infinite, Half, and Bimaterial Spaces: Transverse Isotropy

Green's functions in 3D transversely isotropic piezoelectric infinite, half, and bimaterial spaces were derived by Ding's group at Zhejiang University [26–34] and Dunn's group at the University of Colorado at Boulder [35–37]. However, results presented below are based on the works by the Zhejiang University group using the combined potential function and trial-and-error method (i.e., [32]). To facilitate the discussion, we first present the basic equations with special parameters associated with transversely isotropic materials only.

For the transversely isotropic piezoelectric material with its poling direction along the z-axis, the corresponding constitutive relation is the one given by (2.5) or (2.6). For this material, the characteristic equation is separated into two: one corresponds to the antiplane deformation and another to the in-plane deformation. For the antiplane case, the characteristic root is given by

$$s_0 = \sqrt{c_{66}/c_{44}}. \tag{2.53}$$

For the in-plane case, its three characteristic roots, $s_i (i = 1, 2, 3)$, are the solutions of the following characteristic equation

$$as^6 - bs^4 + cs^2 - d = 0, \tag{2.54}$$

where

$$\begin{aligned}
a &= c_{44}(e_{33}^2 + c_{33}\varepsilon_{33}) \\
b &= c_{33}[c_{44}\varepsilon_{11} + (e_{15} + e_{31})^2] + \varepsilon_{33}[c_{11}c_{33} + (c_{44}^2 - (c_{13} + c_{44})^2)] \\
&\quad + e_{33}[2c_{44}e_{15} + c_{11}e_{33} - 2(c_{13} + c_{44})(e_{15} + e_{31})] \\
c &= c_{44}[c_{11}\varepsilon_{33} + (e_{15} + e_{31})^2] + \varepsilon_{11}[c_{11}c_{33} + (c_{44}^2 - (c_{13} + c_{44})^2)] \\
&\quad + e_{15}[2c_{11}e_{33} + c_{44}e_{15} - 2(c_{13} + c_{44})(e_{15} + e_{31})] \\
d &= c_{11}(e_{15}^2 + c_{44}\varepsilon_{11}).
\end{aligned} \tag{2.55}$$

Other parameters used in this section are:

$$\begin{aligned}
&m_1 = \varepsilon_{11}(c_{13} + c_{44}) + e_{15}(e_{15} + e_{31}) \qquad m_2 = \varepsilon_{33}(c_{13} + c_{44}) + e_{33}(e_{15} + e_{31}) \\
&m_3 = c_{11}\varepsilon_{33} + c_{44}\varepsilon_{11} + (e_{15} + e_{31})^2 \quad m_4 = c_{11}e_{33} + c_{44}e_{15} - (c_{13} + c_{44}) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \times (e_{15} + e_{31})
\end{aligned} \tag{2.56}$$

$$\begin{aligned}
\alpha_{i1} &= \frac{c_{11}\varepsilon_{11} - m_3 s_i^2 + c_{44}\varepsilon_{33}s_i^4}{(m_1 - m_2 s_i^2)s_i} \qquad (\text{for } i = 1, 2, 3) \\
\alpha_{i2} &= \frac{c_{11}e_{15} - m_4 s_i^2 + c_{44}e_{33}s_i^4}{(m_1 - m_2 s_i^2)s_i}
\end{aligned} \tag{2.57}$$

$$\omega_{01} = c_{44}s_0; \qquad \omega_{02} = e_{15}s_0 \qquad \omega_{i1} = c_{44}(s_i + \alpha_{i1}) + e_{15}\alpha_{i2}$$
$$\omega_{i2} = e_{15}(s_i + \alpha_{i1}) - \varepsilon_{11}\alpha_{i2} \qquad \theta_{i1} = (c_{33}\alpha_{i1} + e_{33}\alpha_{i2})s_i - c_{13}$$
$$\theta_{i2} = (e_{33}\alpha_{i1} + \varepsilon_{33}\alpha_{i2})s_i - e_{13}. \tag{2.58}$$

We also define the following position-related parameters (for $i, j = 0, 1, 2, 3$),

$$z_i = s_i z; \qquad h_i = s_i h; \qquad z'_i = s'_i z; \qquad z_{ij} = z_i + h_j;$$
$$R_{ij} = \sqrt{x^2 + y^2 + z_{ij}^2}; \qquad \overline{z}_{ij} = z_i - h_j; \qquad \overline{R}_{ij} = \sqrt{x^2 + y^2 + \overline{z}_{ij}^2};$$
$$z'_{ij} = z'_i - h_j; \qquad R'_{ij} = \sqrt{x^2 + y^2 + (z'_{ij})^2}. \tag{2.59}$$

The prime "′" here and afterwards denotes parameters or quantities in the lower half-space $z < 0$.

2.5.1 Green's Functions for Infinite Space

For a point charge Q and a point force P in the z-direction, applied at the source point $(0, 0, h)$, the elastic displacements and electric potential at the field point (x, y, z) are

$$u = \text{sign}(z - h) \sum_{i=1}^{3} \frac{A_i x}{\overline{R}_{ii}(\overline{R}_{ii} + s_i|z - h|)}$$
$$v = \text{sign}(z - h) \sum_{i=1}^{3} \frac{A_i y}{\overline{R}_{ii}(\overline{R}_{ii} + s_i|z - h|)}$$
$$w = \sum_{i=1}^{3} \frac{\alpha_{i1} A_i}{\overline{R}_{ii}};$$
$$\phi = \sum_{i=1}^{3} \frac{\alpha_{i2} A_i}{\overline{R}_{ii}}. \tag{2.60}$$

For a point force T in the x-direction, applied at the source point $(0, 0, h)$, the elastic displacements and electric potential at the field point (x, y, z) are

$$u = -D_0 \left[\frac{1}{\overline{R}_{00} + s_0|z - h|} - \frac{y^2}{\overline{R}_{00}(\overline{R}_{00} + s_0|z - h|)^2} \right]$$
$$+ \sum_{i=1}^{3} D_i \left[\frac{1}{\overline{R}_{ii} + s_i|z - h|} - \frac{x^2}{\overline{R}_{ii}(\overline{R}_{ii} + s_i|z - h|)^2} \right]$$

$$
\begin{aligned}
v =& -\frac{D_0 xy}{\overline{R}_{00}(\overline{R}_{00})+s_0|z-h|)^2} - xy\sum_{i=1}^{3}\frac{D_i}{\overline{R}_{ii}(\overline{R}_{ii}+s_i|z-h|)^2}\\
w =& -\operatorname{sign}(z-h)x\sum_{i=1}^{3}\frac{\alpha_{i1}D_i}{\overline{R}_{ii}(\overline{R}_{ii}+s_i|z-h|)}\\
\phi =& -\operatorname{sign}(z-h)x\sum_{i=1}^{3}\frac{\alpha_{i2}D_i}{\overline{R}_{ii}(\overline{R}_{ii}+s_i|z-h|)}.
\end{aligned}
\tag{2.61}
$$

In (2.60) and (2.61),

$$
\begin{aligned}
A_1 &= \frac{[P(\theta_{22}-\theta_{32})+Q(\theta_{21}-\theta_{31})]}{b_1}\\
A_2 &= \frac{[P(\theta_{32}-\theta_{12})+Q(\theta_{31}-\theta_{11})]}{b_1}\\
b_1 &= 4\pi[(\theta_{11}-\theta_{31})(\theta_{22}-\theta_{32})-(\theta_{21}-\theta_{31})(\theta_{12}-\theta_{32})]\\
A_3 &= -A_1-A_2=\frac{[P(\theta_{12}-\theta_{22})+Q(\theta_{11}-\theta_{21})]}{b_1}\\
D_0 &= \frac{-T}{(4\pi c_{44}s_0)} \qquad D_1=\frac{(\alpha_{21}\alpha_{32}-\alpha_{31}\alpha_{22})T}{b_2}\\
D_2 &= \frac{(\alpha_{31}\alpha_{12}-\alpha_{11}\alpha_{32})T}{b_2} \qquad D_3=\frac{(\alpha_{11}\alpha_{22}-\alpha_{21}\alpha_{12})T}{b_2}
\end{aligned}
\tag{2.62}
$$

$$
\begin{aligned}
b_2 = 4\pi c_{44}[&s_1(\alpha_{21}\alpha_{32}-\alpha_{31}\alpha_{22})+s_2(\alpha_{31}\alpha_{12}-\alpha_{11}\alpha_{32})\\
&+s_3(\alpha_{11}\alpha_{22}-\alpha_{21}\alpha_{12})].
\end{aligned}
\tag{2.63}
$$

2.5.2 Green's Functions for Half and Bimaterial Spaces

We first present the bimaterial space Green's functions. The Green's functions in the corresponding half-space can be reduced from the former ones. We assume that along the interface $z = 0$ of the two half-places, the elastic traction and the z-component of the electric displacement are continuous across the interface (i.e., perfect interface condition). For a point charge Q and a point force P in the z-direction, applied at the source point $(0, 0, h > 0)$, the elastic displacements and electric potential at the field point $(x, y, z > 0)$ of the upper half-space are

$$
u=\sum_{i=1}^{3}\left[\operatorname{sign}(z-h)\frac{A_i x}{\overline{R}_{ii}(\overline{R}_{ii}+s_i|z-h|)}+\sum_{j=1}^{3}\frac{A_{ij}x}{R_{ij}(R_{ij}+z_{ij})}\right]
$$

$$v=\sum_{i=1}^{3}\left[\operatorname{sign}(z-h)\frac{A_i y}{\overline{R}_{ii}(\overline{R}_{ii}+s_i|z-h|)}+\sum_{j=1}^{3}\frac{A_{ij}y}{R_{ij}(R_{ij}+z_{ij})}\right]$$

$$w=\sum_{i=1}^{3}\alpha_{i1}\left[\frac{A_i}{\overline{R}_{ii}}+\sum_{j=1}^{3}\frac{A_{ij}}{R_{ij}}\right]\qquad \phi=\sum_{i=1}^{3}\alpha_{i2}\left[\frac{A_i}{\overline{R}_{ii}}+\sum_{j=1}^{3}\frac{A_{ij}}{R_{ij}}\right]. \tag{2.64}$$

At the field point $(x, y, z<0)$ of the lower half-space, the elastic displacements and electric potential are (the prime "′"is added to quantities in the lower half-space)

$$u'=\sum_{i=1}^{3}\sum_{j=1}^{3}\frac{A'_{ij}x}{R'_{ij}(R'_{ij}-z'_{ij})};\qquad v'=\sum_{i=1}^{3}\sum_{j=1}^{3}\frac{A'_{ij}y}{R'_{ij}(R'_{ij}-z'_{ij})}$$

$$w'=-\sum_{i=1}^{3}\alpha'_{i1}\sum_{j=1}^{3}\frac{A'_{ij}}{R'_{ij}};\qquad \phi'=-\sum_{i=1}^{3}\alpha'_{i2}\sum_{j=1}^{3}\frac{A'_{ij}}{R'_{ij}}. \tag{2.65}$$

The coefficients A_{ij} and A'_{ij} in (2.64) and (2.65) are solved from the following equations (for $i=1,2,3; m=1,2$).

$$-A_i+\sum_{j=1}^{3}A_{ji}=\sum_{j=1}^{3}A'_{ji}\qquad \alpha_{im}A_i+\sum_{j=1}^{3}\alpha_{jm}A_{ji}=-\sum_{j=1}^{3}\alpha'_{jm}A'_{ji}$$

$$-\omega_{i1}A_i-\sum_{j=1}^{3}\omega_{j1}A_{ji}=-\sum_{j=1}^{3}\omega'_{j1}A'_{ji}\qquad \theta_{im}A_i-\sum_{j=1}^{3}\theta_{jm}A_{ji}=-\sum_{j=1}^{3}\theta'_{jm}A'_{jm}. \tag{2.66}$$

For a point force T in the x-direction, applied at the source point $(0,0,h>0)$, the elastic displacements and electric potential at the field point $(x, y, z>0)$ of the upper half-space are

$$u=-D_0\left[\frac{1}{\overline{R}_{00}+s_o|z-h|}-\frac{y^2}{\overline{R}_{00}(\overline{R}_{00}+s_0|z-h|)^2}\right]-D_{00}\left[\frac{1}{R_{00}+z_0}-\frac{y^2}{R_{00}(R_{00}+z_{00})^2}\right]+\sum_{i=1}^{3}\left\{\begin{array}{l}D_i\left[\frac{1}{\overline{R}_{ii}+s_i|z-h|}-\frac{x^2}{\overline{R}_{ii}(\overline{R}_{ii}+s_i|z-h|)^2}\right]\\+\sum\limits_{j=1}^{3}D_{ij}\left[\frac{1}{R_{ij}+z_{ij}}-\frac{x^2}{R_{ij}(R_{ij}+z_{ij})^2}\right]\end{array}\right\}$$

$$v=-\frac{D_0xy}{\overline{R}_{00}(\overline{R}_{00}+s_0|z-h|)^2}-\frac{D_{00}xy}{R_{00}(R_{00}+z_{00})^2}-xy\sum_{i=1}^{3}\left[\frac{D_i}{\overline{R}_{ii}(\overline{R}_{ii}+s_i|z-h|)^2}+\sum_{j=1}^{3}\frac{D_{ij}}{R_{ij}(R_{ij}+z_{ij})^2}\right]$$

$$w = -x\sum_{i=1}^{3}\alpha_{i1}\left[\text{sign}(z-h)\frac{D_i}{\overline{R}_{ii}(\overline{R}_{ii}+s_{ii}|z-h|)}+\sum_{j=1}^{3}\frac{D_{ij}}{R_{ij}(R_{ij}+z_{ij})}\right]$$
$$\phi = -x\sum_{i=1}^{3}\alpha_{i2}\left[\text{sign}(z-h)\frac{D_i}{\overline{R}_{ii}(\overline{R}_{ii}+s_{ii}|z-h|)}+\sum_{j=1}^{3}\frac{D_{ij}}{R_{ij}(R_{ij}+z_{ij})}\right]. \tag{2.67}$$

The elastic displacements and electric potential at the field point $(x, y, z < 0)$ of the lower half-space are

$$u' = -L'_{00}\left[\frac{1}{R'_{00}-z'_{00}}-\frac{y^2}{R'_{00}(R'_{00}-z'_{00})^2}\right]+\sum_{i=1}^{3}\sum_{j=1}^{3}L'_{ij}\left[\frac{1}{R'_{ij}-z'_{ij}}-\frac{x^2}{R'_{ij}(R'_{ij}-z'_{ij})^2}\right]$$
$$v' = -\frac{L'_{00}xy}{R'_{00}(R'_{00}-z'_{00})^2}-xy\sum_{i=1}^{3}\sum_{j=1}^{3}\frac{L'_{ij}}{R'_{ij}(R'_{ij}-z'_{ij})^2}$$
$$w' = x\sum_{i=1}^{3}\alpha'_{i1}\sum_{j=1}^{3}\frac{L'_{ij}}{R'_{ij}(R'_{ij}-z'_{ij})} \qquad \phi' = x\sum_{i=1}^{3}\alpha'_{i2}\sum_{j=1}^{3}\frac{L'_{ij}}{R'_{ij}(R'_{ij}-z'_{ij})}. \tag{2.68}$$

The involved coefficients in (2.67) and (2.68), D_{ij} and L'_{ij}, are solved from the following equations (for $i = 1, 2, 3; m = 1, 2$).

$$D_0 + D_{00} = L'_{00} \qquad \omega_{01}(D_{00}-D_0) = -\omega'_{01}L'_{00}$$
$$D_i + \sum_{j=1}^{3}D_{ji} = \sum_{j=1}^{3}L'_{ji} \qquad \alpha_{im}D_i - \sum_{j=1}^{3}\alpha_{jm}D_{ji} = \sum\alpha'_{jm}L'_{ji}$$
$$-\omega_{i1}D_i + \sum_{j=1}^{3}\omega_{j1}D_{ji} = -\sum_{j=1}^{3}\omega'_{j1}L'_{ij} \qquad \theta_{im}D_i + \sum_{j=1}^{3}\theta_{jm}D_{ji} = \sum_{j=1}^{3}\theta'_{jm}L'_{ji}. \tag{2.69}$$

The half-space Green's functions with traction-free (i.e., the elastic traction and the z-component of the electric displacement are zero) on the surface $z = 0$ can be directly reduced from the bimaterial space Green's functions presented in this section. Actually, assuming that the half-space is in the $z > 0$ domain and the source is also at $z = h(> 0)$, then the half-space Green's functions will be exactly the same as the bimaterial case in the source half-space, except that the involved coefficients are simply given as

$$
\begin{aligned}
A_{11} &= A_1(\theta_{21}\theta_{32}\omega_{11} - \theta_{22}\theta_{31}\omega_{11} - \theta_{12}\theta_{31}\omega_{21} \\
&\quad + \theta_{11}\theta_{32}\omega_{21} + \theta_{12}\theta_{21}\omega_{31} - \theta_{11}\theta_{22}\omega_{31})/d_a \\
A_{21} &= \frac{2A_1(\theta_{12}\theta_{31} - \theta_{11}\theta_{32})\omega_{11}}{d_a} \\
A_{31} &= \frac{2A_1(\theta_{11}\theta_{22} - \theta_{12}\theta_{21})\omega_{11}}{d_a} \\
A_{12} &= \frac{2A_2(\theta_{21}\theta_{32} - \theta_{22}\theta_{31})\omega_{21}}{d_a} \\
A_{22} &= A_2(\theta_{22}\theta_{31}\omega_{11} - \theta_{21}\theta_{32}\omega_{11} + \theta_{12}\theta_{31}\omega_{21} \\
&\quad - \theta_{11}\theta_{32}\omega_{21} + \theta_{12}\theta_{21}\omega_{31} - \theta_{11}\theta_{22}\omega_{31})/d_a \\
A_{32} &= \frac{2A_2(\theta_{11}\theta_{22} - \theta_{12}\theta_{21})\omega_{21}}{d_a} \\
A_{13} &= \frac{2A_3(\theta_{21}\theta_{32} - \theta_{22}\theta_{21})\omega_{31}}{d_a} \\
A_{23} &= \frac{2A_3(\theta_{12}\theta_{31} - \theta_{11}\theta_{32})\omega_{31}}{d_a} \\
A_{33} &= A_3(\theta_{22}\theta_{31}\omega_{11} - \theta_{21}\theta_{32}\omega_{11} - \theta_{12}\theta_{31}\omega_{21} \\
&\quad + \theta_{11}\theta_{32}\omega_{21} - \theta_{12}\theta_{21}\omega_{31} - \theta_{11}\theta_{22}\omega_{31})/d_a \\
d_a &= \theta_{22}\theta_{31}\omega_{11} - \theta_{21}\theta_{32}\omega_{11} - \theta_{12}\theta_{32}\omega_{21} \\
&\quad + \theta_{11}\theta_{32}\omega_{21} + \theta_{12}\theta_{21}\omega_{31} - \theta_{11}\theta_{22}\omega_{31} \\
D_{00} &= D_0; \qquad D_{ji} = \frac{D_i A_{ij}}{A_i(i,j=1,2,3)}.
\end{aligned}
\tag{2.70}
$$

2.6 Green's Functions in Three-Dimensional Infinite, Half, and Bimaterial Spaces: General Anisotropy

When the piezoelectric material is of general anisotropy, the Green's function solution becomes complicated even for the infinite space case. Therefore, various precise and computationally efficient algorithms have been developed for the evaluation of the infinite space piezoelectric Green's functions [38–40]. A more efficient way to evaluate the piezoelectric Green's function is by calculating the corresponding eigenvalues and eigenvectors [41, 42], as, for example, in [43]. In this section, however, the solutions developed by the author and coworkers are presented. The Green's functions in the infinite space were derived by employing the Radon transform [44] and those in the half and bimaterial spaces were obtained by separating the solution into two parts: the infinite-space Green's function and the complementary part. While the infinite-space Green's function is in an exact closed-form, the

complementary part is expressed by a finite-line integral after utilizing the double Fourier transform [i.e., 46].

2.6.1 Infinite Space

Following Pan and Tonon [44], the Green's displacement in the Jth direction at $\boldsymbol{x}$ due to a point force in the Kth direction at the origin can be found as

$$U_{JK}(\mathbf{x}) = -\frac{\mathrm{Im}}{2\pi r}\sum_{m=1}^{4}\frac{A_{JK}(\mathbf{p}+\zeta_m\mathbf{q})}{a_9(\zeta_m-\overline{\zeta}_m)\prod\limits_{\substack{k=1\\k\neq m}}^{4}(\zeta_m-\zeta_k)(\zeta_m-\overline{\zeta}_k)} \tag{2.71}$$

$$D(\mathbf{p}+\zeta\mathbf{q}) = \sum_{k=0}^{8}a_{k+1}\zeta^k = a_9\prod_{m=1}^{4}(\zeta-\zeta_m)(\zeta-\overline{\zeta}_m), \tag{2.72}$$

where $A_{JK}(\mathbf{m})$ is the adjoint matrix of $\Gamma_{JK}(\mathbf{m})$, and $D(\mathbf{m})$ is the determinant of $\Gamma_{JK}(\mathbf{m})$, defined as

$$\Gamma_{JK}(\mathbf{m}) = C_{iJKq}m_i m_q. \tag{2.73}$$

The vector $\mathbf{m}$ is given by

$$\boldsymbol{m} = \mathbf{p}+\zeta\mathbf{q}, \tag{2.74}$$

where

$$\mathbf{p} = \frac{\mathbf{e}\times\mathbf{v}}{|\mathbf{e}\times\mathbf{v}|}; \qquad \mathbf{q} = \mathbf{e}\times\mathbf{p}; \qquad \mathbf{e} = \frac{\mathbf{x}}{|\mathbf{x}|}, \tag{2.75}$$

with $\mathbf{v}$ being an arbitrary unit vector different from $\mathbf{e}(\mathbf{v}\neq\mathbf{e})$.

There are a couple of features associated with this Green's function expression (2.71). First of all, (2.71) is an explicit expression. It is therefore very accurate and efficient [45]. For a given pair of field and source points, we need only to solve the eighth-order polynomial equation (2.72) numerically once to obtain all the components of the Green's displacement. Secondly, in obtaining (2.71), we have assumed that all the poles are simple. Should the poles be multiple, a slight change in the material constants will result in single poles, with negligible errors in the computed Green's tensor, as for the purely elastic case [20]. Thirdly, because Γ_{JK} is symmetric, so is its adjoint A_{JK}. Therefore, the Green's displacement G_{JK} is symmetric [39] and one needs to calculate only 10 out of its 16 elements. The symmetric property of the extended Green's tensor can also be considered as a consequence of the Betti-type reciprocity as presented in Section 3 of this chapter. Finally, although one can choose the vector $\mathbf{v}(\neq\mathbf{e})$ arbitrarily, it should be one of

the base vectors in the space-fixed Cartesian coordinates, that is, (1, 0, 0) or (0, 1, 0) or (0, 0, 1). The analytical expression for the Green's displacement is much simpler using such a vector **v** than using any other vectors.

2.6.2 Half-Space

For the half-space case ($z > 0$) with traction-free boundary condition at $z = 0$ (i.e., the elastic traction and z-component of the electric displacement are zero), the Green's function solutions at the field point $\boldsymbol{x}$ $(x_1, x_2, z > 0)$ due to a point force at $\boldsymbol{d}$ $(d_1, d_2, d > 0)$ can be obtained first in the Fourier transformed domain, and then invert back to the physical domain. By so doing, the final half-space Green's function in the physical domain, or the generalized Mindlin solution, can be expressed as a sum of an explicit Kelvin-type solution and a complementary part in terms of a line integral over $[0, 2\pi]$. Furthermore, the latter can be reduced to an integral over $[0, \pi]$. For the half-space displacement tensor (4×4), with its row and column indices being the components of the field quantity and the direction of the point source, respectively, it can be expressed as [21]

$$\boldsymbol{U}(\boldsymbol{x};\boldsymbol{d}) = \boldsymbol{U}^{\infty}(\boldsymbol{x};\boldsymbol{d}) + \frac{1}{2\pi^2}\int_0^{\pi} \overline{\boldsymbol{A}}\boldsymbol{G}_1\boldsymbol{A}^T d\theta, \tag{2.76}$$

where

$$(\boldsymbol{G}_1)_{IJ} = \frac{(\overline{\boldsymbol{B}}^{-1}\boldsymbol{B})_{IJ}}{-\overline{p}_I z + p_J d - [(x_1 - d_1)\cos\theta + (x_2 - d_2)\sin\theta]}. \tag{2.77}$$

The first term in (2.76) corresponds to the Green's displacement tensor in an anisotropic and piezoelectric full space, which is given by (2.71) [43, 44]. Consequently, the half-space displacement tensor can be expressed as a sum of an explicit Kelvin tensor and a complementary part in terms of a line integral over $[0, \pi]$. It is emphasized that in (2.76) and (2.77), the eigenvalues p_J and the eigenmatrices $\boldsymbol{A}$ and $\boldsymbol{B}$ are functions of θ, with p_J $(J = 1, 2, 3, 4)$ and $\boldsymbol{A} = [\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3, \boldsymbol{a}_4]$ being the eigensolutions of (2.33) for a given θ. In other words, p and $\mathbf{a}$ satisfy the same eigenequation (2.33) but with

$$\begin{aligned} Q_{IK} &= C_{jIKs} n_j n_s, \qquad R_{IK} = C_{jIKs} n_j m_s, \\ T_{IK} &= C_{jIKs} m_j m_s \end{aligned} \tag{2.78}$$

$$\boldsymbol{n} = \begin{bmatrix} n_1 \\ n_2 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta \\ 0 \end{bmatrix}, \qquad \boldsymbol{m} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \tag{2.79}$$

Matrix $\boldsymbol{B}$ is defined by (2.36), with its vector $\boldsymbol{b}_i$ related to $\boldsymbol{a}_i$ via (2.35).

Equation (2.76) is the generalized Mindlin solution, or the Green's displacement under the traction-free boundary conditions in an anisotropic and piezoelectric half-space. It is remarked that similar Mindlin solutions can be presented for many other homogeneous boundary conditions on the surface $z = 0$ [21]. We also point out that if the source and field points are not simultaneously on the surface, then the line integral in (2.76) can be carried out by employing regular numerical quadrature. If, however, $z = d = 0$, then the half-space Green's function is reduced to the special surface Green's function, where the involved singular integration needs special numerical treatment.

2.6.3 Bimaterial Space

The procedure for solving the bimaterial Green's functions is as follows. First, we apply the double Fourier transform to the two horizontal variables (x, y); second, we solve the Green's function problem in the transformed domain; third, we apply the inverse Fourier transform to obtain the physical domain Green's function. To handle the double infinite integrals in the inverse space, the polar coordinate transform is applied so that the infinite integral with respect to the radial variable can be carried out exactly. Thus, the final bimaterial Green's functions in the physical domain can be expressed in terms of a regular line integral over $[0, 2\pi]$, which can be further reduced to $[0, \pi]$ using certain properties of the Stroh eigenvalues and Stroh matrices.

We assume that the upper half-space ($z > 0$) is occupied by material 1 and the lower half ($z < 0$) by material 2. The interface at $z = 0$ between the two half-spaces is further assumed to be perfect. In other words, the elastic traction and the z-component of the electric displacement are continuous across the interface. We further assume that the point force is in material 1 at $\boldsymbol{d}$ $(d_1, d_2, d > 0)$; then the 4×4 Green's function tensor at $\boldsymbol{x}$ $(x_1, x_2, z > 0)$ in material 1, with its first index for the displacement component and the second for the extended point force direction, is found to be [46]

$$\boldsymbol{U}^{(1)}(\boldsymbol{x};\boldsymbol{d}) = \boldsymbol{U}^{\infty}(\mathbf{x};\mathbf{d}) + \frac{1}{2\pi^2}\left[\int_0^{\pi} \overline{\boldsymbol{A}}^{(1)} \boldsymbol{G}_u^{(1)} (\boldsymbol{A}^{(1)})^T d\theta\right] \tag{2.80}$$

$$(\boldsymbol{G}_u^{(1)})_{IJ} = \frac{(\boldsymbol{G}_1)_{IJ}}{-\overline{p}_I^{(1)} z + p_J^{(1)} d - [(x_1 - d_1)\cos\theta + (x_2 - d_2)\sin\theta]}. \tag{2.81}$$

In (2.80), $\boldsymbol{U}^{\infty}(\mathbf{x};\mathbf{d})$ denotes the Green's function tensor for the displacements in the full space with the material 1 property (i.e., (2.71)).

In material 2, the Green's tensor at $\boldsymbol{x}$ $(x_1, x_2, z < 0)$ is

$$\boldsymbol{U}^{(2)}(\boldsymbol{x};\boldsymbol{d}) = -\frac{1}{2\pi^2}\left[\int_0^{\pi} \boldsymbol{A}^{(2)}\boldsymbol{G}_u^{(2)}(\boldsymbol{A}^{(1)})^T d\theta\right] \tag{2.82}$$

$$(\boldsymbol{G}_u^{(2)})_{IJ} = \frac{(\boldsymbol{G}_2)_{IJ}}{-p_I^{(2)}z + p_J^{(1)}d - [(x_1 - d_1)\cos\theta + (x_2 - d_2)\sin\theta)]}. \tag{2.83}$$

In (2.80) to (2.83), the superscripts "(1)" and "(2)" denote quantities in materials 1 and 2, respectively, and the matrices $\boldsymbol{G}_1$ and $\boldsymbol{G}_2$ are given by

$$\begin{aligned}\boldsymbol{G}_1 &= -(\overline{\boldsymbol{A}}^{(1)})^{-1}(\overline{\boldsymbol{M}}^{(1)} + \boldsymbol{M}^{(2)})^{-1}(\boldsymbol{M}^{(1)} - \boldsymbol{M}^{(2)})\boldsymbol{A}^{(1)}\\ \boldsymbol{G}_2 &= (\boldsymbol{A}^{(2)})^{-1}(\overline{\boldsymbol{M}}^{(1)} + \boldsymbol{M}^{(2)})^{-1}(\boldsymbol{M}^{(1)} - \overline{\boldsymbol{M}}^{(1)})\boldsymbol{A}^{(1)},\end{aligned} \tag{2.84}$$

where $\boldsymbol{M}^{(\alpha)}$ is the modified impedance tensor defined as

$$\boldsymbol{M}^{(\alpha)} = -i\boldsymbol{B}^{(\alpha)}(\boldsymbol{A}^{(\alpha)})^{-1} \qquad (\alpha = 1, 2). \tag{2.85}$$

In summary, in material 1, the bimaterial Green's function is expressed as a sum of the explicit full-space Green's function and a complementary part in terms of a line integral over $[0, \pi]$; In material 2, the bimaterial Green's function is expressed in terms of a line integral over $[0, \pi]$. With regard to these physical domain bimaterial Green's functions ((2.80) and (2.82)), the following important observations can be made.

(a) For the complementary part of the solution in material 1 and the solution in material 2, the dependence of the solutions on the field point $\boldsymbol{x}$ and source point $\boldsymbol{d}$ appears only through matrices $\boldsymbol{G}_u^{(1)}$ and $\boldsymbol{G}_u^{(2)}$ defined in (2.81) and (2.83).

(b) The integrals in (2.80) and (2.82) are regular if $z \neq 0$ or $d \neq 0$, and thus can be easily carried out by a standard numerical integral method such as Gaussian quadrature.

(c) If $z \neq 0$ and $d = 0$, the bimaterial Green's function is still mathematically regular although some of its components may not have a direct and apparent physical meaning (see Pan, [25], for the purely elastic counterpart).

(d) When the field and source points are both on the interface (i.e., $z = d = 0$), the bimaterial Green's function is then reduced to the interfacial Green's function. For this special case, the line integral involved in the Green's function expression becomes singular and the resulting finite part integral needs to be handled with special care [47].

(e) Bimaterial Green's functions can be solved similarly for other (imperfect) interface models. To do so, one need only find the modified Stroh matrices $\boldsymbol{A}^{(\alpha)}$ and $\boldsymbol{B}^{(\alpha)}$ for the given interface models. For detailed discussion, one should refer to [25].

2.7 Green's Functions in Layered Half-Space

We solve the Green's functions in layered half-space in terms of two systems of vector functions combined with the propagator matrix method. The vector function method is essentially equivalent to the double Fourier transform or Hankel transform in the horizontal layer plane, but possesses certain advantages over the latter ones (i.e., [48]). The propagator matrix method is utilized to propagate the field quantities from one layer to the other.

We first introduce the two systems of vector functions. The Cartesian coordinate system of vector functions is defined as [12, 48, 49]

$$\begin{aligned}
\boldsymbol{L}(x,y;\alpha,\beta) &= \boldsymbol{e}_z S(x,y;\alpha,\beta) \\
\boldsymbol{M}(x,y;\alpha,\beta) &= (\boldsymbol{e}_x\partial_x + \boldsymbol{e}_y\partial_y)S(x,y;\alpha,\beta) \\
\boldsymbol{N}(x,y;\alpha,\beta) &= (\boldsymbol{e}_x\partial_y - \boldsymbol{e}_y\partial_x)S(x,y;\alpha,\beta)
\end{aligned} \tag{2.86}$$

with

$$S(x,y;\alpha,\beta) = \frac{e^{-i(ax+\beta y)}}{(2\pi)}\,, \tag{2.87}$$

where $\boldsymbol{e}_x$, $\boldsymbol{e}_y$, and $\boldsymbol{e}_z$ are the unit vectors along the x-, y-, and z-axes, respectively; x and y are horizontal axes, while z-axis points to the problem domain; α and β are the transformation variables corresponding to the two horizontal physical variables x and y.

The corresponding cylindrical system of vector functions is defined as [12, 48, 49]

$$\begin{aligned}
\boldsymbol{L}(r,\theta;\lambda,m) &= e_z S(r,\theta;\lambda,m) \\
\boldsymbol{M}(r,\theta;\lambda,m) &= (e_r\frac{\partial}{\partial r} + e_\theta\frac{\partial}{r\partial\theta})S(r,\theta;\lambda,m) \\
\boldsymbol{N}(r,\theta;\lambda,m) &= (e_r\frac{\partial}{r\partial\theta} - e_\theta\frac{\partial}{\partial r})S(r,\theta;\lambda,m)
\end{aligned} \tag{2.88}$$

with $\boldsymbol{e}_{\mathrm{r}}$, $\boldsymbol{e}_{\theta}$, and $\boldsymbol{e}_{\mathrm{z}}$ as the unit vectors along the r-, θ-, and z-axes, respectively, and

$$S(t,\theta;\lambda,m) = \frac{1}{\sqrt{2\pi}}J_m(\lambda r)e^{im\theta}, \tag{2.89}$$

where $J_m(\lambda r)$ is the Bessel function of order m with $m = 0$ corresponding to the axial symmetric deformation.

There are several important features associated with the two systems of vector functions.

(a) For plane strain deformation in the (x,z)-plane, one needs only to replace 2π by $\sqrt{2\pi}$ and β by 0, respectively.

(b) While the solution in terms of the $\boldsymbol{L}$ & $\boldsymbol{M}$ vectors is contributed to the dilatational deformation, that of the $\boldsymbol{N}$ vector to the rotational part.

Corresponding to the dynamic counterparts, the $\boldsymbol{L}$ & $\boldsymbol{M}$ part is related to the Rayleigh wave and the $\boldsymbol{N}$ part to the Love wave. Here, we name the solution associated with the $\boldsymbol{L}$ & $\boldsymbol{M}$ vectors the LM-type solution and that associated with the $\boldsymbol{N}$ vector the N-type solution.

(c) We remark that the general solution and propagator matrix in the cylindrical system of vector functions are exactly the same as those in the Cartesian system. This feature gives certain numerical advantages when programming these formulations in the two systems of vector functions.

(d) Another advantage is that both 2D and 3D Green's functions can be studied uniformly under these systems [12, 48, 49] because the general solutions in terms of the two systems are the same for both 2D and 3D deformation.

For the Green's function problem, we first express the elastic displacement, electric potential, traction, electric displacements, body force, and negative electric charge density in terms of the cylindrical system of vector functions,

$$\boldsymbol{u}(r,\theta,z) = \sum_m \int_0^{+\infty} [U_L(z)\boldsymbol{L}(r,\theta) + U_M(z)\boldsymbol{M}(r,\theta) + U_N(z)\boldsymbol{N}(r,\theta)]\lambda d\lambda \tag{2.90}$$

$$\phi(r,\theta,z) = \sum_m \int_0^{+\infty} \Phi(z)S(r,\theta)\lambda d\lambda \tag{2.91}$$

$$\begin{aligned}\boldsymbol{t}(r,\theta,z) &\equiv \sigma_{rz}\boldsymbol{e}_r + \sigma_{\theta z}\boldsymbol{e}_\theta + \sigma_{zz}\boldsymbol{e}_z \\ &= \sum_m \int_0^{+\infty} [T_L(z)\boldsymbol{L}(r,\theta) + T_M(z)\boldsymbol{M}(r,\theta) + T_N(z)\boldsymbol{N}(r,\theta)]\lambda d\lambda\end{aligned} \tag{2.92}$$

$$\boldsymbol{D}(r,\theta,z) = \sum_m \int_0^{+\infty} [D_L(z)\boldsymbol{L}(r,\theta) + D_M(z)\boldsymbol{M}(r,\theta) + D_N(z)\boldsymbol{N}(r,\theta)]\lambda d\lambda \tag{2.93}$$

$$\boldsymbol{f}(r,\theta,z) = \sum_m \int_0^{+\infty} [F_L(z)\boldsymbol{L}(r,\theta) + F_M(z)\boldsymbol{M}(r,\theta) + F_N(z)\boldsymbol{N}(r,\theta)]\lambda d\lambda \tag{2.94}$$

$$-q(r,\theta,z) = \sum_m \int_0^{+\infty} Q(z)S(r,\theta)\lambda d\lambda. \tag{2.95}$$

In (2.90)–(2.95), the left-hand side variables in lowercase are the unknown field quantities in the physical domain, and the right-hand side variables in capitals, such as U, Φ, T, ..., are the unknown expansion coefficients in the transformed domain. Making use of corresponding governing equations presented in Section 2, the layered Green's function problems can be

converted into an ordinary differential system of equations for each layer in the transformed domain so that the unknowns in the transformed domain can be obtained. Because the Cartesian and cylindrical systems of vector functions are employed, the problem in the transformed domain can be further separated into two independent problems, which are discussed below.

2.7.1 General N- and LM-Type Solutions in the Transformed Domain

(a) N-type solution

Based on either the Cartesian or cylindrical system of vector functions, one can show easily that the *N*-type solution is independent of the rest, and furthermore, it is independent of the electric quantities. In other words, it is purely elastic and its general solution in each layer can be expressed as

$$[\boldsymbol{E}^N] = [\boldsymbol{Z}^N(z)][\boldsymbol{K}^N], \tag{2.96}$$

where $[\boldsymbol{K}^N]$ is a column coefficient matrix of 2×1 with its elements to be determined by the continuity and/or boundary conditions. Also in (2.96),

$$[\boldsymbol{E}^N(z)] = \left[U_N(z), \frac{T_N(z)}{\lambda}\right]^T, \tag{2.97}$$

and $[\boldsymbol{Z}^N(z)]$ is the solution matrix, the same as that for the purely elastic case [12, 48, 49].

(b) *LM*-type solution

For this type of deformation, the elastic and piezoelectric fields are coupled together. The ordinary differential equations in each layer for this type can be derived as

$$[\boldsymbol{E}]_z = \lambda[\boldsymbol{W}][\boldsymbol{E}]. \tag{2.98}$$

It is remarked that the diagonal elements of $[\boldsymbol{W}]$ are zero and independent of λ. Also in (2.98),

$$[\boldsymbol{E}] = \left[U_L, \lambda U_M, \frac{T_L}{\lambda}, T_M, \Phi, \frac{D_L}{\lambda}\right]^T. \tag{2.99}$$

To find the homogeneous solution of (2.98), we assume that

$$[\boldsymbol{E}(z)] = [\boldsymbol{b}]e^{\lambda\eta z}. \tag{2.100}$$

Substituting (2.100) into (2.98) and noticing that all the diagonal elements of $[\boldsymbol{W}]$ are zero, we obtain the following six-dimensional eigenequations for (2.98)

$$\{[\boldsymbol{W}] - \eta[\boldsymbol{I}]\}[\boldsymbol{b}] = 0, \tag{2.101}$$

where $[\boldsymbol{I}]$ is the 6×6 identity matrix.

It is observed from (2.101) that the eigenvalues and their corresponding eigenvectors are independent of the integral variable λ! Therefore, these eigenequations need to be solved only once for each layer for the given material properties. Let us, therefore, assume that the six eigenvalues are distinct, and the general solution to (2.98) is then obtained as

$$[\boldsymbol{E}(z)] = [\boldsymbol{Z}(z)][\boldsymbol{K}], \tag{2.102}$$

where $[\boldsymbol{K}]$ is a 6×1 coefficient matrix with its elements to be determined by the interface and/or boundary conditions, and

$$[\boldsymbol{Z}(z)] = [\boldsymbol{B}] \left\langle e^{\lambda \eta^* z} \right\rangle \tag{2.103}$$

with

$$\left\langle e^{\lambda \eta^* z} \right\rangle = \text{diag}[e^{\lambda \eta_1 z}, \quad e^{\lambda \eta_2 z}, \quad e^{\lambda \eta_3 z}, \quad e^{-\lambda \eta_1 z}, \quad e^{-\lambda \eta_2 z}, \quad e^{-\lambda \eta_3 z}] \tag{2.104}$$

being associated with the six eigenvalues, and

$$[\boldsymbol{B}] = [\boldsymbol{b}_1, \boldsymbol{b}_2, \boldsymbol{b}_3, \boldsymbol{b}_4, \boldsymbol{b}_5, \boldsymbol{b}_6] \tag{2.105}$$

associated with the corresponding eigenvectors.

2.7.2 Propagator Matrix Method for Multilayered Structures

The propagator matrix method is most suitable to layered structures. Application of this method can help avoid the complicated calculation of a large matrix and also save significant computation resource. At the core of this method is the propagator matrix, which relates the N and LM expansion coefficients $[\boldsymbol{E}^N]$ and $[\boldsymbol{E}]$ at the top interface to the bottom interface of layer j. In other words, for layer j, we have

$$\left[\boldsymbol{E}^N(z_{j-1})\right] = [\boldsymbol{a}^N][\boldsymbol{E}^N(z_j)] \tag{2.106}$$

and

$$[\boldsymbol{E}(z_{j-1})] = [\boldsymbol{a}][\boldsymbol{E}(z_j)] \tag{2.107}$$

where z_{j-1} and z_j ($>z_{j-1}$) are the depths of the top and bottom interfaces of layer j, $[\boldsymbol{a}^N]$ and $[\boldsymbol{a}]$ are the so-called propagator matrix (or layer matrix, or transfer matrix). Propagating the solution from the surface $z=0$ to half-space bottom $z=H$, we obtained,

$$[\boldsymbol{E}^N(0)] = [\boldsymbol{a}_1^N][\boldsymbol{a}_2^N] - - - [\boldsymbol{a}_{p-1}^N][\boldsymbol{Z}_p^N(h)][\boldsymbol{K}_p]$$
$$[\boldsymbol{E}(0)] = [\boldsymbol{a}_1][\boldsymbol{a}_2] - - - [\boldsymbol{a}_{p-1}][\boldsymbol{Z}_p(h)][\boldsymbol{K}_p] \tag{2.108a,b}$$

with the undetermined coefficients having the structure as

$$[\boldsymbol{K}_p^N] = [0,*]^t \qquad [\boldsymbol{K}_p] = [0,0,0,*,*,*]^t \tag{2.109a,b}$$

to satisfy the requirement that the solution vanishes when z approaches $+\infty$. The symbol "*" denotes an unknown coefficient. These coefficients can be solved using the traction-free boundary condition on the surface and the discontinuity condition at the source level due to the point force. After the unknown coefficients in $[\boldsymbol{K}_p^N]$ and $[\boldsymbol{K}_p]$ are determined through the propagator matrix method, the expansion coefficients at any depth (e.g., for $z \geq h$ in layer j, i.e., $z_{j-1} \leq z \leq z_j$) can be derived exactly as

$$[\boldsymbol{E}^N(z)] = [\boldsymbol{a}_j^N(z-z_{j-1})][\boldsymbol{a}_{j+1}^N] - - - [\boldsymbol{a}_{p-1}^N][\boldsymbol{Z}_p^N(H)][\boldsymbol{K}_p^N]$$
$$[\boldsymbol{E}(z)] = [\boldsymbol{a}_j(z-z_{j-1})][\boldsymbol{a}_{j+1}] - - - [\boldsymbol{a}_{p-1}][\boldsymbol{Z}_p(H)][\boldsymbol{K}_p]. \tag{2.110a,b}$$

2.7.3 Physical Domain Solutions

From (2.90)–(2.95), in order to get the field quantities in the physical domain, numerical integration must be carried out. It is noted that the integrands in the infinite integrals for the Green's functions involve Bessel functions that are oscillatory and go to zero slowly when their variable approaches infinity. Thus, the common numerical integral methods, such as the trapezoidal rule or Simpson's rule, are not suitable for such integrations. On the other hand, numerical integration of this type of function via adaptive Gauss quadrature has been found to be very accurate and efficient. In this adaptive quadrature, we express the infinite integral for each Green's function as a summation of partial integration terms:

$$\int_0^{+\infty} f(\lambda,z)J_m(\lambda r)d\lambda = \sum_{n=1}^{N}\int_{\lambda_n}^{\lambda_{n+1}} f(\lambda,z)J_m(\lambda r)d\lambda. \tag{2.111}$$

In each subinterval, a starting three-point Gauss rule is applied to approximate the integral. A combined relative–absolute error criterion is used to check the results. If the error criterion is not satisfied, new Gauss points are added optimally so that only the new integrand values need to be calculated. This procedure continues until the selected error criterion is satisfied.

The methodology presented in this section can also be applied to find the Green's function solutions in many different layered material structures. These include transversely isotropic layered thermoelastic half-space [13], layered poroelastic half-space [14], transversely isotropic layered piezoelectric half-space [50] and transversely isotropic functionally graded and layered piezoelectric half-space [51, 52]. Similar approaches have been also developed to derive the Green's functions in general anisotropic and layered elastic spaces [53, 54].

2.8 Conclusions

This chapter presents a review of the Green's function solutions in piezoelectric anisotropic solids. It is limited to the static case and with infinite or semi-infinite domains. Even in this limited case, the author may have missed some of the references by other experts. For example, one interesting area that the author intentionally omitted is the circular loading on the surface of the layered piezoelectric half-space as this can be obtained from the corresponding point-source Green's functions by integration over the loading domain [52]. The Green's functions in the functionally graded piezoelectric space are not covered (i.e., [51]). There are also Green's functions associated with the finite domain, for example, on the Green's function-related issues in layered piezoelectric spheres (i.e., [55–57]), and in layered piezoelectric cylinders(i.e., [58, 59]). The dynamic and transient Green's functions are not reviewed. These include dynamic and transient problems in layered cylinders [58, 60–62], in layered spheres [63], and in horizontally layered plates (i.e., [64]). Dynamic Green's functions in anisotropic infinite space are not covered either, and contributions to this difficult area can be found, for example, in [65, 66]. A recent special issue of *Engineering Analysis and Boundary Elements* edited by the author also includes many interesting Green's function solutions [67]. Another interesting area that the author hasn't reviewed but is extremely attractive is related to the multiferroic materials/structures. The coupling between the electric and magnetic fields via the induced strain inside the system has potential applications to many semiconductor devices using electric and magnetic fields. Various Green's functions have already been developed and interested readers should refer to [68–76] for details.

References

[1] Yang, J. S. and Hu, Y. T. (2004): Mechanics of electroelastic bodies under biasing fields, *Appl. Mech. Rev.*, 57, 173–189.

[2] Yang, J. S. (2006): A review of a few topics in piezoelectricity. *Applied Mechanics Reviews*, 59, 335–345.

[3] Bimberg, D., Grundmann, M., and Ledentsov, N. N. (1999): *Quantum Dot Heterostructures*. Wiley, New York.
[4] Maranganti, R. and Sharma, P. (2006): A review of strain field calculations in embedded quantum dots and wires. In *Handbook of Theoretical and Computational Nanotechnology*. Edited by M. Reith and W. Schommers.
[5] Saravanos, D. A., Heyliger, P. R., and Hopkins, D. A. (1997): Layerwise mechanics and finite element for the dynamic analysis of piezoelectric composite plates. *Int. J. Solids Struct.*, 34, 359–378.
[6] Kuvshinov, B. N. (2008): Elastic and piezoelectric fields due to polyhedral inclusions. *Int. J. Solids Struct.*, 45, 1352–1384.
[7] Tiersten, H. F. (1969): *Linear Piezoelectric Plate Vibrations*, Plenum, New York.
[8] Barnett, D. M. and Lothe, J. (1975): Dislocations and line charges in anisotropic piezoelectric insulators, *Phys. Stat. Sol.* (*b*), 67, 105–111.
[9] Pan, E. (1999): A BEM analysis of fracture mechanics in 2D anisotropic piezoelectric solids, *Eng. Anal. Bound. Elem*, 23, 67–76.
[10] Steketee, J. A. (1958): On Volterra's dislocation on a semi-infinite elastic medium. *Can. J. Phys.*, 36, 192–205.
[11] Pan, E. (1991): Dislocation in an infinite poroelastic medium. *Acta Mech*, 87, 105–115.
[12] Pan, E. (1989): Static response of a transversely isotropic and layered half-space to general dislocation sources. *Phys. Earth Planet. Inter.*, 58, 103–117.
[13] Pan, E. (1990): Thermoelastic deformation of a transversely isotropic and layered half-space by surface loads and internal sources. *Phys. Earth Planet. Inter.*, 60, 254–264.
[14] Pan, E. (1999): Green's functions in layered poroelastic half-spaces. *Int. J. Numer. Anal. Meth. Geomech.*, 23, 1631–1653.
[15] Mindlin, R. D. (1936): Force at a point in the interior of a semi-infinite solid. *Physics*, 7, 195–202.
[16] Maruyama, T. (1964): Statical elastic dislocations in an infinite and semi-infinite medium. *Bull. Earthq. Res. Inst.*, 42, 289–368.
[17] Maruyama, T. (1966): On two-dimensional elastic dislocations in an infinite and semi-infinite medium. *Bull. Earthq. Res. Inst.*, 44, 811–871.
[18] Yu, H.Y., Sanday, S.C., and Chang, C.I. (1994): Elastic inclusions and inhomogeneities in transversely isotropic solids. *Proc. R. Soc. Lond. A*, 444, 239–252.
[19] Ting, T. C. T. (1996): *Anisotropic Elasticity*, Oxford University Press, Oxford.
[20] Pan, E. (1997): A general boundary element analysis of 2-D linear elastic fracture mechanics. *Int. J. Fract.*, 88, 41–59.
[21] Pan, E. (2002): Mindlin's problem for an anisotropic piezoelectric half space with general boundary conditions. *Proc. R. Soc. Lond. A*, 458, 181–208.
[22] Pan, E. (2003): Three-dimensional Green's functions in an anisotropic half space with general boundary conditions. ASME *J. Appl. Mech.*, 70, 101–110.
[23] Jiang, X. and Pan, E. (2004): Exact solution for 2D polygonal inclusion problem in anisotropic magnetoelectroelastic full-, half-, and bimaterial-planes. *Int. J. Solids Struct.*, 41, 4361–4382.
[24] Pan, E. (2003): Three-dimensional Green's functions in anisotropic elastic bimaterials with imperfect interfaces. ASME *J. Appl. Mech.*, 70, 180–190.
[25] Pan, E. (2003): Some new three-dimensional Green's functions in anisotropic piezoelectric bimaterials. *Electron. J. Bound. Elem.*, 1, 236–269.
[26] Ding, H. J., Chen, B., and Liang, J. (1996): On the general solutions for coupled equation for piezoelectric media. *Int. J. Solids Struct.*, 33, 2283-2298.
[27] Ding, H. J., Wang, G. Q., and Liang, J. (1996): General and fundamental solutions of plane piezoelectroelastic problem. *Acta Mech Sinica*, 28, 441-448.
[28] Ding, H. J., Wang, G. Q., and Chen, W. Q. (1997): Fundamental solutions for plane problem of piezoelectric materials. *Sci. China*, E40, 331–336.

[29] Ding, H. J., Chen, B., and Liang, J. (1997): Fundamental solution for transversely isotropic piezoelectric media. *Sci. China*, A39, 766–775. 27

[30] Ding, H. J., Chi, Y., and Guo, F. (1999): Solutions for transversely isotropic piezoelectric infinite body, semi-infinite body and bimaterial infinite body subjected to uniform ring loading and charge. *Int. J. Solids Struct.*, 36, 2613–2631.

[31] Ding, H. J. and Liang, J. (1999): The fundamental solutions for transversely isotropic piezoelectricity and boundary element method.*Comput. Struct.*, 71, 447–455.

[32] Ding, H. J. and Chen, W. Q. (2001): *Three Dimensional Problems of Piezoelasticity.* Nova Science, Huntington, New York.

[33] Ding, H. J., Chen, W. Q., and Jiang, A. M. (2004): Green's functions and boundary element method for transversely isotropic piezoelectric materials. *Eng. Anal. Bound. Elem*, 28, 975–987.

[34] Ding, H. J., Chen, W. Q., and Zhang, L. (2006): *Elasticity of Transversely Isotropic Materials.* Series: Solid Mechanics and Its Applications (Ed. G. M. L. Galdwell), Vol. 126, Kluwer Academic, Boston.

[35] Dunn, M. L. (1994): Electroelastic Green's functions for transversely isotropic piezoelectric media and their application to the solution of inclusion and inhomogeneity problems. *Int. J. Eng. Sci.*, 32, 119–131.

[36] Dunn, M. L. and Wienecke, H. A. (1996): Green's functions for transversely isotropic piezoelectric solids. *Int. J. Solids Struct.*, 33, 4571–4581.

[37] Dunn, M. L. and Wienecke, H. A. (1999): Half-space Green's functions for transversely isotropic piezoelectric solids. *ASME J. Appl. Mech.*, 66, 675–679.

[38] Barnett, D. M. (1972): The precise evaluation of derivatives of the anisotropic elastic Green's functions, *Phys. Stat. Sol.* (*b*), 49, 741–748.

[39] Chen, T. (1993): Green's functions and the non-uniform transformation problem in a piezoelectric medium, *Mech. Res. Comm.*, 20, 271–278.

[40] Chen, T. and Lin, F. Z. (1993): Numerical evaluation of derivatives of the anisotropic piezoelectric Green's functions. *Mech. Res. Comm.*, 20, 501–506.

[41] Malen, K. (1971): A unified six-dimensional treatment of elastic Green's functions and dislocations. *Phys. Stat. Sol.* (*b*), 44, 661–672.

[42] Wang, C. Y. (1997): Elastostatic fields produced by a point source in solids of general anisotropy. *J. Eng. Math.*, 32, 41–52.

[43] Akamatsu, M. and Tanuma, K. (1997): Green's function of anisotropic piezoelectricity. *Proc. R. Soc. Lond. A*, 453, 473–487.

[44] Pan, E. and Tonon, F. (2000): Three-dimensional Green's functions in anisotropic piezoelectric solids. *Int. J. Solids Struct.*, 37, 943–958.

[45] Tonon, F., Pan, E., and Amadei, B. (2001): Green's functions and boundary element method formulation for 3D anisotropic media. *Comput. Struct.*, 79, 469–482.

[46] Pan, E. and Yuan, F. G. (2000): Three-dimensional Green's functions in anisotropic piezoelectric bimaterials, *Int. J. Eng. Sci.*, 38, 1939–1960.

[47] Pan, E. and Yang, B. (2003): Three-dimensional interfacial Green's functions in anisotropic bimaterials. *Appl. Math. Model.*, 27, 307–326.

[48] Pan, E. (1989): Static response of a transversely isotropic and layered half-space to general surface loads. *Phys. Earth Planet. Inter.*, 54, 353–363.

[49] Pan, E. (1997): Static Green's functions in multilayered half-spaces. *Appl. Math. Model.*, 21, 509–521.

[50] Pan, E. (2003): Three-dimensional fundamental solutions in multilayered piezoelectric solids. *Chinese J. Mech.*, A19, 127–132.

[51] Pan, E. and Han, F. (2005): Green's functions for transversely isotropic piezoelectric functionally graded multilayered half spaces. *Int. J. Solids Struct.*, 42, 3207-3233.

[52] Han, F., Pan, E., Roy, A. K., and Yue, Z. Q. (2006): Responses of piezoelectric, transversely isotropic, functionally graded, and multilayered half spaces to uniform circular surface loadings. *Comput. Model. Eng. Sci.*, 14, 15–29.

[53] Yang, B. and Pan, E. (2002): Three-dimensional Green's functions in anisotropic trimaterials. *Int. J. Solids Struct.*, 39, 2235–2255.

[54] Yang, B. and Pan, E. (2002): Efficient evaluation of three-dimensional Green's functions in anisotropic elastostatic multilayered composites. *Eng. Anal. Bound. Elements*, 26, 355–366.

[55] Heyliger, P. and Wu, Y. C. (1999): Electroelastic fields in layered piezoelectric spheres. *Int. J. Eng. Sci.*, 38, 143–161.

[56] Chen, W. Q., Ding, H. J., and Xu, R. Q. (2001): Three-dimensional static analysis of multi-layered piezoelectric hollow spheres via the state space method. *Int. J. Solids Struct.*, 38, 4921–4936.

[57] Wang, H. M. and Ding, H. J. (2007): Radial vibration of piezoelectric/ magnetostrictive composite hollow sphere. *J. Sound Vib.*, 307, 330–348.

[58] Wang, H. M., Ding, H. J., and Ge, W. (2007): Transient responses in a two-layered elasto-piezoelectric composite hollow cylinder. *Compos. Struct.*, 79, 192–201.

[59] Tanigawa, Y. and Ootao, Y. (2007): Transient piezothermoelasticity of a two-layered composite hollow cylinder constructed of isotropic elastic and piezoelectric layers due to asymmetrical heating. *J. Thermal Stresses*, 30, 1003–1023.

[60] Sun, C. T. and Cheng, N. C. (1974): Piezoelectric waves on a layered cylinder. *J. Appl. Phys.*, 45, 4288–4294.

[61] Bai, H., Taciroglu, E., Dong, S. B., and Shah, A. H. (2004): Elastodynamic Green's functions for a laminated piezoelectric cylinder. *Int. J. Solids Struct.*, 41, 6335–6350.

[62] Bai, H., Shah, A. H., Dong, S. B., and Taciroglu, E. (2006): End reflections in a layered piezoelectric circular cylinder. *Int. J. Solids Struct.*, 43, 6309–6325.

[63] Ding, H. J., Wang, H. M., and Chen, W. Q. (2003): Transient responses in a piezoelectric spherically isotropic hollow sphere for symmetric problems. ASME *J. Appl. Mech.*, 70, 436–445.

[64] Chakraborty, A., Gopalakrishnan, S., and Kausel, E. (2005): Wave propagation analysis in inhomogeneous piezo-composite layer by the thin-layer method. *Int. J. Num. Meth. Eng.*, 64, 567–598.

[65] Norris, A. N. (1994): Dynamic Green's functions in anisotropic piezoelectric, thermoelastic and poroelastic solids. *Proc. R. Soc. Lond.*, A447, 175–188.

[66] Wang, C. Y. and Zhang, Ch. (2005): 3-D and 2-D dynamic Green's functions and time-domain BIEs for piezoelectric solids. *Eng. Anal. Bound. Elem.*, 29, 454–465.

[67] Pan, E. (2005): *Anisotropic Green's Functions and BEMs* (ed.), special issue: *Eng. Analy. Bound. Elem.*, 29, 511–671.

[68] Liu, J.X., Liu, X.L., and Zhao, Y.B. (2001): Green's functions for anisotropic magnetoelectroelastic solids with an elliptical cavity or a crack. *Int. J. Eng. Sci.*, 39, 1405–1418.

[69] Pan, E. (2002): Three-dimensional Green's functions in anisotropic magneto-electroelastic bimaterials. *J. Appl. Math. Phys. ZAMP*, 53, 815–838.

[70] Li, J.Y. (2002): Magnetoelectric Green's functions and their application to the inclusion and inhomogeneity problems. *Int. J. Solids Struct.*, 39, 4201–4213.

[71] Qin, Q. H. (2005): 2D Green's functions of defective magnetoelectroelastic solids under thermal loading. *Eng. Analy. Bound. Elem.*, 29, 577–585.

[72] Li, L. J. and Li, J. Y. (2006): Electromagnetic fields induced in a uniaxial multiferroic material by a point source or an ellipsoidal inclusion. *Phys. Rev.*, B73, 184416.

[73] Qin, Q. H. (2007): *Green's Function and Boundary Elements of Multified Materials.* Elsevier, Oxford, UK.

[74] Lee, J. M. and Ma, C. C. (2007): Analytical full-field solutions of a magnetoelectroelastic layered half-plane. *J. Appl. Phys.*, 101, 083502.

[75] Wang, X. and Pan, E. (2007): Electromagnetic fields induced by a point source in a uniaxial multiferroic full-space, half-space, and bimaterial-space. *J. Mat. Res.*, 22, 2144–2155.

[76] Wang, X. and Pan, E. (2007): Electromagnetic fields induced by a cuboidal inclusion with uniform spontaneous polarization and magnetization. *J. Appl. Phys.*, 101, 103524.

[77] Wang, X., Pan, E., Feng, W. J., and Albrecht, J. D. (2008): Electromagnetic fields induced by a concentrated heat source in multiferroic materials. *Phys. Stat. Sol. (b)*, 245, 206–212.

Springer

http://www.springer.com/978-0-387-89497-3

Special Topics in the Theory of Piezoelectricity

Yang, J. (Ed.)
2009, XII, 329 p. 89 illus., Hardcover
ISBN: 978-0-387-89497-3