2

Powers of Integers

An integer n is a *perfect square* if $n = m^2$ for some integer m. Taking into account the prime factorization, if $m = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$, then $n = p_1^{2\alpha_1} \cdots p_k^{2\alpha_k}$. That is, n is a perfect square if and only if all exponents in its prime factorization are even.

An integer n is a *perfect power* if $n = m^s$ for some integers m and s, $s \geq 2$. Similarly, n is an sth perfect power if and only if all exponents in its prime factorization are divisible by s.

We say that the integer n is *square-free* if for any prime divisor p, p^2 does not divide n. Similarly, we can define the sth power-free integers.

These preliminary considerations seem trivial, but as you will see shortly, they have significant rich applications in solving various problems.

2.1 Perfect Squares

Problem 2.1.1. *Find all nonnegative integers n such that there are integers a and b with the property*

$$n^2 = a + b \text{ and } n^3 = a^2 + b^2.$$

(2004 Romanian Mathematical Olympiad)

Solution. From the inequality $2(a^2 + b^2) \geq (a + b)^2$ we get $2n^3 \geq n^4$, that is, $n \leq 2$. Thus:

for $n = 0$, we choose $a = b = 0$,
for $n = 1$, we take $a = 1$, $b = 0$, and
for $n = 2$, we may take $a = b = 2$.

Problem 2.1.2. *Find all integers n such that $n - 50$ and $n + 50$ are both perfect squares.*

T. Andreescu and D. Andrica, *Number Theory*, DOI: 10.1007/b11856_2,

Solution. Let $n - 50 = a^2$ and $n + 50 = b^2$. Then $b^2 - a^2 = 100$, so $(b - a)(b + a) = 2^2 \cdot 5^2$. Because $b - a$ and $b + a$ are of the same parity, we have the following possibilities: $b - a = 2$, $b + a = 50$, yielding $b = 26$, $a = 24$, and $b - a = 10$, $b + a = 10$ with $a = 0$, $b = 10$. Hence the integers with this property are $n = 626$ and $n = 50$.

Problem 2.1.3. *Let $n \geq 3$ be a positive integer. Show that it is possible to eliminate at most two numbers among the elements of the set $\{1, 2, \ldots, n\}$ such that the sum of the remaining numbers is a perfect square.*

(2003 Romanian Mathematical Olympiad)

Solution. Let $m = \lfloor \sqrt{n(n+1)/2} \rfloor$. From $m^2 \leq n(n+1)/2 < (m+1)^2$ we obtain
$$\frac{n(n+1)}{2} - m^2 < (m+1)^2 - m^2 = 2m + 1.$$

Therefore, we have

$$\frac{n(n+1)}{2} - m^2 \leq 2m \leq \sqrt{2n^2 + 2n} \leq 2n - 1.$$

Since any number k, $k \leq 2n - 1$, can be obtained by adding at most two numbers from $\{1, 2, \ldots, n\}$, we obtain the result.

Problem 2.1.4. *Let k be a positive integer and $a = 3k^2 + 3k + 1$.*

(i) Show that $2a$ and a^2 are sums of three perfect squares.

(ii) Show that if a is a divisor of a positive integer b, and b is a sum of three perfect squares, then any power b^n is a sum of three perfect squares.

(2003 Romanian Mathematical Olympiad)

Solution. (i) $2a = 6k^2 + 6k + 2 = (2k+1)^2 + (k+1)^2 + k^2$ and $a^2 = 9k^4 + 18k^3 + 15k^2 + 6k + 1 = (k^2 + k)^2 + (2k^2 + 3k + 1)^2 + k^2(2k+1)^2 = a_1^2 + a_2^2 + a_3^2$. (ii) Let $b = ca$. Then $b = b_1^2 + b_2^2 + b_3^2$ and $b^2 = c^2a^2 = c^2(a_1^2 + a_2^2 + a_3^2)$. To end the proof, we proceed as follows: for $n = 2p + 1$ we have $b^{2p+1} = (b^p)^2(b_1^2 + b_2^2 + b_3^2)$, and for $n = 2p + 2$, $b^n = (b^p)^2 b^2 = (b^p)^2 c^2 (a_1^2 + a_2^2 + a_3^2)$.

Problem 2.1.5. *(a) Let k be an integer number. Prove that the number*

$$(2k+1)^3 - (2k-1)^3$$

is the sum of three squares. (b) Let n be a positive number. Prove that the number $(2n+1)^3 - 2$ can be represented as the sum of $3n - 1$ squares greater than 1.

(2000 Romanian Mathematical Olympiad)

Solution. (a) It is easy to check that

$$(2k+1)^3-(2k-1)^3=(4k)^2+(2k+1)^2+(2k-1)^2.$$

(b) Observe that

$$(2n+1)^3-1=(2n+1)^3-(2n-1)^3+(2n-1)^3-(2n-3)^3+\cdots+3^3-1^3.$$

Each of the n differences in the right-hand side can be written as a sum of three squares greater than 1, except for the last one:

$$3^3-1^3=4^2+3^2+1^2.$$

It follows that

$$(2n+1)^3-2=3^2+4^2+\sum_{k=2}^{n}\left[(4k)^2+(2k+1)^2+(2k-1)^2\right]$$

as desired.

Problem 2.1.6. *Prove that for any positive integer n the number*

$$\frac{\left(17+12\sqrt{2}\right)^n-\left(17-12\sqrt{2}\right)^n}{4\sqrt{2}}$$

is an integer but not a perfect square.

Solution. Note that $17+12\sqrt{2}=\left(\sqrt{2}+1\right)^4$ and $17-12\sqrt{2}=\left(\sqrt{2}-1\right)^4$, so

$$\begin{aligned}\frac{\left(17+12\sqrt{2}\right)^n-\left(17-12\sqrt{2}\right)^n}{4\sqrt{2}}&=\frac{\left(\sqrt{2}+1\right)^{4n}-\left(\sqrt{2}-1\right)^{4n}}{4\sqrt{2}}\\&=\frac{\left(\sqrt{2}+1\right)^{2n}+\left(\sqrt{2}-1\right)^{2n}}{2}\cdot\frac{\left(\sqrt{2}+1\right)^{2n}-\left(\sqrt{2}-1\right)^{2n}}{2\sqrt{2}}.\end{aligned}$$

Define

$$A=\frac{\left(\sqrt{2}+1\right)^{2n}+\left(\sqrt{2}-1\right)^{2n}}{2}\quad\text{and}\quad B=\frac{\left(\sqrt{2}+1\right)^{2n}-\left(\sqrt{2}-1\right)^{2n}}{2\sqrt{2}}.$$

Using the binomial expansion formula we obtain positive integers x and y such that

$$\left(\sqrt{2}+1\right)^{2n}=x+y\sqrt{2},\quad \left(\sqrt{2}-1\right)^{2n}=x-y\sqrt{2}.$$

Then

$$x=\frac{\left(\sqrt{2}+1\right)^{2n}+\left(\sqrt{2}-1\right)^{2n}}{2}=A$$

and

$$y = \frac{(\sqrt{2}+1)^{2n} - (\sqrt{2}-1)^{2n}}{2\sqrt{2}} = B,$$

and so AB is as integer, as claimed. Observe that

$$A^2 - 2B^2 = (A + \sqrt{2}B)(A - \sqrt{2}B) = (\sqrt{2}+1)^{2n}(\sqrt{2}-1)^{2n} = 1,$$

so A and B are relatively prime. It is sufficient to prove that at least one of them is not a perfect square. We have

$$A = \frac{(\sqrt{2}+1)^{2n} + (\sqrt{2}-1)^{2n}}{2} = \left[\frac{(\sqrt{2}+1)^{n} + (\sqrt{2}-1)^{n}}{\sqrt{2}}\right]^2 - 1 \qquad (1)$$

and

$$A = \frac{(\sqrt{2}+1)^{2n} + (\sqrt{2}-1)^{2n}}{2} = \left[\frac{(\sqrt{2}+1)^{n} - (\sqrt{2}-1)^{n}}{\sqrt{2}}\right]^2 + 1. \qquad (2)$$

Since one of the numbers

$$\frac{(\sqrt{2}+1)^{n} + (\sqrt{2}-1)^{n}}{\sqrt{2}}, \quad \frac{(\sqrt{2}+1)^{n} - (\sqrt{2}-1)^{n}}{\sqrt{2}}$$

is an integer, depending on the parity of n, from the relations (1) and (2) we derive that A is not a square. This completes the proof.

Problem 2.1.7. *The integers a and b have the property that for every nonnegative integer n, the number $2^n a + b$ is a perfect square. Show that $a = 0$.*

(2001 Polish Mathematical Olympiad)

Solution. If $a \neq 0$ and $b = 0$, then at least one of $2^1 a + b$ and $2^2 a + b$ is not a perfect square, a contradiction. If $a \neq 0$ and $b \neq 0$, then each $(x_n, y_n) = (2\sqrt{2^n a + b}, \sqrt{2^{n+2} a + b})$ satisfies

$$(x_n + y_n)(x_n - y_n) = 3b.$$

Hence, $x + n + y_n$ divides $3b$ for each n. But this is impossible because $3b \neq 0$ but $|x_n + y_n| > |3b|$ for large enough n. Therefore, $a = 0$.

Remark. We invite the courageous reader to prove that if $f \in \mathbb{Z}[X]$ is a polynomial and $f(2^n)$ is a perfect square for all n, then there is $g \in \mathbb{Z}[X]$ such that $f = g^2$.

Problem 2.1.8. *Prove that the number*

$$\underbrace{11\ldots11}_{1997}\underbrace{22\ldots22}_{1998}5$$

is a perfect square.

First solution.

$$\begin{aligned}
N &= \underbrace{11\ldots11}_{1997}\cdot 10^{1999} + \underbrace{22\ldots22}_{1998}\cdot 10 + 5 \\
&= \frac{1}{9}(10^{1997}-1)\cdot 10^{1999} + \frac{2}{9}(10^{1998}-1)\cdot 10 + 5 \\
&= \frac{1}{9}(10^{3996} + 2\cdot 5\cdot 10^{1998} + 25) = \left[\tfrac{1}{3}(10^{1998}+5)\right]^2 \\
&= \left(\frac{1\overbrace{00\ldots00}^{1997}5}{3}\right)^2 = \underbrace{33\ldots33}_{1997}5^2.
\end{aligned}$$

Second solution. Note that

$$9N = 1\underbrace{00\ldots00}_{1996}1\underbrace{00\ldots00}_{1997}25 = 10^{3996} + 10^{1999} + 25 = (10^{1998}+5)^2;$$

hence N is a square.

Problem 2.1.9. *Find all positive integers n, $n \geq 1$, such that $n^2 + 3^n$ is a perfect square.*

Solution. Let m be a positive integer such that

$$m^2 = n^2 + 3^n.$$

Since $(m-n)(m+n) = 3^n$, there is $k \geq 0$ such that $m-n = 3^k$ and $m+n = 3^{n-k}$. From $m-n < m+n$ follows $k < n-k$, and so $n-2k \geq 1$. If $n-2k = 1$, then $2n = (m+n)-(m-n) = 3^{n-k} - 3^k = 3^k(3^{n-2k}-1) = 3^k(3^1-1) = 2\cdot 3^k$, so $n = 3^k = 2k+1$. We have $3^m = (1+2)^m = 1 + 2m + 2^2\binom{m}{2} + \cdots > 2m+1$. Therefore $k = 0$ or $k = 1$, and consequently $n = 1$ or $n = 3$. If $n - 2k > 1$, then $n - 2k \geq 2$ and $k \leq n-k-2$. It follows that $3^k \leq 3^{n-k-2}$, and consequently

$$\begin{aligned}
2n &= 3^{n-k} - 3^k \geq 3^{n-k} - 3^{n-k-2} = 3^{n-k-2}(3^2-1) = 8\cdot 3^{n-k-2} \\
&\geq 8[1 + 2(n-k-2)] = 16n - 16k - 24,
\end{aligned}$$

which implies $8k+12 \geq 7n$. On the other hand, $n \geq 2k+2$; hence $7n \geq 14k+14$, contradiction. In conclusion, the only possible values for n are 1 and 3.

Problem 2.1.10. *Find the number of five-digit perfect squares having the last two digits equal.*

Solution. Suppose $n = \overline{abcdd}$ is a perfect square. Then $n = 100\overline{abc} + 11d = 4m + 3d$ for some m. Since all squares have the form $4m$ or $4m + 1$ and $d \in \{0, 1, 4, 5, 6, 9\}$ as the last digit of a square, it follows that $d = 0$ or $d = 4$. If $d = 0$, then $n = 100\overline{abc}$ is a square if $\overline{abc}$ is a square. Hence $\overline{abc} \in \{10^2, 11^2, \dots, 31^2\}$, so there are 22 numbers. If $d = 4$, then $100\overline{abc}+44 = n = k^2$ implies $k = 2p$ and $\overline{abc} = \frac{p^2-11}{25}$. (1) If $p = 5x$, then $\overline{abc}$ is not an integer, false. (2) If $p = 5x + 1$, then $\overline{abc} = \frac{25x^2+10x-1}{25} = x^2 + \frac{2(x-1)}{5} \Rightarrow x \in \{11, 16, 21, 26, 31\}$, so there are 5 solutions. (3) If $p = 5x+2$, then $\overline{abc} = x^2 + \frac{20x-7}{25} \notin \mathbb{N}$, false. (4) If $p = 5x+3$, then $\overline{abc} = x^2 + \frac{30x-2}{25} \notin \mathbb{N}$, false. (5) If $p = 5x+4$ then $\overline{abc} = x^2 + \frac{8x+1}{5}$; hence $x = 5m + 3$ for some $m \Rightarrow x \in \{13, 18, 23, 28\}$, so there are four solutions. Finally, there are $22 + 5 + 4 = 31$ squares.

Problem 2.1.11. *The last four digits of a perfect square are equal. Prove they are all zero.*

(2002 Romanian Team Selection Test for JBMO)

Solution. Denote by k^2 the perfect square and by a the digit that appears in the last four positions. It easily follows that a is one of the numbers 0, 1, 4, 5, 6, 9. Thus $k^2 \equiv a \cdot 1111 \pmod{16}$. (1) If $a = 0$, we are done. (2) Suppose that $a \in \{1, 5, 9\}$. Since $k^2 \equiv 0 \pmod 8$, $k^2 \equiv 1 \pmod 8$ or $k^2 \equiv 4 \pmod 8$ and $1111 \equiv 7 \pmod 8$, we obtain $1111 \equiv 7 \pmod 8$, $5 \cdot 1111 \equiv 3 \pmod 8$, and $9 \cdot 1111 \equiv 7 \pmod 8$. Thus the congruence $k^2 \equiv a \cdot 1111 \pmod{16}$ cannot hold. (3) Suppose $a \in \{4, 6\}$. Since $1111 \equiv 7 \pmod{16}$, $4 \cdot 1111 \equiv 12 \pmod{16}$, and $6 \cdot 1111 \equiv 10 \pmod{16}$, we conclude that in this case the congruence $k^2 \equiv a \cdot 1111 \pmod{16}$ cannot hold. Thus $a = 0$.

Remark. $38^2 = 1444$ ends in three equal digits, so the problem is sharp.

Problem 2.1.12. *Let* $1 < n_1 < n_2 < \cdots < n_k < \cdots$ *be a sequence of integers such that no two are consecutive. Prove that for all positive integers* m *between* $n_1 + n_2 + \cdots + n_m$ *and* $n_2 + n_2 + \cdots + n_{m+1}$ *there is a perfect square.*

Solution. It is easy to prove that between numbers $a > b \geq 0$ such that $\sqrt{a} - \sqrt{b} > 1$ there is a perfect square: take for example $([\sqrt{b}] + 1)^2$. It suffices to prove that

$$\sqrt{n_1 + \cdots + n_{m+1}} - \sqrt{n_1 + \cdots + n_m} > 1, \quad m \geq 1.$$

This is equivalent to

$$n_1 + \cdots + n_m + n_{m+1} > (1 + \sqrt{n_1 + n_2 + \cdots + n_m})^2,$$

and then

$$n_{m+1} > 1 + 2\sqrt{n_1 + n_2 + \cdots + n_m}, \quad m \geq 1.$$

We induct on m. For $m = 1$ we have to prove that $n_2 > 1 + 2\sqrt{n_1}$. Indeed, $n_2 > n_1 + 2 = 1 + (1 + n_1) > 1 + 2\sqrt{n_1}$. Assume that the claim holds for some $m \geq 1$. Then

$$n_{m+1} - 1 > 2\sqrt{n_1 + \cdots + n_m}$$

so $(n_{m+1} - 1)^2 > 4(n_1 + \cdots + n_m)$ hence

$$(n_{m+1} + 1)^2 > 4(n_1 + \cdots + n_{m+1}).$$

This implies

$$n_{m+1} + 1 > 2\sqrt{n_1 + \cdots + n_{m+1}},$$

and since $n_{m+2} - n_{m+1} \geq 2$, it follows that

$$n_{m+2} > 1 + 2\sqrt{n_1 + \cdots + n_{m+1}},$$

as desired.

Problem 2.1.13. *Find all integers x, y, z such that $4^x + 4^y + 4^z$ is a square.*

Solution. It is clear that there are no solutions with $x < 0$. Without loss of generality assume that $x \leq y \leq z$ and let $4^x + 4^y + 4^z = u^2$. Then $2^{2x}(1 + 4^{y-x} + 4^{z-x}) = u^2$. We have two situations.

Case 1. $1 + 4^{y-x} + 4^{z-x}$ is odd, i.e., $1 + 4^{y-x} + 4^{z-x} = (2a + 1)^2$. It follows that

$$4^{y-x-1} + 4^{z-x-1} = a(a + 1),$$

and then

$$4^{y-x-1}(1 + 4^{z-y}) = a(a + 1).$$

We consider two cases. (1) The number a is even. Then $a + 1$ is odd, so $4^{y-x-1} = a$ and $1 + 4^{z-y} = a + 1$. It follows that $4^{y-x-1} = 4^{z-y}$; hence $y - x - 1 = z - y$. Thus $z = 2y - x - 1$ and

$$4^x + 4^y + 4^z = 4^x + 4^y + 4^{2y-x-1} = (2^x + 2^{2y-x-1})^2.$$

(2) The number a is odd. Then $a + 1$ is even, so $a = 4^{z-y} + 1$, $a + 1 = 4^{y-x-1}$ and $4^{y-x-1} - 4^{z-y} = 2$. It follows that $2^{2y-2x-3} = 2^{2x-2y-1} + 1$, which is impossible, since $2x - 2y - 1 \neq 0$.

Case 2. $1 + 4^{y-x} + 4^{z-x}$ is even; thus $y = x$ or $z = x$. Anyway, we must have $y = x$, and then $2 + 4^{z-x}$ is a square, which is impossible, since it is congruent to $2 \pmod 4$ or congruent to $3 \pmod 4$.

Additional Problems

Problem 2.1.14. Let x, y, z be positive integers such that

$$\frac{1}{x} - \frac{1}{y} = \frac{1}{z}.$$

Let h be the greatest common divisor of x, y, z. Prove that $hxyz$ and $h(y-x)$ are perfect squares.

(1998 United Kingdom Mathematical Olympiad)

Problem 2.1.15. Let b an integer greater than 5. For each positive integer n, consider the number

$$x_n = \underbrace{11\ldots1}_{n-1}\underbrace{22\ldots2}_{n}5,$$

written in base b. Prove that the following condition holds if and only if $b = 10$: There exists a positive integer M such that for every integer n greater than M, the number x_n is a perfect square.

(44th International Mathematical Olympiad Shortlist)

Problem 2.1.16. Do there exist three natural numbers greater than 1 such that the square of each, minus one, is divisible by each of the others?

(1996 Russian Mathematical Olympiad)

Problem 2.1.17. (a) Find the first positive integer whose square ends in three 4's. (b) Find all positive integers whose squares end in three 4's. (c) Show that no perfect square ends with four 4's.

(1995 United Kingdom Mathematical Olympiad)

Problem 2.1.18. Let $\overline{abc}$ be a prime. Prove that $b^2 - 4ac$ cannot be a perfect square.

(Mathematical Reflections)

Problem 2.1.19. For each positive integer n, denote by $s(n)$ the greatest integer such that for all positive integer $k \le s(n)$, n^2 can be expressed as a sum of squares of k positive integers. (a) Prove that $s(n) \le n^2 - 14$ for all $n \ge 4$. (b) Find a number n such that $s(n) = n^2 - 14$. (c) Prove that there exist infinitely many positive integers n such that

$$s(n) = n^2 - 14.$$

(33rd International Mathematical Olympiad)

Problem 2.1.20. Let A be the set of positive integers representable in the form $a^2 + 2b^2$ for integers a, b with $b \neq 0$. Show that if $p^2 \in A$ for a prime p, then $p \in A$.

(1997 Romanian International Mathematical Olympiad Team Selection Test)

Problem 2.1.21. Is it possible to find 100 positive integers not exceeding 25000 such that all pairwise sums of them are different?

(42nd International Mathematical Olympiad Shortlist)

Problem 2.1.22. Do there exist 10 distinct integers, the sum of any 9 of which is a perfect square?

(1999 Russian Mathematical Olympiad)

Problem 2.1.23. Let n be a positive integer such that n is a divisor of the sum

$$1 + \sum_{i=1}^{n-1} i^{n-1}.$$

Prove that n is square-free.

(1995 Indian Mathematical Olympiad)

Problem 2.1.24. Let n, p be integers such that $n > 1$ and p is a prime. If $n \mid (p-1)$ and $p \mid (n^3 - 1)$, show that $4p - 3$ is a perfect square.

(2002 Czech–Polish–Slovak Mathematical Competition)

Problem 2.1.25. Show that for any positive integer $n > 10000$, there exists a positive integer m that is a sum of two squares and such that $0 < m - n < 3\sqrt[4]{n}$.

(Russian Mathematical Olympiad)

Problem 2.1.26. Show that a positive integer m is a perfect square if and only if for each positive integer n, at least one of the differences

$$(m+1)^2 - m, \quad (m+2)^2 - m, \quad \ldots, \quad (m+n)^2 - m$$

is divisible by n.

(2002 Czech and Slovak Mathematical Olympiad)

2.2 Perfect Cubes

Problem 2.2.1. *Prove that if n is a perfect cube, then $n^2 + 3n + 3$ cannot be a perfect cube.*

Solution. If $n = 0$, then we get 3 and the property is true. Suppose by way of contradiction that $n^2 + 3n + 3$ is a cube for some $n \neq 0$. Hence $n(n^2 + 3n + 3)$ is a cube. Note that

$$n(n^2 + 3n + 3) = n^3 + 3n^2 + 3n = (n + 1)^3 - 1,$$

and since $(n + 1)^3 - 1$ is not a cube when $n \neq 0$, we obtain a contradiction.

Problem 2.2.2. *Let m be a given positive integer. Find a positive integer n such that $m + n + 1$ is a perfect square and $mn + 1$ is a perfect cube.*

Solution. Choosing $n = m^2 + 3m + 3$, we have

$$m + n + 1 = m^2 + 4m + 4 = (m + 2)^2$$

and

$$mn + 1 = m^3 + 3m^2 + 3m + 1 = (m + 1)^3.$$

Problem 2.2.3. *Which are there more of among the natural numbers from* 1 *to* 1000000*, inclusive: numbers that can be represented as the sum of a perfect square and a (positive) perfect cube, or numbers that cannot be?*

(1996 Russian Mathematical Olympiad)

Solution. There are more numbers not of this form. Let $n = k^2 + m^3$, where $k, m, n \in \mathbb{N}$ and $n \leq 1000000$. Clearly $k \leq 1000$ and $m \leq 100$. Therefore there cannot be more numbers in the desired form than the 100000 pairs (k, m).

Problem 2.2.4. *Show that no integer of the form $\overline{xyxy}$ in base* 10 *can be the cube of an integer. Also find the smallest base $b > 1$ in which there is a perfect cube of the form $xyxy$.*

(1998 Irish Mathematical Olympiad)

Solution. If the 4-digit number $\overline{xyxy} = 101 \times \overline{xy}$ is a cube, then $101 \mid \overline{xy}$, which is a contradiction. Convert $\overline{xyxy} = 101 \times \overline{xy}$ from base b to base 10. We obtain $\overline{xyxy} = (b^2 + 1) \times (bx + y)$ with $x, y < b$ and $b^2 + 1 > bx + y$. Thus for $\overline{xyxy}$ to be a cube, $b^2 + 1$ must be divisible by a perfect square. We can check easily that $b = 7$ is the smallest such number, with $b^2 + 1 = 50$. The smallest cube divisible by 50 is 1000, which is $\overline{2626}$ is base 7.

Additional Problems

Problem 2.2.5. Find all the positive perfect cubes that are not divisible by 10 such that the number obtained by erasing the last three digits is also a perfect cube.

Problem 2.2.6. Find all positive integers n less than 1999 such that n^2 is equal to the cube of the sum of n's digits.

(1999 Iberoamerican Mathematical Olympiad)

Problem 2.2.7. Prove that for any nonnegative integer n the number

$$A = 2^n + 3^n + 5^n + 6^n$$

is not a perfect cube.

Problem 2.2.8. Prove that every integer is a sum of five cubes.

Problem 2.2.9. Show that every rational number can be written as a sum of three cubes.

2.3 *k*th Powers of Integers, *k* at least 4

Problem 2.3.1. *Given* 81 *natural numbers whose prime divisors belong to the set* $\{2, 3, 5\}$*, prove that there exist four numbers whose product is the fourth power of an integer.*

(1996 Greek Mathematical Olympiad)

Solution. It suffices to take 25 such numbers. To each number, associate the triple (x_2, x_3, x_5) recording the parity of the exponents of 2, 3, and 5 in its prime factorization. Two numbers have the same triple if and only if their product is a perfect square. As long as there are 9 numbers left, we can select two whose product is a square; in so doing, we obtain 9 such pairs. Repeating the process with the square roots of the products of the pairs, we obtain four numbers whose product is a fourth power.

Problem 2.3.2. *Find all collections of* 100 *positive integers such that the sum of the fourth powers of every four of the integers is divisible by the product of the four numbers.*

(1997 St. Petersburg City Mathematical Olympiad)

Solution. Such sets must be $n, n, \ldots, n$ or $3n, n, n, \ldots, n$ for some integer n. Without loss of generality, we assume that the numbers do not have a common factor. If u, v, w, x, y are five of the numbers, then uvw divides $u^4+v^4+w^4+x^4$ and $u^4 + v^4 + w^4 + y^4$, and so divides $x^4 - y^4$. Likewise, $v^4 \equiv w^4 \equiv x^4 \pmod{u}$, and from above, $3v^4 \equiv 0 \pmod{u}$. If u has a prime divisor not equal to 3, we conclude that every other integer is divisible by the same prime, contrary to assumption. Likewise, if u is divisible by 9, then every other integer is divisible by 3. Thus all of the numbers equal 1 or 3. Moreover, if one number is 3, the others are all congruent modulo 3, so are all 3 (contrary to assumption) or 1. This completes the proof.

Problem 2.3.3. *Let M be a set of* 1985 *distinct positive integers, none of which has a prime divisor greater than* 26. *Prove that M contains at least one subset of four distinct elements whose product is the fourth power of an integer.*

(26th International Mathematical Olympiad)

Solution. There are nine prime numbers less than 26: $p_1 = 2$, $p_2 = 3, \ldots, p_9 = 23$. Any element x of M has a representation $x = \prod_{i=1}^{9} p_i^{a_i}$, $a_i \geq 0$. If $x, y \in M$ and $y = \prod_{i=1}^{9} p_i^{b_i}$, the product $xy = \prod_{i=1}^{9} p_i^{a_i+b_i}$ is a perfect square if and only if $a_i + b_i \equiv 0 \pmod 2$. Equivalently, $a_i \equiv b_i \pmod 2$ for all $i = 1, 2, \ldots, 9$. Because there are $2^9 = 512$ elements in $(\mathbb{Z}/2\mathbb{Z})^9$, any subset of M having at least 513 elements contains two elements x, y such that xy is a perfect square. Starting from M and eliminating such pairs, one obtains $\frac{1}{2}(1985 - 513) = 736 > 513$ distinct two-element subsets of M having a square as the product of elements. Reasoning as above, we find among these squares at least one pair (in fact many pairs) whose product is a fourth power.

Problem 2.3.4. *Let A be a subset of* $\{0, 1, \ldots, 1997\}$ *containing more than 1000 elements. Prove that A contains either a power of* 2, *or two distinct integers whose sum is a power of* 2.

(1997 Irish Mathematical Olympiad)

Solution. Suppose A did not satisfy the conclusion. Then A would contain at most half of the integers from 51 to 1997, since they can be divided into pairs whose sum is 2048 (with 1024 left over); likewise, A contains at most half of the integers from 14 to 50, at most half of the integers from 3 to 13, and possibly 0, for a total of

$$973 + 18 + 5 + 1 = 997$$

integers.

Problem 2.3.5. *Show that in the arithmetic progression with first term* 1 *and difference* 729, *there are infinitely many powers of* 10.

(1996 Russian Mathematical Olympiad)

First solution. We will show that for all natural numbers n, $10^{81n} - 1$ is divisible by 729. In fact,

$$10^{81n} - 1 = (10^{81})^n - 1^n = (10^{81} - 1) \cdot A,$$

and

$$\begin{aligned}10^{81}-1 &= \underbrace{9\ldots9}_{81}\\ &= (\underbrace{9\ldots9}_{9})\cdots(\underbrace{10\ldots01}_{8}\underbrace{10\ldots01}_{8}\ldots\underbrace{10\ldots01}_{8})\\ &= 9(\underbrace{1\ldots1}_{9})\cdots(\underbrace{10\ldots01}_{8}\underbrace{10\ldots01}_{8}\ldots\underbrace{10\ldots01}_{8}).\end{aligned}$$

The second and third factors have nine digits equal to 1 and the root of digits (if any) 0, so the sum of the digits is 9, and each is a multiple of 9. Hence $10^{81}-1$ is divisible by $9^3=729$, as is $10^{81n}-1$ for any n.

Second solution. In order to prove that $10^{81}-1$ is divisible by 9^3, just write

$$\begin{aligned}10^{81}-1=(9+1)^{81}-1 &= k\cdot 9^3+\binom{81}{2}9^2+\binom{81}{1}\cdot 9\\ &= k\cdot 9^3+81\cdot 40\cdot 9^2+81\cdot 9=(k+361)\cdot 9^3.\end{aligned}$$

Remark. An alternative solution uses Euler's theorem (see Section 7.2). We have $10^{\varphi(729)}\equiv 1 \pmod{729}$; thus $10^{n\varphi(729)}$ is in this progression for any positive integer n.

Additional Problems

Problem 2.3.6. Let p be a prime number and a, n positive integers. Prove that if

$$2^p+3^p=a^n,$$

then $n=1$.

(1996 Irish Mathematical Olympiad)

Problem 2.3.7. Let x, y, p, n, k be natural numbers such that

$$x^n+y^n=p^k.$$

Prove that if $n>1$ is odd and p is an odd prime, then n is a power of p.

(1996 Russian Mathematical Olympiad)

Problem 2.3.8. Prove that a product of three consecutive integers cannot be a power of an integer.

Problem 2.3.9. Show that there exists an infinite set A of positive integers such that for any finite nonempty subset $B\subset A$, $\sum_{x\in B} x$ is not a perfect power.

(Kvant)

Problem 2.3.10. Prove that there is no infinite arithmetic progression consisting only of powers ≥ 2.

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