2. Integral Calculus

The problems of characterizing the class of functions that are Riemann integrable and of discussing discontinuous functions, in particular of understanding for which functions the fundamental theorem of calculus is valid, as well as the need of integrating new functions, led to a new definition of integral due to Henri Lebesgue (1875–1941). Though the main ideas of Lebesgue's integration theory go back to Henri Lebesgue (1875–1941) and Giuseppe Vitali (1875–1932) at the beginning of the 1900's, applications as well as generalizations and extensions followed each other during the past century giving *measure* and *integration theory* a fundamental role in mathematical analysis.

Here we follow the approach of first introducing *Lebesgue's measure* and accordingly *Lebesgue's integral*. In Section 2.1.1 we collect the main results of the theory without proofs,[1] and in the following sections we develop its basic features.

2.1 Lebesgue's Integral

2.1.1 Definitions and properties: a short summary

The area of the subgraph of a nonnegative function can be computed in at least two different ways, see Figure 2.1. We compute the area of trapezoidal approximations determined by subdivisions of the x axis and then we pass to the limit when the lengths of the intervals of the subdivision tend to zero: this leads to Riemann's integral, compare [GM1]. Alternatively, we may subdivide the y axis and proceed similarly. In this second case, by taking equidistributed subdivisions, we may define

[1] The reader may find these proofs in, e.g., M. Giaquinta, G. Modica, *Mathematical Analysis: Foundations and Advanced Techniques for Functions of Several Variables*, Birkhäuser, to which in the sequel we shall refer to as [GM5].

M. Giaquinta and G. Modica, *Mathematical Analysis: An Introduction to Functions of Several Variables*, DOI: 10.1007/978-0-8176-4612-7_2,

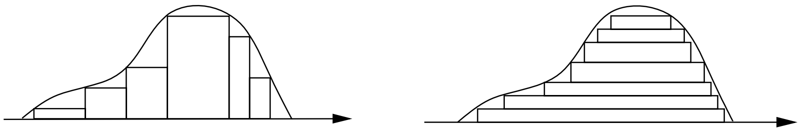


Figure 2.1. The integral: on the left Riemann's approach and on the right Lebesgue's approach.

$$\int_a^b f(x)\,dx := \lim_{N\to\infty} \frac{1}{2^N}\sum_{k=1}^{\infty} |E_{f,k\,2^{-N}}|, \tag{2.1}$$

where

$$E_{f,t} := \Big\{x \,\Big|\, f(x) > t\Big\}, \qquad t \in \mathbb{R},$$

and $|E_{f,t}|$ denotes the "measure" of $E_{f,t}$. Since $t \to |E_{f,t}|$ is nondecreasing, (2.1) defines Lebesgue's integral of f via *Cavalieri's formula* as

$$\text{Lebesgue}\int_a^b f(x)\,dx := \text{Riemann}\int_0^{\infty} |E_{f,t}|\,dt. \tag{2.2}$$

However, in order to proceed this way, we need a "good" notion of "measure" in $\mathbb{R}^n$ that allows us to measure rather wild sets as the sets $E_{f,t}$ may be. This is the role of *Lebesgue's measure*.

a. Lebesgue's measure

An *interval* I in $\mathbb{R}^n$, $n \geq 1$, is the product of n intervals, which for convenience we take left-open and right-closed, $I = \prod_{i=1}^n]a_i, b_i]$. The elementary n-dimensional *volume* of the interval I is by definition $|I| := \prod_{i=1}^n (b_i - a_i)$. The *outer* or *external measure* of an arbitrary subset E of $\mathbb{R}^n$ is defined by

$$\mathcal{L}^{n*}(E) := \inf\Big\{\sum_{k=1}^{\infty} |I_k| \,\Big|\, I_k \text{ intervals}, E \subset \bigcup_{k=1}^{\infty} I_k\Big\}. \tag{2.3}$$

Of course, $\mathcal{L}^{n*}$ defines a map $\mathcal{L}^{n*} : \mathcal{P}(\mathbb{R}^n) \to \overline{\mathbb{R}}_+$. It is easy to see that $\mathcal{L}^{n*}$ extends the elementary volume, in the sense that for every interval I we have $\mathcal{L}^{n*}(I) = |I|$. Intuitively $\mathcal{L}^{n*}(E)$ is computed by covering E in an "optimal" way with intervals $\{I_k\}$ and computing the sum of the series of the volumes of these intervals.

At this point, we would be done were if not for the fact that the outer measure $\mathcal{L}^{n*}$ is not additive: there exist disjoint subsets E, F of $\mathbb{R}^n$ such that $\mathcal{L}^{n*}(E \cup F) < \mathcal{L}^{n*}(E) + \mathcal{L}^{n*}(F)$.[2] We avoid this by selecting a class of special subsets, the class of *Lebesgue measurable sets*, and we define *Lebesgue's measure* as the restriction of $\mathcal{L}^{n*}$ to measurable sets.

[2] *Banach's paradox*: We can divide a ball in two parts each of the measure of the entire initial ball.

2.1 Definition. *A subset $E \subset \mathbb{R}^n$ is said to be* Lebesgue's measurable *or simply* measurable *if, given $\epsilon > 0$, there exists a set P_ϵ that is the union of at most a denumerable set of intervals such that*

$$P_\epsilon \supset E \qquad \textit{and} \qquad \mathcal{L}^{n*}(P_\epsilon \setminus E) < \epsilon.$$

*The class of all Lebesgue measurable subsets of $\mathbb{R}^n$ will be denoted by $\mathcal{M}$. The exterior measure of a measurable set E is its (*Lebesgue*)* measure *and denoted by $\mathcal{L}^n(E)$ or simply by $|E|$.*

Intervals and countable union of intervals, as well as sets for which $\mathcal{L}^{n*}(E) = 0$, are clearly in $\mathcal{M}$. One can also easily see that the interior and the closure of an interval, as well as the countable union of open or closed intervals, are measurable sets. Since *every open set is the denumerable union of disjoint intervals*, we then infer that open sets are measurable. Moreover, though we can show that there exist nonmeasurable sets, see, e.g., [GM5], one shows that $\mathcal{M}$ has the following *closure properties* and that Lebesgue's measure is well behaved on measurable sets.

2.2 Theorem. *We have*

(i) *$\mathcal{M}$ is a σ-*algebra*, i.e., if $E, F \in \mathcal{M}$, then $E \cup F$, $E \setminus F$ and $E \cap F$ are in $\mathcal{M}$ and, if $\{E_k\}$ is a sequence of measurable subsets, then $\cup_k E_k$ e $\cap_k E_k$ are measurable.*

(ii) *$\mathcal{L}^n$ is σ-*additive*, i.e., if $E, F \subset \mathbb{R}^n$ are measurable, then $|E \cup F| + |E \cap F| = |E| + |F|$ and, if $\{E_k\}$ is a sequence of measurable pairwise disjoint subsets of $\mathbb{R}^n$, then*

$$\Big| \bigcup_{k=1}^{\infty} E_k \Big| = \sum_{k=1}^{\infty} |E_k|.$$

(iii) *$\mathcal{L}^n$ is continuous on nondecreasing sequences of measurable sets, i.e., if $\{E_k\}$ is a sequence of measurable sets in $\mathbb{R}^n$ such that $E_k \subset E_{k+1}$ $\forall k$, then $|E_k| \to |\cup_h E_h|$ as $k \to \infty$.*

(iv) *$\mathcal{L}^n$ is continuous on nonincreasing sequences of measurable subsets with finite measure, i.e., if $\{E_k\}$ is a sequence of measurable subsets such that $E_k \supset E_{k+1}$ $\forall k$ and if $|E_1| < +\infty$, then $|E_k| \to |\cap_h E_h|$ as $k \to \infty$.*

For arbitrary sequences of subsets $\{E_k\}$, one shows:

(i) $\mathcal{L}^{n*}(\cup_k E_k) \le \sum_{k=1}^{\infty} \mathcal{L}^{n*}(E_k)$,

(ii) if $E_k \subset E_{k+1}$ $\forall k$, then $\mathcal{L}^{n*}(E_k) \to \mathcal{L}^{n*}(\cup_k E_k)$.

Since open sets are measurable, Theorem 2.2 (i) yields that closed sets are measurable, too. Finally, one shows that a measurable set is the countable intersection of open sets except for a set of zero measure. One also shows that it is a countable union of closed sets union a set of zero measure, compare [GM5].

Figure 2.2. Henri Lebesgue (1875–1941) and Giuseppe Vitali (1875–1932).

b. Measurable functions

Starting from Lebesgue's measure in $\mathbb{R}^n$ and from the class of Lebesgue's measurable sets $\mathcal{M}$ we are now able to build a theory of integration for functions $f : E \subset \mathbb{R}^n \to \overline{\mathbb{R}}$, where E is a measurable set and $\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty, -\infty\}$.

We begin by selecting the class of *measurable functions*, with respect to the $\mathcal{L}^n$ Lebesgue measure.

A function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ is said to be $\mathcal{L}^n$-measurable, in short *measurable* when the measure is understood, if for every $t \in \mathbb{R}$ the set

$$E_{f,t} := f^{-1}(]t, +\infty]) = \left\{ x \in \mathbb{R}^n \,\middle|\, f(x) > t \right\}$$

is $\mathcal{L}^n$-measurable. We then say that $f : E \subset \mathbb{R}^n \to \overline{\mathbb{R}}$ is *measurable on E* if E is measurable and the extension of f to $\mathbb{R}$ as $-\infty$ outsides E produces a function $\widetilde{f} : \mathbb{R}^n \to \overline{\mathbb{R}}$ that is measurable. Of course, a continuous function in $\mathbb{R}^n$ is measurable, and actually, if $E \subset \mathbb{R}^n$ is measurable, a function $f : E \to \mathbb{R}$ continuous in E is measurable.

As there exist nonmeasurable sets, there also exist nonmeasurable functions. However, on the ground of the fact that measurable sets form a σ-algebra, one shows that *all algebraic operations on measurable functions as well as taking pointwise limits of measurable functions produce measurable functions.*

Finally, the possibility of approximating in measure a Lebesgue measurable set from inside with closed sets and from outside with open sets yields the following characterization of Lebesgue measurable functions, see, e.g., [GM5].

2.3 Theorem (Lusin). *Let $f : E \to \mathbb{R}$ be a function defined on a measurable set $E \subset \mathbb{R}^n$. Then f is $\mathcal{L}^n$-measurable if and only if for any $\epsilon > 0$ there exists a closed set $F_\epsilon \subset \mathbb{R}^n$ such that $|E \setminus F_\epsilon| < \epsilon$ and the restriction of f to F_ϵ is continuous.*

c. Lebesgue's integral

We are now ready to define Lebesgue's integral of a measurable nonnegative function via Cavalieri's formula (2.2), using Riemann's integral and Lebesgue's measure:

$$\int_E f(x)\,dx := \int_0^{+\infty} \mathcal{L}^n(\{x \,|\, f(x) > t\})\,dt.$$

However, we prefer to follow a more direct approach and recover Cavalieri's formula later.

Recall that the *characteristic* (or *indicator*) *function* of a subset A of a set X is defined by

$$\chi_A(x) := \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \in X \setminus A. \end{cases}$$

A *simple function* is a measurable function that assumes only a finite number of values all of which are finite. We denote the class of simple functions by $\mathcal{S}$. If $a_1, a_2, \dots, a_k$ are the distinct values of a simple function φ, we can write

$$\varphi(x) = \sum_{j=1}^k a_j \chi_{E_j}(x), \qquad E_j := \Big\{x \,\Big|\, \varphi(x) = a_j\Big\}$$

where the E_j are measurable and pairwise disjoint sets. If φ is a simple nonnegative function, as suggested by intuition, the *integral* of φ is

$$I(\varphi) := \sum_{j=1}^k a_j |E_j|$$

with the agreement that $a_j|E_j| = 0$ if $a_j = 0$ and $|E_j| = +\infty$. The *Lebesgue integral* of a generic measurable and nonnegative function $f : E \to \overline{\mathbb{R}}$ is then defined by

$$\begin{aligned} &\int_E f(x)\,d\mathcal{L}^n(x) &(2.4)\\ &\quad := \sup\Big\{I(\varphi) \,\Big|\, \varphi \in \mathcal{S},\ \varphi(x) \le f(x) \forall x \in E,\ \varphi(x) = 0\ \forall x \in E^c\Big\}. \end{aligned}$$

We also write when necessary $\int_E f(x)\,d\mathcal{L}^n(x)$ instead of $\int_E f(x)\,dx$.

Finally, if $f : E \to \overline{\mathbb{R}}$ is measurable (but not of a constant sign) we decompose f as difference of its positive and negative parts, $f(x) = f_+(x) - f_-(x)$ where

$$f_+(x) := \max(f(x), 0), \qquad f_-(x) := \max(-f(x), 0),$$

and we set the following.


Figure 2.3. Beppo Levi (1875–1962) and Guido Fubini (1879–1943).

2.4 Definition. *Let $f : E \to \overline{\mathbb{R}}$ be measurable on the measurable set $E \subset \mathbb{R}^n$. We say that f is* (Lebesgue-)integrable *if at least one of the two integrals $\int_E f_+(x)\,dx$ and $\int_E f_-(x)\,dx$ is finite. If f is integrable, its* Lebesgue integral *is defined by*

$$\int_E f(x)\,d\mathcal{L}^n(x) := \int_E f_+(x)\,d\mathcal{L}^n(x) - \int_E f_-(x)\,d\mathcal{L}^n(x).$$

When no confusion may arise, we write

$$\int_E f(x)\,dx \qquad \textit{instead of} \qquad \int_E f(x)\,d\mathcal{L}^n(x).$$

*Finally, we say that f is (*Lebesgue-*)*summable *if both the integrals of f_+ and of f_- are finite. We denote the class of summable functions in E by $\mathcal{L}^1(E)$.*

Of course, $\int_{\mathbb{R}^n} \varphi(x)\,dx = I(\varphi)$ if $\varphi \in \mathcal{S}$. Also, notice that the difference in the definitions of the Riemann and Lebesgue integrals consists merely in the choice of the class of simple functions: finite combinations of characteristic functions of *intervals* in Riemann's theory, finite combinations of characteristic functions of *Lebesgue's measurable sets* in Lebesgue's theory.

2.5 ¶. Let $f : E \subset \mathbb{R}^n \to \overline{\mathbb{R}}$ be integrable, and let $\widetilde{f} : \mathbb{R}^n \to \overline{\mathbb{R}}$ denote its extension with zero values outside E. Show that

$$\int_E f(x)\,dx = \int_{\mathbb{R}^n} \widetilde{f}(x)\,dx.$$

d. Basic properties of Lebesgue's integral

The basic properties of Lebesgue's integral are easily inferred from the analogous properties of the integral of simple functions using the denumerable additivity of the Lebesgue measure and the following approximation lemma.

2.6 Lemma. *Let $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ be a nonnegative and measurable function. Then, there exists an increasing sequence $\{\psi_k\}$ of simple functions such that*

$$\psi_k \to f \quad \textit{pointwise}, \qquad \textit{and} \qquad \int_{\mathbb{R}^n} \psi_k(x)\, dx \to \int_{\mathbb{R}^n} f(x)\, dx.$$

In this way we can prove the following.

2.7 Theorem. *We have*

(i) (Monotonicity) *If f and g are integrable on E, and $f \leq g$, then*

$$\int_E f(x)\, dx \leq \int_E g(x)\, dx.$$

(ii) (Linearity) *$\mathcal{L}^1(E)$ is a real vector space, and the integral as a map from $\mathcal{L}^1(E)$ into $\mathbb{R}$ is a linear operator,*

$$\int_E (\alpha f(x) + \beta g(x))\, dx = \alpha \int_E f(x)\, dx + \beta \int_E g(x)\, dx$$

for all $\alpha, \beta \in \mathbb{R}$ and all $f, g \in \mathcal{L}^1(E)$.

(iii) (Continuity) *If f is integrable on E, then*

$$\Big| \int_E f(x)\, dx \Big| \leq \int_E |f(x)|\, dx.$$

(iv) (Beppo Levi Theorem) *Let E be a measurable set, let $f_k : E \to \mathbb{R}_+$ be an increasing sequence of nonnegative and measurable functions in E, and let $f(x) := \lim_{k\to\infty} f_k(x)$ be the pointwise limit of $\{f_k\}$. Then we have*

$$\int_E f(x)\, dx = \lim_{k\to+\infty} \int_E f_k(x)\, dx.$$

Beppo Levi's theorem is also referred to as the monotone convergence theorem for nonnegative functions.

The following claims are easy consequences of the above.

(i) If f is integrable on E, $|f(x)| \leq M$ for all $x \in E$ and $|E| < +\infty$, then f is summable on E and $\int_E |f(x)|\, dx \leq M\, |E|$.

(ii) $f \in \mathcal{L}^1(E)$ if and only if f is measurable and $\int_E |f(x)|\, dx < +\infty$.

(iii) If E and F are measurable sets, and f is integrable in $E \cup F$, then

$$\int_E f(x)\, dx + \int_F f(x)\, dx = \int_{E\cup F} f(x)\, dx + \int_{E\cap F} f(x)\, dx.$$

e. The integral as area of the subgraph

The Lebesgue integral can be equivalently defined as the area of the subgraph or via Cavalieri's formula. In fact the following holds.

2.8 Theorem. *Let $f : E \subset \mathbb{R}^n \to \overline{\mathbb{R}}$ be a nonnegative function. Then, f is measurable on E if and only if its subgraph*

$$SG_{f,E} := \Big\{(x,t) \,\Big|\, x \in E,\ 0 < t < f(x)\Big\} \subset \mathbb{R}^{n+1}$$

is a measurable subset of $\mathbb{R}^{n+1}$. Moreover,

$$\int_E f(x)\,dx = \mathcal{L}^{n+1}(SG_{f,E}). \tag{2.5}$$

2.9 Theorem (Cavalieri's formula). *Let $f : E \subset \mathbb{R}^n \to \overline{\mathbb{R}}$ be a nonnegative measurable function. Then*

$$\int_E f(x)\,dx = \int_0^{+\infty} \mathcal{L}^n(\{x \in E \,|\, f(x) > t\})\,dt.$$

f. Chebyshev's inequality

Let $f : E \to \overline{\mathbb{R}}$ be measurable in $E \subset \mathbb{R}^n$ and nonnegative. Set $E_{f,t} := \{x \in E \,|\, f(x) > t\}$. From the monotonicity of the integral we infer

$$|E_{f,t}| \le \frac{1}{t}\int_{E_{f,t}} f(x)\,dx \qquad \forall t > 0 \tag{2.6}$$

which for its wide use in several contexts has got various names: *weak estimate*, *Markov's inequality*, *Chebyshev's inequality*. It estimates the "size" of f in terms of the integral of f. The nondecreasing function $t \to |E_{f,t}|$ is called the *repartition function* of f.

g. Negligible sets and the integral

We say that the predicate $p(x)$, $x \in E \subset \mathbb{R}^n$ is true *for almost every $x \in E$*, or *almost everywhere in E* (in short *a.e.*), if the Lebesgue measure of the set

$$\Big\{x \in E \,\Big|\, p(x) \text{ is not true}\Big\}$$

is zero. For instance, if $f : E \subset \mathbb{R}^n \to \overline{\mathbb{R}}$ is a function, we say "$f = 0$ a.e. in E" or "$f(x) = 0$ for a.e. $x \in E$" if $\mathcal{L}^n(\{x \,|\, f(x) \neq 0\}) = 0$. Similarly, we say that "$|f(x)| < \infty$ a.e. in E" or "$|f(x)| < +\infty$ for a.e. $x \in E$" if $\mathcal{L}^n(\{x \,|\, |f(x)| \notin \mathbb{R}\}) = 0$. From the denumerable additivity of the Lebesgue measure we can easily deduce the following.

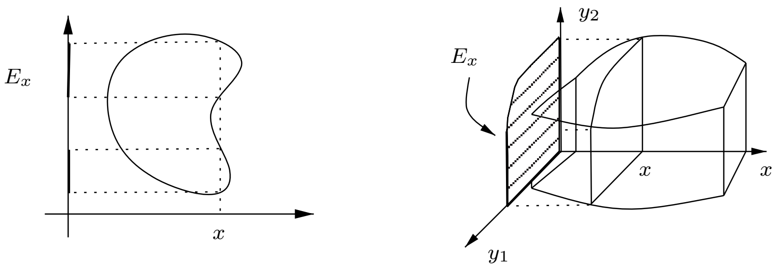


Figure 2.4. The slice E_x of E over x.

2.10 Proposition. *We have*

(i) *If* $f : E \to \mathbb{R}$ *is summable,* $f \in \mathcal{L}^1(E)$*, then* $|f(x)| < +\infty$ *for a.e.* $x \in E$.
(ii) *If* $f : E \to \mathbb{R}$ *is nonnegative, then* $\int_E f(x)\,dx = 0$ *if and only if* $f(x) = 0$ *for a.e.* $x \in E$.

Let $f : E \subset \mathbb{R}^n \to \overline{\mathbb{R}}$ be measurable. The *essential supremum of* f is the number (possibly $+\infty$) defined by

$$||f||_{\infty,E} = \operatorname*{ess\,sup}_E f := \inf\Big\{t \in \mathbb{R} \,\Big|\, f(x) < t \text{ for a.e. } x \in E\Big\}. \qquad (2.7)$$

Of course, $||f||_{\infty,E} = \sup_E |f(x)|$ if f is continuous on E, and

$$\int_E |f(x)|\,dx \le ||f||_{\infty,E}\,|E| \qquad \forall f \in \mathcal{L}^1(E).$$

h. Riemann integrable functions

The Lebesgue integral extends the Riemann integral. In fact, (generalized) Riemann integrable functions are Lebesgue integrable and the Lebesgue integral and Rieamnn integral of one of these functions agree, see [GM5]. This remark gives us a way to compute the Lebesgue integral of a large class of functions. For instance,

$$\int_0^1 \frac{1}{x}\,dx = +\infty, \qquad \int_{-\infty}^{+\infty} \frac{1}{1+x^2}\,dx = \pi, \qquad \text{etc.}$$

On the other hand, the long-standing problem of characterizing Riemann integrable functions was solved by Giuseppe Vitali (1875–1932) in terms of Lebesgue integral: *A bounded function* $f : [a,b] \to \mathbb{R}$ *is Riemann integrable if and only if it is* $\mathcal{L}^1$*-almost-everywhere continuous.*

2.1.2 Fubini's theorem and reduction to iterated integrals

2.11 Example. Let E be a subset of $\mathbb{R}^2$ whose coordinates are denoted by (x, y). For every $x \in \mathbb{R}$, we define the *slice of E at x* (actually the projection of) by

$$E_x := \Big\{y \in \mathbb{R} \,\Big|\, (x, y) \in E\Big\}.$$

If $E =]a, b] \times]c, d]$, then

$$E_x := \begin{cases}]c, d] & \text{if } x \in]a, b], \\ \emptyset & \text{otherwise.} \end{cases}$$

In particular $|E_x| = 0$ if $x \notin]a, b]$ and $|E_x| = d - c$ if $x \in]a, b]$; consequently

$$\mathcal{L}^2(]a, b] \times]c, d]) = (b - a)(d - c) = \int_a^b |E_x|\, dx.$$

Fubini's theorem extends the remark of the previous example to arbitrary measurable subsets of Euclidean spaces. Split the coordinate variables in $\mathbb{R}^{n+k}$ in two groups, for instance the first n coordinates and the remaining ones, which we denote by $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^k$ respectively, so that (x, y) denotes the coordinate variables in $\mathbb{R}^{n+k}$. Let E be a subset of $\mathbb{R}^{n+k}$, and for $x \in \mathbb{R}^n$, let

$$E_x := \Big\{y \in \mathbb{R}^k \,\Big|\, (x, y) \in E\Big\}$$

denote the *slice* of E over x (projected into the coordinate space $\mathbb{R}^k$), see Figure 2.4.

2.12 Theorem (Fubini). *Let $E \subset \mathbb{R}^{n+k}$ be $\mathcal{L}^{n+k}$-measurable in $\mathbb{R}^{n+k}$. Then the following hold:*

(i) *For a.e. $x \in \mathbb{R}^n$ the set $E_x \subset \mathbb{R}^k$ is $\mathcal{L}^k$-measurable.*
(ii) *The function $x \to \mathcal{L}^k(E_x)$ is $\mathcal{L}^n$-measurable.*
(iii) *We have*
$$\mathcal{L}^{n+k}(E) = \int_{\mathbb{R}^n} \mathcal{L}^k(E_x)\, d\mathcal{L}^n(x).$$

A very useful variant of Fubini's theorem is the following theorem that provides a formula that allows us to compute a multiple integral as the iteration of simple integrals.

2.13 Theorem (Reduction to iterated integrals). *Let $f : E \to \overline{\mathbb{R}}$, $E \subset \mathbb{R}^{n+k}$, be an $\mathcal{L}^{n+k}$-integrable function. Then*

(i) *for a.e. $x \in E$ the function $y \to f_x(y) := f(x, y)$ is $\mathcal{L}^k$-integrable in E,*
(ii) *the function $x \to \int_{E_x} f(x, y)\, dy$ is $\mathcal{L}^n$-measurable,*

(iii) *we have*

$$\int_E f(x,y)\,d\mathcal{L}^{n+k}(x,y) := \int_{\mathbb{R}^n} \left(\int_{E_x} f(x,y)\,d\mathcal{L}^k(y) \right) d\mathcal{L}^n(x).$$

We emphasize the fact that the only assumptions in the previous theorems are the $\mathcal{L}^{n+k}$ integrability of f in E in Theorem 2.12 and the $\mathcal{L}^{n+k}$-measurability of E in Theorem 2.13. We recall once again that f is integrable in E in each of the following cases:

(i) f is measurable and has constant sign;
(ii) f is summable in E; this happens in particular if f is measurable in E, $|f|$ is bounded, and $|E| < +\infty$.

We observe that Theorem 2.13 reduces in particular the calculus of a *double* integral to successively computing two simple integrals, the order being irrelevant.

$$\iint_E f(x,y)\,dx\,dy = \int_{-\infty}^{+\infty} \left(\int_{E_x} f(x,y)\,dy \right) dx$$
$$\iint_E f(x,y)\,dx\,dy = \int_{-\infty}^{+\infty} \left(\int_{E_y} f(x,y)\,dx \right) dy$$

where

$$E_x := \left\{ y \in \mathbb{R}^k \,\middle|\, (x,y) \in E \right\}, \qquad E_y := \left\{ x \in \mathbb{R}^n \,\middle|\, (x,y) \in E \right\}.$$

Of course, Theorem 2.13 can be used iteratively, thus reducing the calculus of the integral of an integrable function of n-variables to successively computing n integrals in one variable, the order of them being irrelevant. In other words we can also state the following.

2.14 Theorem (Tonelli). *Let $f : E \subset \mathbb{R}^2 \to \overline{\mathbb{R}}$ be integrable in E. Then, the three integrals*

$$\int_E f(x,y)\,d\mathcal{L}^{n+k}(x,y),$$

$$\int_{\mathbb{R}^n} \left(\int_{E_x} f(x,y)\,d\mathcal{L}^k(y) \right) d\mathcal{L}^n(x), \qquad \int_{\mathbb{R}^k} \left(\int_{E_y} f(x,y)\,d\mathcal{L}^n(x) \right) d\mathcal{L}^n(y)$$

exist and are equal.

2.1.3 Change of variables

The exterior Lebesgue measure $\mathcal{L}^{n*}$ is invariant under isometries; even more, if $T : \mathbb{R}^n \to \mathbb{R}^n$ is linear, then T *maps* $\mathcal{L}^n$*-measurable sets into* $\mathcal{L}^n$*-measurable sets, and*

$$\mathcal{L}^{n*}(T(E)) = |\det T|\, \mathcal{L}^{n*}(E), \qquad \forall E \subset \mathbb{R}^n, \tag{2.8}$$

in particular, *linear maps map set of measure zero into sets of measure zero.* Lipschitz-continuous functions, and consequently C^1 functions do the same, however continuous functions do not; in fact, continuous maps may map null sets into sets of positive Lebesgue measure.

The formula (2.8) extends to diffeomorphisms, i.e., one-to-one transformations of class C^1 with inverse of class C^1, as follows.

2.15 Theorem (Change of variables). *Let A be an open set in $\mathbb{R}^n$, and let $\varphi : A \to \mathbb{R}^n$ be a map of class C^1. Then φ maps measurable sets into measurable sets and negligible sets into negligible sets. Moreover, if $E \subset A$ is measurable, and φ is injective in E, then:*

(i) *We have*

$$\mathcal{L}^n(\varphi(E)) = \int_E |\det \mathbf{D}\varphi(x)|\, dx.$$

(ii) *If $f : \varphi(E) \to \overline{\mathbb{R}}$ is any function, then f is integrable on $\varphi(E)$ if and only if $x \to f(\varphi(x))\, |\det \mathbf{D}\varphi(x)|$ is integrable on E and*

$$\int_{\varphi(E)} f(y)\, dy = \int_E f(\varphi(x))\, |\det \mathbf{D}\varphi(x)|\, dx.$$

Notice that there is no need to assume $\det \mathbf{D}\varphi(x) \neq 0$, yet another relevant consequence of the Lebesgue integrability.

2.1.4 Differentiation and primitives

Let $f, g : \mathbb{R}^n \to \overline{\mathbb{R}}$ be two nonnegative and measurable functions. Of course, $\int_A f(x)\, dx = \int_A g(x)\, dx$ $\forall A \subset \mathbb{R}^n$ if and only if $f(x) = g(x)$ a.e. $x \in \mathbb{R}^n$. Is there a way to characterize $f(x)$ in terms of integrals, or more precisely in terms of the map $A \to \int_A f(x)\, dx$? The theory of *differentiation of integrals* answers this important question in measure theory.

Recall that, if $f : \mathbb{R} \to \mathbb{R}$ is continuous, then the integral mean value theorem yields

$$f(x_0) = \lim_{r\to 0} \frac{1}{2r} \int_{x_0-r}^{x_0+r} f(t)\, dt \qquad \forall x_0 \in \mathbb{R}.$$

We also have the following.

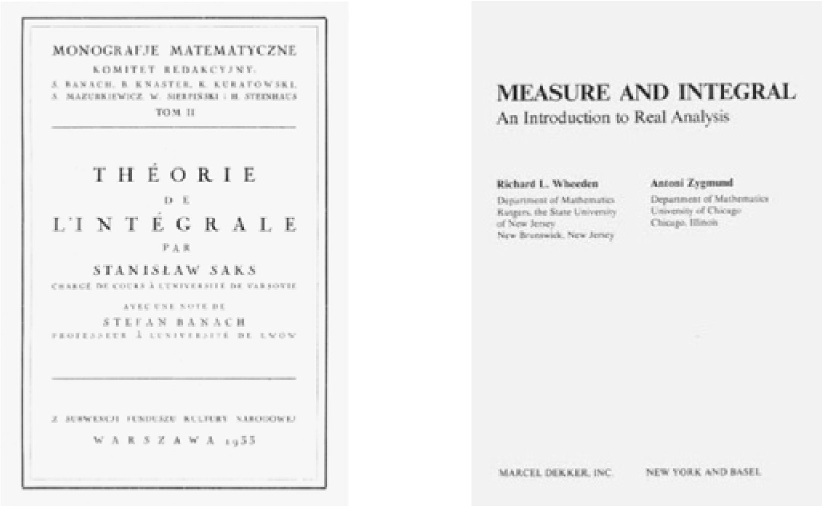


Figure 2.5. Two classic books on Lebesgue's integration.

2.16 Theorem (Lebesgue's differentiation). *Let $f : \mathbb{R}^n \to \mathbb{R}$ be such that $\int |f|^p\,dx < +\infty$ for some $1 \le p < +\infty$. Then, for a.e. $x \in \mathbb{R}^n$ we have*

$$\frac{1}{|B(x_0,r)|}\int_{B(x_0,r)} |f(y)-f(x)|^p\,dy \to 0 \qquad \textit{as } r \to 0^+;$$

in particular, for a.e. $x \in \mathbb{R}^n$,

$$\frac{1}{|B(x,r)|}\int_{B(x,r)} f(y)\,dy \to f(x) \qquad \textit{for a.e. } x \in \mathbb{R}^n.$$

Notice that if $f \in \mathcal{L}^1(E)$, E being measurable in $\mathbb{R}^n$, by applying the previous theorem to the function

$$\widetilde{f}(x) = \begin{cases} f(x) & \text{if } x \in E,\\ 0 & \text{if } x \in E^c \end{cases}$$

we get that for a.e. $x \in E$

$$\frac{1}{|B(x_0,r)|}\int_{E\cap B(x_0,r)} |f(y)-f(x)|^p\,dy \to 0 \qquad \text{as } r \to 0^+;$$

in particular, for a.e. $x \in \mathbb{R}^n$,

$$\frac{1}{|B(x,r)|}\int_{E\cap B(x,r)} f(y)\,dy \to \begin{cases} f(x) & \text{for a.e. } x \in E,\\ 0 & \text{for a.e. } x \in E^c. \end{cases}$$

2.17 Example. If $f \in \mathcal{L}^1(]-1,1[)$, then for $\mathcal{L}^1$-a.e. $x \in]-1,1[$ we have

$$\lim_{r \to 0} \frac{1}{2r} \int_{-r}^{+r} f(y)\, dy = f(x).$$

2.18 Definition. *Let $f : E \subset \mathbb{R}^n \to \overline{\mathbb{R}}$ be summable in E. We say that a point $x \in E$ is a* Lebesgue point *for f if there exists $\lambda \in \mathbb{R}$ such that*

$$\lim_{r \to 0} \frac{1}{|B(x,r)|} \int_{E \cap B(x,r)} |f(y) - \lambda|\, dy \to 0. \tag{2.9}$$

The set of Lebesgue points is then denoted by $\mathcal{L}_f$. Moreover, the value $\lambda = \lambda(x)$ such that (2.9) holds is unique and it is called the *Lebesgue value* of f at the Lebesgue point x of f. Therefore, we have a map $\lambda : \mathcal{L}_f \to \mathbb{R}$ that is called the *Lebesgue representative* of f and, with these notations, the Lebesgue differentiation theorem, Theorem 2.16, reads as follows.

2.19 Theorem (Lebesgue's differentiation). *Let $f \in \mathcal{L}^1(E)$ and let $\mathcal{L}_f$ be the set of Lebesgue points of f. Then $E \setminus \mathcal{L}_f$ has zero Lebesgue measure, $\mathcal{L}^n(E \setminus \mathcal{L}_f) = 0$.*

2.20 Asymmetric differentiation. In the differentiation theorem, Theorem 2.16, we can replace balls with cubes, and actually differentiate with respect to bounded sets A such that for instance

$$A \subset B(0,100), \qquad |A| = c|B_1|.$$

For $x \in \mathbb{R}^n$ and $r > 0$, we set $A_{x,r} := x + rA$. Trivially $A_{x,r} \subset B(x, 100r)$ and $|A_{x,r}| = r^n|A| = cr^n|B_1| = c|B(x,r)|$.

Theorem. *Let $f : E \subset \mathbb{R}^n \to \overline{\mathbb{R}}$ be measurable with $\int_E |f|^p\, dx < \infty$ for some $0 \le p < +\infty$. Then for a.e. $x \in E$ we have*

$$\frac{1}{|A_{x,r}|} \int_{E \cap A_{x,r}} |f(y) - f(x)|^p\, dy \to 0 \qquad \textit{as } r \to 0^+.$$

Example. If $f \in \mathcal{L}^1(\mathbb{R})$, then for a.e. $x \in \mathbb{R}$ we have

$$\lim_{r \to 0^+} \frac{1}{r} \int_0^r f(y)\, dy = \lim_{r \to 0^+} \frac{1}{r} \int_{-r}^0 f(y)\, dy = f(x)$$

and also

$$\lim_{r \to 0^+} \frac{1}{8r} \int_{2r}^{10r} f(y)\, dy = f(x).$$

We conclude by collecting a few relevant consequences of the differentiation theorem.

2.21 Theorem (Vitali). *Monotone real-valued functions $f : \mathbb{R} \to \mathbb{R}$ are a.e. differentiable in the classic sense. Moreover, $h' \in \mathcal{L}^1((a,b))$ $\forall a, b \in \mathbb{R}$, h' is nonnegative if h is nondecreasing and*

$$0 \le \int_x^y h'(t)\,dt \le h(y) - h(x) \qquad \forall x < y.$$

A function $f : \mathbb{R} \to \mathbb{R}$ is said to be *absolutely continuous* if for any $\epsilon > 0$ there exists $\delta > 0$ such that $\sum_{k=1}^{\infty} |f(x_k) - f(y_k)| < \epsilon$ whenever $\{x_k\}$ and $\{y_k\}$ are such that $\sum_{k=1}^{\infty} |x_k - y_k| < \delta$. Trivially Lipschitz-continuous functions are absolutely continuous, absolutely continuous functions are continuous, and there exist functions that are continuous but not absolutely continuous. A celebrated example is the so-called Cantor–Vitali function, see [GM5]. We have the following.

2.22 Theorem (Vitali). *A function $f : [a,b] \to \mathbb{R}$ is absolutely continuous in $[a,b]$ if and only if f is a.e. differentiable in $[a,b]$, $f' \in \mathcal{L}^1([a,b])$ and*

$$\int_x^y h'(t)\,dt = h(y) - h(x) \qquad \forall x, y \in [a,b],\ x < y. \tag{2.10}$$

The above implies that Lipschitz-continuous functions from $\mathbb{R}$ into $\mathbb{R}$ are a.e. differentiable and that the equality (2.10) holds for them. For Lipschitz-continuous functions of several variables we state the following.

2.23 Theorem (Rademacher). *Every Lipschitz-continuous function $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable in the classic sense for a.e. $x \in \mathbb{R}^n$. Moreover, the components of the map $x \to \mathbf{D}f(x)$ are measurable and*

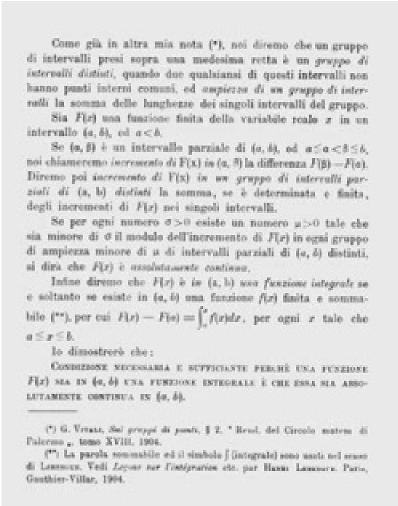
$$||\mathbf{D}f(x)||_{\infty,\mathbb{R}^n} = \mathrm{Lip}\,(f).$$

2.2 Convergence Theorems

In many respects and especially for the applications, the main results of Lebesgue's integration theory are contained in Beppo Levi's monotone convergence theorem, Theorem 2.7 (iv), and in Proposition 2.10. In this section we discuss some important, useful consequences.

a. Monotone convergence

First we state in a more general form Beppo Levi's theorem, weakening the positivity assumption and taking advantage of the fact that a.e. equal functions have the same integral.



Come già in altra mia nota (*), noi diremo che un gruppo di intervalli presi sopra una medesima retta è un *gruppo di intervalli distinti*, quando due qualsiansi di questi intervalli non hanno punti interni comuni, ed *ampiezza di un gruppo di intervalli* la somma delle lunghezze dei singoli intervalli del gruppo.

Sia $F(x)$ una funzione finita della variabile reale x in un intervallo (a, b), ed $a < b$.

Se (α, β) è un intervallo parziale di (a, b), ed $a \leq \alpha < \beta \leq b$, noi chiameremo *incremento di* $F(x)$ *in* (α, β) la differenza $F(\beta) - F(\alpha)$. Diremo poi *incremento di* $F(x)$ *in un gruppo di intervalli parziali di* (a, b) *distinti* la somma, se è determinata e finita, degli incrementi di $F(x)$ nei singoli intervalli.

Se per ogni numero $\sigma > 0$ esiste un numero $\mu > 0$ tale che sia minore di σ il modulo dell'incremento di $F(x)$ in ogni gruppo di ampiezza minore di μ di intervalli parziali di (a, b) distinti, si dirà che $F(x)$ è *assolutamente continua*.

Infine diremo che $F(x)$ è in (a, b) una *funzione integrale* se e soltanto se esiste in (a, b) una funzione $f(x)$ finita e sommabile (**), per cui $F(x) - F(a) = \int_a^x f(x)dx$, per ogni x tale che $a \leq x \leq b$.

Io dimostrerò che:

Condizione necessaria e sufficiente perchè una funzione $F(x)$ sia in (a, b) una funzione integrale è che essa sia assolutamente continua in (a, b).

(*) G. Vitali, *Sui gruppi di punti*, § 2. "Rend. del Circolo matem. di Palermo", tomo XVIII, 1904.

(**) La parola sommabile ed il simbolo $\int$ (integrale) sono usati nel senso di Lebesgue. Vedi *Leçons sur l'intégration* etc. par Henri Lebesgue. Paris, Gauthier-Villar, 1904.

Sul problema della misura dei gruppi di punti di una retta

NOTA

DI

G. VITALI

BOLOGNA

TIP. GAMBERINI E PARMEGGIANI

1905

Figure 2.6. The first page of the paper *Sulle funzioni integrali*, Acad. Sci. Torino 1905, by Giuseppe Vitali (1875–1932) and the frontispiece of the paper, again by Giuseppe Vitali, where for the first time the example of a set that is not Lebesgue measurable is presented.

2.24 Theorem (Beppo Levi). *Let $\{f_k\}$ be a nondecreasing sequence of integrable functions on $E \subset \mathbb{R}^n$ such that $f_k(x) \to f(x)$ for a.e. $x \in E$. If there exists a function $\phi \in \mathcal{L}^1(E)$ such that $f_k(x) \geq \phi(x)$ for all k and a.e. $x \in E$, then*

$$\int_E f(x)\,dx = \lim_{k\to\infty} \int_E f_k(x)\,dx.$$

Proof. We apply Beppo Levi's theorem to the nondecreasing sequence of nonnegative functions $\{f_k - \phi\}$ to get

$$\int_E (f_k(x) - \phi(x))\,dx \to \int_E (f(x) - \phi(x))\,dx.$$

The result then follows on account of the fact that ϕ has finite integral. □

2.25 ¶. Notice that the assumption $f_k \geq \phi$, $\phi \in \mathcal{L}^1(E)$, that is, the assumption that the lower envelope of the $f_k's$ is summable, cannot be omitted, as shown by the sequence

$$f_k(x) := \begin{cases} -1 & \text{if } x > k, \\ 0 & \text{otherwise.} \end{cases}$$

As a trivial consequence of Beppo Levi's theorem we can state the following.

2.26 Corollary (Total convergence of series). *Let $f_k : E \to \overline{\mathbb{R}}$, $k = 1, 2 \ldots$, be nonnegative measurable functions on E. Then*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PARIS,

GAUTHIER-VILLARS, IMPRIMEUR-LIBRAIRE

Quai des Grands-Augustins, 55.

1904

Figure 2.7. Frontispiece of the first edition of the treatise on integration by Henri Lebesgue (1875–1941) and a page from the second edition of 1928.

$$\int_E \sum_{k=1}^{\infty} f_k(x)\, dx = \sum_{k=1}^{\infty} \int_E f_k(x)\, dx.$$

2.27 Corollary. *Let $f_k : E \to \overline{\mathbb{R}}$, $k = 1, 2, \dots$ be measurable functions on E. If $\{f_k\}$ is nonincreasing and there exists $\phi \in \mathcal{L}^1(E)$ such that $f_k(x) \le \phi(x)$ for all k and a.e. $x \in E$, then*

$$\int_E \lim_{k\to\infty} f_k(x)\, dx = \lim_{k\to\infty} \int_E f_k(x)\, dx.$$

2.28 ¶. Notice that the assumption $f_k \le \phi$, $\phi \in \mathcal{L}^1(E)$ cannot be omitted as shown by the sequence

$$f_k(x) := \begin{cases} 1 & \text{if } x < -k, \\ 0 & \text{otherwise.} \end{cases}$$

2.29 ¶. Let $f : E \to \overline{\mathbb{R}}$ be integrable on E and let $\{E_k\}$, $k = 1, 2, \dots$, be a sequence of denumerable pairwise disjoint measurable subsets such that $E = \cup_k E_k$. Show that

$$\int_E f(x)\, dx = \sum_{k=1}^{\infty} \int_{E_k} f(x)\, dx.$$

b. Dominated convergence

2.30 Lemma (Fatou). *Let $\{f_k\}$ be a sequence of nonnegative and measurable functions on E. Then*

$$\int_E \liminf_{k\to\infty} f_k(x)\,dx \le \liminf_{k\to\infty} \int_E f_k(x)\,dx.$$

Proof. The functions $g_n(x) := \inf_{k\ge n} f_k(x)$, are nonnegative, measurable on E, and form a nondecreasing sequence; moreover

$$0 \le g_n(x) \le f_k(x),\ k \ge n, \qquad \liminf_{\kappa\to\infty} f_k(x) := \lim_{n\to\infty} g_n(x).$$

Thus $\int_E g_n(x)\,dx \le \inf_{k\ge n} \int_E f_k(x)\,dx$, and we infer, using Beppo Levi's theorem,

$$\int_E \liminf_{k\to\infty} f_k(x)\,dx = \lim_{n\to\infty} \int_E g_n(x)\,dx \le \lim_{n\to\infty} \inf_{k\ge n} \int_E f_k(x)\,dx =: \liminf_{k\to\infty} \int_E f_k(x)\,dx.$$

□

As previously, we can weaken the positivity condition to get the following result.

2.31 Corollary (Fatou lemma). *Let $\{f_k\}$ be a sequence of integrable functions on E and let $\phi \in \mathcal{L}^1(E)$.*

(i) *If $f_k(x) \ge \phi(x)$ for all k and a.e. $x \in E$, then*

$$\int_E \liminf_{k\to\infty} f_k(x)\,dx \le \liminf_{k\to\infty} \int_E f_k(x)\,dx.$$

(ii) *If $f_k(x) \le \phi(x)$ for all k and a.e. $x \in E$, then*

$$\limsup_{k\to\infty} \int_E f_k(x)\,dx \le \int_E \limsup_{k\to\infty} f_k(x)\,dx.$$

2.32 Theorem (Lebesgue dominated convergence theorem). *Let $\{f_k\}$ be a sequence of measurable functions on $E \subset \mathbb{R}^n$. If*

(i) *$f_k(x) \to f(x)$ for a.e. $x \in E$,*
(ii) *there exists $\phi \in \mathcal{L}^1(E)$ such that $|f_k(x)| \le \phi(x)$ for all k and a.e. $x \in E$,*

then

$$\int_E |f_k(x) - f(x)|\,dx \to 0,$$

in particular

$$\int_E f_k(x)\,dx \to \int_E f(x)\,dx.$$

Proof. By the assumptions $|f_k(x)-f(x)| \to 0$ for a.e. $x \in E$ and $|f_k(x)-f(x)| \le 2\phi(x)$ for all k and a.e. $x \in E$. Fatou's lemma, Corollary 2.31 (ii), then yields

$$\limsup_{k\to\infty} \int_E |f_k(x) - f(x)|\,dx \le \int_E \limsup_{k\to\infty} |f_k(x) - f(x)|\,dx = \int_E 0\,dx = 0.$$

The second part of the claim follows since

$$\left| \int_E f_k(x)\,dx - \int_E f(x)\,dx \right| = \left| \int_E (f_k(x) - f(x))\,dx \right| \le \int_E |f_k(x) - f(x)|\,dx.$$

□

2.33 ¶. Notice that the assumption (ii) amounts to requiring that the *envelope* of the functions $|f_k|$, defined as $\phi(x) := \sup_k |f_k(x)|$ is a summable function on E. Notice that (ii) cannot be omitted as it is shown by the sequence

$$f_k(x) = \begin{cases} k & \text{if } 0 < x < 1/k, \\ 0 & \text{otherwise.} \end{cases}$$

Finally, we state the following important convergence theorem for series of functions.

2.34 Theorem (Lebesgue). *Let $\{f_n\}$ be a sequence of measurable functions on E such that*

$$\sum_{k=0}^{\infty} \int_E |f_n(x)|\, dx < +\infty.$$

Then the series of functions $\sum_{k=0}^{\infty} f_n(x)$ converges absolutely for a.e. $x \in E$ to a function $f \in \mathcal{L}^1(E)$ and

$$\int_E \Big| f(x) - \sum_{k=0}^{p} f_k(x) \Big|\, dx \to 0 \qquad p \to \infty. \tag{2.11}$$

In particular,

$$\int_E f(x)\, dx = \sum_{k=0}^{\infty} \int_E f_k(x)\, dx.$$

Proof. For all $x \in E$, we let $g(x) \in \overline{\mathbb{R}}_+$ be the sum of the series $\sum_{k=0}^{\infty} |f_k(x)|$ with positive terms. From Beppo Levi's theorem and the assumptions we have

$$\int_E g(x)\, dx = \sum_{k=0}^{\infty} \int_E |f_k(x)|\, dx < +\infty.$$

Hence g is summable on E. Proposition 2.10 yields $g(x) < +\infty$ for a.e. $x \in E$. Therefore, for these x the series $\sum_{k=0}^{\infty} f_k(x)$ converges absolutely to a real-valued function $f(x) := \sum_{k=0}^{\infty} f_k(x)$ and for all integers $p \geq 1$ we have

$$\Big| \sum_{k=p}^{\infty} f_k(x) \Big| \leq \sum_{k=p}^{\infty} |f_k(x)|, \tag{2.12}$$

hence

$$|f(x)| \leq \sum_{k=0}^{\infty} |f_k(x)| = g(x)$$

for a.e. $x \in E$. This yields $f \in \mathcal{L}^1(E)$. Integrating (2.12) we also infer

$$\int_E \Big| f(x) - \sum_{k=0}^{p-1} f_k(x) \Big|\, dx = \int_E \Big| \sum_{k=p}^{\infty} f_k(x) \Big|\, dx \leq \int_E \sum_{k=p}^{\infty} |f_k(x)|\, dx = \sum_{k=p}^{\infty} \int_E |f_k(x)|\, dx,$$

hence the first part of the claim, when p tends to infinity. The second part easily follows as

$$\Big| \int_E f(x)\, dx - \sum_{k=0}^{p-1} \int_E f_k(x)\, dx \Big| \leq \int_E \Big| f(x) - \sum_{k=0}^{p-1} f_k(x) \Big|\, dx \to 0 \qquad \text{as } p \to \infty.$$

□

c. Absolute continuity of the integral

2.35 Theorem (Absolute continuity of the integral). *Suppose $f \in \mathcal{L}^1(E)$. Then for every $\epsilon > 0$ there exists $\delta > 0$ such that for every measurable subset $F \subset E$ with $|F| < \delta$ we have $\int_F |f|\,dx < \epsilon$. Equivalently*

$$\int_E f(x)\,dx \to 0 \qquad \textit{as } |E| \to 0.$$

Proof. Let $\epsilon > 0$. We set

$$f_k(x) = \begin{cases} k & \text{if } f(x > k, \\ f(x) & \text{if } -k \le f(x) \le k, \\ -k & \text{if } f(x < -k. \end{cases}$$

Trivially $|f_k(x) - f(x)| \to 0$ for every $x \in E$ as $k \to \infty$, and $|f_k(x) - f(x)| \le 2|f(x)| \in \mathcal{L}^1(E)$; the theorem of dominated convergence, Theorem 2.32, then yields that there exists $N = N_\epsilon$ such that

$$\int_E |f(x) - f_N(x)|\,dx < \epsilon/2.$$

We now choose $\delta := \epsilon/(2N)$; clearly for any $F \subset E$ with $|F| \le \delta$ we find

$$\int_F |f_N(x)|\,dx \le N\,|F| \le N\,\frac{\epsilon}{N} = \epsilon/2$$

hence

$$\int_F |f(x)|\,dx \le \int_F |f_N(x)|\,dx + \int_E |f - f_N|\,dx \le \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

□

2.36 ¶. Let f be summable in $\mathbb{R}^n$. Show that the function

$$F(x,r) := \int_{B(x,r)} f(t)\,dt, \qquad x \in \mathbb{R}^n,\ r \ge 0,$$

is continuous on $\mathbb{R}^n \times [0, +\infty[$.

d. Differentiation under the integral sign

Let E be a measurable set in $\mathbb{R}^n$ and let A be an open set in $\mathbb{R}^k$. If $f(t,x)$ is a function defined in $A \times E$ and integrable on E for each fixed $t \in A$, we may consider the function

$$F(t) := \int_E f(t,x)\,dx, \qquad t \in A.$$

2.37 Proposition. *Let $A \subset \mathbb{R}^k$ be open and $E \subset \mathbb{R}^n$ measurable. If $f : A \times E \to \mathbb{R}$ is such that*

(i) *for a.e. $x \in E$ the function $t \to f(t,x)$ is continuous on A,*
(ii) *$\forall t \in A$ the function $x \to f(t,x)$ is summable on E,*
(iii) *there exists $\phi \in \mathcal{L}^1(E)$ such that*

$$|f(t,x)| \le \phi(x) \qquad \textit{for all } t \in A \textit{ and a.e. } x \in E, \tag{2.13}$$

then the function

$$F(t) := \int_E f(t,x)\,dx, \qquad t \in A,$$

is continuous on A.

Proof. Let $t_0 \in A$ and let $\{t_k\}$ be a sequence in A converging to t_0. If $g_k(x) := f(t_k, x)$, then $g_k(x) \to f(t_0, x)$ as $k \to \infty$ for a.e. $x \in E$ and $|g_k(x)| \le \phi(x) \in \mathcal{L}^1(E)$ for all k and a.e. $x \in E$. The dominated convergence theorem, Theorem 2.32, then yields

$$F(t_k) = \int_E f(t_k,x)\,dx = \int_E g_k(x)\,dx \to \int_E f(t_0,x)\,dx = F(t_0),$$

i.e., the conclusion since the point t_0 and the sequence $\{t_k\}$ were arbitrary. □

2.38 ¶. Notice the following:

(i) The hypotheses of Proposition 2.37 hold if A and E are bounded domains and $f \in C^0(\overline{A \times E})$.

(ii) Consider the family of functions $x \to f_t(x) := f(t,x)$ when t varies in A. The estimate (2.13) amounts to the summability of the envelope $h(x) := \sup_{t\in A} |f_t(x)|$ of the family $\{|f_t(x)|\}_{t\in A}$.

(iii) The assumption (2.13) cannot be omitted. Indeed, if

$$f(t,x) = \begin{cases} \frac{|t|-|x|}{t^2} & \text{if } |x| < t, \\ 0 & \text{if } |x| \ge t, \end{cases}$$

we have $F(t) = 1$ for $t \ne 0$ and $F(0) = 0$.

The following claim is a simple extension of Proposition 2.37.

2.39 Proposition. *Let $A \subset \mathbb{R}^k$ be open and let $f : A \times]c,d[\to \mathbb{R}$ be a function such that*

(i) *$x \to f(t,x)$ is summable for all $t \in A$,*

(ii) *$t \to f(t,x)$ is continuous on A for a.e. x,*

(iii) *there exists $\phi \in L^1(]c,d[)$ such that $|f(t,x)| \le \phi(x)$ for all $t \in A$ and a.e. $x \in]c,d[$,*

Then the function $F : A \times]c,d[\times]c,d[\to \mathbb{R}$ defined by

$$F(t,r,s) := \int_r^s f(t,x)\,dx$$

is continuous on $A \times]c,d[\times]c,d[$.

Proof. Let $t, t_0 \in A$ and $r, s, r_0, s_0 \in]c,d[$. According to Proposition 2.37 we have

$$F(t, r_0, s_0) - F(t_0, r_0, s_0) = o(1) \qquad \text{as } t \to t_0$$

while

$$\begin{aligned} |F(t,r,s) - F(t,r_0,s_0)| &\le \left| \int_{r_0}^r |f(t,x)|\,dx \right| + \left| \int_{s_0}^s |f(t,x)|\,dx \right| \\ &\le \left| \int_{r_0}^r \phi(x)\,dx \right| = \left| \int_{s_0}^s \phi(x)\,dx \right| = o(1), \end{aligned}$$

uniformly in t as $r \to r_0$ and $s \to s_0$ by the absolute continuity of the integral, Theorem 2.35. Therefore we conclude

$$\begin{aligned}|F(t,r,s) - F(t_0,r_0,s_0)|\\ \le |F(t,r,s) - F(t,r_0,s_0)| + |F(t,r_0,s_0) - F(t_0,r_0,s_0)| \to 0\end{aligned}$$

as $(t,r,s) \to (t_0,r_0,s_0)$. □

Now let us state the theorem of derivation under the integral sign.

2.40 Theorem. *Let $A \subset \mathbb{R}^k$ be open and $E \subset \mathbb{R}^n$ be measurable. Denote by $t = (t_1, t_2, \dots, t_k)$ and $x = (x_1, x_2, \dots, x_n)$ the coordinates in A and E respectively. Suppose that $f : A \times E \to \mathbb{R}$, $f = f(t,x)$, satisfies the following:*

(i) *$x \to f(t,x)$ is $\mathcal{L}^n$-summable on E for all $t \in A$,*
(ii) *f has a partial derivative in the variable t_j at (t,x) for all t and for a.e. $x \in E$,*
(iii) *there exists $\phi \in \mathcal{L}^1(E)$ such that*

$$\Big|\frac{\partial f}{\partial t_j}(t,x)\Big| \le \phi(x) \qquad \text{for all } t \in A \text{ and a.e. } x \in E. \tag{2.14}$$

Then the function

$$F(t) := \int_E f(t,x)\,dx, \qquad t \in A,$$

has a partial derivative with respect to t_j at t for all $t \in A$ and

$$\frac{\partial F}{\partial t_j}(t) = \int_E \frac{\partial f}{\partial t_j}(t,x)\,dx \qquad \forall t \in A.$$

Proof. Let $t_0 \in A$ and let $t_k \to t_0$. Since A is open, we assume without loss of generality that $t_k \in B(t_0, \delta)$ for some $\delta > 0$. We have

$$\frac{F(t_k) - F(t_0)}{t_k - t_0} = \int_E \frac{f(t_k,x) - f(t_0,x)}{t_k - t_0}\,dx.$$

Also

$$\frac{f(t_k,x) - f(t_0,x)}{t_k - t_0} \to \frac{\partial f}{\partial t_j}(t_0,x) \qquad \text{as } k \to \infty$$

for a.e. $x \in E$, thus $\frac{\partial f}{\partial t_j}(t_0,x)$ is measurable on E. Applying Lagrange's theorem and the assumption (iii), we find $\xi_k \in A$ such that

$$\Big|\frac{f(t_k,x) - f(t_0,x)}{t_k - t_0}\Big| = \Big|\frac{\partial f}{\partial t}(\xi_k,x)\Big| \le \phi(x)$$

for all k and for a.e. $x \in E$. Therefore, the dominated convergence theorem yields that $x \to \frac{\partial f}{\partial t}(t_0,x)$ is summable on E and that

$$\frac{F(t_k) - F(t_0)}{t_k - t_0} = \int_E \frac{f(t_k,x) - f(t_0,x)}{t_k - t_0}\,dx \to \int_E \frac{\partial f}{\partial t}(t_0,x)\,dx,$$

i.e., the conclusion, since the point t_0 and the sequence $\{t_k\}$ were arbitrary. □

2.41 Corollary. *Let $A \subset \mathbb{R}^k$ be open and let $f : A\times]c,d[\to \overline{\mathbb{R}}$ be a continuous function such that*

(i) *for all $t \in A$, $x \to f(t,x)$ is summable on $]c,d[$,*
(ii) *$t \to f(t,x)$ is of class $C^1(A)$ for a.e. $x \in]c,d[$,*
(iii) *there exists $\phi \in L^1(]c,d[)$ such that*

$$\sum_{j=1}^{k} \Big|\frac{\partial f}{\partial t_j}(t,x)\Big| \le g(x) \qquad \text{for all } t \text{ and a.e. } x \in]c,d[,$$

Then the function

$$F(t,r,s) := \int_r^s f(t,x)\,dx$$

is of class $C^1(A\times]c,d[\times]c,d[)$. In particular, if $\alpha, \beta \in C^1(A)$ take value in $]c,d[$, then the map

$$G(t) := \int_{\alpha(t)}^{\beta(t)} f(t,x)\,dx$$

is of class $C^1(A)$ and, for $j = 1,\dots,k$ we have

$$\frac{\partial G}{\partial t_j}(t) = \int_{\alpha(t)}^{\beta(t)} \frac{\partial f}{\partial t_j}(t,x)\,dx + f(t,\beta(t))\frac{\partial \beta}{\partial t_j}(t) - f(t,\alpha(t))\frac{\partial \alpha}{\partial t_j}(t).$$

Proof. Theorem 2.40 yields the existence of the partial derivatives of $F(t,r,s)$ with respect to the t's variables, which are continuous by Proposition 2.39. On the other hand, by the fundamental theorem of calculus,

$$\frac{\partial F}{\partial s}(t,r,s) = f(t,s), \qquad \frac{\partial F}{\partial r} = -f(t,r)$$

that are continuous by assumptions. Thus $F(t,r,s)$ is of class $C^1(A\times]c,d[\times]c,d[)$. The chain rule yields the second part of the claim, since

$$G(t) = F(t,\alpha(t),\beta(t)), \qquad \forall t \in A.$$

□

2.3 Mollifiers and Approximations

a. C^0-approximations and Lusin's theorem

From Theorem 2.3 and Tietze's extension theorem, see [GM3], we readily infer the following.

2.42 Theorem (Lusin). *Let $f : E \subset \mathbb{R}^n \to \mathbb{R}$ be measurable on E. For every $\epsilon > 0$ there exists a continuous function $g_\epsilon : \mathbb{R}^n \to \mathbb{R}$ such that*

$$\mathcal{L}^n\Big(\Big\{x \in E \,\Big|\, f(x) \neq g_\epsilon(x)\Big\}\Big) < \epsilon \qquad \textit{and} \qquad ||g||_{\infty,E} \leq ||f||_{\infty,E}.$$

Moreover, if $f = 0$ outside an open set Ω of finite measure, then for every $\epsilon > 0$ there exists a function $g \in C^0_c(\Omega)$ such that

$$|\{x \in \Omega \,|\, f(x) \neq g(x)\}| < \epsilon, \qquad \textit{and} \qquad ||g||_{\infty,\Omega} \leq ||f||_{\infty,\Omega}.$$

Proof. Theorem 2.3 yields a closed set $C \subset E$ such that $f_{|C}$ is continuous and $\mathcal{L}^n(E \setminus C) < \epsilon$. By Tietze's theorem $f_{|C}$ admits a continuous extension $g : \mathbb{R}^n \to \mathbb{R}$ with

$$||g||_{\infty,\mathbb{R}^n} \leq \sup_{x \in C} |f(x)| = ||f||_{\infty,C} \leq ||f||_{\infty,E}$$

and, since $\{x \in E \,|\, f(x) \neq g(x)\} \subset E \setminus C$, we have

$$\Big|\Big\{x \in E \,\Big|\, f(x) \neq g(x)\Big\}\Big| < \epsilon.$$

The second part of the claim can be proved similarly. Since Ω has finite measure, Lusin's theorem, Theorem 2.3, yields a compact set $K \subset\subset \Omega$ such that $|\Omega \setminus K| < \epsilon$ and $f_{|K}$ is continuous. If $\epsilon_0 > 0$ is such that $K \subset\subset \Omega_{\epsilon_0}$, where

$$\Omega_{\epsilon_0} := \Big\{x \in \Omega \,\Big|\, \operatorname{dist}(x, \partial\Omega) > \epsilon_0\Big\},$$

the function $\overline{f} : K \cup \Omega^c_{\epsilon_0} \to \mathbb{R}$ defined by

$$\overline{f}(x) = \begin{cases} f(x) & x \in K, \\ 0 & \text{se } x \in \Omega^c_{\epsilon_0} \end{cases}$$

is continuous on the closed set $K \cup \Omega^c_{\epsilon_0}$, hence by Tietze's extension theorem admits a continuous extension to the whole of $\mathbb{R}^n$ with

$$||g||_{\infty,\mathbb{R}^n} \leq ||\overline{f}||_{\infty,\Omega^c_{\epsilon_0} \cup K} \leq ||f||_{\infty,\Omega}.$$

Clearly $g \in C^0_c(\Omega)$ and, since $\{x \in \Omega \,|\, \overline{f}(x) \neq g(x)\} \subset \Omega \setminus K$, we conclude that

$$\Big|\Big\{x \in A \,\Big|\, f(x) \neq g(x)\Big\}\Big| < \epsilon.$$

□

As a consequence we find the following.

2.43 Theorem. *Let Ω be an open set in $\mathbb{R}^n$ and let $f : \Omega \to \mathbb{R}$ be summable on Ω. There exists a sequence $\{\varphi_n\}$ of functions of class $C^0_c(\Omega)$ such that*

$$\int_\Omega |f(x) - \varphi_n(x)|\, dx \to 0 \qquad \textit{as } n \to \infty.$$

Proof. It suffices to show that for any $\epsilon > 0$, there exists $g \in C^0_c(\Omega)$ such that $\int_\Omega |f-g| < 2\epsilon$. Given $\epsilon > 0$ we choose N large so that, by setting

$$f_N(x) := \begin{cases} N & \text{if } f(x) > N \text{ and } |x| \le N, \\ f(x) & \text{if } |f(x)| \le N \text{ and } |x| \le N, \\ -N & \text{if } f(x) < -N \text{ and } |x| \le N, \\ 0 & \text{if } |x| > N, \end{cases}$$

we have $\int_\Omega |f - f_N|\,dx < \epsilon$. This can be done since $\int_\Omega |f - f_N|\,dx \to 0$ as $N \to \infty$ by the dominated convergence theorem.

According to Theorem 2.42 there exists a function $g \in C^0_c(\Omega)$ such that

$$||g||_\infty \le ||f_N||_\infty \le N \qquad \text{and} \qquad |\{x \in \Omega \,|\, g(x) \ne f_N(x)\}| < \frac{\epsilon}{2N}.$$

Consequently we find

$$\int_\Omega |f-g|\,dx \le \int_\Omega |f - f_N|\,dx + \int_\Omega |f_N - g|\,dx \le \epsilon + 2N\frac{\epsilon}{2N} = 2\epsilon.$$

□

2.44 Proposition (Mean continuity). *Let $f \in \mathcal{L}^1(\mathbb{R}^n)$. Then*

$$\int_{\mathbb{R}^n} |f(x+h) - f(x)|\,dx \to 0 \qquad \text{as } h \to 0.$$

Proof. (i) If $f \in C^0_c(\mathbb{R}^n)$, $\operatorname{spt} f \subset B(0,R)$, and $|h| < 1$, we have

$$\begin{aligned} &|f(x+h) - f(x)| \to 0 \qquad \text{for all } x \in E \\ &|f(x+h) - f(x)| \le 2||f||_\infty \chi_{B(0,R+1)}(x). \end{aligned}$$

Therefore, in this case, the claim follows from the dominated convergence theorem.

(ii) In the general case we proceed by approximation. Given $\epsilon > 0$, there exists $g \in C^0_c(\mathbb{R}^n)$ such that $\int_{\mathbb{R}^n} |f-g|\,dx < \epsilon$. Since

$$\int_{\mathbb{R}^n} |f(x+h) - f(x)|\,dx \le \int_{\mathbb{R}^n} |g(x+h) - g(x)|\,dx + 2\int_{\mathbb{R}^n} |f-g|\,dx$$

when $h \to 0$, by (i) we conclude

$$\limsup_{h \to 0} \int_{\mathbb{R}^n} |f(x+h) - f(x)|\,dx \le 2\epsilon.$$

□

b. Mollifying in $\mathbb{R}^n$

A function $k(x) \in C^\infty(\mathbb{R}^n)$ such that

$$k(x) = k(-x), \quad k(x) \ge 0, \quad k(x) = 0 \text{ if } |x| \ge 1 \quad \text{and} \quad \int_{\mathbb{R}^n} k(x)\,dx = 1,$$

is called a *mollifying* (or *regularizing*) *kernel.* The family

$$k_\epsilon(x) := \epsilon^{-n} k\Big(\frac{x}{\epsilon}\Big), \qquad \epsilon > 0,$$

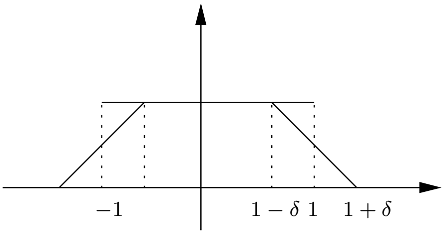


Figure 2.8. The function $\int_{\mathbb{R}} \chi_{[-1,1]}(y)\chi_{[-\delta,\delta]}(x-y)\,dx$, $\delta \le 1$.

is the family of *mollifiers* generated by k. Clearly $k_\epsilon(x) = k_\epsilon(-x)$, $k_\epsilon(x) \ge 0$ and $k_\epsilon(x) = 0$ outside $B(0,\epsilon)$. Moreover, by the change of variables $y = x/\epsilon$,

$$\int_{\mathbb{R}^n} k_\epsilon(y)\,dy = 1 \qquad \forall \epsilon > 0.$$

2.45 Example. The function $\varphi(x) := g(|x|)$, $x \in \mathbb{R}^n$, where

$$\varphi(x) := \begin{cases} \exp\left(-\frac{1}{1-|x|^2}\right) & \text{if } |x| < 1, \\ 0 & \text{otherwise} \end{cases}$$

is symmetric, nonnegative, of class $C^\infty(B(0,1))$, and nonzero exactly on $B(0,1)$. Consequently, if

$$C := \int_{\mathbb{R}^n} \varphi(x)\,dx,$$

the function $k(x) := \frac{1}{C}\varphi(x)$ is a mollifying kernel in $\mathbb{R}^n$.

A function $f : \mathbb{R}^n \to \mathbb{R}$ is *locally summable* and we write $f \in \mathcal{L}^1_{loc}(\mathbb{R}^n)$, if $f \in \mathcal{L}^1(A)$ for any bounded set $A \subset \mathbb{R}^n$. If f is locally summable in $\mathbb{R}^n$, the function

$$f_\epsilon(x) = f * k_\epsilon(x) := \int_{\mathbb{R}^n} k_\epsilon(x-y) f(y)\,dy$$

is called the ϵ-*regularized*, or ϵ-*mollified*, of f, and the operators $S_\epsilon(f) := f_\epsilon$ are called the *regularizing operators* associated to k. Notice that

$$\begin{aligned} f * k_\epsilon(x) &= \int_{\mathbb{R}^n} k_\epsilon(x-y) f(y)\,dy = \int_{B(x,\epsilon)} f(y) k_\epsilon(x-y)\,dy \\ &= \int_{B(0,\epsilon)} f(x-z) k_\epsilon(z)\,dz \end{aligned}$$

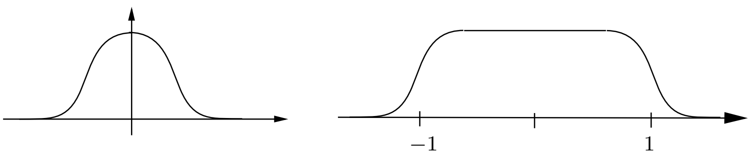


Figure 2.9. A convolution kernel and a regularized of $f(x) = \chi_{[-1,1]}(x)$.

since k_ϵ vanishes outside $B(0,\epsilon)$; the last inequality follows by changing y into $z := x - y$.

Finally, given $\Omega \subset \mathbb{R}^n$ and $\epsilon > 0$, we define

$$\Omega_\epsilon := \Big\{x \in \Omega \,\Big|\, \operatorname{dist}(x, \partial\Omega) > \epsilon\Big\}, \qquad \Omega_{-\epsilon} := \Big\{x \in \mathbb{R}^n \,\Big|\, \operatorname{dist}(x, \Omega) < \epsilon\Big\}.$$

Of course Ω_ϵ is nonempty for ϵ small if and only if Ω has nonempty interior.

2.46 Proposition. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a summable function. Then*

(i) *For any $\epsilon > 0$, the function $f_\epsilon(x) := f * k_\epsilon(x)$, $x \in \mathbb{R}^n$, is of class $C^\infty(\mathbb{R}^n)$. If f is constant in Ω, $f(x) = c$, then $f_\epsilon(x) = c$ in Ω_ϵ. In particular, if f vanishes outside Ω, then f_ϵ vanishes outside $\Omega_{-\epsilon}$.*

(ii) *We have*

$$\int_{\mathbb{R}^n} |f_\epsilon(x)|\, dx \le \int_{\mathbb{R}^n} |f(x)|\, dx \qquad \text{and} \qquad \int_{\mathbb{R}^n} |f_\epsilon - f|\, dx \to 0.$$

(iii) *For every compact $K \subset \mathbb{R}^n$ we have*

$$\sup_{x \in K} |f_\epsilon(x)| \le ||f||_{\infty, K_{-\epsilon}}.$$

Proof. (i) The theorem of differentiation under the integral sign yields that $f * k_\epsilon(x)$ has continuous partial derivatives and for $i = 1, \dots, n$,

$$D_i(f * k_\epsilon)(x) = \int_{\mathbb{R}^n} f(y) D_i k_\epsilon(x - y)\, dy.$$

By induction we conclude that $f_\epsilon = f * k_\epsilon \in C^\infty(\mathbb{R}^n)$. The second part of the claim follows since k_ϵ has support in $\overline{B(0,\epsilon)}$.

(ii) Changing the order of integration, Fubini's theorem, we infer

$$\int_{\mathbb{R}^n} |g_\epsilon(x)|\, dx \le \int_{\mathbb{R}^n} dx \int_{\mathbb{R}^n} |g(y)|\, k_\epsilon(x - y)\, dy = \int_{\mathbb{R}^n} |g(y)| \Big(\int_{\mathbb{R}^n} k_\epsilon(x - y)\, dx \Big) dy.$$

Since $\int_{\mathbb{R}^n} k_\epsilon(x - y)\, dx = 1\ \forall y \in \mathbb{R}^n$, we conclude that

$$\int_{\mathbb{R}^n} |g_\epsilon(x)|\, dx \le \int_{\mathbb{R}^n} |g(y)|\, dy.$$

To prove the second part of the claim, first, we notice that we have

$$\begin{aligned} |f_\epsilon(x) - f(x)| &= \Big| \int_{\mathbb{R}^n} f(y) k_\epsilon(x - y)\, dy - f(x) \Big| = \Big| \int_{\mathbb{R}^n} (f(y) - f(x)) k_\epsilon(x - y)\, dy \Big| \\ &\le \int_{\mathbb{R}^n} k_\epsilon(z) |f(x - z) - f(x)|, \end{aligned} \tag{2.15}$$

and, integrating we conclude

$$\int_{\mathbb{R}^n} |f_\epsilon(x) - f(x)|\, dx \le \int_{\mathbb{R}^n} k_\epsilon(z) \Big(\int_{\mathbb{R}^n} |f(x - z) - f(x)|\, dx \Big)\, dz.$$

Given $\sigma > 0$, by Proposition 2.44 there exists $\epsilon_0 > 0$ such that

$$\int_{\mathbb{R}^n} |f(x - z) - f(x)|\, dx < \sigma$$

for all z with $|z| < \epsilon_0$. Therefore, for every $\epsilon < \epsilon_0$ we have

$$\int_{\mathbb{R}^n} k_\epsilon(z)\Big(\int_{\mathbb{R}^n} |f(x-z) - f(x)|\,dx\Big)\,dz \le \sigma \int_{\mathbb{R}^n} k_\epsilon(z)\,dz = \sigma.$$

(iii) For all $x \in \mathbb{R}^n$ we have

$$|f_\epsilon(x)| \le \int_{B(x,\epsilon)} |f(y)| k_\epsilon(x-y)\,dy \le ||f||_{\infty,K_{-\epsilon}} \int_{B(x,\epsilon)} k_\epsilon(x-y)\,dy = ||f||_{\infty,K_{-\epsilon}}.$$

□

c. Mollifying in Ω

Let $\Omega \subset \mathbb{R}^n$ be an open set and let $f : \Omega \to \mathbb{R}$ be summable. We can extend f as a function $\overline{f}$ defined on all of $\mathbb{R}^n$ and summable on $\mathbb{R}^n$ in several ways, for example as

$$\overline{f}(x) = \begin{cases} f(x) & \text{if } x \in \Omega, \\ 0 & \text{if } x \in \Omega^c. \end{cases} \tag{2.16}$$

The mollified of $\overline{f}$, *a priori* depend on the value of $\overline{f}$ on Ω^c; however, for every $\epsilon > 0$, if $\Omega_\epsilon \neq \emptyset$, the value of the ϵ-mollified of $\overline{f}$ at point $x \in \Omega_\epsilon$ depends merely on f, since $f = \overline{f}$ on $B(x,\epsilon)$ and

$$\overline{f}_\epsilon(x) = \int_{B(x,\epsilon)} \overline{f}(y) k_\epsilon(x-y)\,dy = \int_\Omega \overline{f}(y) k_\epsilon(x-y)\,dy. \tag{2.17}$$

We therefore define the *ϵ-mollified*, or *ϵ-regularized*, of f in Ω by setting for $x \in \Omega_\epsilon$

$$f_\epsilon(x) := \int_\Omega f(y) k_\epsilon(x-y)\,dy = \int_{B(x,\epsilon)} f(y) k_\epsilon(x-y)\,dy$$

so that (2.17) writes also as $f_\epsilon(x) = \overline{f}_\epsilon(x)$ $\forall x \in \Omega_\epsilon$.

2.47 Proposition. *Let Ω be an open set in $\mathbb{R}^n$ and let $f : \Omega \to \mathbb{R}$ be a summable function. For all $\epsilon > 0$, the ϵ-mollified $f_\epsilon := f * k_\epsilon$ is well defined in Ω_ϵ and we have the following.*

(i) *If $\widetilde{\Omega} \subset\subset \Omega$, then*

$$\int_{\widetilde{\Omega}} |f_\epsilon|\,dx \le \int_\Omega |f|\,dx, \qquad ||f_\epsilon||_{\infty,\widetilde{\Omega}} \le ||f||_{\infty,\Omega} \qquad \forall \epsilon < \operatorname{dist}(\widetilde{\Omega}, \partial\Omega)$$

and

$$\int_{\widetilde{\Omega}} |f_\epsilon(x) - f(x)|\,dx \to 0 \qquad \text{as } \epsilon \to 0.$$

(ii) *If $f \in C^0(\Omega)$, then $f_\epsilon \to f$ uniformly on compact sets of Ω.*

(iii) *If* $f \in C^k(\Omega)$, *then for every* α, $|\alpha| \le k$,

$$D^\alpha(f * k_\epsilon)(x) = (D^\alpha f) * k_\epsilon(x) \qquad \forall x \in \Omega_\epsilon,$$

and $D^\alpha f_\epsilon \to D^\alpha f$ *uniformly on compact sets of* Ω.

(iv) *If* $f \in Lip(\Omega)$, *then* f_ϵ *is Lipschitz-continuous in* Ω_ϵ *and*

$$\sup_{x,y\in\Omega_\epsilon} \frac{|f_\epsilon(x) - f_\epsilon(y)|}{|x-y|} \le \mathrm{Lip}\,(f,\Omega).$$

(v) *If* $\varphi : E \to \mathbb{R}$, $||\varphi||_{\infty,E} < +\infty$ *and* $\operatorname{spt}\varphi \subset \Omega_{2\epsilon}$, *then* $f\varphi_\epsilon$ *and* $f_\epsilon\varphi$ *are summable on* Ω *and*

$$\int_\Omega f(x)\varphi_\epsilon(x)\,dx = \int_\Omega f_\epsilon(x)\varphi(x)\,dx.$$

Proof. Let $\overline{f}$ be as in (2.16).

(i) Trivially, it follows from Proposition 2.46.

(ii) Let K be compact and let $\epsilon_0 := \mathrm{dist}\,(K, \partial\Omega)$. The set $\overline{K_{\epsilon_0/2}}$ is again a compact in Ω, and using (2.15) we infer

$$\begin{aligned}|f_\epsilon(x) - f(x)| = |\overline{f}_\epsilon(x) - \overline{f}(x)| &\le \sup_{z\in B(0,\epsilon)} |\overline{f}(x-z) - \overline{f}(x)| \\ &= \sup_{z\in B(0,\epsilon)} |f(x-z) - f(x)| \le \sup_{\substack{x\in K, y\in K_{\epsilon_0/2} \\ |x-y|<\epsilon}} |f(y) - f(x)| \qquad (2.18)\end{aligned}$$

for all $\epsilon < \epsilon_0/2$. The uniform continuity of f on $\overline{K_{\epsilon_0/2}}$ yields, for $\sigma > 0$, a $\delta > 0$ such that $|f(x) - f(y)| < \sigma$ if $x, y \in \overline{K_{\epsilon_0/2}}$ and $|x-y| < \delta$. Therefore we find

$$|f_\epsilon(x) - f(x)| \le \sigma \qquad \forall \epsilon \le \min(\delta, \epsilon_0/2) \text{ and } \forall x \in K,$$

i.e., $f_\epsilon \to f$ uniformly on K.

(iii) Changing variables, $z = x - y$, we find

$$f_\epsilon(x) = \int_{B(0,\epsilon)} f(x-z)k_\epsilon(z)\,dz,$$

and differentiating under the integral sign,

$$D_i(f*k_\epsilon)(x) = \int_{B(0,\epsilon)} D_i f(x-z)k_\epsilon(z)\,dz = \int_{B(x,\epsilon)} D_i f(y)k_\epsilon(x-y)\,dy = (D_i f)*k_\epsilon(x).$$

From (ii) we then infer that $D_i f_\epsilon \to D_i f$ uniformly on the compact sets of Ω.

(iv) In fact, if $x, y \in \Omega_\epsilon$, then

$$\begin{aligned}|f_\epsilon(x) - f_\epsilon(y)| &= \left| \int_{B(0,\epsilon)} (f(x-z) - f(y-z))k_\epsilon(z)\,dz \right| \\ &\le \int_{B(0,\epsilon)} |f(x-z) - f(y-z)|k_\epsilon(z)\,dz \le \mathrm{Lip}\,(f,\Omega)|x-y|.\end{aligned}$$

(v) Using (i) we find

$$\int_\Omega |f(x)|\,|\varphi_\epsilon(x)|\,dx = \int_{\Omega_\epsilon} |f(x)|\,|\varphi_\epsilon(x)|\,dx \leq ||\varphi_\epsilon||_{\infty,\Omega_\epsilon} \int_\Omega |f|\,dx$$
$$\leq ||\varphi||_{\infty,\Omega} \int_\Omega |f|\,dx < +\infty$$

and

$$\int_\Omega |f_\epsilon(x)|\,|\varphi(x)|\,dx = \int_{\Omega_{2\epsilon}} |f_\epsilon(x)|\,|\varphi(x)|\,dx \leq ||\varphi||_{\infty,\Omega} \int_\Omega |f|\,dx < +\infty.$$

The function $f(x)\varphi(y)k_\epsilon(x-y)$, $(x,y) \in \Omega \times \Omega$, is summable; therefore, by changing the order of integration, we find

$$\begin{aligned}
\int_\Omega f(x)\varphi_\epsilon(x)\,dx &= \int_\Omega f(x)\Big(\int_\Omega \varphi(y)k_\epsilon(x-y)dy\Big)\,dx \\
&= \int_\Omega dx \int_\Omega f(x)\varphi(y)k_\epsilon(x-y)\,dy \\
&= \int_\Omega \varphi(y)\Big(\int_\Omega f(x)k_\epsilon(x-y)\,dx\Big)\,dy \\
&= \int_\Omega \varphi(y)\Big(\int_\Omega f(x)k_\epsilon(y-x)\,dx\Big)\,dy \\
&= \int_\Omega \varphi(y)f_\epsilon(y)\,dy.
\end{aligned}$$

□

2.4 Calculus of Integrals

The aim of this section is to familiarize the reader with the calculus of multiple integrals and with the theorem of derivation under the integral sign.

2.4.1 Calculus of multiple integrals

As we have seen, the calculus of a *double integral*, i.e., of the integral of a function of two independent variables, can be reduced to the successive calculus of two simple integrals, i.e., of a function of one variable, and this can be done in two different ways that are equivalent. Moreover, if it is useful, we may at each stage change variables. For the calculus of a *triple integral*, i.e., the integral of a function of three variables, there are 12 different ways of using the formula of reduction of integrals, *a priori* all praticable, and at each step we can change variables. In short, any strategy that uses all possible combinations of Fubini's theorem in one of its forms and of the theorem of change variables, even the most unlikely, is possible as long as it leads to the end.

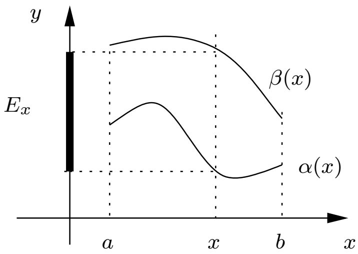


Figure 2.10. A normal subset in $\mathbb{R}^2$.

The aim of exercises is that of learning how to choose an optimal strategy for the calculus of integrals on the basis, for instance, of symmetries of the domain of integration and/or of the function to be integrated.

We recall that, for the formula of reduction of integrals to be valid, see Tonelli's theorem, Theorem 2.14, the summability of the involved functions is required.

2.48 ¶. Show that the assumption of integrability in Tonelli's theorem is essential. For instance, show that the following iterated integrals

$$\int_0^1 dy \int_1^\infty (e^{-xy} - 2e^{-2xy})\, dx, \qquad \int_1^\infty dx \int_0^1 (e^{-xy} - 2e^{-2xy})\, dy$$

both exist and are different.

We repeat that f is integrable on E in each of the following two cases.

(i) f is measurable on E and has constant sign, for instance if E is measurable and f is a.e. continuous on E and nonnegative. This applies in particular for $|f|$.
(ii) f is summable in E, $f \in \mathcal{L}^1(E)$, in particular if $|E| < +\infty$ and f is bounded on E; for instance if E is compact and f is continuous on E.

In other cases the measurability of f and the application of the reduction formula to $|f|$ or to f_+ and f_- (that are nonnegative) suffice to decide on the integrability of f.

a. Normal sets

2.49 Normal sets in $\mathbb{R}^2$. We say that a set $E \subset \mathbb{R}^2$ is *normal with respect to the y axis* if E can be written as

$$E := \{(x, y) \,|\, a < x < b, \alpha(x) < y < \beta(x)\}$$

where $\alpha, \beta :]a, b[\to \mathbb{R}$ are functions with $\alpha(x) < \beta(x)$ $\forall x \in]a, b[$, see Figure 2.10. What makes normal sets useful is the fact that the slice of E over x is a possibly empty interval

$$E_x := \{y \in \mathbb{R} \,|\, \alpha(x) < y < \beta(x)\} = \begin{cases}]\alpha(x), \beta(x)[& \text{if } x \in A, \\ \emptyset & \text{otherwise.} \end{cases}$$

If E is measurable in $\mathbb{R}^2$ and $f : E \to \overline{\mathbb{R}}$ is an integrable function on E, Fubini's theorem yields that $x \to \int_{\alpha(x)}^{\beta(x)} f(x,y)\,dy$ is measurable on $]a,b[$ and

$$\iint_E f(x,y)\,dx\,dy = \int_{\mathbb{R}} \left(\int_{E_x} f(x,y)\,dy \right) dx = \int_A dx \int_{\alpha(x)}^{\beta(x)} f(x,y)\,dy.$$

Notice that one also proves that E is measurable if $\alpha, \beta : A \to \overline{\mathbb{R}}$ are measurable functions, for instance if α and β are continuous, see, e.g., [GM5].

2.50 Normal sets in $\mathbb{R}^n$. Similarly, we say that a set $E \subset \mathbb{R}^n$ is *normal* with respect to a coordinate axis, say x_n, if E can be written as

$$E := \Big\{ x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} \,\Big|\, x' \in A,\ \alpha(x') < x_n < \beta(x') \Big\}.$$

where $A \subset \mathbb{R}^{n-1}$ and $\alpha, \beta : A \to \overline{\mathbb{R}}$ are functions with $\alpha(x) < \beta(x)$ $\forall x \in A$. The slice of E over $x' \in \mathbb{R}^{n-1}$ is a possibly empty interval

$$E_{x'} := \{t \in \mathbb{R} \,|\, \alpha(x') < t < \beta(x')\} = \begin{cases}]\alpha(x'), \beta(x')[& \text{if } x' \in A \\ \emptyset & \text{otherwise.} \end{cases}$$

If E is $\mathcal{L}^n$-measurable and $f : E \to \overline{\mathbb{R}}$ is an integrable function on E, Fubini's theorem then yields that $x' \to \int_{\alpha(x')}^{\beta(x')} f(x', t)\,dt$ is measurable on A and

$$\int_E f(x)\,dx = \int_{\mathbb{R}^{n-1}} dx' \left(\int_{E'_x} f(x', t)\,dt \right) = \int_A dx' \int_{\alpha(x')}^{\beta(x')} f(x', t)\,dt.$$

Notice that one also proves that E is $\mathcal{L}^n$-measurable if A is $\mathcal{L}^{n-1}$-measurable and α and β are measurable functions on A, see, e.g., [GM5]. A typical case would be the one in which A is an open or closed set in $\mathbb{R}^{n-1}$ and α, β are continuous functions on A.

2.51 Example. Compute $\int_T x^2\,dxdy$ where T is the triangle in $\mathbb{R}^2$ of vertices $(0,0)$, $(0,2)$, and $(1,0)$.

The function x^2 is continuous and nonnegative, the domain T is compact, thus x^2 is summable on T so that we can use the reduction formulas. The domain T is normal both with respect to the x-axis and the y-axis, as

$$T = \Big\{0 \le x \le 1, 0 \le y \le -2x + 2\Big\} = \Big\{0 \le y \le 1, 0 \le x \le -y/2 + 1\Big\}.$$

Since the function to be integrated, x^2, depends only on the variable x, it is convenient to leave integration in x as the last, and look at T as a normal domain with respect to the x-axis to obtain

$$\iint_T x^2\,dxdy = \int_0^1 dx \int_0^{-2x+2} x^2\,dy = \int_0^1 x^2(-2x+2)dx = \frac{1}{6}.$$

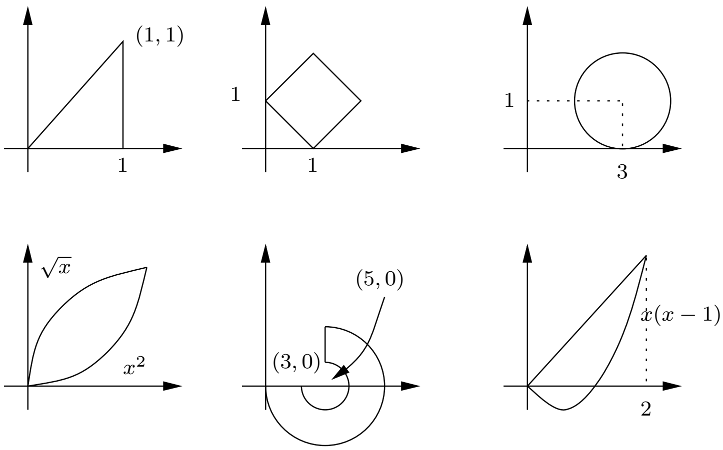


Figure 2.11. Some normal sets or union of normal sets in $\mathbb{R}^2$.

2.52 ¶. Integrate $f(x,y) = x^2$ on each of the domains $E \subset \mathbb{R}^2$ in Figure 2.11.

b. Rotational figures

2.53 Rotational solids. Let $f :]a,b[\subset \mathbb{R} \to \mathbb{R}_+$ be a nonnegative and measurable (for example, continuous) function. By rotating in $\mathbb{R}^3$ the graph of $x = f(z)$ around the z-axis we find the solid

$$E := \Big\{(x,y,z) \,\Big|\, x^2 + y^2 < f^2(z)\Big\}.$$

The slice of E by the plane through $(0,0,z)$ and orthogonal to the z-axis is

$$E_z := \begin{cases} \{(x,y) \in \mathbb{R}^2 \,|\, x^2 + y^2 < f^2(z)\} & \text{if } a < z < b, \\ \emptyset & \text{otherwise,} \end{cases}$$

i.e., E_z is the disk on the plane (x,y) of radius $f(z)$ around the origin if $z \in]a,b[$ and the empty set otherwise. If E is measurable and g is integrable on E, Fubini's theorem yields that $z \to \iint_{E_z} g(x,y,z)\,dz$ is measurable and

$$\int_E g(x,y,z)\,dxdydz = \int_a^b dz \iint_{E_z} g(x,y,z)\,dx\,dy. \qquad (2.19)$$

Notice that, since $x^2 + y^2 - f^2(z)$ is measurable on $\mathbb{R}^3$ if f is measurable on $]a,b[$, E is $\mathcal{L}^3$-measurable if $f :]a,b[\to \overline{\mathbb{R}}_+$ is measurable.

If $g = 1$ we get in particular that $z \to \mathcal{L}^2(E_z)$ is measurable and

$$\mathcal{L}^3(E) = \int_E 1\,dxdydz = \int_{-\infty}^{+\infty} \mathcal{L}^2(E_z)\,dz = \int_a^b \pi f^2(z)\,dz = \pi \int_a^b f^2(z)\,dz.$$

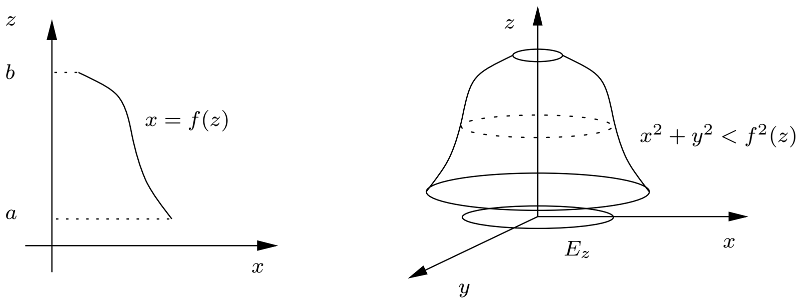


Figure 2.12. A rotational solid and E_z.

Formula (2.19) is particularly convenient when g depends only on z, $g(x,y,z) := g(z)$, as

$$\begin{aligned}\int_E g(z)\,dxdydz &= \int_a^b g(z)\iint_{E_z} 1\,dxdy\\ &= \int_a^b g(z)\mathcal{L}^2(E_z)\,dz = \pi\int_a^b g(z)f^2(z)\,dz.\end{aligned}$$

c. Changes of coordinates

2.54 Polar coordinates in $\mathbb{R}^2$. The map $\varphi(\rho,\theta) := (\rho\cos\theta, \rho\sin\theta)$ is of class $C^1(\mathbb{R}^2)$ with $|\det \mathbf{D}\varphi(\rho,\theta)| = \rho$ and is injective on the set $A :=]0,+\infty[\times]0,2\pi[$. Moreover $\mathcal{L}^2(\partial A) = 0$ and $\mathcal{L}^2(\varphi(\partial A)) = 0$. Therefore, for every measurable $E \subset \overline{A} = [0\times+\infty[\times[0,2\pi]$ and every integrable function f on $\varphi(E)$ we have

$$\int_{\varphi(E)} f(x,y)\,dx\,dy = \int_E f(\rho\cos\theta, \rho\sin\theta)\rho\,d\rho\,d\theta.$$

Since φ is injective on each interval $]0,+\infty[\times]a,a+2\pi[$, $a\in\mathbb{R}$, the same conclusion holds for E measurable, $E \subset]0,\infty[\times[a,a+2\pi[$.

2.55 Polar coordinates in $\mathbb{R}^3$. The map $\phi : \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$\phi(\rho,\theta,\varphi) := \begin{cases} x = \rho\sin\varphi\cos\theta,\\ y = \rho\sin\varphi\sin\theta,\\ z = \rho\cos\varphi\end{cases}$$

is of class $C^1(\mathbb{R}^3)$ with $|\det \mathbf{D}\phi(\rho,\theta,\varphi)| = \rho^2\sin\varphi$ and injective on $A :=]0,+\infty[\times]0,2\pi[\times]0,\pi[$. Moreover $\mathcal{L}^3(\partial A) = 0$ and $\mathcal{L}^3(\varphi(\partial A)) = 0$. Therefore, for every measurable set $E \subset \overline{A} = [0\times+\infty[\times[0,2\pi]\times[0,\pi]$ and every integrable function f on $\phi(E)$ we have

$$\int_{\phi(E)} f(x,y)\,dx\,dy\,dz$$
$$= \int_E f(\rho\sin\theta\cos\varphi, \rho\sin\theta\sin\varphi, \rho\cos\theta)\rho^2\sin\varphi\,d\rho\,d\theta\,d\varphi.$$

2.56 Cylindrical coordinates in $\mathbb{R}^3$. The map $\phi : \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$\begin{cases} x = \rho\cos\theta, \\ y = \rho\sin\theta, \\ z = z \end{cases}$$

is of class $C^1(\mathbb{R}^3)$ with $|\det \mathbf{D}\phi(\rho,\theta,z)| = \rho$ and injective on $A :=]0,+\infty[\times]a,a+2\pi[\times\mathbb{R}$. Moreover, $\mathcal{L}^3(\partial A) = 0$ and $\mathcal{L}^3(\varphi(\partial A)) = 0$. Therefore, for every measurable set $E \subset \overline{A} = [0\times+\infty[\times[a,a+2\pi]\times\mathbb{R}$ and every integrable function f on $\phi(E)$ we have

$$\int_{\phi(E)} f(x,y,z)\,dx\,dy\,dz = \int_E f(\rho\cos\theta,\rho\sin\theta,z)\rho\,d\rho\,d\theta\,dz.$$

2.57 Example. Compute $\iint_E \sqrt{x^2+y^2}\,dx\,dy$ where $E \subset \mathbb{R}^2$ is the disk of radius 1 around $(1,0)$.

We notice that E is compact and $\sqrt{x^2+y^2}$ is summable on E, therefore we can use both Fubini's and the change of variables theorems. The disk E has equation $(x-1)^2+y^2 \le 1$ that is, $x^2+y^2-2x \le 0$, and in polar coordinates, we get

$$\rho \ge 0, \quad \theta \in [-\pi,\pi], \quad \rho^2 - 2\rho\cos\theta \le 0,$$

i.e.,

$$\rho \ge 0, \quad \theta \in [-\pi/2,\pi/2], \quad \rho \le 2\cos\theta.$$

If φ denotes the polar coordinates map and

$$F := \Big\{(\rho,\theta)\,\Big|\, -\pi/2 \le \theta \le \pi/2, 0 \le \rho \le 2\cos\theta\Big\},$$

then the set F is contained in a strip of periodicity of φ and $E = \varphi(F)$. Therefore by a change of variables and taking into account that F is normal with respect to ρ, we find

$$\iint_E \sqrt{x^2+y^2}\,dx\,dy = \iint_F \rho^2\,d\rho\,d\theta = \int_{-\pi/2}^{\pi/2} d\theta \int_0^{2\cos\theta} \rho^2\,d\rho$$
$$= \frac{8}{3}\int_{-\pi/2}^{\pi/2} \cos^3\theta\,d\theta = \frac{32}{9}.$$

2.58 Example (Rotational solids). Let $f : [a,b] \to \mathbb{R}_+$ be a measurable function and let E be the set obtained by rotating the subgraph of f in the plane (y,z) around the z-axis,

$$E := \Big\{(x,y,z)\,\Big|\, a \le z \le b,\ x^2+y^2 \le f^2(z)\Big\}.$$

By parameterizing E with the cylindrical coordinates,

$$\phi(r,\theta,z) := \begin{cases} x = \rho\cos\theta, \\ y = \rho\sin\theta, \\ z = z \end{cases}$$

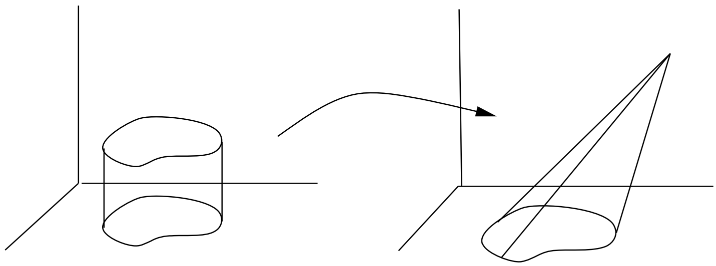


Figure 2.13. Conical coordinates.

so that $E \setminus \{(0,0,z)\}$ is the one-to-one image of the set

$$F := \Big\{(\rho,\theta,z)\,\Big|\,0 \le \theta < 2\pi,\ a \le z \le b,\ 0 < \rho \le f(z)\Big\}$$

and changing variables, we find that E is measurable and

$$\mathcal{L}^3(E) = \mathcal{L}^3(E \setminus \{(0,0,z)\}) = \int_F |\det \mathbf{D}\phi|\, d\rho d\theta dz$$
$$= \int_a^b dz \int_0^{2\pi} d\theta \int_0^{f(z)} \rho\, d\rho = \pi \int_a^b f^2(z)\, dz.$$

2.59 Example (Guldin's formula). Let $f, g : [a,b] \to \mathbb{R}_+$ be measurable functions with $g \le f$. The set

$$E := \Big\{(x,y,z)\,\Big|\,z \in [a,b],\ g(z) \le \sqrt{x^2+y^2} \le f(z)\Big\},$$

obtained by rotating the set

$$A := \Big\{(x,y,z)\,\Big|\,x = 0,\ z \in [a,b],\ f(z) \le y \le g(z)\Big\}$$

around the z-axis, has as volume

$$\mathcal{L}^3(E) = \pi \int_a^b (g^2(z) - f^2(z))\, dz.$$

The *center of mass*, or *barycenter*, of A (the density is assumed to be one) is the point $(\overline{x}, \overline{y}) \in \mathbb{R}^2$ given by

$$\overline{y} := \frac{1}{\mathcal{L}^2(A)} \int_A y\, dydz, \qquad \overline{z} := \frac{1}{\mathcal{L}^2(A)} \int_A z\, dydz.$$

Guldin's formula writes as: *The volume of the rotational solid E is the product of section A times the length of the circle of revolution of the barycenter of section A*, i.e.,

$$\mathcal{L}^3(E) = \mathcal{L}^2(A)\, 2\pi\, \overline{y}.$$

In fact, we have

$$2\pi\overline{y}\mathcal{L}^2(A) = 2\pi \int_A y\, dydz = \int_a^b dz \int_{f(z)}^{g(z)} ydy = \pi \int_a^b (g^2(z) - f^2(z))\, dz = \mathcal{L}^3(E).$$

2.60 Conical coordinates in $\mathbb{R}^3$. Consider $\mathbb{R}^2$ as the coordinate plane $z = 0$ of $\mathbb{R}^3$, $\mathbb{R}^2 = \{(x, y, z) \,|\, z = 0\}$, let A be an open set in $\mathbb{R}^2$ and let $P_0 := (x_0, y_0, z_0)$. By definition, a point P is in the cone $C(P_0, A)$ of vertex P_0 and basis A if there are $(\alpha, \beta, 0) \in A$ and $t \in [0, 1]$ such that

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = (1-t) \begin{pmatrix} \alpha \\ \beta \\ 0 \end{pmatrix} + t \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}.$$

The function $\phi(\alpha, \beta, t) : \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$\begin{cases} x = (1-t)\alpha + t\,x_0, \\ y = (1-t)\beta + t\,y_0, \\ z = t\,z_0 \end{cases}$$

is a map of class $C^1(\mathbb{R}^3)$ with $\det \mathbf{D}\varphi(\alpha, \beta, t)| = (1-t)^2$ and one-to-one from $A \times [0, 1[$ onto $C(P_0, A) \setminus \{P_0\}$. Consequently,

$$\begin{aligned} \int_{C(P_0,A)} f(x, y, z)\, dx\, dy\, dz &= \int_{A\times[0,1]} f(\phi(\alpha, \beta, t))(1-t)^2\, d\alpha\, d\beta\, dt \\ &= \int_0^1 (1-t^2)\Big(\int_A f(\phi(\alpha, \beta, t))\, d\alpha\, d\beta \Big)\, dt. \end{aligned}$$

In particular, if $f = 1$, we get the 3-dimensional measure of the cone $C(P_0, A)$,

$$\mathcal{L}^3(C(P_0, A)) = \frac{z_0}{3}\mathcal{L}^2(A).$$

2.61 Example. Suppose we want to compute the measure of

$$E := \Big\{ (x, y) \in \mathbb{R}^2 \,\Big|\, 0 < x < y < 2x,\ 1 < xy < 2 \Big\}.$$

We set $u = xy$ and $v = y/x$; then we have

$$E = \varphi(F)$$

where $\varphi(u, v) := (\sqrt{u/v}, \sqrt{uv})$ and

$$F := \Big\{ (u, v) \in \mathbb{R}^2 \,\Big|\, 1 < u < 2,\ 1 < v < 2 \Big\}.$$

Since $\det \mathbf{D}\varphi = \frac{1}{2v} > 0$ on F, we find

$$\mathcal{L}^2(E) = \mathcal{L}^2(\varphi(F)) = \int_F \frac{1}{2v}\, du\, dv = \frac{1}{2} \int_1^2 du \int_1^2 \frac{dv}{v} = \frac{1}{2} \log 2.$$

d. Measure of the n-dimensional ball

Let ω_n be the n-dimensional measure of the n-dimensional ball

$$\omega_n := \mathcal{L}^n(B^n(0,1)), \qquad B^n(0,1) := \{x \in \mathbb{R}^n \,\big|\, |x| \le 1\}.$$

2.62 Proposition. *We have*

$$\omega_{2k} = \frac{\pi^k}{k!}, \qquad \omega_{2k+1} = \frac{2^{k+1}\pi^k}{(2k+1)!!}. \tag{2.20}$$

Proof. We split the coordinates $x = (x_1, x_2, \dots, x_n)$ of $\mathbb{R}^n$ as $x = (y,t)$ where $y := (x_1, x_2, \dots, x_{n-1}) \in \mathbb{R}^{n-1}$ and $t = x_n \in \mathbb{R}$. The unit ball is then described as

$$B^n(0,1) := \{(y,t) \in \mathbb{R}^{n-1} \times \mathbb{R} \,\big|\, |y|^2 + t^2 < 1\}.$$

Now we slice $\mathbb{R}^{n-1} \times \mathbb{R}$ with $(n-1)$-planes perpendicular to the t-axis. The slice of $B^n(0,1)$ at the level t is then

$$E_t := \Big\{y \in \mathbb{R}^{n-1} \,\Big|\, |y|^2 < 1 - t^2\Big\} = \begin{cases} B^{n-1}(0, \sqrt{1-t^2})) & \text{if } t \in [-1,1], \\ \emptyset & \textit{if } |t| > 1. \end{cases}$$

By homogeneity $B^{n-1}(0, \sqrt{1-t^2}) = \omega_{n-1}(1-t^2)^{(n-1)/2}$, and, since $B^n(0,1)$ is open, Fubini's theorem yields

$$\omega_n = \int_{-\infty}^{+\infty} \mathcal{L}^{n-1}(E_t)\,dt = \omega_{n-1} \int_{-1}^{1} (1-t^2)^{\frac{n-1}{2}}\,dt = 2\omega_{n-1} \int_0^1 (1-t^2)^{\frac{n-1}{2}}\,dt.$$

Since $\int_0^1 (1-t^2)^{\frac{n-1}{2}}\,dt = \int_0^{\pi/2} \cos^n(t)\,dt$, and

$$\int_0^{\pi/2} \cos^n(t)\,dt = \frac{n-1}{n} \int_0^{\pi/2} \cos^{n-2}(t)\,dt,$$

we find, see [GM2],

$$\int_0^{\pi/2} \cos^{2k}(t)\,dt = \frac{(2k-1)!!}{(2k)!!}\,\frac{\pi}{2}, \qquad \int_0^{\pi/2} \cos^{2k+1}(t)\,dt = \frac{(2k)!!}{(2k+1)!!}.$$

As $\omega_1 = 2$ and $\omega_2 = \pi$, we get the result. □

As a curiosity, notice that $\omega_n \to 0$ as $n \to \infty$. On the other hand the measure of the n-dimensional cube of side 2 that circumscribes the unit ball is 2^n and tends to infinity as $n \to \infty$.

The measure of the n-ball is tied to Euler's Γ function, see Example 2.67.

2.63 ¶. Let $a > 0$. Compute the measure of the n-dimensional set

$$E := \Big\{x = (x_1, x_2, \dots, x_n) \,\Big|\, \sum_{i=1}^n x_i \le a,\ x_i \ge 0\ \forall i\Big\}.$$

e. Isodiametric inequality

2.64 Proposition (Isodiametric inequality). *Let E be a measurable bounded set in $\mathbb{R}^n$. Then*

$$\mathcal{L}^n(E) \le \omega_n\Big(\frac{\operatorname{diam} E}{2}\Big)^n.$$

Notice that, whereas in $\mathbb{R}$ every set E is contained in an interval of radius half the diameter of E, this is not true anymore if $n \ge 2$: think of the equilateral triangle in $\mathbb{R}^2$. Of course every set E is contained in a ball of radius the diameter of E so that

$$\mathcal{L}^n(E) \le \omega_n(\operatorname{diam} E)^n. \tag{2.21}$$

But proving the isodiametric inequality requires some effort. However, it is trivial for special sets. For instance, if E is symmetric with respect to the origin, that is $x \in E$ iff $-x \in E$, then we have $2|x| = |x-(-x)| \le \operatorname{diam} E$, hence $E \subset B(0, \operatorname{diam} E/2)$ which yields the isodiametric inequality.

For generic sets, we shall use *Steiner's symmetrization method.* Given a direction $a \in S^{n-1}$, we denote by $P(a)$ the $(n-1)$-dimensional subspace of $\mathbb{R}^n$ orthogonal to a so that every $x \in \mathbb{R}^n$ writes uniquely as $x = y + ta$ with $y \in P(a)$ and $t \in \mathbb{R}$. For every $y \in P(a)$ we then set

$$E_{a,y} = \Big\{t \in \mathbb{R} \,\Big|\, ta + y \in E\Big\} \qquad \text{e} \qquad \ell_a(y) := \mathcal{L}^1(E_{a,y})$$

and define the *Steiner symmetrization* of E in the direction a by

$$S_a(E) := \Big\{(y,t) \in \mathbb{R}^{n-1} \times \mathbb{R} \,\Big|\, |t| \le \frac{\ell_a(y)}{2}\Big\}.$$

We have

2.65 Lemma. *If E is bounded and measurable, then*

(i) *$S_a(E)$ is measurable,*
(ii) *if E is symmetric with respect to a k-plane orthogonal to a, $1 \le k \le n-1$, then $S_a(E)$ has the same symmetry,*
(iii) *$|S_a(E)| = |E|$,*
(iv) *$\operatorname{diam}(S_a(E)) \le \operatorname{diam}(E)$.*

Proof. After a rotation that does not change the measurability, the measure, and the diameter of E, see (2.8), we can assume $a = (0, 0, \dots, 1)$. Consequently $P(a) = \{x = (y, 0),\ y \in \mathbb{R}^{n-1}\}$, every point $x \in \mathbb{R}^n$ writes as $x = (y, t)$, and $E_{a,y}$ is the slice of E over y. Fubini's theorem then yields that $E_{a,y}$ is measurable for a.e. $y \in \mathbb{R}^{n-1}$ and $y \to \ell_a(y) := \mathcal{L}^1(E_{a,y})$ is a measurable function, hence $S_a(E)$ is measurable, see Theorem 2.8, and

$$|E| = \int_{\mathbb{R}^{n-1}} \mathcal{L}^1(E_{a,y})\, dy = \int_{\mathbb{R}^{n-1}} \Big(\int_{-\ell_a(y)/2)}^{\ell_a(y)/2} 1\, dt\Big) dy = |S_a(E)|.$$

A symmetry of E with respect to a k-plane orthogonal to $(0, 0, \dots, 1)$ yields a similar symmetry for the function $\ell_a(y)$ hence of $S_a(E)$. Finally, from the elementary inequality

$$\mathcal{L}^1(I_1) + \mathcal{L}^1(I_2) \le \operatorname{diam}(I_1 \cup I_2)$$

for subsets of $\mathbb{R}$, we readily infer that $\operatorname{diam}(S_a(E)) \le \operatorname{diam}(E)$. $\square$

Proof of Proposition 2.64. Let $(e_1, e_2, \dots, e_n)$ be the standard basis of $\mathbb{R}^n$ and let $E_1 := S_{e_1}(E)$, $E_2 := S_{e_2}(E_1), \dots, E_n := S_{e_n}(E_{n-1})$. Applying iteratively Lemma 2.65, we deduce that

$$|E| = |E_1| = \dots |E_n|, \qquad \operatorname{diam}(E_n) \le \operatorname{diam}(E_{n-1}) \le \dots \le \operatorname{diam} E,$$

and E_1 is symmetric with respect to the plane perpendicular to e_1, E_2 is symmetric with respect to the plane perpendicular to e_1 and $e_2, \dots$, E_n is symmetric with respect to the coordinate axes, hence with respect to the origin. Therefore, E_n is contained in a ball of radius $\operatorname{diam} E_n/2$, thus concluding

$$|E| = |E_n| \le \omega_n \Big(\frac{\operatorname{diam} E_n}{2}\Big)^n \le \omega_n \Big(\frac{\operatorname{diam} E}{2}\Big)^n.$$

□

f. Euler's Γ function

2.66 Example. We have

$$\int_{-\infty}^{+\infty} e^{-x^2}\, dx = \sqrt{\pi}. \tag{2.22}$$

In fact, since $e^{-x^2-y^2}$ is integrable on $\mathbb{R}^2$, using Fubini's theorem and passing to polar coordinates, we find

$$\begin{aligned}\Big(\int_{-\infty}^{+\infty} e^{-x^2}\, dx\Big)^2 &= \int_{-\infty}^{\infty} e^{-x^2}\, dx \int_{-\infty}^{\infty} e^{-y^2}\, dy = \iint_{\mathbb{R}^2} e^{-x^2-y^2}\, dx\, dy \\ &= \int_0^{2\pi} d\theta \int_0^{\infty} e^{-\rho^2} \rho\, d\rho = 2\pi \frac{1}{2} \int_0^{\infty} e^{-\sigma}\, d\sigma = \pi.\end{aligned}$$

If we change variable in (2.22), we also get

$$\int_{-\infty}^{+\infty} e^{-\lambda x^2}\, dx = \sqrt{\frac{\pi}{\lambda}}, \qquad \lambda > 0. \tag{2.23}$$

2.67 Example (Euler's Γ function and the measure of $B^n(0,1)$). The function Γ was defined by Euler in 1729,

$$\Gamma(\alpha) := \int_0^{\infty} t^{\alpha-1} e^{-t}\, dt, \qquad \alpha > 0, \tag{2.24}$$

It is an important *special function* that surprisingly appears in many contexts.

Trivially $\Gamma(1) = 1$ and, on account of Example 2.66,

$$\Gamma(1/2) = 2\int_0^{\infty} e^{-s^2}\, ds = \sqrt{\pi}.$$

Integrating by parts we see that

$$\Gamma(\alpha + 1) = \alpha\, \Gamma(\alpha) \qquad \forall \alpha > 0.$$

It follows by induction

$$\Gamma(n+1) = n!, \qquad \Gamma(n+1/2) = \frac{(2n-1)!!}{2^n}\Gamma(1/2) = \sqrt{\pi}\frac{(2n-1)!!}{2^n} = \sqrt{\pi}\,\frac{(2n)!}{4^n n!},$$

that, by comparison with (2.20) yields

$$\omega_n = \mathcal{L}^n(B^n(0,1)) = \frac{\pi^{n/2}}{\Gamma\big(\frac{n}{2}+1\big)} \qquad \forall n \ge 1. \tag{2.25}$$

We presented some of the properties of the Γ-function in [GM2]. Further properties of the Γ-function will be discussed in the following Example 2.68 and Section 2.4.3.

2.68 Example (Euler's Beta function). *Euler's Beta* function is defined by

$$B(p,q) := \int_0^1 x^{p-1}(1-x)^{q-1}\,dx, \qquad p,q>0.$$

Changing variables $y = 1-x$, we see that

$$B(p,q) = B(q,p) \qquad \forall p,q>0, \tag{2.26}$$

while, writing

$$x^{p-1} = \frac{x^{p+q-2}}{x^{q-1}}, \qquad x^{p+q-2} = D\frac{x^{p+q-1}}{p+q-1}$$

and integrating by parts, we find

$$B(p,q) = \frac{q-1}{p+q-1}B(p,q-1) \qquad \forall p>0,\ q>1, \tag{2.27}$$

and, because of the symmetry,

$$B(p,q) = \frac{p-1}{p+q-1}B(p-1,q) \qquad \forall p>1,\ q>0. \tag{2.28}$$

Changing variables, $x = z/(1+z)$, we also find

$$B(p,q) = \int_0^\infty \frac{z^{p-1}}{(1+z)^{p+q}}\,dz. \tag{2.29}$$

We can compute the B-function in terms of the Γ-function as

$$B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}, \qquad p,q>0. \tag{2.30}$$

To prove this, we begin by noticing that if we change variables $x = \lambda z$, $\lambda > 0$, then

$$\frac{\Gamma(\alpha)}{\lambda^\alpha} = \lambda^{-\alpha}\int_0^\infty x^{\alpha-1}e^{-x}\,dx = \int_0^\infty z^{\alpha-1}e^{-\lambda z}\,dz, \qquad \alpha>0. \tag{2.31}$$

Now, applying Fubini's theorem, changing variables $(\lambda, y) \to x = \lambda y, y = y$, and taking into account (2.31) and (2.29), we find

$$\begin{aligned}
\Gamma(p)\Gamma(q) &= \int_0^\infty\int_0^\infty x^{p-1}y^{q-1}e^{-(x+y)}\,dx\,dy \\
&= \int_0^\infty \lambda^{p-1}\left(\int_0^\infty y^{p+q-1}e^{-(1+\lambda)y}\,dy\right)d\lambda \\
&= \int_0^\infty \lambda^{p-1}\frac{\Gamma(p+q)}{(1+\lambda)^{p+q}}\,d\lambda = \Gamma(p+q)B(p,q).
\end{aligned}$$

The beta function is useful when computing several interesting integrals. For instance, if

$$I_\alpha := \int_{-1}^1 (1-x^2)^\alpha\,dx, \qquad \alpha > -1,$$

and we change variables, we find

$$I_\alpha = \int_0^1 (1-t)^\alpha t^{-1/2}\,dt = B(1/2,\alpha+1) = \sqrt{\pi}\frac{\Gamma(\alpha+1)}{\Gamma(\alpha+3/2)}. \tag{2.32}$$

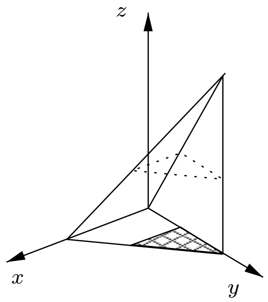


Figure 2.14. A tetrahedron.

2.69 Example. Let $p \geq 1$ and $||x||_p := \left(\sum_{i=1}^n |x_i|^p\right)^{1/p}$, $x \in \mathbb{R}^n$. We want to compute $\gamma_{n,p} := \mathcal{L}^n(\{x \,|\, ||x||_p \leq 1\}$.

By slicing with planes orthogonal to a chosen coordinate axis, we find the following recursive relation for $\gamma_{n,p}$,

$$\gamma_{n,p} = \gamma_{n-1,p} \cdot 2\int_0^1 (1-t^p)^{(n-1)/p}\,dt.$$

By (2.32) we get

$$\frac{\gamma_{n,p}}{\gamma_{n-1,p}} = \frac{2}{p}B\Big(\frac{n+p-1}{p}, \frac{1}{p}\Big) = \frac{2}{p}\,\Gamma\Big(\frac{1}{p}\Big)\,\frac{\Gamma\Big(\frac{n-1+p}{p}\Big)}{\Gamma\Big(\frac{n+p}{p}\Big)},$$

hence, since $\gamma_1 = 2$,

$$\begin{aligned}
\gamma_{n,p} &= \gamma_{1,p}\,\frac{\gamma_{2,p}}{\gamma_{1,p}} \cdots \frac{\gamma_{n-1,p}}{\gamma_{n-2,p}}\,\frac{\gamma_{n,p}}{\gamma_{n-1,p}} = \gamma_{1,p}\prod_{i=1}^n \frac{\gamma_{i,p}}{\gamma_{i-1,p}} \\
&= 2\Big(\frac{2}{p}\Big)^{n-1}\Gamma\Big(\frac{1}{p}\Big)^{n-1}\prod_{i=2}^n \frac{\Gamma\Big(\frac{i-1+p}{p}\Big)}{\Gamma\Big(\frac{i+p}{p}\Big)} = 2\Big(\frac{2}{p}\Big)^{n-1}\Gamma\Big(\frac{1}{p}\Big)^{n-1}\frac{\Gamma\Big(\frac{p+1}{p}\Big)}{\Gamma\Big(\frac{n+p}{p}\Big)} \\
&= 2\Big(\frac{2}{p}\Big)^{n-1}\Gamma\Big(\frac{1}{p}\Big)^{n-1}\frac{\frac{1}{p}\Gamma\Big(\frac{1}{p}\Big)}{\frac{n}{p}\Gamma\Big(\frac{n}{p}\Big)} = \frac{p}{n}\,\Big(\frac{2}{p}\Big)^n\,\frac{\Gamma(1/p)^n}{\Gamma(n/p)}.
\end{aligned}$$

g. Tetrahedrons

2.70 Example (Tetrahedrons, I). Consider the tetrahedron $T \subset \mathbb{R}^3$ of vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, and $(0,2,2)$, see Figure 2.14. Let us compute

$$\int_T \frac{z}{1+z}\,dz.$$

A face of the tetrahedron is on the plane $z = 0$ and, if we slice the tetrahedron with planes parallel to the basis, we get slices that are congruent to the basis; moreover, the function to be integrated depends only on the variable z. Therefore, we decide to slice with planes orthogonal to the z-axis. If T_z is the slice of T at the level z, we see that $T_z \neq \emptyset$ if and only if $0 \leq z \leq 2$. Since T is measurable and $z/(1+z)$ is continuous on $\overline{T}$, Fubini's theorem yields

$$\int_T \frac{z}{1+z}\,dz = \int_0^2 \frac{z}{1+z}\mathcal{L}^2(T_z)\,dz.$$

Since, by Thales theorem, $\mathcal{L}^2(T_z) = \mathcal{L}^2(T_0)(\frac{2-z}{2})^2$, we conclude

$$\int_T \frac{z}{1+z}\,dz = \frac{1}{8}\int_0^2 \frac{z(2-z)^2}{1+z}\,dz = \dots$$

2.71 Example (Tetrahedrons, II). Consider the tethrahedron $T \subset \mathbb{R}^3$ of Example 2.70 and let us compute

$$\int_T \frac{xz}{1+z}\,dz$$

that is well defined since T is compact and the integrand is continuous on $\overline{T}$. We slice as in Example 2.70 and, with the same notation, we find

$$\int_T \frac{xz}{1+z}\,dz = \int_0^2 \frac{z}{1+z}\iint_{T_z} x\,dxdy.$$

Now we compute $\iint_{T_z} x\,dxdy$. The domain T_z is a triangle in $\mathbb{R}^2$ congruent to the basis of T. Its vertices $P(z)$, $Q(z)$, $R(z)$ are the projections on the (x,y)-plane of the intersections of the plane perpendicular to the z-axis through $(0,0,z)$ and the straight line respectively through $(0,2,2)$ and $(0,0,0)$, $(0,2,2)$ and $(1,0,0)$, and $(0,2,2)$ and $(0,1,0)$. Again by Thales theorem, the coordinates $x(z)$ and $y(z)$ of $P(z)$ depend linearly on z, i.e.,

$$\begin{cases} x(z) = mz+q, \\ x(0)=0, x(2)=0, \end{cases} \qquad \begin{cases} y(z) = mz+q, \\ y(0)=0, y(2)=2, \end{cases}$$

hence $P(z) = (0,z)$. Similarly, one computes $Q(z) = (1-z/2, z)$ and $R(z) = (0, 1+z/2)$. Points $P(z)$ and $R(z)$ have the same abscissa, hence the triangle T_z is normal with respect to the x-axis. Writing the equation for the straight line through $P(z)$ and $Q(z)$, and $R(z)$ and $Q(z)$ respectively,

$$\alpha_z(x) = z, \qquad \beta_z(x) = -(x + z/2 - 1) + z = 1 + z/2 - x,$$

we find

$$T_z := \Big\{(x,y)\in\mathbb{R}^2 \,\Big|\, 0 \le x \le 1 - z/2,\ \alpha_z(x) \le y \le \beta_z(x)\Big\}$$

$$\int_{T_z} xdxdy = \int_o^{1-z/2} x\,dx \int_{\alpha_z(x)}^{\beta_z(x)} dy = \int_0^{1-z/2} x(1-z/2-x)\,dx = \frac{1}{6}\Big(1-\frac{z}{2}\Big)^3.$$

In conclusion

$$\int_T \frac{z}{1+z}\,dx\,dy\,dz = \frac{1}{6}\int_0^2 \frac{z(1-z/2)^3}{1+z}\,dz = \dots$$

We may proceed differently. We regard the tetrahedron as a cone over a face and let the formula of change of variables operate the details. The map $\varphi(t,a,b) : \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$\begin{cases} x = ta + (1-t)\,0, \\ y = tb + (1-t)\,2, \\ z = t\cdot 0 + (1-t)\,2 \end{cases}$$

maps the prism $T_0 \times [0,1]$ onto the cone-tetrahedron T with basis T_0 defined by the vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$ and vertex $(0,2,2)$. It is easily seen that φ is one-to-one from $T_0 \times]0,1]$ onto $T \setminus \{(0,2,2)\}$ and that $\det \mathbf{D}\varphi(t,a,b) = -2t^2$. Thus,

$$\begin{aligned}\int_T \frac{xz}{1+z}\,dx\,dy\,dz &= \int_{T_0\times[0,1]} \frac{2ta(1-t)}{1+2(1-t)}2t^2\,da\,db\,dt = \int_0^1 \frac{4t^3(1-t)}{3-2t}\iint_{T_0} a\,da\,db \\ &= \int_0^1 \frac{4t^3(1-t)}{3-2t}\int_0^1 a\Big(\int_0^{1-a} db\Big)\,da = \frac{2}{3}\int_0^1 \frac{t^3(1-t)}{3-2t}\,dt = \dots\end{aligned}$$


Figure 2.15. Pierre Fatou (1878–1929) and Felix Hausdorff (1869–1942).

2.4.2 Monte Carlo method

Suppose we want to evaluate

$$f_Q := \int_Q f(x)\,dx, \qquad Q = [0,1]^n$$

for a function $f \in C^0(Q)$. We may use the analog of the one-dimensional *Simpson's rule*, see [GM2]. We subdivide the cube Q into k^n subcubes of side $1/k$, on each of those cubes we choose a point x_i and then compute $\frac{1}{k^n}\sum_{i=1}^{k^n} f(x_i)$; in particular, we need to compute f in k^n points: an enormous value already if $k = 100$ and $n = 4$.

During the Second World War, Enrico Fermi (1901–1954), John von Neumann (1903–1957), and Stanislaw Ulam (1909–1984) invented a probabilistic method, nowadays known as the *Monte Carlo method.* This method with probability close to 1 allows us to compute the value of the integral except for a small error by means of relatively few cubes.

Notice that Lebesgue's measure $\mathcal{L}^n$ on Q is a probability measure on Q and actually the equidistributed probability measure. Let $\{X_k\}$ be a sequence of points that are equidistributed and independently chosen on Q, i.e., a sequence on independent random variables on Q. If $f : Q \to \mathbb{R}$, then the *expectation* and the *variance* are defined respectively by

$$\mathbf{E}\,(f(X_j)) = \int_Q f(x)\,d\mathcal{L}^n(x) = f_Q, \qquad \mathrm{Var}\,(f(X_j)) = \int_Q |f(x) - f_Q|^2\,dx$$

for all integers j. Since the variables $\{X_j\}$ are independent

$$\mathrm{Var}\Big(\sum_{j=1}^k f(X_k)\Big) = \sum_{j=1}^k \mathrm{Var}\,(f(X_j)) = k\int_Q |f(x) - f_Q|^2\,dx \le 4kM^2$$

where $M := ||f||_\infty$. Hence

$$\int_{Q^k}\Big(\frac{1}{k}\sum_{j=1}^k f(x_j) - f_Q\Big)^2 dx_1\ldots dx_k = \mathrm{Var}\Big(\frac{1}{k}\sum_{j=1}^k f(X_j)\Big) \le \frac{4}{k}||f||_\infty^2.$$

If $A \subset Q \times \cdots \times Q = Q^k$ is the event

$$A := \Big\{ (X_1, \dots, X_k) \,\Big|\, \Big| \frac{1}{k} \sum_{i=1}^{k} f(X_i) - f_Q \Big| > \epsilon \Big\}$$

then Chebyshev's inequality yields

$$P(A_k) := \mathcal{L}^{nk}(A_k) \le \frac{1}{\epsilon^2} \int_{Q^k} \Big(\frac{1}{k} \sum_{j=1}^{k} f(x_j) - f_Q \Big)^2 dx_1 \dots dx_k \le \frac{4M^2}{\epsilon^2\, k},$$

i.e., the probability that, choosing randomly k equidistributed points $\{X_i\}$, the event that $\frac{1}{k}\sum_{i=1}^k f(X_i)$ has distance from f_Q more than ϵ has a probability to happen less than $4M^2/k\epsilon^2$. For instance, if $M \le 1$ and we choose $k = 10^6$, in 99% of the cases we find an error less than 2%.

2.4.3 Differentiation under the integral sign

2.72 Example. Let us compute

$$F(t) := \int_0^{+\infty} \exp\left(-x^2 - t^2/x^2\right) dx, \qquad t \in \mathbb{R}.$$

It is easily seen that F is even, $F(0) = \int_0^\infty e^{-x^2/2}\, dx = \sqrt{\pi}/2$, see Example 2.66, and we have

$$|f(t,x)| \le e^{-x^2} \qquad \forall t \in \mathbb{R},\ \forall x \ge 0.$$

Therefore $F(t)$ is continuous in $\mathbb{R}$, see Proposition 2.37. Moreover, for $t > 0$ we have

$$\Big|\frac{\partial f}{\partial t}(t,x)\Big| = \frac{2e^{-x^2}}{t}\, \frac{t^2}{x^2}\, e^{-t^2/x^2} \le \frac{2}{t}\, e^{-x^2} \sup_{\mathbb{R}_+}(se^{-s}) = \frac{2}{e\,t}\, e^{-x^2},$$

thus

$$\Big|\frac{\partial f}{\partial t}(t,x)\Big| \le \frac{2}{e\,\epsilon}\, e^{-x^2} \in \mathcal{L}^1(\mathbb{R}_+)$$

for all $t > \epsilon > 0$ and $x \ge 0$. Theorem 2.40 then yields that $F(t)$ is differentiable for all $t > \epsilon$, and therefore for all $t > 0$, since ϵ is arbitrary, and

$$F'(t) = \int_0^\infty \frac{\partial f}{\partial t}(t,x)\, dx = -2t \int_0^\infty \frac{1}{x^2} \exp\left(-x^2 - t^2/x^2\right) dx = -2\,t\,F(t) \qquad \forall t > 0,$$

where the last equality follows by changing variables $y = t/x$. It follows

$$F(t) = F(0)e^{-2t}, \qquad t > 0,$$

i.e.,

$$F(t) = \frac{\sqrt{\pi}}{2}\, e^{-2\,|t|}.$$

2.73 Example. We shall write in terms of elementary functions the following oscillatory integral

$$g(\omega) = \int_{-\infty}^{+\infty} e^{-x^2/2}\cos(\omega x)\,dx, \qquad \omega \in \mathbb{R}.$$

As usual, it is convenient to use the complex notation. Since

$$\int_{-\infty}^{+\infty} e^{-x^2/2}\sin(\omega x)\,dx = 0,$$

we have

$$g(\omega) = \int_{-\infty}^{+\infty} e^{-x^2/2}e^{-i\omega x}\,dx.$$

Since

$$\Big|\frac{\partial}{\partial\omega}(e^{-x^2/2}e^{-i\omega x})\Big| = \Big| -ixe^{-x^2/2}e^{-i\omega x}\Big| \le |x|\,e^{-x^2/2} \in \mathcal{L}^1(\mathbb{R}),$$

the function $g(\omega)$ is differentiable and

$$g'(\omega) = -i\int_{-\infty}^{+\infty} xe^{-x^2/2}e^{-i\omega x}\,dx.$$

Writing $-xe^{-x^2/2} = D(e^{-x^2/2})$ and integrating by parts, we find

$$g'(\omega) = -\omega\, g(\omega),$$

hence, by integration,

$$g(\omega) = g(0)\,e^{-\omega^2/2} = \sqrt{2\pi}\,e^{-\omega^2/2}.$$

Alternatively, we may also proceed as follows. Since

$$\cos(\omega x) = \sum_{n=0}^{\infty}(-1)^n\frac{(\omega x)^{2n}}{(2n)!},\ \ x \in \mathbb{R},$$

we consider the functions

$$f_n(x) := (-1)^n\frac{\omega^{2n}x^{2n}}{(2n)!}e^{-x^2/2}$$

and compute

$$\begin{aligned}(-1)^n\int_{-\infty}^{+\infty} f_n(x)\,dx &= \frac{\omega^{2n}}{(2n)!}\,2\int_0^{\infty} e^{-x^2}x^{2n}\,dx = (\text{by changing variables } y = x^2)\\ &= \sqrt{2}\,\omega^{2n}\,\frac{\Gamma(n+1/2)}{(2n)!} = \omega^{2n}\,\sqrt{2\pi}\,\frac{(2n-1)!!}{(2n)!}\\ &= \sqrt{2\pi}\,\frac{(\omega^2/2)^n}{n!}.\end{aligned}$$

We infer

$$\sum_{n=0}^{\infty}\int_{-\infty}^{+\infty}|f_n(x)|\,dx < +\infty$$

and, on account of Lebesgue's theorem,

$$\begin{aligned}\int_{-\infty}^{+\infty} e^{-x^2/2}\cos(\omega x)\,dx &= \int_{-\infty}^{+\infty}\sum_{n=0}^{\infty} f_n(x)\,dx = \sum_{n=0}^{\infty}\int_{-\infty}^{+\infty} f_n(x)\,dx\\ &= \sum_{n=0}^{\infty}(-1)^n\sqrt{2\pi}\,\frac{(\omega^2/2)^n}{n!} = \sqrt{2\pi}\sum_{n=0}^{\infty}\frac{(-\omega^2/2)^n}{n!}\\ &= \sqrt{2\pi}\,e^{-\omega^2/2}.\end{aligned}$$

2.74 Example (Derivatives of Γ). We already observed, see Examples 2.66 and 2.67, that for Euler's Γ-function

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1}\, e^{-t}\, dt, \qquad \alpha > 0,$$

we have

$$\begin{cases} \Gamma(\alpha+1) = \alpha\Gamma(\alpha) & \forall \alpha > 0, \\ \Gamma(n+1) = n!, \\ \Gamma(1/2) = \sqrt{\pi}. \end{cases}$$

Moreover, we discussed some characteristic properties of Euler functions in [GM2]. Here we want to compute the derivatives of Γ.

We prove that Γ is of class C^∞ in its domain $E := \{\alpha \,|\, \alpha > 0\}$. Choose $\alpha_0 > 0$ and set $h(t) := \max(1, t^{\alpha_0/2-1}, t^{2\alpha_0-1})$, $t > 0$. For $k = 0, 1, \dots,$ the functions $h(t)|\log t|^k e^{-t}$ are summable on E and, for all $\alpha \in]\alpha_0/2, 2\alpha_0[$, we have

$$t^{\alpha-1} \le h(t) \qquad \forall t > 0, \forall \alpha,\ \alpha_0/2 < \alpha < 2\alpha_0.$$

If follows for $f(\alpha, t) := t^{\alpha-1}e^{-t}$, $t > 0$, that

$$\Big|\frac{\partial f}{\partial \alpha}(\alpha, t)\Big| \le h(t)\,|\log t|\, e^{-t}$$

and by induction

$$\Big|\frac{\partial^k f}{\partial \alpha^k}(\alpha, t)\Big| \le h(t)|\log t|^k\, e^{-t}$$

for all $t > 0$ and for all $\alpha \in]a_0/2, 2\alpha_0[$. Applying the theorem of differentiation under the integral sign, we conclude that Γ has derivatives of any order at α_0, and

$$\Gamma^{(k)}(\alpha_0) = \int_0^\infty t^{\alpha_0-1}(\log t)^k e^{-t}\, dt, \tag{2.33}$$

consequently,

$$\Gamma'(\alpha) = \int_0^\infty t^{\alpha-1} \log t\, e^{-t}\, dt \qquad \forall \alpha > 0.$$

Since $\Gamma'(\alpha) > 0$ for $\alpha \ge 2$, Γ is increasing for $\alpha \ge 2$. Since $\Gamma(n) \to +\infty$ as $n \to \infty$, also $\Gamma(\alpha) \to +\infty$ as $\alpha \to +\infty$. On the other hand, from $\Gamma(\alpha+1) = \alpha\,\Gamma(\alpha)$ we infer that $\Gamma(a) \sim 1/a$ as $\alpha \to 0^+$. Moreover,

$$\Gamma''(\alpha) = \int_0^\infty t^{\alpha-1}(\log t)^2 e^{-t}\, dt > 0 \qquad \forall \alpha > 0,$$

thus Γ is strictly convex on $[0, \infty[$. Since $\Gamma(1) = \Gamma(2) = 1$, we conclude that Γ has a unique minimum point and it is contained in the interval $]1, 2[$. Moreover, as $|\log t| \le 1 + \log^2 t\ \forall t > 0$, we also get $(\Gamma')^2(x) \le \Gamma(x)\Gamma''(x)$, that is, $\log\Gamma(x)$ is convex.

2.75 Example. We have

$$\Gamma(\alpha)\Gamma(1-\alpha) = \frac{\pi}{\sin \pi\alpha}, \qquad 0 < a < 1. \tag{2.34}$$

In fact, in terms of the Beta function, see (2.29)(2.30), for $0 < \alpha < 1$ we have

$$\Gamma(\alpha)\Gamma(1-\alpha) = \Gamma(1)\, B(\alpha, 1-\alpha) = \int_0^\infty \int_0^\infty \frac{t^{\alpha+1}}{1+t}\, dt.$$

On the other hand, if $\alpha := (2m+1)/(2n)$, $n, m \in \mathbb{N}$, we find by changing variables $t = x^{2n}$

$$\int_0^\infty \frac{t^{\frac{2m+1}{2n}-1}}{(1+t)}\, dt = 2n \int_0^\infty \frac{x^{2m}}{1+x^{2n}}\, dx = \frac{\pi}{\sin\left(\frac{2m+1}{2n}\pi\right)},$$

see, e.g., [GM2, 5.36]. This yields (2.34) when $\alpha = (2m+1)/2n$ for some $n, m \in \mathbb{N}$. The claim now follows for all $\alpha \in]0, 1[$ since the numbers of type $(2m+1)/(2n)$ are dense in $[0, 1]$ and both functions on the left and on the right side of (2.34) are continuous.

2.5 Measure and Area

We are interested in computing not only volumes of n-dimensional objects in $\mathbb{R}^n$ but also the "k-dimensional *area*" of "k-dimensional surfaces" in $\mathbb{R}^n$, $k < n$, as for example the two-dimensional area of the graph of a function $f : \mathbb{R}^2 \to \mathbb{R}_+$.

This is a question that can be treated at various levels of difficulty finding formulas that apply to more or less general objects, or using measure theory to define the k-measure of a subset of $\mathbb{R}^n$, $k < n$. In fact, in contrast with the n-dimensional measure that is *essentially* unique (one can show that the $\mathcal{L}^n$ Lebesgue measure is the only measure that is invariant under rotations and translations, is homogeneous of degree n, and for which the unit cube has measure 1), there are several k-measures suited to measure subsets of $\mathbb{R}^n$, $k < n$: they are different on *nonregular* subsets but agree on "regular surfaces". Among these measures, the *Hausdorff k-dimensional measure* appears as the most suited in many contexts.

2.5.1 Hausdorff's measures

It is convenient to define Hausdorff s-dimensional measure $\mathcal{H}^s(E)$ of a set $E \subset \mathbb{R}^n$ also for noninteger $s \geq 0$. For $s \in \mathbb{R}$, $s \geq 0$, we set

$$\omega_s := \frac{\pi^{s/2}}{\Gamma(1 + s/2)},$$

recalling that, if s is an integer, then ω_s is the $\mathcal{L}^s$ measure of the s-dimensional ball $B(0,1) \subset \mathbb{R}^s$, $\omega_s = \mathcal{L}^s(B(0,1))$. For $E \subset \mathbb{R}^n$ and $\delta > 0$, we define

$$\mathcal{H}^s_\delta(E) := \inf\Big\{\omega_s \sum_{j=1}^{\infty} \Big(\frac{\operatorname{diam} E_j}{2}\Big)^s \,\Big|\, E \subset \cup_j E_j,\ \operatorname{diam}(E_j) \leq \delta\Big\}$$

and, since $\mathcal{H}^s_\delta$ is nondecreasing in $\delta > 0$, we set

$$\mathcal{H}^s(E) := \lim_{\delta \to 0} \mathcal{H}^s_\delta(E).$$

The set-function $\mathcal{H}^s : \mathcal{P}(X) \to \overline{\mathbb{R}}_+$ is by definition the (*exterior*) *s-dimensional Hausdorff measure* in $\mathbb{R}^n$. We then say that a set $E \subset \mathbb{R}^n$ is *$\mathcal{H}^s$-measurable* if E satisfies the *Carathéodory criterion* for measurability: for any set $A \subset \mathbb{R}^n$ we have

$$\mathcal{H}^s(A) = \mathcal{H}^s(A \cap E) + \mathcal{H}^s(A \setminus E).$$

Methods of *measure theory*, see [GM5], allow us to prove the following.

(i) The class $\mathcal{M}$ of $\mathcal{H}^s$-measurable sets is a σ-algebra of sets and $\mathcal{H}^s$ is σ-addditive on $\mathcal{M}$.

(ii) Open and closed sets of $\mathbb{R}^n$ are $\mathcal{H}^s$-measurable.
(iii) In $\mathbb{R}^n$, $\mathcal{H}^n$ and the Lebesgue n-dimensional measure $\mathcal{L}^n$ agree.

Moreover, it is not difficult to prove the following.

(i) For $\delta > 0, \mathcal{H}^s_\delta(E) < +\infty$ for all bounded sets.
(ii) $\mathcal{H}^s$ is not necessarily finite on compact sets. For example, if $E \subset \mathbb{R}^n$ has a nonempty interior and $s < n$, then $\mathcal{H}^s(E) = +\infty$.
(iii) In the definition of $\mathcal{H}^s_\delta(E)$ we may replace the generic sets E_j with closed, or closed and convex sets, or with open sets without changing the definition of $\mathcal{H}^s$. However, we cannot replace the E_j's by balls. If we do it, for the new measure $\mathcal{H}^s_{\text{sph}}(E)$ we have $\mathcal{H}^s_{\text{sph}}(E) > \mathcal{H}^s(E)$ for some subsets $E \subset \mathbb{R}^n$.
(iv) $\mathcal{H}^0$ is the *counting measure*, $\mathcal{H}^0(E) = \#$ points of E.
(v) $\mathcal{H}^s$ is invariant under orthogonal transformations: if $E \subset \mathbb{R}^n$ and $\mathbf{R}^T\mathbf{R} =$ Id, then $\mathcal{H}^s(\mathbf{R}(E)) = \mathcal{H}^s(E)$.
(vi) $\mathcal{H}^s$ is positively homogeneous of degree s, i.e., for all $\lambda > 0$ and $E \subset \mathbb{R}^n$, we have $\mathcal{H}^s(\lambda E) = \lambda^s \mathcal{H}^s(E)$.
(vii) $\mathcal{H}^s = 0$ if $s > n$.
(viii) If $0 \le t < s \le n$, then $\mathcal{H}^s \le \mathcal{H}^t$. Moreover, $\mathcal{H}^s(E) > 0$ implies $\mathcal{H}^t(E) = +\infty$ and $\mathcal{H}^t(E) < \infty$ implies $\mathcal{H}^s(E) = 0$.
(ix) If $f : \mathbb{R}^n \to \mathbb{R}^k$ is Lipschitz-continuous, then $\forall 0 \le s \le n$ we have

$$\mathcal{H}^s(f(E)) \le (\mathrm{Lip}\, f)^s\, \mathcal{H}^s(E).$$

2.76 Remark. Notice the following:

(i) The claim in (vii) shows that $\mathcal{H}^s(E)$, $E \subset \mathbb{R}^n$, is finite and nonzero for at most one value of s, $0 \le s \le n$, which is called the *Hausdorff dimension of* E, defined in general as

$$\begin{aligned}\dim_{\mathcal{H}}(E) :&= \sup\Big\{s \,\Big|\, \mathcal{H}^s(E) > 0\Big\} = \sup\Big\{s \,\Big|\, \mathcal{H}^s(E) = +\infty\Big\} \\ &= \inf\Big\{s \,\Big|\, \mathcal{H}^s(E) < \infty\Big\} = \inf\Big\{s \,\Big|\, \mathcal{H}^s(E) = 0\Big\}\end{aligned}$$

where the equalities follow from (viii).
(ii) The estimate (ix) is useful to estimate from below the Hausdorff measure of a set. Estimates from above are usually obtained by estimating from above $\mathcal{H}^s_\delta$ by choosing suitable coverings of E with sets of diameter less than δ.

Finally, observe that we may construct an integral with respect to the Haussdorff measure with the same procedure we used to define the Lebesgue integral from Lebesgue's measure, compare [GM5]. From now on, for a given $\mathcal{H}^s$-measurable set of $\mathbb{R}^n$ and an $\mathcal{H}^s$-integrable map, the symbol

$$\int_E f(x)\, d\mathcal{H}^s(x)$$

is well-understood.

2.5.2 Area formula

We did not need any measure theory to define the length of a curve in $\mathbb{R}^n$. If $\gamma : [a,b] \to \mathbb{R}^n$ is a curve, for any subdivision $\sigma := \{t_j\}$, $a = t_0 < t_1 < \dots < t_N = b$, of $[0,1]$ we compute

$$\sum_{i=1}^{N} |\gamma(t_{i-1}) - \gamma(t_i)|$$

and define the length of γ as

$$L(\gamma) := \sup\Big\{ \sum_{i=1}^{n} |\gamma(t_{i-1}) - \gamma(t_i)| \,\Big|\, a = t_0 < t_1 < \dots < t_N = b \Big\},$$

the supremum being taken over all possible subdivisions, see [GM2].

If we want to imitate the previous procedure to define the area of a C^1-image of an open set of $\mathbb{R}^2$ into $\mathbb{R}^3$, we may think of triangularizing the space of parameters and, associated to it, considering the polyhedral surface in $\mathbb{R}^3$ with triangular faces whose vertices are the images of the vertices of the triangulation of the space of parameters. Then, we may compute the area of these approximating polyhedral surfaces and define the area of the surface as the supremum of the areas of the inscribed polyhedral surfaces when the triangulation of the parameters varies. The following example due to Hermann Schwarz (1843–1921) shows how illusory it is to imagine being able to come to a reasonable definition of the area in this way.

2.77 Example (Schwarz). Consider the map $\varphi : [0, 2\pi[\times[0,1] \to \mathbb{R}^3$ $\varphi(\theta, z) := (\cos\theta, \sin\theta, z)$ that maps one-to-one the square $[0, 2\pi[\times[0,1]$ onto a portion of a cylinder $S := \{(x,y,z) \,|\, x^2+y^2 = 1, 0 \le z \le 1\}$. Trivially the elementary area of S is $A(S) = 2\pi$.

We divide the side of the square $[0, 2\pi] \times [0,1]$ in n and m parts, respectively, then we divide each rectangle obtained in this way in four triangles by means of its diagonals, obtaining a triangulation of $[0, 2\pi] \times [0,1]$ in $4nm$ triangles. We construct a polyhedral surface S_{mn} with triangular faces inscribed to the cylinder using the images of the vertices of the triangulation as vertices. A tedious computation yields the area A_{mn} of the inscribed polyhedral surface S_{mn},

$$A_{mn} = 2n \sin\frac{\pi}{2n} + \left[\frac{1}{4} + \frac{4m^2}{n^4}\Big(n \sin\frac{\pi}{2n}\Big)^4\right]^{1/2} \cdot 2n \sin\frac{\pi}{n}.$$

Now, if we choose $m = n$, then $A_{mn} \to 2\pi$ as suggested by intuition; but, if $m = n^3$, then $A_{mn} \to +\infty$. The supremum of the areas of all polyhedral surfaces obtained by triangulations on the space of parameters is therefore $+\infty$ and not the area of S. The intuitive reason for this behavior is the following: If m, the number of subdivisions of the z-axis, is large with respect to the number of subdivisions of the angle, then the triangles of the inscribed polyhedral surface to the cylinder tend to become closer to the orthogonal to the surface of the cylinder. Consequently, the area of the polyhedral surface is large.

This example motivated a flourishing of possible definitions (and, consequently, of treatises) for the area of two-dimensional surfaces in $\mathbb{R}^3$. Among those definitions, the most effective, at least for elementary purposes, has proved to be the one based on Hausdorff measure.

Let $\Omega \subset \mathbb{R}^n$ and $f : \Omega \to \mathbb{R}^N$, $n \leq N$, be a map of class C^1. For any measurable set $A \subset \mathbb{R}^n$, we say that the image $f(A) \subset \mathbb{R}^N$ *is parameterized by* f, and we think of the area of $f(A)$ as of $\mathcal{H}^n(f(A))$ when f is injective. An important formula, called the *area formula*, allows us to compute $\mathcal{H}^n(f(A))$.

2.78 Theorem. *Let $\Omega \subset \mathbb{R}^n$ be an open set and let $f \in C^1(\Omega, \mathbb{R}^N)$, $N \geq n$. If $A \subset \Omega$ is a $\mathcal{L}^n$-measurable set and f is injective on A, then $f(A)$ is $\mathcal{H}^n$-measurable and*

$$\int_A J(\mathbf{D}f(x))\, dx = \mathcal{H}^n(f(A)) \tag{2.35}$$

where $J_f(x) := J(\mathbf{D}f(x)) = \sqrt{\det \mathbf{D}f^T(x)\mathbf{D}f(x)}$ is the Jacobian *of f.*

2.79 Remark. Notice the following:

(i) If $n = 1$, then $\mathbf{D}f = f'$ and $J(\mathbf{D}f) = |f'|$: the area formula (2.35) says that the length of a curve agrees with the one-dimensional Hausdorff measure of the trajectory.

(ii) If f is linear, $f(x) = \mathbf{L}x$, (2.35) simply reads as

$$\mathcal{H}^n(\mathbf{L}(A)) = J(\mathbf{L})\mathcal{L}^n(A).$$

Actually, this is the starting point for the proof and follows from the invariance of the Hausdorff measure under rotations. In fact, using the polar decomposition of $\mathbf{L}$, and identifying $\mathbb{R}^n$ with the n coordinate plane of the first n coordinates of $\mathbb{R}^N$, $\mathbf{L}$ writes as $\mathbf{L} = \mathbf{U}\Delta\mathbf{S}$ where $\mathbf{S} \in M_{n,n}$ is symmetric, $\mathbf{U} \in M_{N,N}$ is orthogonal, and

$$\Delta = \begin{pmatrix} \mathrm{Id} \\ 0 \end{pmatrix}.$$

Using the invariance of $\mathcal{H}^n$ and the change of variable formula for $\mathcal{L}^n$, we find

$$\begin{aligned} \mathcal{H}^n(\mathbf{U}\Delta\mathbf{S}(A)) &= \mathcal{H}^n(\Delta\mathbf{S}(A)) = \mathcal{H}^n(\mathbf{S}(A)) = \mathcal{L}^n(\mathbf{S}(A)) \\ &= |\det \Delta\mathbf{S}|\mathcal{L}^n(A) = |\det \mathbf{S}|\mathcal{L}^n(A) = J(\mathbf{L})\mathcal{L}^n(A). \end{aligned}$$

(iii) The area formula implies that the *image of the points at which the linear tangent map is not injective has zero $\mathcal{H}^n$-measure.* In fact, for any $\mathbf{T} \in M_{N,n}$ $N \geq n$, $\ker \mathbf{T} \neq \{0\}$ if and only if $J(\mathbf{T}) = 0$. Hence from (2.35) we get $\mathcal{H}^n(f(A)) = 0$ if

$$A := \Big\{x \in \Omega \,\Big|\, \ker \mathbf{D}f(x) \neq 0\Big\} = \Big\{x \in \Omega \,\Big|\, J(\mathbf{D}f(x)) = 0\Big\}.$$

(iv) From the area formula we also get the following: *Let Ω, Δ be open sets of $\mathbb{R}^n$ and $\phi : \Omega \to \mathbb{R}^N$ and $\psi : \Delta \to \mathbb{R}^N$ be two maps of class C^1 that are injective, respectively in $A \subset \Omega$ and $B \subset \Delta$, A and B being $\mathcal{L}^n$-measurable. If $\phi(A) = \psi(B)$, then*

$$\int_A J(\mathbf{D}\phi(x))\, dx = \int_B J(\mathbf{D}\psi(y))\, dy.$$

(v) If $n = N$, then (2.35) is simply the change of variable formula for $\mathcal{L}^n$.

The area formula extends in several ways. First, we can drop the injectivity hypothesis by introducing the *multiplicity function* or *Banach*'s *indicatrix*

$$y \to N(f, A, y) := \mathcal{H}^0(A \cap f^{-1}(y))$$

which counts the points of A in the inverse image of y. Under the hypotheses that $f \in C^1(\Omega)$, one shows that the multiplicity function is $\mathcal{H}^N$-measurable and

$$\int_A J(\mathbf{D}f)\, dx = \int_{\mathbb{R}^N} N(f, A, y)\, d\mathcal{H}^n(y). \tag{2.36}$$

Moreover, we can also relax the regularity of the map f: One can in fact prove that (2.36) holds also if f is Lipschitz-continuous (recall that, if f is Lipschitz-continuous, then $J(\mathbf{D}f)$ is defined $\mathcal{L}^n$-a.e. since f is differentiable $\mathcal{L}^n$-almost-everywhere by the Rademacher theorem).

Starting from (2.36), by approximating $\mathcal{L}^n$-measurable functions u by simple functions and then passing to the limit by means of the monotone convergence theorem of Beppo Levi, we also get the following.

2.80 Theorem (Change of variables formula). *Let $\Omega \subset \mathbb{R}^n$ be open, let $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}^N$, $n \le N$, be of class $C^1(\Omega)$ (or, more generally, locally Lipschitz-continuous in Ω), and let $u : \Omega \to \mathbb{R}$ be $\mathcal{L}^n$-measurable and nonnegative, or such that $|u|\, J(\mathbf{D}f)$ is $\mathcal{L}^n$-summable. Then the function*

$$y \to \sum_{x \in f^{-1}(y)} u(x)$$

is $\mathcal{H}^n$-measurable and

$$\int_{\mathbb{R}^n} u(x)\, J(\mathbf{D}f)(x)\, dx = \int_{\mathbb{R}^N} \Big(\sum_{x \in f^{-1}(y)} u(x) \Big)\, d\mathcal{H}^n(y). \tag{2.37}$$

In particular, if $v : \mathbb{R}^N \to \mathbb{R}$ is $\mathcal{H}^n$-measurable and nonnegative, then

$$\int_A v(f(x))\, J(\mathbf{D}f)(x)\, dx = \int_{\mathbb{R}^N} v(y)\, N(f, A, y)\, d\mathcal{H}^n(y)\ . \tag{2.38}$$

a. Calculus of the area of a surface

Parameterizing, at least locally, a k-dimensional surface in $\mathbb{R}^n$ by a C^1 map, we can easily compute its area by means of the area formula (2.35).

In this procedure we need to compute the Jacobian of the parameterization, and the following information may be useful.

(i) Let $\mathbf{A} \in M_{N,n}$. The alternative theorem yields

$$\operatorname{Rank}(\mathbf{A}^T\mathbf{A}) = \operatorname{Rank}\mathbf{A}^T = \operatorname{Rank}\mathbf{A} = \operatorname{Rank}\mathbf{A}\mathbf{A}^T \leq \min(n, N). \tag{2.39}$$

It follows that $\det \mathbf{A}\mathbf{A}^T = 0$ if $N > n$.

(ii) We have $\ker \mathbf{A}^T\mathbf{A} = \ker \mathbf{A}$, consequently the three claims
   (a) $J(\mathbf{A}) = (\det \mathbf{A}^T\mathbf{A})^{1/2} = 0$,
   (b) $\ker \mathbf{A} \neq \{0\}$,
   (c) $\operatorname{Rank}\mathbf{A}$ is not maximal,

are equivalent.

(iii) (Area and metric tensor) Let $\{A_1, A_2, \dots, A_n\}$ denote the columns of $\mathbf{A}$, $\mathbf{A} = [A_1|A_2|\dots|A_n]$. Then

$$\mathbf{A}^T\mathbf{A} = \mathbf{G}, \qquad \text{where} \qquad \mathbf{G} = (g_{ij}),\ g_{ij} := A_i \bullet A_j\,.$$

Consequently, if $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}^N$, $N \geq n$, is of class C^1, then

$$J(\mathbf{D}f) = \sqrt{\det \mathbf{G}}, \qquad \mathbf{G} = (g_{ij}),\ g_{ij} = f_{x^i} \bullet f_{x^j}\,,$$

and the area formula becomes

$$\mathcal{H}^n(f(\Omega)) = \int_\Omega \sqrt{g(x)}\,dx, \qquad g(x) := \det \mathbf{G}(x).$$

(iv) (The Cauchy–Binet formula) Let $\mathbf{A} \in M_{N,n}$, $N \geq n$. For every multiindex $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$, $1 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_n \leq N$, let $\mathbf{A}^\alpha$ be the $n \times n$-submatrix of $\mathbf{A}$ made of the rows $\alpha_1, \alpha_2, \dots, \alpha_n$ of $\mathbf{A}$. Then, see, e.g., [GM5], the following *Cauchy–Binet formula* holds

$$J(\mathbf{A})^2 = \sum_{\alpha \in I(n,N)} (\det(\mathbf{A}^\alpha))^2.$$

2.81 Example (Two-dimensional parameterized surfaces in $\mathbb{R}^3$). Let $n = 2$ and $N = 3$. Then

$$\mathbf{D}f := \begin{pmatrix} a & d \\ b & e \\ c & f \end{pmatrix}$$

where the columns are the vectors with components the partial derivatives of f with respect to x and y, $f_x := (a, b, c)^T$ and $f_y := (d, e, f)^T$. If we set

$$E := |f_x|^2, \qquad F := f_x \bullet f_y\,, \qquad G := |f_y|^2,$$

we find

$$\mathbf{D}f^T\mathbf{D}f = \begin{pmatrix} E & F \\ F & G \end{pmatrix}$$

and
$$\mathcal{H}^2(f(\Omega)) = \int_\Omega J(\mathbf{D}f)\,dxdy = \int_\Omega \sqrt{EG-F^2}\,dxdy.$$

Alternatively, we can compute $J(\mathbf{D}f)$ and the area of $u(\Omega)$ by means of the Cauchy–Binet formula. If
$$\mathbf{A}^{12} := \begin{pmatrix} a & d \\ b & e \end{pmatrix}, \qquad \mathbf{A}^{23} := \begin{pmatrix} b & e \\ c & f \end{pmatrix}, \qquad \mathbf{A}^{13} := \begin{pmatrix} a & d \\ c & f \end{pmatrix},$$
then
$$J(\mathbf{D}f)^2 = (ae-bd)^2 + (bf-ec)^2 + (af-dc)^2.$$
Notice that the three numbers $ae-bd$, $-(bf-ec)$, and $af-dc$ are the three components of the *vector product*
$$f_x \times f_y$$
of the columns of $\mathbf{D}f$, hence
$$\mathcal{H}^2(f(\Omega)) = \int_\Omega |f_x \times f_y|\,dxdy.$$

2.82 Example (Graphs of codimension 1). Let $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ be a function of class C^1 and
$$\mathcal{G}_{u,\Omega} := \Big\{(x,y) \in \Omega \times \mathbb{R} \,\Big|\, y = u(x)\Big\}$$
be its graph. $\mathcal{G}_{u,\Omega}$ is the image of the injective map $f(x) = (x, u(x))$ from Ω into $\mathbb{R}^{n+1}$. Since
$$\mathbf{D}f(x) = \begin{pmatrix} \boxed{\quad \mathrm{Id} \quad} \\ \mathbf{D}u(x) \end{pmatrix},$$
the Cauchy–Binet formula gives $J(\mathbf{D}f(x)) = \sqrt{1+|\mathbf{D}u(x)|^2}$, hence
$$\mathcal{H}^n(\mathcal{G}_{u,\Omega}) = \int_\Omega \sqrt{1+|\mathbf{D}u(x)|^2}\,dx.$$

2.83 Example (Parameterized hypersurfaces). Let $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^{n+1}$ be an injective map of class C^1. The Jacobian matrix of u has $n+1$ rows and n columns, and its $n \times n$ submatrices can be indexed by the missing row. If
$$\frac{\partial(u^1,\dots,u^{i-1},u^{i+1},\dots,u^n)}{\partial(x^1,\,x^2,\dots,\,x^n)}$$
denote the determinant of the submatrix obtained by removing the ith row, we then get
$$\mathcal{H}^n(u(\Omega)) = \int_\Omega \left(\sum_{i=1}^n \left(\frac{\partial(u^1,\dots,u^{i-1},u^{i+1},\dots,u^n)}{\partial(x^1,\,x^2,\dots,\,x^n)}\right)^2\right)^{1/2} dx.$$

2.84 Example (Rotational surfaces). A rotational surface around an axis is well described by its perpendicular sections to its axis that are circles. We can describe its points P by means of two parameters: the orthogonal projection of P on the rotational axis and a parameter describing the points on the circle in the perpendicular plane to the axis through P. This way, if S is a rotational surface around the axis z, S is the one-to-one image of
$$A := [a,b] \times [0, 2\pi[$$
by a map $\phi : [a,b] \times \mathbb{R} \to \mathbb{R}^3$ of the type

$$\phi(z,\theta) := \begin{cases} x = \rho(z)\cos\theta, \\ y = \rho(z)\sin\theta, \\ z = z \end{cases}$$

where $\rho(z)$ is the radius of the section at level z. Assuming $\rho(z) \in C^1([a,b])$, we have

$$\mathbf{D}\phi(z,\theta) := \begin{pmatrix} \rho'\cos\theta & -\rho\sin\theta \\ \rho'\sin t & -\rho\cos\theta \\ 1 & 0 \end{pmatrix}, \qquad (z,t) \in A := [a,b] \times [0,2\pi[$$

hence, by the Cauchy–Binet formula,

$$J(\mathbf{D}\phi)(z,\theta) = \rho(z)\sqrt{1+\rho'^2(z)};$$

therefore

$$\mathcal{H}^2(S) = \mathcal{H}^2(\phi(A)) = \int_0^{2\pi}\int_a^b \rho(z)\sqrt{1+(\rho'(z))^2}\,dz = 2\pi\int_a^b \rho(z)\sqrt{1+(\rho'(z))^2}\,dz.$$

2.5.3 The coarea formula

Consider a function $f : \mathbb{R}^n \to \mathbb{R}^N$, $N \leq n$ and, for $y \in \mathbb{R}^N$ its inverse image $f^{-1}(y)$. When y varies, the family $\{f^{-1}(y)\}$ yields a sort of *foliation* of $\mathbb{R}^n$, for example think of $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = x^2 + y^2$, for which $f^{-1}(t) := \{(x,y) = x^2 + y^2 = t\}$. As we shall see in Chapter 5, if $\mathbf{D}f(x)$ is of maximal rank N, then the *leaf* $f^{-1}(y)$ is an $(n-N)$-dimensional submanifold of $\mathbb{R}^n$. The coarea formula provides a formula that allows us to express the $\mathcal{L}^n$-integration on a set $A \subset \mathbb{R}^n$ as the integration with respect to y of an $\mathcal{H}^{n-N}$-integration over $f^{-1}(y)$.

2.85 Theorem (Coarea formula). *Let Ω be an open set and $A \subset \Omega$ be $\mathcal{L}^n$-measurable, let $f : \Omega \to \mathbb{R}^N$ be a map of class C^1, and assume $N \leq n$. For $\mathcal{L}^N$-a.e. y the set $A \cap f^{-1}(y)$ is $\mathcal{H}^{n-N}$-measurable, the function $y \to \mathcal{H}^{n-N}(A \cap f^{-1}(y))$ is $\mathcal{L}^N$-measurable and*

$$\int_A J(\mathbf{D}f(x))\,d\mathcal{L}^n(x) = \int_{\mathbb{R}^N} \mathcal{H}^{n-N}(A \cap f^{-1}(y))\,d\mathcal{L}^N(y); \tag{2.40}$$

here

$$J_f(x) := J(\mathbf{D}f(x)) = \sqrt{\det(\mathbf{D}f(x)\mathbf{D}f(x)^T)}$$

denotes the Jacobian of f.

Actually the previous theorem can be generalized in several ways. First, it suffices to assume that f be locally Lipschitz-continuous. Moreover, by approximating measurable maps u with simple functions, one shows the following.

2.86 Theorem. *Let $\Omega \subset \mathbb{R}^n$ be open, let $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}^N$, $N \leq n$, be a map of class $C^1(\Omega)$ (or merely locally Lipschitz-continuous) and let $u : \Omega \to \mathbb{R}$ be a measurable function on Ω such that $|u|\, J(\mathbf{D}f)$ is $\mathcal{L}^n$-integrable. Then*

$$\int_\Omega u(x)\, J(\mathbf{D}f)(x)\, dx = \int_{\mathbb{R}^N} \left(\int_{f^{-1}(y)} u(x)\, d\mathcal{H}^{n-N}(x) \right) d\mathcal{L}^N(y). \tag{2.41}$$

2.87 Remark. We notice the following:

(i) If we split $\mathbb{R}^n$ as $\mathbb{R}^n = \mathbb{R}^{n-N} \times \mathbb{R}^N$ denoting its coordinates by (x, y), $x \in \mathbb{R}^{n-N}$, $y \in \mathbb{R}^N$, and we choose $f(x, y) := y$, then $J(\mathbf{D}f)(x, y) = |\mathbf{D}f(x, y)| = 1$, $A \cap f^{-1}(y) = \{(x, z) \in A \,|\, z = y\} = A_y$. Therefore, Theorem 2.85 simply reduces to Fubini's theorem.

(ii) Let $f : \mathbb{R}^n \to \mathbb{R}$. Since

$$J(\mathbf{D}f) = \sqrt{\det(\mathbf{D}f)(\mathbf{D}f)^T} = |\mathbf{D}f|,$$

we then obtain

$$\int_A |Df| dx = \int_{-\infty}^{+\infty} \mathcal{H}^{n-1}(A \cap f^{-1}(t))\, dt$$

and

$$\int_A g(x)|Df(x)| dx = \int_{-\infty}^{+\infty} \left(\int_{A \cap f^{-1}(t)} g(z)\, d\mathcal{H}^{n-1}(z) \right) dt$$

for any measurable $A \subset \mathbb{R}^n$ and any measurable $g : A \to \mathbb{R}$ such that $g(x)|Df(x)|$ is $\mathcal{L}^n$-integrable.

2.88 Example (Measure of the unit sphere in $\mathbb{R}^n$). The volume of the ball of radius r in $\mathbb{R}^n$ is $\omega_n r^n$ where $\omega_n = \mathcal{L}^n(B(0,1))$, and the measure of the sphere of radius t, $\mathcal{H}^{n-1}(\partial B(0,t))$ is positively homogeneous of degree $n-1$, in particular it is continuous in t. If we choose $f(x) = |x|$, then $J(\mathbf{D}f(x)) = |\mathbf{D}f(x)| = 1$: from the coarea formula

$$\int_{B(0,r+h)\setminus B(0,r)} dx = \int_r^{r+h} \mathcal{H}^{n-1}(\partial B(0,t))\, dt$$

and on account of the fundamental theorem of calculus we infer

$$\begin{aligned} \mathcal{H}^{n-1}(\partial B(0,r)) &= \lim_{h\to 0} \frac{1}{h} \int_r^{r+h} \mathcal{H}^{n-1}(\partial B(0,t))\, dt = \lim_{h\to 0} \frac{1}{h} \int_{B(0,r+h)\setminus B(0,r)} dx \\ &= \frac{d\mathcal{L}^n(B(0,r))}{dr}(r) = n\omega_n r^{n-1}. \end{aligned}$$

2.89 Example. Notice that if $f(x) := |x|$, then $f^{-1}(y) = \partial B(0,y)$ and Theorem 2.86 yields the well-known formula of integration on polar coordinates

$$\int_{\{s<|x|<t\}} u\, dx = \int_r^s \left(\int_{\partial B(0,\rho)} u(y)\, d\mathcal{H}^{n-1}(y) \right) d\rho.$$

In particular, for $h \neq 0$ we have

$$\frac{1}{h}\int_{\{r<|x|<r+h\}} u\,dx = \frac{1}{h}\int_r^{r+h}\Big(\int_{\partial B(0,\rho)} u(y)\,d\mathcal{H}^{n-1}(y)\Big)\,d\rho,$$

hence, by the theorem of differentiation of the integral, for $\mathcal{L}^1$-a.e. r (for all r if, for instance, u is continuous),

$$\frac{d}{dr}\Big(\int_{B(0,r)} u\,dx\Big) = \int_{\partial B(0,r)} u\,d\mathcal{H}^{n-1}.$$

For $u = 1$ we therefore find again

$$\mathcal{H}^{n-1}(\partial B(0,r)) = \frac{d}{dr}(\omega_n r^n) = n\omega_n r^{n-1}.$$

2.90 Example (Measure of the unit ball in $\mathbb{R}^n$). The coarea formula yields also an alternative way to compute the measure ω_n of the unit ball $B(0,1)$ of $\mathbb{R}^n$. Using Fubini's theorem we find

$$\int_{\mathbb{R}^n} e^{-|x|^2}\,dx = \int_{-\infty}^{+\infty} e^{-x_1^2}\,dx_1 \int_{-\infty}^{+\infty} e^{-x_2^2}\,dx_2 \dots \int_{-\infty}^{+\infty} e^{-x_n^2}\,dx_n = \pi^{n/2}.$$

On the other hand, the coarea formula yields

$$\begin{aligned}\int_{\mathbb{R}^n} e^{-|x|^2}\,dx &= \int_0^{+\infty}\Big(\int_{\partial B(0,t)} e^{-t^2}\,d\mathcal{H}^{n-1}(t)\Big)\,dt = \frac{n}{2}\,|B(0,1)|\int_0^{+\infty} e^{-t^2}t^{n-1}\,dt\\ &= \frac{n}{2}\,|B(0,1)|\int_0^{+\infty} s^{\frac{n}{2}-1}e^{-s}\,ds = \frac{n}{2}\,|B(0,1)|\Gamma\Big(\frac{n}{2}\Big)\end{aligned}$$

in terms of Euler's Γ-function. If follows

$$\omega_n = |B(0,1)| = \frac{2}{n}\frac{\pi^{n/2}}{\Gamma\big(\frac{n}{2}\big)} = \frac{\pi^{n/2}}{\Gamma\big(\frac{n}{2}+1\big)}.$$

2.91 Example. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function of class C^1 (or merely locally Lipschitz-continuous). From the coarea formula, for any positive h we have

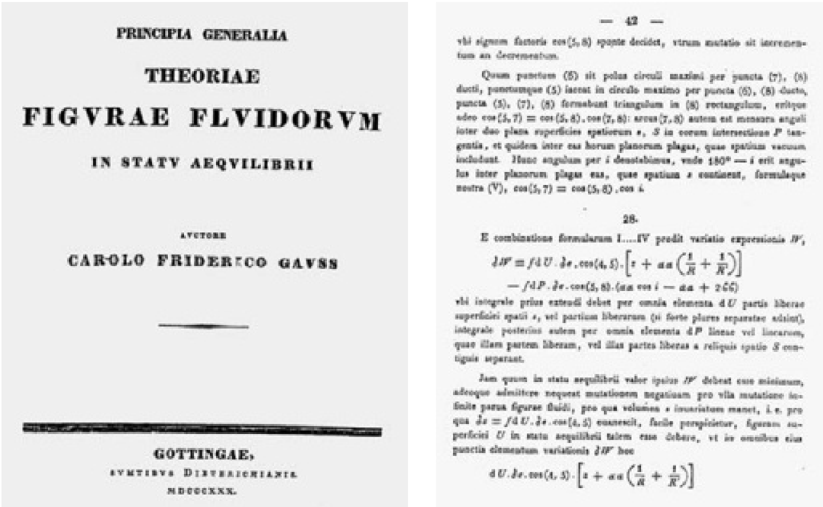
$$\begin{aligned}\frac{1}{h}\int_{\{t<f<t+h\}} |Df|\,dx &= \frac{1}{h}\int_t^{t+h} \mathcal{H}^{n-1}(\{f=t\})\,dt\\ \frac{1}{h}\int_{\{t-h<f<t\}} |Df|\,dx &= \frac{1}{h}\int_{t-h}^{t} \mathcal{H}^{n-1}(\{f=t\})\,dt\end{aligned}$$

hence, by the theorem of differentiation of integral, for $\mathcal{L}^1$-a.e. $t \in \mathbb{R}$

$$\frac{d}{dt}\int_{\{f<t\}} |Df|\,dx = -\frac{d}{dt}\int_{\{f>t\}} |Df|\,dx = \mathcal{H}^{n-1}(\{f=t\}).$$

2.6 Gauss–Green Formulas

In this section, we state the *Gauss–Green theorem* and discuss the *divergence theorem.* These topics are of fundamental relevance for the development of the calculus for functions of several variables. In particular, Gauss–Green formulas extend the fundamental theorem of calculus



PRINCIPIA GENERALIA

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AVCTORE

CAROLO FRIDERICO GAVSS

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SVMTIBVS DIETERICHIANIS.

MDCCCXXX.

— 42 —

vbi signum factoris cos(5, 8) sponte decidet, vtrum mutatio sit incrementum an decrementum.

Quum punctum (5) sit polus circuli maximi per puncta (7), (8) ducti, punctumque (5) iaceat in circulo maximo per puncta (5), (8) ducto, puncta (5), (7), (8) formabunt triangulum in (8) rectangulum, eritque adeo $\cos(5,7) = \cos(5,8).\cos(7,8)$: arcus (7, 8) autem est mensura anguli inter duo plana superficies spatiorum s, S in eorum intersectione P tangentia, et quidem inter eas horum planorum plagas, quae spatium vacuum includunt. Hunc angulum per i denotabimus, vnde $180° - i$ erit angulus inter planorum plagas eas, quae spatium s continent, formulaque nostra (V), $\cos(5,7) = \cos(5,8).\cos i$.

28.

E combinatione formularum I....IV prodit variatio expressionis W,

$$\delta W = \int dU.\delta e.\cos(4,5).\left[z + \alpha\alpha\left(\frac{1}{R} + \frac{1}{R'}\right)\right] - \int dP.\delta e.\cos(5,8).(\alpha\alpha\cos i - \alpha\alpha + 2\beta\beta)$$

vbi integrale prius extendi debet per omnia elementa dU partis liberae superficiei spatii s, vel partium liberarum (si forte plures separatae adsint), integrale posterius autem per omnia elementa dP lineae vel linearum, quae illam partem liberam, vel illas partes liberas a reliquis spatio S contiguis separant.

Jam quum in statu aequilibrii valor ipsius W debeat esse minimum, adeoque admittere nequeat mutationem negatiuam pro vlla mutatione infinite parua figurae fluidi, pro qua volumen s inuariatum manet, i. e. pro qua $\delta s = \int dU.\delta e.\cos(4,5)$ euanescit, facile perspicitur, figuram superficiei U in statu aequilibrii talem esse debere, vt in omnibus eius punctis elementum variationis δW hoc

$$dU.\delta e.\cos(4,5).\left[z + \alpha\alpha\left(\frac{1}{R} + \frac{1}{R'}\right)\right]$$

Figure 2.16. Two pages from the paper by Carl Friedrich Gauss (1777–1855), in which the Gauss–Green formula appears.

$$f(b) - f(a) = \int_a^b f'(t)\,dt$$

to functions of several variables.

a. Two simple situations

We begin with a very simple situation.

2.92 Proposition. *Let $A \subset \mathbb{R}^n$ be open and $f \in C^1_c(A)$. Then*

$$\int_A D_i f(x)\,dx = 0 = \int_{\partial A} f\,d\mathcal{H}^{n-1} \qquad \forall i = 1, \dots, n.$$

Proof. We extend f to all $\mathbb{R}^n$ as zero outside A; we call it f. It is not restrictive to assume that its support is contained in the unit cube. Since for every $x_1, \dots, x_{i-1}$, $x_{i+1}, \dots, x_n$ we have

$$\int_{-1}^1 D_i f(x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_n)\,dx_i = \\ = f(x_1, \dots, x_{i-1}, 1, x_{i+1}, \dots, x_n) - f(x_1, \dots, x_{i-1}, -1, x_{i+1}, \dots, x_n) = 0,$$

integrating with respect to the remaining variables we get

$$\int_A D_i f(x)\,dx = \int_Q D_i f(x)\,dx = 0.$$

$\square$

Split $\mathbb{R}^n$ as $\mathbb{R}^{n-1} \times \mathbb{R}$ and let $x = (x', x_n)$, $x' = (x_1, x_2, \dots, x_{n-1}) \in \mathbb{R}^{n-1}$, $x_n \in \mathbb{R}$, be its coordinates. Let Q be a bounded open set in $\mathbb{R}^{n-1}$ and let $\alpha : Q \to]a, b[$ be a function of class C^1; set

$$A := \Big\{ x = (x', x_n) \in Q \times [a, b] \,\Big|\, a < x_n < \alpha(x') \Big\}.$$

Since the vector $(-\nabla\alpha(x'), 1)$ is perpendicular to the plane tangent to the graph of α at $(x', \alpha(x'))$, the exterior normal vector to A at $(x', \alpha(x'))$ has components $\nu = (\nu_1, \nu_2, \dots, \nu_n)$ given by

$$\begin{cases} \nu_i := \dfrac{-D_i\alpha}{\sqrt{1 + |D\alpha(x')|^2}} & \forall i = 1, \dots, n-1, \\ \nu_n := \dfrac{1}{\sqrt{1 + |D\alpha(x')|^2}}. \end{cases}$$

2.93 Proposition. *Let $f \in C^1(A) \cap C^0(\overline{A})$ with $|Df| \in \mathcal{L}^1(A)$. Suppose that f vanishes near $\partial(Q \times [a, b]) \cap \overline{A}$, trivially*

$$\int_A D_i f(x)\, dx = \int_{\partial A} f \nu_i \mathcal{H}^{n-1}.$$

Proof. Since f vanishes on $\partial(Q \times [a, b]) \cap \overline{A}$, trivially

$$\int_{\partial A} f\nu_i d\mathcal{H}^{n-1} = \int_{\mathcal{G}_{\alpha,Q}} f\nu_i d\mathcal{H}^{n-1}$$

and, since the element of area on $\mathcal{G}_{\alpha,Q}$ is $d\mathcal{H}^{n-1} = \sqrt{1 + |D\alpha(x')|^2}\, dx'$, we have

$$\int_{\partial A} f\nu_i \mathcal{H}^{n-1} = \begin{cases} -\displaystyle\int_Q f(x', \alpha(x')) \frac{\partial\alpha}{\partial x_i}(x')\, dx' & \text{if } i = 1, \dots, n-1, \\ \displaystyle\int_Q f(x', \alpha(x'))\, dx' & \text{if } i = n. \end{cases} \tag{2.42}$$

From the fundamental theorem of calculus, since $f = 0$ near $\partial(Q \times [a, b]) \cap \overline{A}$ and $|Df| \in \mathcal{L}^1(A)$, we infer

$$\begin{aligned} \int_A D_n f(x', x_n)\, dx' dx_n &= \int_Q \int_a^{\alpha(x')} D_n f(x', x_n)\, dx_n dx' \\ &= \int_Q (f(x', \alpha(x')) - f(x', a))\, dx' = \int_Q f(x', \alpha(x'))\, dx'. \end{aligned} \tag{2.43}$$

A comparison of (2.42) and (2.43) yields the result for $i = n$.

Now set

$$F(x') := \int_a^{\alpha(x')} f(x', x_n)\, dx_n, \qquad x' \in Q.$$

Differentiating under the integral sign with respect to x_i, we infer

$$D_i F(x') = \int_a^{\alpha(x')} D_i f(x', x_n)\, dx_n + f(x', \alpha(x')) D_i\alpha(x');$$

on the other hand

$$\int_A D_i f(x', x_n)\, dx' dx_n = \int_Q \left(\int_a^{\alpha(x')} D_i f(x', x_n)\, dx_n \right) dx'$$
$$= \int_Q D_i F(x')\, dx' - \int_Q f(x', \alpha(x')) D_i \alpha(x')\, dx', \tag{2.44}$$

and, since f vanishes if $x \in \partial(Q \times [a,b]) \cap \overline{A}$, $F(x') = 0$ if $x' \in \partial Q$ and

$$\int_Q D_i F(x')\, dx' = \int_Q D_i F(x')\, dx_1 \dots dx_{n-1}$$
$$= \int \int \Big(\int_{-1}^{1} D_i F(x') dx_i \Big) dx_1 \dots dx_{i-1}\, dx_{i+1} \dots dx_{n-1}$$
$$= \int \int \Big(F(x_1, \dots, x_{i-1}, 1, x_{i+1}, \dots, x_{n-1}) - F(x_1, \dots, x_{i-1}, -1, x_{i+1}, \dots, x_{n-1}) \Big)$$
$$dx_1 \dots dx_{i-1}\, dx_{i+1} \dots dx_{n-1}$$
$$= \int \int 0\, dx_1 \dots dx_{i-1}\, dx_{i+1} \dots dx_{n-1} = 0.$$

Therefore, (2.44) becomes

$$\int_A D_i f(x', x_n)\, dx' dx_n = - \int_Q f(x', \alpha(x')) D_i \alpha(x')\, dx'. \tag{2.45}$$

From (2.45) and (2.42), we infer the result for $i = 1, \dots, n-1$. □

b. Admissible sets

In the sequel we shall limit ourselves to prove Gauss–Green formulas for a class of sets, which we now introduce, sufficiently large for the applications. Actually, measure theory would allow us to prove them for a much larger class.

Let $A \subset \mathbb{R}^n$ be an open set. In this context, we say that $x \in \partial A$ is a *regular point* for ∂A if there exists an open cube with center at x and sides parallel to the axes (that we write as $Q \times [a, b]$ where Q is a cube on $\mathbb{R}^{n-1}$) and a function $\alpha : Q \to]a, b[$ of class $C^1(Q)$ such that

(i) $U_x \cap A = \{(x', x_n) \,|\, a < x_n < \alpha(x'),\ x' \in Q\}$,
(ii) $U_x \cap \partial A = \{(x', x_n) \,|\, x_n = \alpha(x'),\ x' \in Q\}$.

The set $r(A) \subset \partial A$ of regular points for ∂A is open relatively to ∂A, and for every $x \in \partial A$, the *exterior unit vector* to A at x is given by the vector $\nu = (\nu_1, \nu_2, \dots, \nu_n)$ given by

$$\nu_i := \frac{-D_i \alpha}{\sqrt{1 + |D\alpha(x')|^2}} \quad \forall i = 1, \dots, n-1, \qquad \nu_n := \frac{1}{\sqrt{1 + |D\alpha(x')|^2}}.$$

Obviously $|\nu| = 1$, ν is perpendicular to the tangent plane to the graph of α at $x = (x', \alpha(x'))$ and $x - t\nu(x) \in A$ for all $t > 0$ sufficiently small positive t. Of course, neither the cube nor the function α are uniquely defined by A; however, it is not difficult to show, compare Chapter 5, that the exterior unit normal is uniquely defined at the points $x \in r(A)$. Moreover, one sees, compare Chapter 5, that $x \in \partial A$ is regular if and only if there exist an open cube Q centered at x and a function $\varphi : U_x \to \mathbb{R}$ of class C^1 such that

(i) $U_x \cap A = \{y \in U_x \,|\, \varphi(y) < 0\}$,
(ii) $U_x \cap \partial A = \{y \in U_x \,|\, \varphi(y) = 0\}$,
(iii) $\nabla\varphi \neq 0$ in $U_x \cap \partial A$.

In this case the exterior normal vector at $x \in r(A)$ at $x \in r(A)$ is

$$\nu(x) = \frac{\nabla\varphi(x)}{|\nabla\varphi(x)|}.$$

2.94 Definition. *We shall say that an open set $A \subset \mathbb{R}^n$ is* admissible *if A is open, $\mathcal{H}^{n-1}(\partial A) < +\infty$ and $\mathcal{H}^{n-1}(\partial A \setminus r(A)) = 0$.*

For example, an open set in $\mathbb{R}^2$ whose boundary is the union of a finite number of closed and disjoint piecewise regular curves is an admissible set of $\mathbb{R}^2$. Also a bounded set whose boundary is a polyhedron with a finite number of faces is an admissible set. Actually, it is easily seen that A is admissible if A is bounded and ∂A is a finite disjoint union $\partial A = \cup_{i=0}^N \Gamma_i$ where Γ_0 is closed with $\mathcal{H}^{n-1}(\Gamma_0) = 0$, and, for $i = 1, \dots, N$, Γ_i is a $(n-1)$-submanifold of $\mathbb{R}^n$, see Chapter 5.

c. Decomposition of unity

The *decomposition* (or *partition*) *of unity* is a useful tool when we want to transfer local information to global ones.

2.95 Theorem. *Let $\{V_\alpha\}$ be a family of open sets in $\mathbb{R}^n$ and $\Omega := \cup_\alpha V_\alpha$. There exists a locally finite covering of Ω with balls $B_j \subset\subset \Omega$ such that for every j we have $\overline{B_j} \subset V_\alpha$ for some α.*

Proof. For all $j = 1, 2, \dots,$ we choose a sequence $\{H_j\}$ of compact sets contained in Ω with $H_j \subset\subset \operatorname{int}(H_{j+1})$ and $\Omega = \cup_j H_j$; we also set $H_{-1} = H_{-2} := \emptyset$. For $j := 0, 1, \dots$ we consider the compact sets $K_j := H_j \setminus \operatorname{int}(H_{j-1})$ and the open sets $A_j := \operatorname{int} H_{j+1} \setminus H_{j-2}$. We have $K_j \subset\subset A_j$, $\Omega = \cup_j A_j$ and $A_i \cap A_j = \emptyset$ except for $i = j-1, j$ o $j+1$. Now, for every $x \in K_j$ choose $\lambda = \lambda(x)$ such that $x \in V_{\lambda(x)}$ and a ball $B(x, r(x))$ with closure in $A_j \cap V_{\lambda(x)}$. The family $\{B(x, r(x))\}_{x \in K_j}$ is clearly an open covering of the compact set K_j from which we can choose a finite covering $\{B_{j,1}, B_{j,2}, \dots B_{j,N_j}\}$. The family

$$\mathcal{B} := \Big\{ B_{j,i} \,\Big|\, j = 0, 1, \dots, i = 1, \dots N_j \Big\}$$

has the required properties. □

2.96 Lemma. *The function*

$$\varphi(x) := \begin{cases} \exp\left(\frac{1}{1-|x|^2}\right) & \text{if } |x| < 1, \\ 0 & \text{if } |x| \geq 1 \end{cases}$$

is of class C^∞ and nonzero exactly on $B(0,1)$.

2.97 Theorem. *Let $\{B_j\}$ be a locally finite covering of $\Omega = \cup_j B_j$, B_j being balls. There exists functions $w_j : \mathbb{R}^n \to \mathbb{R}$ of class C^∞ such that*

(i) $0 \le \alpha_j(x) \le 1\ \forall x \in \mathbb{R}^n$,
(ii) $\alpha_j(x) > 0$ *if and only if* $x \in B_j$,
(iii) $\sum_{j=1}^\infty \alpha_j(x) = 1\ \forall x \in \Omega$.

Proof. For $j = 1, 2, \dots$, we choose $\varphi_j \in C^\infty(\mathbb{R}^n)$ with $\varphi_j > 0$ on B_j and $\varphi_j = 0$ outside B_j. The function $\sum_{j=1}^\infty \varphi_j(x)$ is well defined on $\mathbb{R}^n$ since locally it is a finite sum ($\{B_j\}$ being locally finite) and positive in Ω, since $\{B_j\}$ is a covering of Ω. Thus, we readily see that the functions

$$\alpha_j(x) := \frac{\varphi_j(x)}{\sum_{j=0}^\infty \varphi_j(x)}$$

have the desired properties. □

We notice that the number of functions α_j of the decomposition of unity that are nonzero at each x is finite and that they are exactly the nonzero functions of the decomposition of unity that are nonzero at y if y is suffciently close to x. Consequently, we also have

$$\sum_{j=1}^\infty D\alpha_j(x) = D\Big(\sum_{j=1}^\infty \alpha_j(x)\Big) = 0 \qquad \forall x \in \Omega$$

and

$$\int \sum_{j=1}^\infty \alpha_j(x)\, d\mu = \sum_{j=1}^\infty \int \alpha_j(x)\, d\mu$$

for $\mu = \mathcal{L}^n$ or $\mathcal{H}^{n-1}$.

d. Gauss–Green formulas

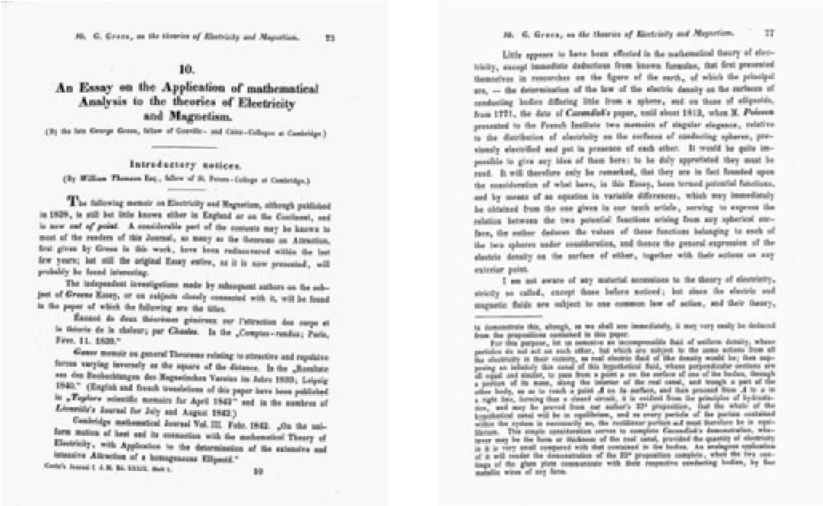
2.98 Theorem (Gauss–Green formulas). *Let $A \subset \mathbb{R}^n$ be an admissible open set and let f be a function of class C^1 in a neighborhood of A with $|\mathbf{D}f| \in \mathcal{L}^1(A)$. Denote by $\nu : r(A) \to \mathbb{R}^n$ the field of exterior unit normal vectors to A. Then ν is defined $\mathcal{H}^{n-1}$-a.e. on ∂A. We have*

$$\int_A D_i f(x)\, dx = \int_{\partial A} f\nu_i\, d\mathcal{H}^{n-1} \qquad \forall i = 1, \dots, n.$$

Proof. Recall that $r(A)$ is the set of regular points for ∂A. We set $s(A) := \partial A \setminus r(A)$, Δ to be an open set so that $\Delta \supset \overline{A}$, and, finally, $\Omega := \Delta \setminus s(A)$. Since $s(A)$ is closed, Ω is open. Now, for $x \in \Omega$ we can choose an open neighborhood U_x of x so that

(i) if $x \in \Omega \setminus \overline{A}$, then U_x is a cube centered at x and contained in $\Omega \setminus \overline{A}$,
(ii) if $x \in A \cap \Omega$, then U_x is a cube centered at x and contained in $\Omega \cap A$,
(iii) if $x \in \partial A \cap \Omega$, i.e., $x \in r(\partial A)$, then we choose U_x as in the definition of regular points and, without loss of generality, we assume that U_x is small enough so that $U_x \subset \Omega$.

The family $\{U_x\}$ covers Ω. Therefore, there exists a denumerable locally finite refinement $\{B_j\}$ of $\{U_x\}$, Theorem 2.95, with the associated decomposition of unity $\{\alpha_j\}$, Theorem 2.97, and we distinguish the following three cases:



10.

An Essay on the Application of mathematical Analysis to the theories of Electricity and Magnetism.

Figure 2.17. Two pages from the *Essay* by George Green (1793–1841), which appeared in 1828 and was reprinted in 1850 in *Crelle's Journal* where Gauss–Green formulas appear.

- B_j is exterior to $\overline{A}$. Then $\alpha_j = 0$ in $\overline{A}$ hence
$$\int_A D_i(f\alpha_j)\,dx = 0 = \int_{\partial A} f\nu_i\alpha_j\,d\mathcal{H}^{n-1}.$$
- B_j is interior to A. Then from Proposition 2.92 and $\alpha_j = 0$ in ∂A
$$\int_A D_i(f\alpha_j)\,dx = 0 = \int_{\partial A} f\nu_i\alpha_j\,d\mathcal{H}^{n-1}.$$
- $B_j \cap \partial A \neq 0$, then B_j is contained in some U_x of type (iii) and $f\alpha_j : U_x \to \mathbb{R}$ satisfies the assumptions of Proposition 2.93. If follows
$$\int_A D_i(f\alpha_j)\,dx = \int_{U_x} D_i(f\alpha_j)\,dx = \int_{\partial A\cap U_x} f\nu_i\alpha_j\,d\mathcal{H}^{n-1} = \int_{\partial A} f\nu_i\alpha_j\,d\mathcal{H}^{n-1}.$$

Summing on $j = 1, \dots,$ since $\sum_{j=1}^{\infty}\alpha_j = 1$ in Ω, $\{B_j\}$ is locally finite and
$$\mathcal{H}^{n-1}(\partial A\cap\Omega) = \mathcal{H}^{n-1}(r(A)) = \mathcal{H}^{n-1}(\partial A),$$
we conclude
$$\begin{aligned}\int_A D_i f\,dx &= \int_{A\cap\Omega} D_i f\,dx = \int_A \sum_{j=1}^{\infty}(D_i f)\alpha_j\,dx = \sum_{j=1}^{\infty}\int_A D_i(f\alpha_j)\,dx\\ &= \sum_{j=1}^{\infty}\int_{\partial A} f\nu_i\alpha_j\,d\mathcal{H}^{n-1} = \int_{\partial A} f\nu_i\sum_{j=1}^{\infty}\alpha_j\,d\mathcal{H}^{n-1} = \int_{\partial A\cap\Omega} f\nu_i\,d\mathcal{H}^{n-1}\\ &= \int_{\partial A} f\nu_i\,d\mathcal{H}^{n-1}.\end{aligned}$$

□

e. Integration by parts

As stated, Gauss–Green formulas may be thought of as the fundamental theorem of calculus for functions of several variables. Applying them to the product of two functions f and g, we deduce the *formulas of integration by parts.*

2.99 Proposition. *Let A be an admissible domain, $\nu : \partial A \to \mathbb{R}^n$ the field of exterior unit normal vectors to A, and let $f, g \in C^0(\overline{A}) \cap C^1(A)$ be such that $|\mathbf{D}f|$ and $|\mathbf{D}g|$ are summable in A. Then*

$$\int_A D_i f(x) g(x)\, dx = \int_{\partial A} f(y) g(y) \nu_i(y)\, d\mathcal{H}^{n-1}(y) - \int_A f(x) D_i g(x)\, dx \tag{2.46}$$

for $i = 1, 2, \dots, n$.

f. The divergence theorem

Let A be an admissible domain and $E : \overline{A} \to \mathbb{R}^n$, $E = (E^1, E^2, \dots, E^n)$ a field of class $C^0(\overline{A}) \cap C^1(A)$ with summable Jacobian matrix $\mathbf{D}E$. The *divergence* of E at $x \in A$ is the number

$$\operatorname{div} E(x) := \operatorname{tr} \mathbf{D}E(x) = \sum_{i=1}^n \frac{\partial E^i}{\partial x^i}(x) = \sum_{i=1}^n D_i E^i(x).$$

Since the functions $D_i E^i : A \to \mathbb{R}$, $i = 1, \dots, n$, are summable, if we apply the Gauss–Green formulas to them, we find in particular

$$\int_A D_i E^i\, dx = \int_{\partial A} E^i \nu_i\, d\mathcal{H}^{n-1} \qquad \forall i = 1, \dots, n$$

and, summing over i, the *divergence theorem*

$$\int_A \operatorname{div} E(x)\, dx = \int_{\partial A} E \bullet \nu\; d\mathcal{H}^{n-1}. \tag{2.47}$$

The quantity

$$\phi(E, A) := \int_{\partial A} E \bullet \nu\; d\mathcal{H}^{n-1}$$

is called the *flux* of E outgoing from A.

g. Geometrical meaning of the divergence

Let $E : A \to \mathbb{R}^n$ be a field that we assume of class $C^1(A)$. For every ball $B(x, r) \subset\subset A$ we denote by $\phi(E, r)$ the flux of E outgoing from $B(x, r)$,

$$\phi(E, r) := \int_{\partial B(x,r)} E \bullet \nu\; d\mathcal{H}^{n-1}, \qquad \nu(x) := x/|x|.$$

The divergence theorem yields

$$\phi(E,r) = \int_{B(x,r)} \operatorname{div} E(x)\, dx$$

hence, if we divide by $|B(x,r)| = \omega_n r^n$ and let $r \to 0$, we infer

$$\lim_{r\to 0} \frac{\phi(E,r)}{\omega_n r^n} = \lim_{r\to 0} \frac{1}{|B(x,r)|} \int_{B(x,r)} \operatorname{div} E(y)\, dy = \operatorname{div} E(x),$$

because of the continuity of $\operatorname{div} E(x)$, or

$$\phi(E,r) = \omega_n \operatorname{div} E(x) r^n + o(r^n) \qquad \text{as } r \to 0.$$

In other words, $\operatorname{div} E(x)$ represents the (rescaled) flow outgoing from an infinitesimal ball centered at x.

h. Divergence and transport of volume

Let $A \subset \mathbb{R}^n$ be open and $F : \mathbb{R} \times A \to \mathbb{R}^n$ be smooth. A curve $\gamma(t) : I \to A$ satisfying the differential equation $\gamma'(t) = F(t, \gamma(t))$, i.e., a curve $t \to (t, \gamma(t))$ with velocity $(1, F(t, \gamma(t)))$, is called a *flux line* or an *integral line* of F. As we shall see in Chapter 6, for every $x \in A$ there exists a unique flux line defined for small times that at time $t = 0$ is at x. If we denote by $\phi(x,t)$ these flux lines, i.e.,

$$\begin{cases} \dfrac{\partial}{\partial t}\phi(t,x) = F(t, \phi(t,x)), \\ \phi(0,x) = x, \end{cases}$$

and set $\phi_t(x) := \phi(t,x)$, then $D\phi_0(x) = \mathrm{Id}$, and, for $K \subset\subset \Omega$ there exists ϵ_0 such that $\phi(t,x)$ is defined on $]-\epsilon_0, \epsilon_0[\times K$ with $\det \mathbf{D}\phi_t(x) > 0$. From (1.28) with $A(t) := \mathbf{D}\phi_t(x)$, we then infer

$$\begin{aligned} \frac{\partial}{\partial t}[\det \mathbf{D}\phi_t(x)](t) &= \det \mathbf{D}\phi_t(x) \operatorname{tr} \Big(\mathbf{D}\phi_t(x)^{-1} \frac{\partial}{\partial t} \mathbf{D}\phi_t(x) \Big) \\ &= \det \mathbf{D}\phi_t(x) \operatorname{tr} \Big(\mathbf{D}\phi_t(x)^{-1} \mathbf{D} \frac{\partial}{\partial t} \phi(t,x) \Big) \\ &= \det \mathbf{D}\phi_t(x) \operatorname{tr} \Big(\mathbf{D}\phi_t(x)^{-1}\, \mathbf{D}F(t, \phi(t,x))\, \mathbf{D}\phi_t(x) \Big) \\ &= \det \mathbf{D}\phi_t(x) \operatorname{tr} \mathbf{D}F(t, \phi(t,x)) \\ &= \det \mathbf{D}\phi_t(x) \operatorname{div} F(t, \phi(t,x)). \end{aligned}$$

If $\Omega \subset\subset A$ and $\Omega_t := \phi_t(\Omega)$ is the image of Ω at time t transported by the flow, then the area formula says

$$\mathcal{L}^n(\Omega_t) = \int_\Omega |\det \mathbf{D}\phi_t(x)| dx = \int_\Omega \det \mathbf{D}\phi_t(x) dx$$

and, differentiating under the integral sign,

$$\frac{d\mathcal{L}^n(\Omega_t)}{dt}(t) = \int_\Omega \frac{\partial}{\partial t} \det \mathbf{D}\phi_t(x)dx$$
$$= \int_\Omega \det \mathbf{D}\phi_t(x) \operatorname{div} F(t, \phi(t,x))\, dx = \int_{\Omega_t} \operatorname{div} F(t,x)\, dx.$$

In the so-called *autonomous case*, $F = F(x)$, and for $t = 0$, we get

$$\frac{1}{\mathcal{L}^n(\Omega)} \frac{d\mathcal{L}^n(\Omega_t)}{dt}(0) = \frac{1}{\mathcal{L}^n(\Omega)} \int_\Omega \operatorname{div} F(x)\, dx,$$

i.e., $\operatorname{div} E(x)$ is the percentage variation of the infinitesimal volume when transported by the flow at time $t = 0$.

2.7 Exercises

2.100 ¶. Let $C(A)$ be the cone of basis $A = \{(x,y) \in \mathbb{R}^2 \,|\, x^2 < y < 1\}$, and vertex $(0,0,1)$. Compute the volume of $C \setminus B((0,0,1), 1/2)$.

2.101 ¶. Prove Schwarz's theorem, Theorem 1.34, for functions of class $C^2(\Omega)$ by using the theorem of differentiation under the integral sign. [*Hint:* Differentiate at (t_0, x_0) the identity

$$f(t, x_0 + h) - f(t, x_0) = \int_0^h \frac{\partial f}{\partial x}(t,s)\, dt$$

for $|t - t_0|, |h|$ small enough, and then use the fundamental theorem of calculus.]

2.102 ¶. Show that *Airy's function* $\phi(t) := \frac{1}{\sqrt{\pi}} \int_0^\infty \cos\left(tx + \frac{x^3}{3}\right) dx$ solves the equation

$$\varphi''(t) - t\varphi(t) = 0.$$

2.103 ¶. Show a sequence $\{f_n\}$ of nonnegative summable functions on $[0,1]$ such that

$$\lim_{n\to\infty} \int_0^1 f_n(x)\, dx = 0 \qquad \text{and} \qquad \limsup_{n\to\infty} f_n(x) = +\infty \quad \forall x \in [0,1].$$

2.104 ¶. Show that $f' : [0,1] \to \mathbb{R}$ is measurable if $f : [0,1] \to \mathbb{R}$ is differentiable.

2.105 ¶. Show that

$$\int_0^1 \frac{x^{1/3}}{1-x} \log\frac{1}{x}\, dx = \sum_{n=0}^\infty \frac{9}{(3n+4)^2},$$
$$\int_0^\infty e^{-x} \cos\sqrt{x}\, dx = \sum_{n=0}^\infty (-1)^n \frac{n!}{(2n)!},$$
$$\int_0^\pi \sum_{n=1}^\infty \frac{n^2 \sin nx}{a^n} = \frac{2a(1+a^2)}{(a^2-1)^2} \qquad \forall a > 1.$$

2.106 ¶. Show that for $p, q > 0$ we have

$$\int_0^1 \frac{x^{p-1}}{1+x^q} = \sum_{n=0}^{\infty} \frac{(-1)^n}{p+nq};$$

infer that $\frac{\pi}{4} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$.

2.107 ¶. Show that for $|a| < 1$,

$$\int_0^1 \frac{1-t}{1-at^3}\, dt = \sum_{n=0}^{\infty} \frac{a^n}{(3n+1)(3n+2)};$$

infer that

$$\frac{\pi}{3\sqrt{3}} = \sum_{n=0}^{\infty} \frac{1}{(3n+1)(3n+2)}.$$

2.108 ¶. Compute $\int_D \frac{e^{-xy}}{y}\, dxdy$ where $D := \{(x,y) \in \mathbb{R}^2 \,|\, x \geq 0,\ x^2 \leq y, 0 \leq y \leq 2\}$.

2.109 ¶. Show that

$$\int_0^{\infty} \frac{x^{\alpha-1}}{e^{at}-1}\, dt = \frac{\Gamma(\alpha)}{a^{\alpha}} \sum_{j=0}^{\infty} \frac{1}{(n+1)^{\alpha}},$$

and

$$\int_0^1 \frac{\arctan t}{\sqrt{1-t^2}}\, dt = \frac{\pi}{2} \log(1+\sqrt{2}).$$

2.110 ¶. Let $\mathbf{A}$ be a positive $n \times n$ symmetric matrix. Show that

$$\int_{\mathbb{R}^n} \exp\left(-\,\mathbf{A}x \bullet x\,\right) dx = \sqrt{\frac{\pi^n}{\det \mathbf{A}}}.$$

2.111 ¶. Let $f : [0,1] \to \mathbb{R}$ be a continuous function. Show that $\mathcal{L}^2(\mathcal{G}_{f,[0,1]}) = 0$.

2.112 ¶. Let $E \subset \mathbb{R}^n$. Show that E is measurable if $\mathcal{L}^{n*}(\partial E) = 0$.

2.113 ¶. Compute

$$\lim_{t\to+\infty} \int_{-1}^{4} \frac{t^2 + \sqrt{|x|}}{1+t^2x^2}\, dx, \qquad \lim_{t\to 0+} \int_0^1 \frac{x+\sqrt{tx}}{t+x}\, dx.$$

2.114 ¶. Show that for $\alpha > 0$

$$\lim_{n\to\infty} \int_0^n \left(1 - \frac{x}{n}\right)^n x^{\alpha-1}\, dx = \int_0^{\infty} e^{-x} x^{\alpha-1}\, dx.$$

2.115 ¶ Astroid. Compute the area and the length of the boundary of the astroid

$$A := \left\{(x,y) \in \mathbb{R}^2 \,\middle|\, x^{2/3} + y^{2/3} \leq 1\right\}.$$

2.116 ¶. If $S^2 := \{(x,y,z) \,|\, x^2+y^2+z^2 = 1\}$, compute

$$\int_{S^2} x^2\, d\mathcal{H}^2.$$

2.117 ¶. Let T be the triangle in $\mathbb{R}^3$ with vertices $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$. Compute

$$\int_T x\, d\mathcal{H}^2.$$

2.118 ¶. If $G \subset \mathbb{R}^3$ is the graph of the function $f : [0,1]\times[-1,1] \to \mathbb{R}$, $f(x,y) = x^2+y$, compute

$$\int_G x\, d\mathcal{H}^2.$$

2.119 ¶. For $a, L > 0$, let $C \subset \mathbb{R}^3$ be the truncated cone

$$C := \Big\{(x,y,z) \in \mathbb{R}^3 \,\Big|\, z^2 = a\,(x^2+y^2),\ 0 \le z \le L\Big\}.$$

Compute the volume of C and the area of the boundary of C.

2.120 ¶ The Viviani solid. Let

$$V := \Big\{(x,y,z) \,\Big|\, x^2+y^2+z^2 \le 1,\ x^2+y^2 \le x\Big\}$$

be the intersection of the unit ball in $\mathbb{R}^3$ with the vertical cylinder $\{(x,y,z) \in \mathbb{R}^3 \,|\, x^2 + y^2 - x \le 0\}$.

(i) Compute the colume of V.

(ii) Show that $S := \partial V = S_1 \cup S_2$ where

$$S_1 := \Big\{(x,y,z) \in \mathbb{R}^3 \,\Big|\, x^2+y^2+z^2 = 1,\ x^2+y^2 \le x\Big\},$$
$$S_2 := \Big\{(x,y,z) \in \mathbb{R}^3 \,\Big|\, x^2+y^2+z^2 \le 1,\ x^2+y^2 = x\Big\},$$

and compute the area of S_1 and S_2.

(iii) Show that the curve $s(\alpha) := (\cos^2\alpha, \cos\alpha\sin\alpha, \sin\alpha)$ maps the interval $]-\pi,\pi[$ onto $S_1 \cap S_2$, and compute the length of $S_1 \cap S_2$.

2.121 ¶. Compute

$$\int_{\mathbb{R}^{n-1}} \frac{1}{(1+|x|^2)^n}\, dx.$$

2.122 ¶. Compute $\mathcal{H}^{n-1}(\Sigma_{n-1})$ where

$$\Sigma_{n-1} := \Big\{x \in \mathbb{R}^n, \,\Big|\, \sum_{i=1}^n x_i = 1,\ 0 \le x_i \le 1\ \forall i\Big\}.$$

2.123 ¶ Feynman's formula. Let $a \in \mathbb{R}^n$ be a point with positive coordinates. Show that

$$\int_{S^{n-1}_+} \frac{1}{a \bullet x}\, d\mathcal{H}^{n-1}(x) = \frac{1}{(n-1)!\prod_{1\le j\le n} a_j}.$$

2.124 ¶. Let $f : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ be positively homogeneous of degree d, $f(tx) = t^d f(x)$ $\forall x \in \mathbb{R}^n \setminus \{0\}$ $\forall t > 0$. Prove that

$$\int_{B(0,1)} \Delta f(x)\, dx = d \int_{\partial B(0,1)} f(x)\, d\mathcal{H}^{n-1}(x).$$

In particular, if $x = (x_1, \dots, x_n)$, show that $\forall j = 1, \dots, n$

$$\mathcal{L}^n(B(0,1)) = \int_{S^{n-1}} x_j^2\, d\mathcal{H}^{n-1}.$$

2.125 ¶. If $B_R \subset \mathbb{R}^n$ denotes the ball of radius R around 0 in $\mathbb{R}^n$, show that for every $f \in C^1(\overline{B_R})$ we have

$$\int_{B_R} \Big(\sum_{i=1}^n x^i D_i f(x) + n f(x) \Big)\, dx = R \int_{\partial B_R} f(x)\, d\mathcal{H}^{n-1}(x).$$

2.126 ¶. If $f \in C^3(\overline{\Omega})$ and $\nabla f = 0$ on $\partial\Omega$, show that

$$\int_\Omega (\Delta f)^2\, dx = \int_\Omega \sum_{1 \le i,j \le n} (D_i D_j f)^2\, dx.$$

2.127 ¶. Compute the outgoing flux from the unit ball in $\mathbb{R}^3$ centered at 0 of the field $E := (2x, y^2, z^2)$.

2.128 ¶. Compute the outgoing flux from the lateral surface of the cylinder

$$C := \Big\{ (x, y, z) \in \mathbb{R}^3 \,\Big|\, x^2 + y^2 \le 1,\ -1 \le z \le 1 \Big\}$$

of the field $E = (xy^2, xy, y)$.

2.129 ¶. Let Ω be an open admissible set. Then

$$\mathcal{L}^n(\Omega) = \frac{1}{n} \int_{\partial\Omega} x \bullet \nu_\Omega \; d\mathcal{H}^{n-1}.$$

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