Chapter 2

Fourier-Mukai functors

Introduction

According to a fundamental theorem due to D. Orlov, any equivalence between derived categories of coherent sheaves of two smooth projective varieties is an integral functor. This "representability" result — which lies at the heart of the current chapter — opens the way to the investigation of the geometric consequences of the equivalence between the derived categories of two varieties.

Section 2.1 is quite technical; it presents and develops the notions of *spanning class, ample sequence* and *convolution* (of a complex of objects in the derived category). These will be mainly used in the next section and may at first be given little attention, concentrating on definitions and main results, leaving proofs and details to a second reading. Section 2.2 pivots around Orlov's theorem. An important ingredient of the proof that we provide for this result is a construction of the resolution of the diagonal of a projective variety which generalizes Beĭlinson's resolution of the diagonal of projective space. This generalization is originally due to Kapustin, Kuznetsov and Orlov and has been further formalized by Kawamata. In Section 2.3 we specialize our attention to those integral functors which establish an equivalence of categories. We call such functors *Fourier-Mukai functors*, reserving the term *Fourier-Mukai transforms* to the cases where the associated kernel is a concentrated complex (i.e., it is a sheaf). One of the main objectives of the section is to state and prove (basically following [61]) a criterion for testing whether a fully faithful integral functor is a Fourier-Mukai functor. The existence of an equivalence between the derived categories of two varieties of course severely constrains their geometry, and indeed one proves that whenever two smooth projective varieties have equivalent derived categories, and one of them has an ample canonical bundle, then they are isomorphic. The section also provides other geo-

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metric applications, some of them concerning moduli spaces of sheaves.

2.1 Spanning classes and equivalences

We introduce the notion (due to Bridgeland [61] but already implicit in [48]) of spanning class for a triangulated category.

Definition 2.1. Let $\mathfrak{A}$ be a triangulated category. A subclass $\Sigma \subset \mathrm{Ob}(\mathfrak{A})$ is a *spanning class* if the following two properties are satisfied:

1. if $\mathrm{Hom}^i_{\mathfrak{A}}(\sigma, a) = 0$ for all $\sigma \in \Sigma$ and all $i \in \mathbb{Z}$, then $a = 0$;
2. if $\mathrm{Hom}^i_{\mathfrak{A}}(b, \sigma) = 0$ for all $\sigma \in \Sigma$ and all $i \in \mathbb{Z}$, then $b = 0$.

△

Example 2.2. The skyscraper sheaves $\mathcal{O}_x$ form a spanning class for the derived category $D^b(X)$ of a smooth projective variety, as we shall see in Proposition 2.52. Moreover, if $\mathcal{L}$ is an ample sheaf, then $\{\mathcal{L}^i\}_{i\in\mathbb{Z}}$ is also a spanning class for $D^b(X)$ by virtue of Proposition 2.9. In particular $\{\mathcal{O}_{\mathbb{P}^n}(i)\}_{i\in\mathbb{Z}}$ is a spanning class for the bounded derived category of the projective n-space. (Actually, for any $m \in \mathbb{Z}$ the collection $\{\mathcal{L}^i\}_{i<m}$ is a spanning class as well.) △

Definition 2.3. A triangulated category $\mathfrak{A}$ is *decomposable* if there exist two triangulated nontrivial full subcategories $\mathfrak{A}_1$, $\mathfrak{A}_2$ such that

1. For every object a in $\mathfrak{A}$ there exist objects a_i in $\mathfrak{A}_i$ and a triangle
$$a_1 \to a \to a_2 \xrightarrow{0} a_1[1]\,;$$
2. For every pair of objects a_1, a_2 in $\mathfrak{A}_1, \mathfrak{A}_2$ one has
$$\mathrm{Hom}^i_{\mathfrak{A}}(a_1, a_2) = \mathrm{Hom}^i_{\mathfrak{A}}(a_2, a_1) = 0\,, \quad \text{for any } i \in \mathbb{Z}.$$

We then write $\mathfrak{A} \simeq \mathfrak{A}_1 \oplus \mathfrak{A}_2$. A triangulated category $\mathfrak{A}$ is *indecomposable* if it is not decomposable. △

Lemma 2.4. *Let $F\colon \mathfrak{A} \to \mathfrak{B}$ be an exact fully faithful functor with a right adjoint H and a left adjoint G. Assume that $\mathfrak{B}$ is indecomposable and that $\mathfrak{A}$ is nontrivial. Then F is an equivalence if and only if the condition $H(c) = 0$ for an object c in $\mathfrak{B}$ implies $G(c) = 0$.*

Proof. If F is an equivalence, then $G \simeq H$, and there is nothing to prove. For the converse, define full subcategories $\mathfrak{B}_1$, $\mathfrak{B}_2$ consisting of the objects b in $\mathfrak{B}$ such that $FH(b) \simeq b$ and $H(b) = 0$ respectively. If b_1, b_2 are objects in $\mathfrak{B}_1, \mathfrak{B}_2$, one has

$$\mathrm{Hom}^i_{\mathfrak{B}}(b_1, b_2) \simeq \mathrm{Hom}^i_{\mathfrak{B}}(FH(b_1), b_2) \simeq \mathrm{Hom}^i_{\mathfrak{A}}(H(b_1), H(b_2)) = 0$$

and

$$\mathrm{Hom}^i_{\mathfrak{B}}(b_2, b_1) \simeq \mathrm{Hom}^i_{\mathfrak{B}}(b_2, FH(b_1)) \simeq \mathrm{Hom}^i_{\mathfrak{A}}(G(b_2), H(b_1)) = 0$$

because $G(b_2) = 0$ by hypothesis. Moreover, for any object b in $\mathfrak{B}$ the counit morphism $\epsilon(b) \colon FH(b) \to b$ can be embedded into a triangle

$$FH(b) \xrightarrow{\epsilon(b)} b \to c \xrightarrow{\delta} FH(b)[1]\,.$$

Applying H we get $H(c) = 0$; in fact, $H(\epsilon(b)) \colon HFH(b) \to H(b)$ is an isomorphism since F is fully faithful. Then $FH(b)$ is an object of $\mathfrak{B}_1$ and c is in $\mathfrak{B}_2$. Moreover the composition of $0 = H(c) \xrightarrow{H(\delta)} HFH(b)[1] \simeq Hb[1]$ is the morphism corresponding by adjunction to $c \xrightarrow{\delta} FH(b)[1]$, and then $\delta = 0$. Since $\mathfrak{B}$ is indecomposable, either $\mathfrak{B}_1$ or $\mathfrak{B}_2$ is trivial. If $\mathfrak{B}_1$ is trivial, then $\mathfrak{B}_2 = \mathfrak{B}$ so that $H = 0$; then for every object a of $\mathfrak{A}$ we have $\mathrm{Hom}_{\mathfrak{A}}(a, a) \simeq \mathrm{Hom}_{\mathfrak{B}}(F(a), F(a)) \simeq \mathrm{Hom}_{\mathfrak{A}}(a, H(F(a))) = 0$ and this implies that $\mathfrak{A}$ is trivial. But this is impossible, and therefore $\mathfrak{B}_2$ is trivial, which means that $c = 0$ and $\epsilon(b) \colon FH(b) \xrightarrow{\sim} b$ for any object b in $\mathfrak{B}$. Thus $FH \simeq \mathrm{Id}_{\mathfrak{B}}$, and F is an equivalence. $\square$

As shown in Corollary 1.18, exact equivalences of categories intertwine Serre functors. Whenever this intertwining property holds true for all objects in a spanning class, under some additional assumptions one has a converse statement.

Proposition 2.5. *Let $F \colon \mathfrak{A} \to \mathfrak{B}$ be an exact fully faithful functor of triangulated categories with Serre functors $S_{\mathfrak{A}}$, $S_{\mathfrak{B}}$. Assume that $\mathfrak{B}$ is indecomposable, $\mathfrak{A}$ is not trivial and that F has a right adjoint H. Then F is an equivalence if and only if $S_{\mathfrak{B}}F(\sigma) = FS_{\mathfrak{A}}(\sigma)$ for all σ in some spanning class $\Sigma \subset \mathrm{Ob}(\mathfrak{A})$.*

Proof. By Lemma 1.17, F has a left adjoint given by $G = S_{\mathfrak{A}}^{-1} \circ H \circ S_{\mathfrak{B}}$. For any object b in $\mathfrak{B}$, any $\sigma \in \Sigma$ and any $i \in \mathbb{Z}$, we have

$$\begin{aligned}\mathrm{Hom}^i_{\mathfrak{A}}(\sigma, G(b)) &\simeq \mathrm{Hom}^i_{\mathfrak{A}}(G(b), S_{\mathfrak{A}}(\sigma))^* \simeq \mathrm{Hom}^i_{\mathfrak{A}}(b, FS_{\mathfrak{A}}(\sigma))^* \\ &\simeq \mathrm{Hom}^i_{\mathfrak{A}}(b, S_{\mathfrak{B}}F(\sigma))^* \simeq \mathrm{Hom}^i_{\mathfrak{A}}(F(\sigma), b) \simeq \mathrm{Hom}^i_{\mathfrak{A}}(\sigma), H(b))\,.\end{aligned}$$

Then $G(b) = 0$ if and only if $H(b) = 0$, so that F is an equivalence by Lemma 2.4. $\square$

Spanning classes may be used to test whether a functor is fully faithful.

Theorem 2.6. [61] *Let $F\colon \mathfrak{A} \to \mathfrak{B}$ be an exact functor of triangulated categories, admitting a left and a right adjoint. The functor F is fully faithful if and only if the morphism*

$$F\colon \mathrm{Hom}^i_{\mathfrak{A}}(\sigma, \tau) \to \mathrm{Hom}^i_{\mathfrak{B}}(F(\sigma), F(\tau))$$

is an isomorphism for every σ, τ in some spanning class Σ for $\mathfrak{A}$.

Proof. Let H, G be a right and a left adjoint to F and consider the corresponding units and counits

$$\eta\colon \mathrm{Id}_{\mathfrak{A}} \to H \circ F \qquad \epsilon\colon F \circ H \to \mathrm{Id}_{\mathfrak{B}}$$
$$\xi\colon \mathrm{Id}_{\mathfrak{B}} \to F \circ G \qquad \delta\colon G \circ F \to \mathrm{Id}_{\mathfrak{A}}$$

We have a commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}^i_{\mathfrak{A}}(\sigma, \tau) \arrow[r, "\eta(\tau)_*"] \arrow[d, "\delta(\sigma)^*"'] \arrow[dr, "F"] & \mathrm{Hom}_{\mathfrak{A}}(\sigma, H \circ F(\tau)) \arrow[d, "\beta"', "\simeq"] \\
\mathrm{Hom}^i_{\mathfrak{A}}(G \circ F(\sigma), \tau) \arrow[r, "\simeq", "\alpha"'] & \mathrm{Hom}^i_{\mathfrak{B}}(F(\sigma), F(\tau))
\end{tikzcd} \tag{2.1}$$

where $\alpha = \xi(F(\sigma))^* \circ F$ and $\beta = \epsilon(F(\tau))_* \circ F$ are the isomorphisms given by adjunction. Since F is an isomorphism for all $\sigma, \tau \in \Sigma$, all morphisms in diagram 2.1 are isomorphisms for all $\sigma, \tau \in \Sigma$. Taking this into account we proceed in three steps:

(1) We prove that *$\delta(\sigma)\colon G \circ F(\sigma) \to \sigma$ is an isomorphism for every $\sigma \in \Sigma$.* Indeed, $\delta(\sigma)$ fits into the triangle $G \circ F(\sigma) \xrightarrow{\delta(\sigma)} \sigma \to \rho \to G \circ F(\sigma)[1]$. For every $\tau \in \mathfrak{A}$ we have an exact sequence

$$\cdots \to \mathrm{Hom}^{-1}_{\mathfrak{A}}(\sigma, \tau) \xrightarrow{\delta(\sigma)^*} \mathrm{Hom}^{-1}_{\mathfrak{A}}(G \circ F(\sigma), \tau) \to$$
$$\mathrm{Hom}_{\mathfrak{A}}(\rho, \tau) \to \mathrm{Hom}_{\mathfrak{A}}(\sigma, \tau) \xrightarrow{\delta(\sigma)^*} \mathrm{Hom}_{\mathfrak{A}}(G \circ F(\sigma), \tau) \to \dots$$

If $\tau \in \Sigma$, then all morphisms $\delta(\sigma)^*$ in this exact sequence are isomorphisms, so that $\mathrm{Hom}^i_{\mathfrak{A}}(\rho, \tau) = 0$ for every $i \in \mathbb{Z}$ and $\rho = 0$. Thus $\delta(\sigma)$ is an isomorphism.

(2) Next we show that *$\eta(\tau)\colon \tau \to H \circ F(\tau)$ is an isomorphism for every $\tau \in \mathrm{Ob}(\mathfrak{A})$*. We can embed $\eta(\tau)$ into the triangle $\tau \xrightarrow{\eta(\tau)} H \circ F(\tau) \to \rho \to \tau[1]$ and get for every $\sigma \in \Sigma$ the exact sequence

$$\cdots \to \mathrm{Hom}_{\mathfrak{A}}(\sigma, \tau) \xrightarrow{\eta(\tau)_*} \mathrm{Hom}_{\mathfrak{A}}(\sigma, H \circ F(\tau)) \to \mathrm{Hom}_{\mathfrak{A}}(\sigma, \rho) \to$$
$$\mathrm{Hom}^1_{\mathfrak{A}}(\sigma, \tau) \xrightarrow{\eta(\tau)_*} \mathrm{Hom}^1_{\mathfrak{A}}(\sigma, H \circ F(\tau)) \to \dots$$

As $\sigma \in \Sigma$, by (1) $\delta(\sigma)$ is an isomorphism, hence $\delta(\sigma)^*$ is an isomorphism. By diagram (2.1) F is an isomorphism, so that $\eta(\tau)_*$ is an isomorphism as well for every $\sigma \in \Sigma$, $\tau \in \mathrm{Ob}(\mathfrak{A})$. By the above exact sequence one has $\mathrm{Hom}^i_{\mathfrak{A}}(\sigma, \rho) = 0$ for every $i \in \mathbb{Z}$, $\sigma \in \Sigma$ and thus $\rho = 0$ and $\eta(\tau)$ is an isomorphism.

(3) Finally, we prove that F is fully faithful. Since $\eta(\tau)$ is an isomorphism for every $\tau \in \mathrm{Ob}(\mathfrak{A})$, $\eta(\tau)_*$ is an isomorphism for every $\sigma, \tau \in \mathrm{Ob}(\mathfrak{A})$, and then F is an isomorphism by diagram (2.1). □

2.1.1 Ample sequences

Let $\mathfrak{A}$ be a $\Bbbk$-linear Abelian category.

Definition 2.7. A sequence $\{P_i\}_{i\in\mathbb{Z}}$ of objects in $\mathfrak{A}$ is said to be *ample* if for every object C of $\mathfrak{A}$ there is an integer $i_0 = i_0(C)$ such that the following conditions hold for $i < i_0$:

1. the natural morphism $\mathrm{Hom}_{\mathfrak{A}}(P_i, C) \otimes P_i \to C$ is surjective;
2. $\mathrm{Hom}^j_{D^b(\mathfrak{A})}(P_i, C) = 0$ for $j \neq 0$;
3. $\mathrm{Hom}_{\mathfrak{A}}(C, P_i) = 0$.

△

Condition 1 is equivalent to the existence of an exact sequence

$$P_j^{\oplus s} \xrightarrow{\alpha} P_i^{\oplus k} \xrightarrow{\rho} A \to 0\,.$$

The most important example is provided by the sequence $\{\mathcal{L}^i\}_{i\in\mathbb{Z}}$ in the category of quasi-coherent sheaves on a projective variety where $\mathcal{L}$ is an ample line bundle (here $P_i = \mathcal{L}^{-i}$).

The following results are used in the proof of Orlov's representability theorem 2.15.

Lemma 2.8. [242] *Let $\{P_i\}_{i\in\mathbb{Z}}$ be an ample sequence in $\mathfrak{A}$. An object $\mathcal{A}^\bullet$ of $D^b(\mathfrak{A})$ is isomorphic to an object of $\mathfrak{A}$ (i.e., it is isomorphic in $D^b(\mathfrak{A})$ to a complex concentrated in degree zero) if and only if $\mathrm{Hom}^j_{D^b(\mathfrak{A})}(P_i, \mathcal{A}^\bullet) = 0$ for $j \neq 0$ and $i \ll 0$.*

Proof. The "only if" part is clear by the definition of ample sequence. Now assume that $\mathrm{Hom}^j_{D^b(\mathfrak{A})}(P_i, \mathcal{A}^\bullet) = 0$ for $j \neq 0$ and $i \ll 0$. Embed $\mathfrak{A}$ into an Abelian category with enough injectives, so that $\mathrm{Hom}^j_{D^b(\mathfrak{A})}(P_i, C) = \mathrm{Ext}^j(P_i, C)$, where the Exts are computed in the larger category. Since $\mathcal{A}^\bullet$ has bounded cohomology, we can find $i \ll 0$ such that $\mathrm{Ext}^j(P_i, H^q(\mathcal{A}^\bullet)) = 0$ for every q. Then there is a convergent spectral sequence $E_2^{p,q} = \mathrm{Ext}^p(P_i, H^q(\mathcal{A}^\bullet))$ with $E_\infty^{p+q} = \mathrm{Hom}^{p+q}_{D^b(\mathfrak{A})}(P_i, \mathcal{A}^\bullet)$. If $H^q(\mathcal{A}^\bullet) \neq 0$, we have $E_2^{0,q} = \mathrm{Hom}(P_i, H^q(\mathcal{A}^\bullet)) \neq 0$ for $i \ll 0$ by the first condition in Definition 2.7, so that any nonzero element in $E_2^{0,q}$ survives to infinity yielding a nonzero element in $E_\infty^q = \mathrm{Hom}^q_{D^b(\mathfrak{A})}(P_i, \mathcal{A}^\bullet)$. Thus $q = 0$ and $\mathcal{A}^\bullet \simeq H^0(\mathcal{A}^\bullet)$ in $D^b(\mathfrak{A})$. □

Proposition 2.9. *Let $\{P_i\}_{i\in\mathbb{Z}}$ be an ample sequence in $\mathfrak{A}$. Assume that $D^b(\mathfrak{A})$ has a Serre functor. Then the class $\Sigma = \{P_i\}_{i\in\mathbb{Z}} \subset \mathrm{Ob}(D^b(\mathfrak{A}))$ is a spanning class for $D^b(\mathfrak{A})$.*

Proof. Let $\mathcal{A}^\bullet$ be an object of $D^b(\mathfrak{A})$ and assume that $\mathrm{Hom}^j_{D^b(\mathfrak{A})}(P_i, \mathcal{A}^\bullet) = 0$ for every i and j. By Lemma 2.8, $\mathcal{A}^\bullet$ is isomorphic to an object A in $\mathfrak{A}$. By the first condition in Definition 2.7, one has $A = 0$. Now take a complex $\mathcal{A}^\bullet$ in $D^b(\mathfrak{A})$ such that $\mathrm{Hom}^j_{D^b(\mathfrak{A})}(\mathcal{A}^\bullet, P_i) = 0$ for every i and j. If S is the Serre functor of the category $D^b(\mathfrak{A})$, then $\mathrm{Hom}^j_{D^b(\mathfrak{A})}(P_i, S(\mathcal{A}^\bullet)) \simeq \mathrm{Hom}_{D^b(\mathfrak{A})}(P_i, S(\mathcal{A}^\bullet[j])) \simeq \mathrm{Hom}_{D^b(\mathfrak{A})}(\mathcal{A}^\bullet[j], P_i)^* \simeq \mathrm{Hom}^{-j}_{D^b(\mathfrak{A})}(\mathcal{A}^\bullet, P_i)^* = 0$ for every i and j. Then by the previous argument we have $S(\mathcal{A}^\bullet[j]) = 0$ for every j, so that $\mathcal{A}^\bullet = 0$. □

We shall also denote by Σ the full subcategory of $D^b(\mathfrak{A})$ whose objects are $\{P_i\}_{i\in\mathbb{Z}}$.

Proposition 2.10. [242, Prop. 2.16] *Let $\{P_i\}_{i\in\mathbb{Z}}$ be an ample sequence of objects in $\mathfrak{A}$. Let $F\colon D^b(\mathfrak{A}) \to D^b(\mathfrak{A})$ be an exact equivalence. Every isomorphism $h\colon \mathrm{Id}_\Sigma \xrightarrow{\sim} F_{|\Sigma}$ can be extended to an isomorphism $\mathrm{Id}_{D^b(\mathfrak{A})} \xrightarrow{\sim} F$ on $D^b(\mathfrak{A})$.*

Proof. We have for every i an isomorphism $h_{P_i}\colon P_i \xrightarrow{\sim} F(P_i)$ depending functorially on P_i. Our task is to extend these isomorphisms to functorial isomorphisms $h_{\mathcal{A}^\bullet}\colon \mathcal{A}^\bullet \to F(\mathcal{A}^\bullet)$ for every object $\mathcal{A}^\bullet$ in $D^b(\mathfrak{A})$. We divide this rather long proof into four steps.

(1) $\mathcal{A}^\bullet \simeq \oplus_{k=1}^n P_{i_k}$. In this case $F(\mathcal{A}^\bullet) \simeq \oplus_{k=1}^n P_{i_k}$ and we simply set $h_{\mathcal{A}^\bullet}$ to be the direct sum of the given $h_{P_{i_k}}$.

(2) *$\mathcal{A}^\bullet$ is isomorphic to an object A of $\mathfrak{A}$.* We first note that $F(A)$ is an object of $\mathfrak{A}$ since

$$\mathrm{Hom}^j_{D^b(\mathfrak{A})}(P_i, F(A)) \simeq \mathrm{Hom}^j_{D^b(\mathfrak{A})}(F(P_i), F(A)) \simeq \mathrm{Hom}^j_{D^b(\mathfrak{A})}(P_i, A)$$

due to $P_i \simeq F(P_i)$, and we can apply Lemma 2.8. By definition of ample sequence there is an exact sequence

$$P_j^{\oplus s} \xrightarrow{\alpha} P_i^{\oplus k} \xrightarrow{\rho} A \to 0$$

for $i, j \ll 0$. We then have a diagram

$$\begin{tikzcd} P_j^{\oplus s} \arrow[r, "\alpha"] \arrow[d, "h^s_{P_j}", "\simeq"'] & P_i^{\oplus k} \arrow[r, "\rho"] \arrow[d, "h^k_{P_i}", "\simeq"'] & A \arrow[r] & 0 \\ F(P_j^{\oplus s}) \arrow[r, "F(\alpha)"] & F(P_i)^{\oplus k} \arrow[r, "F(\rho)"] & F(A)\,. \end{tikzcd}$$

Since $F(\rho) \circ h^k_{P_i} \circ \alpha = F(\rho) \circ F(\alpha) \circ h^s_{P_j} = F(\rho \circ \alpha) \circ h^s_{P_j} = 0$, there is an isomorphism $h_A\colon A \xrightarrow{\sim} F(A)$ which completes the diagram. Moreover h_A is unique as $\operatorname{Hom}(A, F(A)) \hookrightarrow \operatorname{Hom}(F(P_i)^{\oplus k}, F(A))$. This also implies that h_A is independent of the choice of the surjection $P_i^{\oplus k} \xrightarrow{\rho} A \to 0$. One can easily check that h_A depends functorially on A.

(3) $\mathcal{A}^\bullet \simeq A[n]$ *for an object* A *of* $\mathfrak{A}$. Since $F(A[n]) \simeq F(A)[n]$ we simply set $h_{A[n]} = h_A[n]$.

(4) $\mathcal{A}^\bullet$ *is any object in* $D^b(\mathfrak{A})$. We proceed by induction on the length $N = \ell(\mathcal{A}^\bullet)$, which is the number of nonzero cohomology objects of $\mathcal{A}^\bullet$. The case $N = 1$ corresponds to the objects of $\mathfrak{A}$ and has been considered in the previous steps. Let us then take $N > 1$.

Let q be the maximum of the integers such that $H^q(\mathcal{A}^\bullet) \neq 0$. Then we can find an index i and a surjective morphism $P_i^{\oplus k} \to \ker d_q$ inducing a morphism $\phi\colon P_i^{\oplus k}[-q] \to \mathcal{A}^\bullet{}_{\leq q} \simeq \mathcal{A}^\bullet$ in the derived category. We can also choose the index i so that:

(a) the induced morphism $H^q(\phi)\colon P_i^{\oplus k} \to H^q(\mathcal{A}^\bullet)$ is surjective;

(b) $\operatorname{Hom}^j_{D^b(\mathfrak{A})}(P_i, H^p(\mathcal{A}^\bullet)) = 0$ for all $j \neq 0$ and for all p;

(c) $\operatorname{Hom}^j_{D^b(\mathfrak{A})}(H^q(\mathcal{A}^\bullet), P_i) = 0$ for all $j \neq 0$ and for all q.

We can now embed ϕ into an exact triangle $\mathcal{B}^\bullet \to P_i^{\oplus k}[-q] \xrightarrow{\phi} \mathcal{A}^\bullet \to \mathcal{B}^\bullet[1]$. Since $\ell(\mathcal{B}^\bullet) = N - 1$, induction provides a commutative diagram

$$\begin{tikzcd} \mathcal{B}^\bullet[-1] \arrow[r] \arrow[d, "h_{\mathcal{B}^\bullet}[-1]", "\simeq"'] & P_i^{\oplus k}[-q] \arrow[r, "\phi"] \arrow[d, "h_{P_i^{\oplus k}}[-q]", "\simeq"'] & \mathcal{A}^\bullet \arrow[r, "\psi"] & \mathcal{B}^\bullet \arrow[d, "h_{\mathcal{B}^\bullet}", "\simeq"'] \\ F(\mathcal{B}^\bullet)[-1] \arrow[r] & F(P_i^{\oplus k})[-q] \arrow[r, "F(\phi)"] & F(\mathcal{A}^\bullet) \arrow[r, "F(\psi)"] & F(\mathcal{B}^\bullet)\,. \end{tikzcd}$$

Then there exists an isomorphism $h_{\mathcal{A}^\bullet}\colon \mathcal{A}^\bullet \xrightarrow{\sim} F(\mathcal{A}^\bullet)$ making the above diagram into a morphism of triangles. Moreover, this morphism is *the only one* satisfying the condition $F(\psi) \circ h_{\mathcal{A}^\bullet} = h_{\mathcal{B}^\bullet} \circ \psi$, as

$$\begin{aligned} \operatorname{Hom}_{D^b(\mathfrak{A})}(\mathcal{A}^\bullet, F(P_i^{\oplus k})[-q]) &\simeq \operatorname{Hom}_{D^b(\mathfrak{A})}(\mathcal{A}^\bullet, P_i^{\oplus k}[-q]) \\ &\simeq \operatorname{Hom}^q_{D^b(\mathfrak{A})}(\mathcal{A}^\bullet, P_i^{\oplus k}) = 0 \end{aligned}$$

Again, one easily proves that the morphism $h_{\mathcal{A}^\bullet}$ does not depend on the choice of the morphism $\phi\colon P_i^{\oplus k}[-q] \to \mathcal{A}^\bullet{}_{\leq q} \simeq \mathcal{A}^\bullet$.

Finally, we prove that this construction is functorial. Let us take a morphism $\varpi\colon \mathcal{A}^\bullet \to \mathcal{C}^\bullet$ with $\ell(\mathcal{C}^\bullet) \leq N$. We must prove that the diagram

$$\begin{tikzcd} \mathcal{A}^\bullet \arrow[r, "\varpi"] \arrow[d, "\simeq"', "h_{\mathcal{A}^\bullet}"] & \mathcal{C}^\bullet \arrow[d, "\simeq"', "h_{\mathcal{C}^\bullet}"] \\ F(\mathcal{A}^\bullet) \arrow[r, "F(\varpi)"] & F(\mathcal{C}^\bullet) \end{tikzcd} \tag{2.2}$$

is commutative. Let q be as above the maximum of the integers such that $H^q(\mathcal{A}^\bullet) \neq 0$ and p the maximum of the integers such that $H^p(\mathcal{C}^\bullet) \neq 0$. We consider separately the cases $p < q$ and $p \geq q$.

The case $p < q$. We take as above a morphism $\phi\colon P_i^{\oplus k} \to \mathcal{A}^\bullet$ and the corresponding exact triangle

$$P_i^{\otimes k}[-q] \xrightarrow{\phi} \mathcal{A}^\bullet \xrightarrow{\psi} \mathcal{B}^\bullet \to P_i^{\otimes k}[-q+1]\,.$$

We can assume $\operatorname{Hom}_{D^b(\mathfrak{A})}(P_i, H^j(\mathcal{C}^\bullet)) = 0$ for all j, so that $\operatorname{Hom}_{D^b(\mathfrak{A})}(P_i, \mathcal{C}^\bullet) = 0$. Then ϖ factors through a morphism $\rho\colon \mathcal{B}^\bullet \to \mathcal{C}^\bullet$. Since $\ell(\mathcal{B}^\bullet) = \ell(\mathcal{A}^\bullet) - 1$, the diagram

$$\begin{tikzcd} \mathcal{B}^\bullet \arrow[r, "\rho"] \arrow[d, "\simeq"', "h_{\mathcal{B}^\bullet}"] & \mathcal{C}^\bullet \arrow[d, "\simeq"', "h_{\mathcal{C}^\bullet}"] \\ F(\mathcal{B}^\bullet) \arrow[r, "F(\rho)"] & F(\mathcal{C}^\bullet) \end{tikzcd}$$

commutes by induction. Then (2.2) is commutative as well.

The case $p \geq q$. We proceed by induction on $p - q$, the case $p - q = -1$ being included in the preceding case. We can find an index i and a surjective morphism $P_i^{\oplus k} \to \ker d_p$. This gives a morphism $\phi\colon P_i^{\oplus k}[-p] \to \mathcal{C}^\bullet{}_{\leq p} \simeq \mathcal{C}^\bullet$ in the derived category. Moreover, the induced morphism $H^p(\phi)\colon P_i^{\oplus k} \to H^p(\mathcal{C}^\bullet)$ is surjective. We can also assume that $\operatorname{Hom}_{D^b(\mathfrak{A})}(H^j(\mathcal{A}^\bullet), P_i^{\oplus k}) = 0$ for all j, so that $\operatorname{Hom}_{D^b(\mathfrak{A})}(\mathcal{A}^\bullet, P_i^{\oplus k}[-p]) = 0$. We have an exact triangle

$$P_i^{\oplus k}[-p] \xrightarrow{\phi} \mathcal{C}^\bullet \xrightarrow{\beta} \mathcal{M}^\bullet \to P_i^{\oplus k}[-p+1]$$

and the maximum of the integers $\tilde{p}$ with $H^{\tilde{p}}(\mathcal{M}^\bullet) \neq 0$ is smaller than p. Let us consider the diagram

$$\begin{tikzcd}
 & \mathcal{C}^\bullet \arrow[dr, "\beta"] \arrow[dd, "\simeq"', "h_{\mathcal{C}^\bullet}"] & \\
\mathcal{A}^\bullet \arrow[ur, "\varpi"] \arrow[rr, "\beta\circ\varpi"] \arrow[dd, "\simeq"', "h_{\mathcal{A}^\bullet}"] & & \mathcal{M}^\bullet \arrow[dd, "\simeq"', "h_{\mathcal{M}^\bullet}"] \\
 & F(\mathcal{C}^\bullet) \arrow[dr, "F(\beta)"] & \\
F(\mathcal{A}^\bullet) \arrow[ur, "F(\varpi)"] \arrow[rr, "F(\beta\circ\varpi)"] & & F(\mathcal{M}^\bullet)\,.
\end{tikzcd}$$

By induction, the rectangular-shaped subdiagram is commutative, and by the construction of the morphisms $h_{\mathcal{C}^\bullet}$ the lozenge-shaped subdiagram on the right is commutative as well. Then $F(\beta)\circ(h_{\mathcal{C}^\bullet}\circ\varpi - F(\varpi)\circ h_{\mathcal{A}^\bullet}) = 0$ and since $\operatorname{Hom}_{D^b(\mathfrak{A})}(H^p(\mathcal{A}^\bullet), P_i^{\oplus k}) = 0$ we obtain that $h_{\mathcal{C}^\bullet}\circ\varpi - F(\varpi)\circ h_{\mathcal{A}^\bullet} = 0$, that is, the diagram (2.2) is commutative. □

Proposition 2.11. [176, Lemma 6.5] *Let $F, \bar{F}\colon D^b(\mathfrak{A}) \to \mathfrak{B}$, be exact functors, where $\mathfrak{B}$ is a $\Bbbk$-linear triangulated category. Assume that:*

1. *$D^b(\mathfrak{A})$ has a Serre functor;*
2. *$F, \bar{F}$ have left adjoint functors G and $\bar{G}$ and right adjoint functors H, $\bar{H}$ (note that by Lemma 1.17, if $\mathfrak{B}$ has a Serre functor, then right adjoints if and only if left adjoints exist);*
3. *F is fully faithful;*
4. *there is an ample sequence $\{P_i\}_{i\in\mathbb{Z}}$ of objects of $\mathfrak{A}$ and an isomorphism of functors $f_\Sigma\colon F_{|\Sigma} \xrightarrow{\sim} \bar{F}_{|\Sigma}$, where Σ is the full subcategory of $D^b(\mathfrak{A})$ whose objects are $\{P_i\}_{i\in\mathbb{Z}}$.*

Then there is an isomorphism of functors $f\colon F \xrightarrow{\sim} \bar{F}$ extending f_Σ.

Proof. Since the restrictions of F and $\bar{F}$ to Σ are isomorphic and $\{P_i\}_{i\in\mathbb{Z}}$ is a spanning class by Proposition 2.9, Theorem 2.6 implies that $\bar{F}$ is fully faithful. Since F and $\bar{F}$ are fully faithful, there are functor isomorphisms $\mathrm{Id}_{D^b(\mathfrak{A})} \xrightarrow{\sim} HF$ and $\bar{G}\bar{F} \xrightarrow{\sim} \mathrm{Id}_{D^b(\mathfrak{A})}$.

Now, $\bar{G}F$ is left adjoint to $H\bar{F}$. Since both composed functors are isomorphic to the identity when rectricted to Σ, again by Theorem 2.6 they are fully faithful. Whenever a fully faithful functor admits a left adjoint which is fully faithful, then it is an equivalence of categories. Thus, $H\bar{F}$ is an exact equivalence. By Proposition

2.10, the isomorphism $\mathrm{Id}_\Sigma \xrightarrow{\sim} H\bar{F}_{|\Sigma}$ can be extended to a functor isomorphism $\mathrm{Id}_{D^b(\mathfrak{A})} \xrightarrow{\sim} H\bar{F}$. Composing from the left with F and using adjunction, we have a morphism $f\colon F \to \bar{F}$ extending f_Σ.

Let us check that f is an isomorphism. For any object $\mathcal{A}^\bullet$ in $D^b(\mathfrak{A})$, let

$$F(\mathcal{A}^\bullet) \xrightarrow{f_{\mathcal{A}^\bullet}} \bar{F}(\mathcal{A}^\bullet) \to c \to F(\mathcal{A}^\bullet)[1]$$

be an exact triangle. Since $H(f_{\mathcal{A}^\bullet})$ is an isomorphism we have $H(c) = 0$ and then

$$\begin{aligned}\mathrm{Hom}_{D^b(\mathfrak{A})}(P_i, \bar{H}(c)) &\simeq \mathrm{Hom}_{\mathfrak{B}}(\bar{F}(P_i), c)\\ &\simeq \mathrm{Hom}_{\mathfrak{B}}(F(P_i), c) \simeq \mathrm{Hom}_{D^b(\mathfrak{A})}(P_i, H(c)) = 0\end{aligned}$$

for every i, so that $\bar{H}(c) = 0$. Thus $\mathrm{Hom}_{\mathfrak{B}}(\bar{F}(\mathcal{A}^\bullet), c) = 0$, and $F(\mathcal{A}^\bullet) \simeq \bar{F}(\mathcal{A}^\bullet) \oplus c$ from the above exact triangle. Since $\mathrm{Hom}_{\mathfrak{B}}(F(\mathcal{A}^\bullet), c) \simeq \mathrm{Hom}_{D^b(\mathfrak{A})}(\mathcal{A}^\bullet, H(c)) = 0$ we deduce that $c = 0$ and then $f_{\mathcal{A}^\bullet}$ is an isomorphism. □

2.1.2 Convolutions

Given an object in Abelian category, we are used to associate with it a complex of objects in the derived category; we also know how to associate an object of the derived category with a double complex. Sometimes, dealing with (bounded) complexes of objects in the derived category

$$(\mathcal{A}^\bullet)^{-m} \xrightarrow{d_{-m}} (\mathcal{A}^\bullet)^{-m+1} \xrightarrow{d_{-m+1}} \dots \to (\mathcal{A}^\bullet)^{-1} \xrightarrow{d_{-1}} (\mathcal{A}^\bullet)^0\,,$$

one wishes to construct objects a of the derived category which somehow represent them; one also requires that when the objects of the complex we start with are in the original Abelian category,

$$\mathcal{A}^\bullet \equiv A^{-m} \xrightarrow{d_{-m}} A^{-m+1} \xrightarrow{d_{-m+1}} \dots \to A^{-1} \xrightarrow{d_{-1}} A^0$$

the new object a is just the image $\mathcal{A}^\bullet$ of the complex in the derived category.

This is possible under very mild requirements, and the process is called *convolution.* Let us consider at first a complex

$$\mathcal{A}^\bullet \equiv A^{-1} \xrightarrow{d_{-1}} A^0$$

of objects of $\mathfrak{A}$. If $\mathrm{Cone}(d_{-1})$ is the cone of d_{-1}, the natural morphism $A^0 \to \mathrm{Cone}(d_{-1})$ induces an isomorphism in the derived category

$$\mathcal{A}^\bullet \xrightarrow{\sim} \mathrm{Cone}(d_{-1})\,.$$

Then $\operatorname{Cone}(d_{-1})$ represents the complex $\mathcal{A}^\bullet$ in the derived category $D^b(\mathfrak{A})$. This is definitely a trivial observation, but this "cone construction" may be straightforwardly extended to complexes of objects in the derived category: given a complex

$$(\mathcal{A}^\bullet)^{-1} \xrightarrow{d_{-1}} (\mathcal{A}^\bullet)^0$$

of objects of $D^b(\mathfrak{A})$, we can take a cone $\operatorname{Cone}(d_{-1})$ of d_{-1} and we have a natural morphism $(\mathcal{A}^\bullet)^0 \to \operatorname{Cone}(d_{-1})$ and an exact triangle

$$(\mathcal{A}^\bullet)^{-1} \xrightarrow{d_{-1}} (\mathcal{A}^\bullet)^0 \to \operatorname{Cone}(d_{-1}) \to (\mathcal{A}^\bullet)^{-1}[1]\,.$$

We can iterate this process to define the right convolution of a bounded complex of objects of the derived category. Actually, we do not need to work with a derived category, since any triangulated category will do.

Let then $\mathfrak{B}$ be a *triangulated* category and

$$a^{-m} \xrightarrow{d_{-m}} a^{-(m-1)} \xrightarrow{d_{-(m-1)}} a^{-(m-2)} \to \cdots \to a^{-1} \xrightarrow{d_{-1}} a^0 \tag{2.3}$$

a complex of objects of $\mathfrak{B}$ (that is, the composition of any two consecutive morphisms vanishes). Assume also that one has

$$\operatorname{Hom}_{\mathfrak{B}}(a^{-p}[r], a^{-q}) = 0\,, \quad \text{for every } p > q \text{ and } r > 0. \tag{2.4}$$

Following Orlov and Kawamata [242, 176] we can define the *right convolution* of the complex 2.3 as the pair formed by the object $\mathbf{a}$ of $\mathfrak{B}$ and the morphism $d_0\colon a_0 \to \mathbf{a}$ constructed by induction on the length m as follows:

- If $m = 0$, then $a = a_0$ and d_0 is the identity.
- If $m \geq 1$, we let $\mathbf{a}^{-(m-1)} = \operatorname{Cone}(d_{-m})$, so that there is an exact triangle

$$a^{-m} \xrightarrow{d_{-m}} a^{-(m-1)} \xrightarrow{g_{-(m-1)}} \mathbf{a}^{-(m-1)} \to a^{-m}[1]\,.$$

After taking homomorphisms we have an exact sequence

$$\operatorname{Hom}_{\mathfrak{B}}(a^{-m}[1], a^{-(m-2)}) \to \operatorname{Hom}_{\mathfrak{B}}(\mathbf{a}^{-(m-1)}, a^{-(m-2)}) \to \operatorname{Hom}_{\mathfrak{B}}(a^{-(m-1)}, a^{-(m-2)}) \to \operatorname{Hom}_{\mathfrak{B}}(a^{-m}, a^{-(m-2)})\,.$$

Since $d_{m-2} \circ d_{m-1} = 0$ there is a morphism $\mathbf{d}_{m-1}\colon \mathbf{a}^{-(m-1)} \to a^{-(m-2)}$ such that $\mathbf{d}_{m-1} \circ g_{m-1} = d_{m-1}$; due to condition (2.4) one also has $\operatorname{Hom}_{\mathfrak{B}}(a^{-m}[1], a^{-(m-2)}) = 0$ and then the morphism is *unique*. Hence, we obtain a new complex

$$\mathbf{a}^{-(m-1)} \xrightarrow{\mathbf{d}_{-(m-1)}} a^{-(m-2)} \to \cdots \to a^{-1} \xrightarrow{d_{-1}} a^0 \tag{2.5}$$

which also fulfils condition (2.4).

- We iterate the previous steps from this new complex.

Note that a^0 remains unchanged during the process. Summing up, we have

Lemma 2.12. *Let*

$$a^{-m} \xrightarrow{d_{-m}} a^{-(m-1)} \xrightarrow{d_{-(m-1)}} a^{-(m-2)} \to \cdots \to a^{-1} \xrightarrow{d_{-1}} a^0$$

be a complex of objects of $\mathfrak{B}$ *that fulfil the condition* (2.4)*. There exists a right convolution* $d_0 \colon a^0 \to \mathbf{a}$ *in* $\mathfrak{B}$*, which is uniquely determined up to isomorphism.*

When one works with the derived category $\mathfrak{B} = D^b(\mathfrak{A})$ of an Abelian category and the objects a^{-p} of $D^b(\mathfrak{A})$ are just objects A^{-p} of the Abelian category $\mathfrak{A}$, then any complex

$$\mathcal{A}^\bullet \equiv A^{-m} \xrightarrow{d_{-m}} A^{-(m-1)} \xrightarrow{d_{-(m-1)}} A^{-(m-2)} \to \cdots \to A^{-1} \xrightarrow{d_{-1}} A^0$$

fulfils the condition (2.4) and the right convolution $\mathbf{a}$ of $\mathcal{A}^\bullet$ is the complex $\mathcal{A}^\bullet$ itself, together with the obvious morphism $A^0 \to \mathbf{a} = \mathcal{A}^\bullet$.

Lemma 2.13. *Let*

$$\begin{tikzcd}
a^{-m} \arrow[r, "d_{-m}"] \arrow[d, "f_{-m}"] & a^{-(m-1)} \arrow[r, "d_{-(m-1)}"] \arrow[d, "f_{-(m-1)}"] & \cdots \arrow[r] & a^{-1} \arrow[r, "d_{-1}"] \arrow[d, "f_{-1}"] & a^0 \arrow[d, "f_0"] \\
b^{-m} \arrow[r, "\widetilde{d}_{-m}"] & b^{-(m-1)} \arrow[r, "\widetilde{d}_{-(m-1)}"] & \cdots \arrow[r] & b^{-1} \arrow[r, "\widetilde{d}_{-1}"] & b^0
\end{tikzcd}$$

be a morphism between complexes of objects of $\mathfrak{B}$ *fulfilling condition* (2.4) *and let* $d_0 \colon a_0 \to \mathbf{a}$*,* $\widetilde{d}_0 \colon b_0 \to \mathbf{b}$ *be right convolutions. If one has*

$$\operatorname{Hom}_{\mathfrak{B}}(a^{-p}[r], b^{-q}) = 0 \quad \textit{for every } p > q \textit{ and } r > 0\,, \tag{2.6}$$

then for any morphism $h \colon \mathbf{b} \to b'$ *there exists a morphism* $f \colon \mathbf{a} \to b'$ *in* $\mathfrak{B}$ *such that the diagram*

$$\begin{tikzcd}
a_0 \arrow[r, "d_0"] \arrow[d, "f_0"] & \mathbf{a} \arrow[dr, "f"] & \\
b_0 \arrow[r, "\widetilde{d}_0"] & \mathbf{b} \arrow[r, "h"] & b'
\end{tikzcd} \tag{2.7}$$

is commutative. Moreover, if

$$\operatorname{Hom}_{\mathfrak{B}}(a^{-p}[r], b') = 0 \quad \textit{for every } p > 0 \textit{ and } r > 0\,, \tag{2.8}$$

then the morphism f *is the only satisfying that property.*

Proof. The morphism f is constructed inductively. If $m = 0$, then $f = hf_0$. If $m \geq 1$, we have a commutative diagram

$$\begin{tikzcd}
a^{-m} \arrow[r] \arrow[d, "f_{-m}"] & a^{-(m-1)} \arrow[r, "g_{-(m-1)}"] \arrow[d, "f_{-(m-1)}"] & \mathbf{a}^{-(m-1)} \arrow[r] & a^{-m}[1] \arrow[d, "f_{-m}[1]"] \\
b^{-m} \arrow[r] & b^{-(m-1)} \arrow[r, "\widetilde{g}_{-(m-1)}"] & \mathbf{b}^{-(m-1)} \arrow[r] & b^{-m}[1]
\end{tikzcd}$$

where $\mathbf{a}^{-(m-1)}$ and $\mathbf{b}^{-(m-1)}$ are cones of the corresponding differentials as before. By the axioms of the triangulated categories, there exists a morphism (not uniquely determined) $\mathbf{f}^{-(m-1)}\colon \mathbf{a}^{-(m-1)} \to \mathbf{b}^{-(m-1)}$ completing the diagram. If we consider the morphisms $\mathbf{d}^{-(m-1)}\colon \mathbf{a}^{-(m-1)} \to a^{-(m-2)}$ and $\widetilde{\mathbf{d}}^{-(m-1)}\colon \mathbf{b}^{-(m-1)} \to b^{-(m-2)}$ constructed above, we have

$$\begin{aligned}\widetilde{\mathbf{d}}_{-(m-1)}\mathbf{f}_{-(m-1)}g_{-(m-1)} &= \widetilde{\mathbf{d}}_{-(m-1)}\widetilde{g}_{-(m-1)}f_{-(m-1)} = \widetilde{d}_{-(m-1)}f_{-(m-1)}\\ &= f_{-(m-2)}d_{-(m-1)} = f_{-(m-2)}\widetilde{\mathbf{d}}_{-(m-1)}g_{-(m-1)}\end{aligned}$$

and then $\widetilde{\mathbf{d}}_{-(m-1)}\mathbf{f}_{-(m-1)} = f_{-(m-2)}\widetilde{\mathbf{d}}_{-(m-1)}$. Thus, we have a morphism of complexes

$$\begin{tikzcd}
\mathbf{a}^{-(m-1)} \arrow[r, "\mathbf{d}_{-(m-1)}"] \arrow[d, "\mathbf{f}_{-(m-1)}"] & a^{-(m-2)} \arrow[r, "d_{-(m-2)}"] \arrow[d, "f_{-(m-2)}"] & \cdots \arrow[r] & a^{-1} \arrow[r, "d_{-1}"] \arrow[d, "f_{-1}"] & a^0 \arrow[d, "f_0"] \\
\mathbf{b}^{-(m-1)} \arrow[r, "\widetilde{\mathbf{d}}_{-(m-1)}"] & b^{-(m-2)} \arrow[r, "\widetilde{d}_{-(m-2)}"] & \cdots \arrow[r] & b^{-1} \arrow[r, "\widetilde{d}_{-1}"] & b^0 .
\end{tikzcd}$$

One easily checks that this morphism of complexes fulfils condition (2.6), and then we obtain the morphism $f\colon \mathbf{a} \to b'$ by induction. If the condition (2.8) is satisfied, f is uniquely determined by the commutativity of diagram (2.7) because $f_0\colon a^0 \to b^0$ does not change during the process. □

Since right convolutions are constructed out of exact triangles and compositions of morphisms, they are compatible with exact functors.

Remark 2.14. Let

$$a^{-m} \xrightarrow{d_{-m}} a^{-(m-1)} \xrightarrow{d_{-(m-1)}} a^{-(m-2)} \to \cdots \to a^{-1} \xrightarrow{d_{-1}} a^0$$

be a complex of objects of $\mathfrak{B}$ fulfilling condition (2.4) and let $d_0\colon a^0 \to \mathbf{a}$ be the right convolution. If $F\colon \mathfrak{B} \to \mathfrak{C}$ is an exact functor to another triangulated category and the complex

$$F(a^{-m}) \xrightarrow{F(d_{-m})} F(a^{-(m-1)}) \to \cdots \to F(a^{-1}) \xrightarrow{F(d_{-1})} F(a^0)$$

of objects of $\mathfrak{C}$ also fulfils condition (2.4), then its right convolution is $F(d_0)\colon F(a^0) \to F(\mathbf{a})$. This happens for instance if F is fully faithful, because in this case

$$\operatorname{Hom}_{\mathfrak{C}}(F(a^{-p})[r], F(a^{-q})) \simeq \operatorname{Hom}_{\mathfrak{B}}(a^{-p}[r], a^{-q}) = 0\,.$$

△

2.2 Orlov's representability theorem

This section is devoted to the proof of the following fundamental result by Orlov [242, Thm. 2.2] and some related issues.

Theorem 2.15. *Let X and Y be smooth projective varieties. Any fully faithful exact functor $\Psi\colon D^b(X) \to D^b(Y)$ is an integral functor.*

This is indeed a deep result since in general functors between triangulated categories are quite difficult to describe. It should be pointed out that much stronger results than Orlov's theorem hold true in the more flexible setting of dg-categories, which provide an enhancement of the usual derived categories. In this framework, roughly speaking, all functors are integral functors, as Theorem A.57 and, more in particular, Equation A.10 show (see Section A.4.4 and the "Notes and further reading" at the end of the current chapter).

Remark 2.16. In our proof of Theorem 2.15 we shall use the fact that any exact functor $D^b(X) \to D^b(Y)$ has a right adjoint (and hence, since the categories $D^b(X)$ and $D^b(Y)$ have Serre functors, a left adjoint as well). This has been proved by Bondal and Van den Bergh [52]. The original result by Orlov was weaker in that this property was assumed in the hypotheses of the theorem. △

2.2.1 Resolution of the diagonal

One of the first important results about derived categories is Beĭlinson's construction of a resolution of the structure sheaf of the diagonal of the product $\mathbb{P}^n \times \mathbb{P}^n$ of two copies of the projective space [33, 34], and its implications for the computation of the derived category $D^b(\mathbb{P}^n)$. We present here Beĭlinson's resolution as a particular case of the Koszul sequence associated to certain sections of a locally free sheaf.

Let $\mathcal{E}$ be a locally free sheaf of rank n on an algebraic variety X and $e\colon \mathcal{O}_X \to \mathcal{E}$ a global section. The zero locus of e is the closed subvariety Z of X defined by the exact sequence

$$\mathcal{E}^* \xrightarrow{e^*} \mathcal{O}_X \to \mathcal{O}_Z \to 0\,.$$

We have a Koszul complex

$$0 \to \bigwedge^n \mathcal{E}^* \xrightarrow{i_e} \bigwedge^{n-1} \mathcal{E}^* \xrightarrow{i_e} \dots \xrightarrow{i_e} \mathcal{E}^* \xrightarrow{e^*} \mathcal{O}_X \to \mathcal{O}_Z \to 0 \tag{2.9}$$

where i_e is the inner product with e, i.e., for every affine open subset U and sections $e_1, \dots, e_p$ on U of $\mathcal{E}$ and Ω^p on U of $\bigwedge^p \mathcal{E}^*$, we have

$$(i_e\Omega^p)(e_1, \dots, e_p) = \Omega^p(e, e_1, \dots, e_p)\,.$$

As a consequence of the theory of Koszul complexes we get the following result.

Corollary 2.17. *Assume that every point x of Z has an affine neighborhood in X where $\mathcal{E}^*$ has a local basis $(\omega_1, \dots, \omega_n)$ such that $(\omega_1(e), \dots, \omega_n(e))$ is a regular sequence in $\mathcal{O}_X(U)$. Then the Koszul complex* (2.9) *is exact, thus providing a finite resolution of the structure sheaf $\mathcal{O}_Z$ of the zero locus of e by locally free sheaves.*

Now take $X = \mathbb{P}^n \times \mathbb{P}^n$ and write $\mathbb{P}^n$ as the projective spectrum of the symmetric algebra $\operatorname{Sym}(V)$, where V is a $\Bbbk$-vector space of dimension $n+1$. Let us write for simplicity $\mathcal{O}(m) = \mathcal{O}_{\mathbb{P}^n}(m)$ and $\Omega = \Omega_{\mathbb{P}^n}$. One has the Euler exact sequence

$$0 \to \Omega(1) \xrightarrow{\alpha} V \otimes_{\Bbbk} \mathcal{O} \to \mathcal{O}(1) \to 0 \tag{2.10}$$

and taking duals

$$0 \to \mathcal{O}(-1) \to V^* \otimes_{\Bbbk} \mathcal{O} \xrightarrow{\alpha^*} \Omega^*(-1) \to 0\,.$$

This gives $\Gamma(\mathbb{P}^n, \mathcal{O}(1)) \simeq V$ and $\Gamma(\mathbb{P}^n, \Omega^*(-1)) \simeq V^*$ so that

$$\Gamma(X, \pi_1^*\mathcal{O}(1) \otimes \pi_2^*\Omega^*(-1)) \simeq V \otimes_{\Bbbk} V^* \simeq \operatorname{End}(V)\,,$$

where π_1, π_2 are the projections onto the two factors. It follows that the identity on V defines a global section e of $\mathcal{E} = \pi_1^*\mathcal{O}(1) \otimes \pi_2^*\Omega^*(-1)$ to which we can apply the precedent discussion on Koszul complexes.

If we take a basis $(x_0, \dots, x_n)$ of V, in the open subset U_i where the homogenous coordinate x_j do not vanish, we have affine coordinates $y_h^{(i)} = x_h/x_i$. On U_i the morphism α is given by $dy_h^{(i)} \otimes x_i^* \mapsto x_h - y_h^{(i)} x_i$ $(h \neq i)$ and on $U_i \times U_j$ a local basis for $\mathcal{E}^*$ is $\{\pi_1^*(x_i^*) \otimes \pi_2^*(dy_h^{(j)} \otimes x_j)\}$. Since the identity e on V as a section of $\mathcal{E}$ is $e = \sum_\ell \pi_1^*(x_\ell) \otimes \pi_2^*((\delta_{\ell h} - y_h^{(j)}\partial_{y_h^{(j)}} \otimes x_j^*)$, computing the morphism $e^*\colon \mathcal{E}^* \to \mathcal{O}_X$ on $U_i \times U_j$ we see that the ideal of the zero set Z of e is generated by $\pi_1^*(y_h^{(i)}) \otimes 1 - \pi_1^*(y_j^{(i)}) \otimes \pi_2^*(y_h^{(j)})$ $(h \neq j)$, and then Z is the diagonal Δ of $X = \mathbb{P}^n \times \mathbb{P}^n$. Moreover, on $U_i \times U_i$ we get that those elements are $\pi_1^*(y_h^{(i)}) \otimes 1 - 1 \otimes \pi_2^*(y_h^{(i)})$, and they form a regular sequence. Corollary 2.17 implies the existence of a resolution of the diagonal, usually called *Beĭlinson resolution of the diagonal of the projective space.*

Proposition 2.18. *There is an exact sequence*

$$\begin{aligned} 0 \to \pi_1^*\mathcal{O}(-n) \otimes \pi_2^*\Omega^n(n) \to \pi_1^*\mathcal{O}(-(n-1)) \otimes \pi_2^*\Omega^{n-1}(n-1) \to \dots \\ \to \pi_1^*\mathcal{O}(-1) \otimes \pi_2^*\Omega(1) \to \mathcal{O}_X \xrightarrow{\delta^*} \mathcal{O}_\Delta \to 0\,. \end{aligned}$$

This provides a resolution of the structure sheaf of the diagonal $\Delta \hookrightarrow X = \mathbb{P}^n \times \mathbb{P}^n$ by locally free sheaves.

Let $j > 0$ be a positive integer and consider the exact sequence

$$0 \to \pi_1^*\mathcal{O}(-n) \otimes \pi_2^*\Omega^n(n+j) \to \pi_1^*\mathcal{O}(-(n-1)) \otimes \pi_2^*\Omega^{n-1}(n-1+j) \to \dots$$
$$\to \pi_1^*\mathcal{O}(-1) \otimes \pi_2^*\Omega(1+j) \to \pi_1^*\mathcal{O} \otimes \pi_2^*\mathcal{O}(j) \xrightarrow{\delta^*} \mathcal{O}_\Delta \otimes \pi_2^*\mathcal{O}(j) \to 0\,,$$

obtained as the tensor product of the Beĭlinson resolution by $\pi_2^*\mathcal{O}(j)$. Since all the sheaves $\Omega^p(p+j)$ $(0 \le p \le n)$ are acyclic, after taking direct images by π_1 we have an exact sequence

$$0 \to \mathcal{O}(-n) \otimes_{\Bbbk} H^0(\mathbb{P}^n, \Omega^n(n+j)) \to$$
$$\mathcal{O}(-(n-1)) \otimes_{\Bbbk} H^0(\mathbb{P}^n, \Omega^{n-1}(n-1+j)) \to$$
$$\dots \to \mathcal{O}(-1) \otimes_{\Bbbk} H^0(\mathbb{P}^n, \Omega(1+j))$$
$$\to \mathcal{O} \otimes_{\Bbbk} H^0(\mathbb{P}^n, \mathcal{O}(j)) \to \mathcal{O}(j) \to 0\,,$$

so that there is an exact sequence

$$0 \to \mathcal{O}(-j) \to V_0^j \otimes_{\Bbbk} \mathcal{O} \to V_1^j \otimes_{\Bbbk} \mathcal{O}(1) \to \dots$$
$$\to V_{n-1}^j \otimes_{\Bbbk} \mathcal{O}(n-1) \to V_n^j \otimes_{\Bbbk} \mathcal{O}(n) \to 0 \quad (2.11)$$

where $V_p^j = H^0(\mathbb{P}^n, \Omega^p(p+j))^*$. We shall use this resolution of $\mathcal{O}(-j)$ later on.

Beĭlinson's resolution has been generalized by Kapustin, Kuznetsov and Orlov in [171] and further formalized and studied by Kawamata [176, Theorem 3.2]. We shall consider here Kawamata's formulation in a way which is sufficient to our purposes. In order to state and prove this generalization we need some preliminaries. Let $A = \oplus_{n\in\mathbb{N}} A_n$ be a graded $\Bbbk$-algebra, with $A_0 = \Bbbk$. Let us define recursively vector spaces B_n, with $n \in \mathbb{N}$, as the kernels

$$B_n = \ker(B_{n-1} \otimes A_1 \to B_{m-2} \otimes A_2)$$

for $n \ge 2$, and $B_n = A_n$ for $n = 0, 1$. There are natural homomorphisms

$$B_n \otimes A[-n] \to B_{n-1} \otimes A[-n+1]$$

where $A[n]$ is the shifted module $A[n]_m = A_{n+m}$.

Definition 2.19. The graded algebra A is *Koszul* if the sequence

$$\dots \to B_n \otimes A[-n] \to B_{n-1} \otimes A[-n+1] \to$$
$$\dots \to B_1 \otimes A[-1] \to A \to \Bbbk \to 0 \quad (2.12)$$

is exact. △

A line bundle $\mathcal{L}$ on a projective variety X is said to be *Koszul* if its associated homogeneous coordinate ring

$$A = \bigoplus_{n\in\mathbb{N}} H^0(X, n\mathcal{L})$$

is Koszul. Theorem 2 in [33] implies that $\mathcal{L}^k$ is Koszul for k big enough if $\mathcal{L}$ is ample.

By composing the morphisms $B_n \to B_{n-1}\otimes A_1$ (tensored by $\mathcal{O}_X$) and $A_1 \otimes \mathcal{O}_X \to \mathcal{L}$, one has morphisms $\psi_n \colon B_n \otimes \mathcal{O}_X \to B_{n-1}\otimes\mathcal{L}$. So we can define sheaves $\mathcal{R}_n$ on X (with $n\in\mathbb{N}$) as $\mathcal{R}_n = \ker\psi_n$. We have an exact sequence

$$0 \to \mathcal{R}_n \to B_n \otimes \mathcal{O}_X \to B_{n-1}\otimes\mathcal{L} \to \cdots \to B_1 \otimes \mathcal{L}^{n-1} \to \mathcal{L}^n \to 0\,. \tag{2.13}$$

Lemma 2.20. [176] *There is an exact sequence*

$$0 \to A_0 \otimes \mathcal{R}_n \to A_1 \otimes \mathcal{R}_{n-1} \to \cdots \to A_{n-1}\otimes\mathcal{R}_1 \to A_n \otimes \mathcal{R}_0 \to \mathcal{L}^n \to 0\,. \tag{2.14}$$

Proof. For any given $n \geq 0$ we introduce a double complex of sheaves

$$\mathcal{G}^{p,q}_{(n)} = \begin{cases} A_p \otimes B_{n-p-q}\otimes\mathcal{L}^q & \text{for} \quad p,\, q,\, n-p-q \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

The differentials δ_1, δ_2 are induced by the morphisms in the sequences (2.12) and (2.13). If one studies the associated spectral sequences $'E_{(n)}$, $''E_{(n)}$ one sees that the second spectral sequence degenerates at the first step, and

$$''E^{p,q}_{(n)1} = \begin{cases} \mathcal{L}^n & \text{for } p=0,\ q=n \\ 0 & \text{otherwise.} \end{cases}$$

Therefore the cohomology of the total complex $\mathcal{T}^\bullet$ is $H^n(\mathcal{T}^\bullet) = \mathcal{L}^n$ in degree n and 0 otherwise.

The first spectral sequence at first step is

$$'E^{p,q}_{(n)1} = \begin{cases} A_p \otimes \mathcal{R}_{n-p} & \text{for } q=0 \\ 0 & \text{otherwise} \end{cases}$$

and degenerates at the second step. This implies that

$$\ker(A_p\otimes\mathcal{R}_{n-p} \xrightarrow{\delta_1} A_{p+1}\otimes\mathcal{R}_{n-p-1}) = \mathrm{im}(A_{p-1}\otimes\mathcal{R}_{n-p+1} \xrightarrow{\delta_1} A_p \otimes \mathcal{R}_{n-p})$$

for $p < n$, and

$$\mathcal{L}^n \simeq (A_n\otimes\mathcal{R}_0)/\,\mathrm{im}(A_{n-1}\otimes\mathcal{R}_1 \xrightarrow{\delta_1} A_n\otimes\mathcal{R}_0)$$

so that the exactness of (2.14) follows. □

Proposition 2.21. [176] *Let X be a projective variety, $\Delta \hookrightarrow X \times X$ the diagonal of the product, and let $\mathcal{L}$ be a Koszul line bundle on X. Define sheaves $\mathcal{R}_m$ for every integer $m \geq 0$ as before. Then there is an exact sequence*

$$\cdots \to \pi_1^*\mathcal{L}^{-m} \otimes \pi_2^*\mathcal{R}_m \xrightarrow{d_m} \pi_1^*\mathcal{L}^{-(m-1)} \otimes \pi_2^*\mathcal{R}_{m-1} \xrightarrow{d_{m-1}} \dots$$
$$\xrightarrow{d_2} \pi_1^*\mathcal{L}^{-1} \otimes \pi_2^*\mathcal{R}_1 \xrightarrow{d_1} \pi_1^*\mathcal{O}_X \otimes \pi_2^*\mathcal{O}_X \xrightarrow{\delta} \mathcal{O}_\Delta \to 0$$

of coherent sheaves on $X \times X$. Moreover, we can choose $\mathcal{L}$ so that $H^i(X, \mathcal{L}^s) = 0$ for ever $i > 0$ and $s \geq 1$.

Proof. Let us define the complex $\mathcal{F}^\bullet$

$$\begin{aligned} \mathcal{F}^m &= \pi_1^*\mathcal{L}^{m+1} \otimes \pi_2^*\mathcal{R}_{-m-1} \quad \text{for} \quad m \leq -1 \\ \mathcal{F}^0 &= \mathcal{O}_\Delta \\ \mathcal{F}^m &= 0 \quad \text{for} \quad m > 0\,. \end{aligned}$$

Let m_0 be a (nonpositive) integer such that $\mathcal{H}^{m_0}(\mathcal{F}^\bullet) \neq 0$ and let k be an integer which is big enough to ensure that

$$\begin{aligned} &R^p\pi_{2*}(\mathcal{H}^q(\mathcal{F}^\bullet) \otimes \pi_1^*\mathcal{L}^k) = 0 \quad \text{for} \quad p > 0, q \geq m_0 - \dim X \\ &H^p(X, \mathcal{L}^{k+q}) = 0 \quad \text{for} \quad p > 0, q \geq m_0 - \dim X \\ &R^0\pi_{2*}(\mathcal{H}^{m_0}(\mathcal{F}^\bullet) \otimes \pi_1^*\mathcal{L}^k) \neq 0. \end{aligned}$$

We may associate two spectral sequences to these data. The first has second term

$$'E_2^{p,q} = R^p\pi_{2*}(\mathcal{H}^q(\mathcal{F}^\bullet) \otimes p_1^*\mathcal{L}^k)$$

and converges to $R^\bullet\pi_{2*}(\mathcal{F}^\bullet \otimes \pi_1^*\mathcal{L}^k)$. One has $'E_2^{p,q} = 0$ for $p > 0$ and $q \geq m_0 - \dim X$ while $'E_2^{0,m_0} \neq 0$, so that $R^{m_0}\pi_{2*}(\mathcal{F}^\bullet \otimes \pi_1^*\mathcal{L}^k) \neq 0$.

The second sequence has first term

$$''E_1^{p,q} = R^q\pi_{2*}(\mathcal{F}^p \otimes \pi_1^*\mathcal{L}^k)$$

and converges to $R^\bullet\pi_{2*}(\mathcal{F}^\bullet \otimes \pi_1^*\mathcal{L}^k)$. One has

$$''E_1^{p,q} = \begin{cases} \mathcal{L}^k & \text{for } p = 0 \text{ and } q = 0 \\ H^q(X, \mathcal{L}^{k+p+1}) \otimes \mathcal{R}_{-p-1} & \text{for } p \leq -1 \\ 0 & \text{otherwise.} \end{cases}$$

As a consequence, $''E_1^{p,0} = A_{k+p+1} \otimes \mathcal{R}_{-p-1}$ for $p \leq -1$ and $''E_1^{p,q} = 0$ for $p+1 \geq m_0 - \dim X$ and $q > 0$. However Lemma 2.20 implies that $''E_2^{p,q} = 0$ for $p+q = m_0$, a contradiction. □

By reasons that will be evident later on, we need to consider the truncated complex

$$\mathcal{C}^\bullet{}_{(m)} \equiv \pi_1^*\mathcal{L}^{-m} \otimes \pi_2^*\mathcal{R}_m \xrightarrow{d_m} \dots \xrightarrow{d_2} \pi_1^*\mathcal{L}^{-1} \otimes \pi_2^*\mathcal{R}_1 \xrightarrow{d_1} \pi_1^*\mathcal{O}_X \otimes \pi_2^*\mathcal{O}_X\,. \quad (2.15)$$

Let $T_m = \ker d_m$ $(m \geq 1)$ and let $\alpha_m\colon T_m[m] \to \mathcal{C}^\bullet{}_{(m)}$ be the morphism induced by the immersion $T_m \hookrightarrow \pi_1^*\mathcal{L}^{-m} \otimes \pi_2^*\mathcal{R}_m$. Then the cone $\mathrm{Cone}(\alpha_m)$ is isomorphic to the complex

$$T_m \hookrightarrow \pi_1^*\mathcal{L}^{-m} \otimes \pi_2^*\mathcal{R}_m \xrightarrow{d_m} \dots \xrightarrow{d_2} \pi_1^*\mathcal{L}^{-1} \otimes \pi_2^*\mathcal{R}_1 \xrightarrow{d_1} \pi_1^*\mathcal{O}_X \otimes \pi_2^*\mathcal{O}_X$$

with T_m at the $-(m+1)$-th place, and is then quasi-isomorphic to $\mathcal{O}_\Delta$.

Assume now that X is smooth and choose $m \geq 2\dim X$. Then in the exact triangle

$$T_m[m] \xrightarrow{\alpha_m} \mathcal{C}^\bullet{}_{(m)} \to \mathrm{Cone}(\alpha_m) \simeq \mathcal{O}_\Delta \to T_m[m+1]$$

the last morphism vanishes, since $\mathrm{Hom}_{D(X)}(\mathcal{O}_\Delta, T_m[m+1]) \simeq \mathrm{Ext}^{m+1}(\mathcal{O}_\Delta, T_m) = 0$ because $m+1 > 2\dim X$ and X is smooth. As a consequence, $\mathcal{C}^\bullet{}_{(m)}$ is a biproduct of $\mathcal{O}_\Delta$ and $T_m[m]$ in the derived category,

$$\mathcal{C}^\bullet{}_{(m)} \simeq \mathcal{O}_\Delta \oplus T_m[m]\,.$$

Moreover if we call $d_0\colon \mathcal{O}_{X\times X} \to \mathbf{c}_{(m)}$ the convolution of $\mathcal{C}^\bullet{}_{(m)}$ in $D^b(X \times X)$ (which is $\mathcal{C}^\bullet{}_{(m)}$ itself, see Section 2.1.2), we can write the above formula in the following form, which we shall shortly use.

$$\mathbf{c}_{(m)} \simeq \mathcal{O}_\Delta \oplus T_m[m]\,. \quad (2.16)$$

Let us consider now the complex

$$\pi_1^*\mathcal{L}^{sk} \otimes \mathcal{C}^\bullet{}_{(m)} \simeq \pi_1^*\mathcal{L}^{sk-m} \otimes \pi_2^*\mathcal{R}_m \xrightarrow{d'_m} \dots \xrightarrow{d'_2} \pi_1^*\mathcal{L}^{sk-1} \otimes \pi_2^*\mathcal{R}_1 \xrightarrow{d'_1} \pi_1^*\mathcal{L}^{sk} \otimes \pi_2^*\mathcal{O}_X\,, \quad (2.17)$$

where $d'_p = 1 \otimes d_p$. By Remark 2.14 we have

$$\begin{aligned}\pi_1^*\mathcal{L}^{sk} \otimes \pi_2^*\mathcal{O}_X \xrightarrow{d'_0} & \pi_1^*\mathcal{L}^{sk} \otimes \mathbf{c}_{(m)} \\ \simeq & (\pi_1^*\mathcal{L}^{sk} \otimes \mathcal{O}_\Delta) \oplus ((\pi_1^*\mathcal{L}^{sk} \otimes T_m)[m])\,,\end{aligned}$$

where $d'_0 = 1 \otimes d_0$, is a convolution of $\pi_1^*\mathcal{L}^{sk} \otimes \mathcal{C}^\bullet{}_{(m)}$.

Now fix a value of m (to be specified later) and let $s > m$. Then all sheaves $\mathcal{L}^{sk-p}$ $(0 \leq p \leq m)$ are acyclic, so that after applying $\mathbf{R}\pi_{2*}$ to the complex $\pi_1^*\mathcal{L}^{sk} \otimes \mathcal{C}^\bullet{}_{(m)}$ we obtain the complex

$$\begin{aligned}\mathbf{R}\pi_{2*}(\pi_1^*\mathcal{L}^{sk} \otimes \mathcal{C}^\bullet{}_{(m)}) \simeq \Gamma(X, \mathcal{L}^{sk-m}) \otimes_{\Bbbk} \mathcal{R}_m \xrightarrow{\pi_{2*}(d'_m)} \dots \\ \xrightarrow{\pi_{2*}(d'_2)} \Gamma(X, \mathcal{L}^{sk-1}) \otimes_{\Bbbk} \mathcal{R}_1 \xrightarrow{\pi_{2*}(d'_1)} \Gamma(X, \mathcal{L}^{sk}) \otimes_{\Bbbk} \mathcal{O}_X \,. \end{aligned} \tag{2.18}$$

Again by Remark 2.14,

$$\begin{aligned}\Gamma(X, \mathcal{L}^{sk}) \otimes_{\Bbbk} \mathcal{O}_X \xrightarrow{\mathbf{R}\pi_{2*}(d'_0)} \mathbf{R}\pi_{2*}(\pi_1^*\mathcal{L}^{sk} \otimes \mathbf{c}_{(m)}) \\ \simeq \mathcal{L}^{sk} \oplus \mathbf{R}\pi_{2*}(\pi_1^*\mathcal{L}^{sk} \otimes T_m[m])\end{aligned}$$

is a convolution of $\mathbf{R}\pi_{2*}(\pi_1^*\mathcal{L}^{sk} \otimes \mathcal{C}^\bullet{}_{(m)})$. Recall that the isomorphism in the formula holds when X is smooth and $m > 2\dim X - 1$.

We finish this section with a lemma that generalizes the argument used in the proof of (2.16).

Lemma 2.22. *Let X be a smooth variety.*

1. *If $\mathcal{E}^\bullet$, $\mathcal{F}^\bullet$ are objects of $D^b(X)$ such that $\mathcal{H}^i(\mathcal{E}^\bullet) = 0$ for $i > z$ and $\mathcal{H}^q(\mathcal{F}^\bullet) = 0$ for $q \leq z + 2\dim X$ for an integer z, then* $\operatorname{Hom}_{D(X)}(\mathcal{F}^\bullet, \mathcal{E}^\bullet) = 0$.

2. *Let $\mathcal{E}^\bullet$ be an object of $D^b(X)$. If there exist integers z and $s > 2\dim X$ such that $\mathcal{H}^i(\mathcal{E}^\bullet) = 0$ for $z < i \leq z + s$, then for every $p \in [z, z+s]$ one has an isomorphism*
$$\mathcal{E}^\bullet \simeq \mathcal{E}^\bullet{}_{\leq p} \oplus \mathcal{E}^\bullet{}_{\geq p}$$
in $D^b(X)$.

Proof. 1. We proceed by induction on the sum $n = \ell(\mathcal{E}^\bullet) + \ell(\mathcal{F}^\bullet)$ of the lengths of $\mathcal{E}^\bullet$ and $\mathcal{F}^\bullet$ (recall that the length is the number of nonzero cohomology sheaves of $\mathcal{E}^\bullet$ and $\mathcal{F}^\bullet$). The first case is $n = 2$, since we can assume that $\mathcal{E}^\bullet$ and $\mathcal{F}^\bullet$ are nonzero. Then $\mathcal{E}^\bullet \simeq \mathcal{E}[-q]$ for $q \leq z$ and $\mathcal{F}^\bullet \simeq \mathcal{F}[-m]$ for $m > z + 2\dim X$, where $\mathcal{E}$ and $\mathcal{F}$ are sheaves. It follows that $\operatorname{Hom}_{D(X)}(\mathcal{F}^\bullet, \mathcal{E}^\bullet) \simeq \operatorname{Hom}_{D(X)}(\mathcal{F}, \mathcal{E}[m-q]) \simeq \operatorname{Ext}_X^{m-q}(\mathcal{F}, \mathcal{E})$ and this is zero because $m - q > 2\dim X$ and X is smooth.

Take $n > 2$. Then either $\mathcal{E}^\bullet$ or $\mathcal{F}^\bullet$ have at least two nonzero cohomology sheaves. If $\ell(\mathcal{F}^\bullet) \geq 2$, let q_0 be the first of the indexes q such that $\mathcal{H}^q(\mathcal{F}^\bullet) \neq 0$. Then there is a exact triangle

$$\mathcal{F}^\bullet{}_{\leq q_0} \xrightarrow{\alpha} \mathcal{F}^\bullet \to \operatorname{Cone}(\alpha) \to \mathcal{F}^\bullet{}_{\leq q_0}[1] \,.$$

Now $\mathcal{H}^q(\mathcal{F}^\bullet{}_{\le q_0}) = 0$ for $q \ne q_0$ and $\mathcal{H}^q(\mathrm{Cone}(\alpha)) = 0$ for $q \le q_0$, so that they are in the same hypotheses as $\mathcal{F}^\bullet$. Moreover $\ell(\mathcal{E}^\bullet)+\ell(\mathcal{F}^\bullet{}_{\le q_0}) = \ell(\mathcal{E}^\bullet)+1 < n$ and $\ell(\mathcal{E}^\bullet)+\ell(\mathrm{Cone}(\alpha) = n-1$. Then $\mathrm{Hom}_{D(X)}(\mathrm{Cone}(\alpha), \mathcal{E}^\bullet) = 0$ and $\mathrm{Hom}_{D(X)}(\mathcal{F}^\bullet{}_{\le q_0}, \mathcal{E}^\bullet) = 0$ by induction so that $\mathrm{Hom}_{D(X)}(\mathcal{F}^\bullet, \mathcal{E}^\bullet) = 0$. If $\ell(\mathcal{E}^\bullet) \ge 2$, we take i_0 as the first of the integers i such that $\mathcal{H}^i(\mathcal{E}^\bullet) \ne 0$ and proceed as above using the exact triangle

$$\mathcal{E}^\bullet{}_{\le i_0} \xrightarrow{\alpha} \mathcal{E}^\bullet \to \mathrm{Cone}(\alpha) \to \mathcal{E}^\bullet{}_{\le i_0}[1]\,.$$

2. There is a exact triangle

$$\mathcal{E}^\bullet{}_{\le p} \xrightarrow{\alpha} \mathcal{E}^\bullet \to \mathrm{Cone}(\alpha) \simeq \mathcal{E}^\bullet{}_{\ge p} \to \mathcal{E}^\bullet{}_{\le p}[1]\,,$$

where the isomorphism $\mathrm{Cone}(\alpha) \simeq \mathcal{E}^\bullet{}_{\ge p}$ is a consequence of $\mathcal{H}^p(\mathcal{E}^\bullet) = 0$. Moreover $\mathcal{H}^i(\mathcal{E}^\bullet{}_{\le p}[1]) \simeq \mathcal{H}^{i+1}(\mathcal{E}^\bullet{}_{\le p}) = 0$ for $i > z$ and $\mathcal{H}^q(\mathcal{E}^\bullet{}_{\ge p}) = 0$ for $q \le z + s \le z + 2\dim X$. Thus $\mathrm{Hom}_{D(X)}(\mathcal{E}^\bullet{}_{\ge p}, \mathcal{E}^\bullet{}_{\le p}[1]) = 0$ by the first part, and then $\mathcal{E}^\bullet$ decomposes as a biproduct in the derived category as claimed. □

2.2.2 Uniqueness of the kernel

As a first step in the proof of Orlov's theorem, and for later use, we want to prove that an integral functor completely determines its kernel. More precisely, let $\mathcal{K}^\bullet$ be a kernel in $D^-(X \times Y)$; we denote simply by Φ the associated integral functor $\Phi^{\mathcal{K}^\bullet}_{X\to Y}$. By using Proposition 1.10 and the base change compatibility of the integral functors (Proposition 1.8), we shall prove that $\mathcal{K}^\bullet$ is completely determined by Φ.

To do so we slightly change notation from Section 1.2. Let us write π_{ij} for the projection of $X \times X \times Y$ onto its (i, j)-th factor. Then by Proposition 1.10 the kernel $\mathcal{K}^\bullet$ can be recovered as

$$\mathcal{K}^\bullet \simeq \Phi^{\pi_{23}^*\mathcal{K}^\bullet}_{X\times X\to X\times Y}(\mathcal{O}_\Delta) = \Phi^{\mathcal{K}^\bullet_X}(\mathcal{O}_\Delta)$$

where we have written $\Phi^{\mathcal{K}^\bullet_X} = \Phi^{\pi_{23}^*\mathcal{K}^\bullet}_{X\times X\to X\times Y}$ for simplicity.

In the rest of this section we assume that X is smooth and that $\mathcal{K}^\bullet$ is of finite Tor-dimension as a complex of $\mathcal{O}_X$-modules (cf. Section 1.2), so that Φ maps $D^b(X)$ into $D^b(Y)$. Then $\Phi^{\mathcal{K}^\bullet_X}$ maps $D^b(X \times X)$ into $D^b(X \times Y)$.

Lemma 2.23. *There exist integer numbers p, m_0 such that for every $m > m_0$, one has a natural isomorphism*

$$\mathcal{K}^\bullet \simeq (\Phi^{\mathcal{K}^\bullet_X}(\mathbf{c}_{(m)}))_{\ge p}\,.$$

Proof. By (2.16), we have $\Phi^{\mathcal{K}^\bullet_X}(\mathbf{c}_{(m)}) \simeq \Phi^{\mathcal{K}^\bullet_X}(\mathcal{O}_\Delta)\oplus\Phi^{\mathcal{K}^\bullet_X}(T_M[m])$ for $m \ge 2\dim X$. Now, Proposition 1.4 for $\Phi^{\mathcal{K}^\bullet_X}$ implies the existence of integer numbers z and

$n \geq 0$ depending only on Φ such that the cohomology sheaves $\mathcal{H}^i(\Phi^{\mathcal{K}^\bullet_X}(\mathcal{O}_\Delta))$ vanish for $i \notin [z, z+n]$ and the cohomology sheaves $\mathcal{H}^i(\Phi^{\mathcal{K}^\bullet_X}(T_m[m]))$ vanish for $i \notin [z-m, z+n-m]$. Now take $p < z$ and $m_0 = \max\{2\dim X, z+n-p\}$. Then the natural morphism $\Phi^{\mathcal{K}^\bullet_X}(\mathcal{O}_\Delta) \to (\Phi^{\mathcal{K}^\bullet_X}(\mathcal{O}_\Delta))_{\geq p}$ is an isomorphism in the derived category and $\Phi^{\mathcal{K}^\bullet_X}(\mathbf{c}_{(m)}) \to \Phi^{\mathcal{K}^\bullet_X}(\mathcal{O}_\Delta)$ induces an isomorphism $\Phi^{\mathcal{K}^\bullet_X}(\mathbf{c}_{(m)})_{\geq p} \simeq (\Phi^{\mathcal{K}^\bullet_X}(\mathcal{O}_\Delta))_{\geq p}$ for $m \geq m_0$ as desired. □

Let us prove that $\Phi^{\mathcal{K}^\bullet_X}(\mathbf{c}_{(m)})$ depends only on Φ. Since $\mathbf{c}_{(m)}$ is a convolution of $\mathcal{C}^\bullet{}_{(m)}$, we consider the complex

$$\Phi^{\mathcal{K}^\bullet_X}(\pi_1^*\mathcal{L}^{-m} \otimes \pi_2^*\mathcal{R}_m) \xrightarrow{\Phi^{\mathcal{K}^\bullet_X}(d_m)} \cdots \to \Phi^{\mathcal{K}^\bullet_X}(\pi_1^*\mathcal{L}^{-1} \otimes \pi_2^*\mathcal{R}_1) \xrightarrow{\Phi^{\mathcal{K}^\bullet_X}(d_1)} \Phi^{\mathcal{K}^\bullet_X}(\pi_1^*\mathcal{O}_X \otimes \pi_2^*\mathcal{O}_X)$$

of objects of $D^b(X \times Y)$, obtained by applying $\Phi^{\mathcal{K}^\bullet_X}$ to the complex $\mathcal{C}^\bullet{}_{(m)}$. By base change (Proposition 1.8) one has isomorphisms

$$\Phi^{\mathcal{K}^\bullet_X}(\pi_1^*\mathcal{L}^{-i} \otimes \pi_2^*\mathcal{R}_i) \simeq \pi_1^*\mathcal{L}^{-i} \otimes \pi_2^*\Phi(\mathcal{R}_i)$$

so that one also has a complex

$$\pi_1^*\mathcal{L}^{-m} \otimes \pi_2^*\Phi(\mathcal{R}_m) \xrightarrow{\Phi(d_m)} \cdots \to \pi_1^*\mathcal{L}^{-1} \otimes \pi_2^*\Phi(\mathcal{R}_1) \xrightarrow{\Phi(d_1)} \pi_1^*\mathcal{O}_X \otimes \pi_2^*\Phi(\mathcal{O}_X)\,. \tag{2.19}$$

The objects of the complex depend only on Φ and not on $\Phi^{\mathcal{K}^\bullet_X}$; we are going to show that under appropriate conditions, even the morphisms depend only on Φ.

For every pair of indices $p > q$ and integer $r \geq 0$, we have

$$\begin{aligned} &\operatorname{Hom}_{D^b(X\times Y)}(\pi_1^*\mathcal{L}^{-p} \otimes \pi_2^*\Phi(\mathcal{R}_p)[r], \pi_1^*\mathcal{L}^{-q} \otimes \pi_2^*\Phi(\mathcal{R}_q)) \\ &\qquad \simeq \operatorname{Hom}_{D^b(X\times Y)}(\pi_2^*\Phi(\mathcal{R}_p)[r], \pi_1^*\mathcal{L}^{p-q} \otimes \pi_2^*\Phi(\mathcal{R}_q)) \\ &\qquad \simeq \operatorname{Hom}_{D^b(Y)}(\Phi(\mathcal{R}_p)[r], \pi_{2*}(\pi_1^*\mathcal{L}^{p-q}) \otimes \Phi(\mathcal{R}_q)) \\ &\qquad \simeq \operatorname{Hom}_{D^b(Y)}(\Phi(\mathcal{R}_p)[r], \Gamma(X, \mathcal{L}^{p-q}) \otimes_{\Bbbk} \Phi(\mathcal{R}_q)) \\ &\qquad \simeq \operatorname{Hom}_{D^b(Y)}(\Phi(\mathcal{R}_p)[r], \Phi(\Gamma(X, \mathcal{L}^{p-q}) \otimes_{\Bbbk} \mathcal{R}_q)) \end{aligned} \tag{2.20}$$

where we have used adjunction between inverse and direct images and the projection formula. Analogously one has

$$\begin{aligned} &\operatorname{Hom}_{D^b(X\times Y)}(\pi_1^*\mathcal{L}^{-p} \otimes \pi_2^*\mathcal{R}_p[r], \pi_1^*\mathcal{L}^{-(q)} \otimes \pi_2^*\mathcal{R}_q) \\ &\qquad \simeq \operatorname{Hom}_{D^b(X\times Y)}(\pi_2^*\mathcal{R}_p[r], \pi_1^*\mathcal{L}^{p-q} \otimes \pi_2^*\mathcal{R}_q) \\ &\qquad \simeq \operatorname{Hom}_{D^b(Y)}(\mathcal{R}_p[r], \pi_{2*}(\pi_1^*\mathcal{L}^{p-q}) \otimes \mathcal{R}_q) \\ &\qquad \simeq \operatorname{Hom}_{D^b(Y)}(\mathcal{R}_p[r], \Gamma(X, \mathcal{L}^{p-q}) \otimes_{\Bbbk} \mathcal{R}_q)\,. \end{aligned} \tag{2.21}$$

Then the natural morphism

$$\begin{multline}\operatorname{Hom}_{D^b(X)}(\mathcal{R}_p[r], \Gamma(X, \mathcal{L}^{p-q}) \otimes_{\Bbbk} \mathcal{R}_q) \xrightarrow{\Phi} \\ \operatorname{Hom}_{D^b(Y)}(\Phi(\mathcal{R}_p)[r], \Phi(\Gamma(X, \mathcal{L}^{p-q}) \otimes_{\Bbbk} \mathcal{R}_q)) \quad (2.22)\end{multline}$$

induces a morphism

$$\begin{multline}\operatorname{Hom}_{D^b(X\times Y)}(\pi_1^*\mathcal{L}^{-p} \otimes \pi_2^*\mathcal{R}_p[r], \pi_1^*\mathcal{L}^{-q} \otimes \pi_2^*\mathcal{R}_q) \xrightarrow{\Phi} \\ \operatorname{Hom}_{D^b(X\times Y)}(\pi_1^*\mathcal{L}^{-p} \otimes \pi_2^*\Phi(\mathcal{R}_p)[r], \pi_1^*\mathcal{L}^{-q} \otimes \pi_2^*\Phi(\mathcal{R}_q))\,. \quad (2.23)\end{multline}$$

Taking in particular $q = p - 1$ and $r = 0$, we see that $\Phi^{\mathcal{K}^\bullet_X}(d_p) = \Phi(d_p)$. We have thus proved the following result.

Lemma 2.24. *There is an isomorphism*

$$\begin{tikzcd} \pi_1^*\mathcal{L}^{-m} \otimes \pi_2^*\Phi(\mathcal{R}_m) \arrow[r, "\Phi(d_m)"] \arrow[d, "\simeq"] & \cdots \arrow[r] & \pi_1^*\mathcal{L}^{-1} \otimes \pi_2^*\Phi(\mathcal{R}_1) \arrow[r, "\Phi(d_1)"] \arrow[d, "\simeq"] & \pi_1^*\mathcal{O}_X \otimes \pi_2^*\Phi(\mathcal{O}_X) \arrow[d, "\simeq"] \\ \Phi^{\mathcal{K}^\bullet_X}(\pi_1^*\mathcal{L}^{-m} \otimes \pi_2^*\mathcal{R}_m) \arrow[r, "\Phi^{\mathcal{K}^\bullet_X}(d_m)"] & \cdots \arrow[r] & \Phi^{\mathcal{K}^\bullet_X}(\pi_1^*\mathcal{L}^{-1} \otimes \pi_2^*\mathcal{R}_1) \arrow[r, "\Phi^{\mathcal{K}^\bullet_X}(d_1)"] & \Phi^{\mathcal{K}^\bullet_X}(\pi_1^*\mathcal{O}_X \otimes \pi_2^*\mathcal{O}_X) \end{tikzcd}$$

of complexes of objects of $D^b(X \times Y)$. As a consequence, the image under $\Phi^{\mathcal{K}^\bullet_X}$ of the complex ${\mathcal{C}^\bullet}_{(m)}$ given by (2.15) *depends only on Φ and not on $\Phi^{\mathcal{K}^\bullet_X}$.*

Assume that Φ is fully faithful. Then (2.23) implies that the complex

$$\pi_1^*\mathcal{L}^{-m} \otimes \pi_2^*\Phi(\mathcal{R}_m) \xrightarrow{\Phi(d_m)} \ldots \xrightarrow{\pi_1^*} \mathcal{L}^{-1} \otimes \pi_2^*\Phi(\mathcal{R}_1) \xrightarrow{\Phi(d_1)} \pi_1^*\mathcal{O}_X \otimes \pi_2^*\Phi(\mathcal{O}_X)$$

has a convolution, which we denote $\pi_1^*\mathcal{O}_X \otimes \pi_2^*\Phi(\mathcal{O}_X) \xrightarrow{\mathbf{d}_0} (1 \otimes \Phi)(\mathbf{c}_{(m)})$. Moreover, by Remark 2.14 a convolution of the complex (2.19) exists and is given by

$$\Phi^{\mathcal{K}^\bullet_X}(\pi_1^*\mathcal{O}_X \otimes \pi_2^*\mathcal{O}_X) \xrightarrow{\Phi^{\mathcal{K}^\bullet_X}(d_0)} \Phi^{\mathcal{K}^\bullet_X}(\mathbf{c}_{(m)})\,.$$

Now by Lemma 2.12 there is commutative diagram

$$\begin{tikzcd} \pi_1^*\mathcal{O}_X \otimes \pi_2^*\Phi(\mathcal{O}_X) \arrow[r, "\mathbf{d}_0"] \arrow[d, "\simeq"] & (1 \otimes \Phi)(\mathbf{c}_{(m)}) \arrow[r, "\alpha_{\geq p}"] \arrow[d, "\gamma"', "\simeq"] & ((1 \otimes \Phi)(\mathbf{c}_{(m)}))_{\geq p} \arrow[d, "\gamma_{\geq p}"', "\simeq"] \\ \Phi^{\mathcal{K}^\bullet_X}(\pi_1^*\mathcal{O}_X \otimes \pi_2^*\mathcal{O}_X) \arrow[r, "\Phi^{\mathcal{K}^\bullet_X}(d_0)"] & \Phi^{\mathcal{K}^\bullet_X}(\mathbf{c}_{(m)}) \arrow[r, "\alpha_{\geq p}"] & (\Phi^{\mathcal{K}^\bullet_X}(\mathbf{c}_{(m)}))_{\geq p} \simeq \mathcal{K}^\bullet \end{tikzcd}$$

for a certain isomorphism γ (not uniquely determined). Here the last isomorphism in the bottom row is due to Lemma 2.23 and the morphisms $\alpha_{\geq p}$ are the natural epimorphisms to the truncations. Since

$$\begin{aligned}
&\mathrm{Hom}_{D^b(X\times Y)}(\pi_1^*\mathcal{L}^{-p}\otimes\pi_2^*\Phi(\mathcal{R}_p)[r],\mathcal{K}^\bullet)\\
&\qquad\simeq \mathrm{Hom}_{D^b(X\times Y)}(\pi_2^*\Phi(\mathcal{R}_p[r]),\pi_1^*\mathcal{L}^p\otimes\mathcal{K}^\bullet)\\
&\qquad\simeq \mathrm{Hom}_{D^b(Y)}(\Phi(\mathcal{R}_p)[r],\mathbf{R}\pi_{2,*}(\pi_1^*\mathcal{L}^p\otimes\mathcal{K}^\bullet))\\
&\qquad= \mathrm{Hom}_{D^b(Y)}(\Phi(\mathcal{R}_p)[r],\Phi(\mathcal{L}^p))\simeq \mathrm{Hom}_{D^b(X)}(\mathcal{R}_p[r],\mathcal{L}^p)=0
\end{aligned}$$

we deduce from Lemma 2.12 that the composed diagonal morphism $\alpha_{\geq p}\circ\gamma_{\geq p}=\alpha_{\geq p}\circ\gamma\colon(1\otimes\Phi)(\mathbf{c}_{(m)})\to\mathcal{K}^\bullet$ is the unique morphism making the diagram commutative. Then, the isomorphism $\gamma_{\geq p}$ is uniquely characterized by the commutativity of the diagram

$$\begin{tikzcd}
\pi_1^*\mathcal{O}_X\otimes\pi_2^*\Phi(\mathcal{O}_X) \arrow[r, "\alpha_{\geq p}\circ\mathbf{d}_0"] \arrow[d, "\simeq"] & ((1\otimes\Phi)(\mathbf{c}_{(m)}))_{\geq p} \arrow[d, "\gamma_{\geq p}"', "\simeq"] \\
\Phi^{\mathcal{K}^\bullet_X}(\pi_1^*\mathcal{O}_X\otimes\pi_2^*\mathcal{O}_X) \arrow[r, "\alpha_{\geq p}\circ\Phi^{\mathcal{K}^\bullet_X}(d_0)"] & (\Phi^{\mathcal{K}^\bullet_X}(\mathbf{c}_{(m)}))_{\geq p}\simeq\mathcal{K}^\bullet\,.
\end{tikzcd} \tag{2.24}$$

This eventually implies the desired uniqueness result. Let $f^{\mathcal{K}^\bullet}=\gamma\geq p\circ\alpha_{\geq p}\circ\mathbf{d}_0$.

Theorem 2.25. *Let X, Y be projective varieties, $\mathcal{K}^\bullet$ be a kernel in $D^-(X\times Y)$ and let $\Phi=\Phi^{\mathcal{K}^\bullet}_{X\to Y}$ be the corresponding integral functor. Assume that X is smooth, $\mathcal{K}^\bullet$ is of finite Tor-dimension as a complex of $\mathcal{O}_X$-modules and that Φ is fully faithful. Then the kernel $\mathcal{K}^\bullet$ is uniquely determined by the functor Φ. Moreover if $\tilde{\mathcal{K}}^\bullet$ is another kernel in $D^-(X\times Y)$, of finite Tor-dimension as a complex of $\mathcal{O}_X$-modules, such that $\Phi=\Phi^{\tilde{\mathcal{K}}^\bullet}_{X\to Y}$, there is a unique isomorphism $\eta\colon\mathcal{K}^\bullet\simeq\tilde{\mathcal{K}}^\bullet$ in $D^b(X\times Y)$ making the diagram*

$$\begin{tikzcd}
\pi_1^*\mathcal{O}_X\otimes\pi_2^*\Phi(\mathcal{O}_X) \arrow[r, "f^{\mathcal{K}^\bullet}"] \arrow[dr, "f^{\tilde{\mathcal{K}}^\bullet}"'] & \mathcal{K}^\bullet \arrow[d, "\eta"', "\simeq"] \\
& \tilde{\mathcal{K}}^\bullet
\end{tikzcd}$$

commutative.

Proof. Since $((1\otimes\Phi)(\mathbf{c}_{(m)}))_{\geq p}\simeq\mathcal{K}^\bullet$ one has that $\mathcal{K}^\bullet$ is uniquely determined by Φ. The second part follows straightforwardly from the above discussion. □

2.2.3 Existence of the kernel

In this section we conclude the proof of Orlov's Theorem 2.15 by constructing the kernel that realizes the given fully faithful functor as an integral functor. Let X,

Y be *smooth* projective varieties and $F\colon D^b(X)\to D^b(Y)$ an exact fully faithful functor. As we shall see in Proposition 2.31, the functor F is *bounded*, so that there exist integer numbers z and $n\geq 0$ such that for every coherent sheaf $\mathcal{F}$ on X, the cohomology sheaves $\mathcal{H}^i(F(\mathcal{F}))$ vanish for $i\notin[z,z+n]$. Then F can be extended to a functor $D(X)\to D(Y)$. We can consider the complex

$$\begin{aligned} F(\mathbf{R}\pi_{2*}(\pi_1^*\mathcal{L}^{sk}\otimes\mathcal{C}^\bullet{}_{(m)})) \equiv \Gamma(X,\mathcal{L}^{sk-m})\otimes_{\Bbbk} F(\mathcal{R}_m) \xrightarrow{F(\pi_{2*}(d'_m))} \dots \\ \xrightarrow{F(\pi_{2*}(d'_2))} \Gamma(X,\mathcal{L}^{sk-1})\otimes_{\Bbbk} F(\mathcal{R}_1) \xrightarrow{F(\pi_{2*}(d'_1))} \Gamma(X,\mathcal{L}^{sk})\otimes_{\Bbbk} F(\mathcal{O}_X) \end{aligned} \tag{2.25}$$

of objects in $D^b(Y)$. By Remark 2.14, the morphism

$$\Gamma(X,\mathcal{L}^{sk})\otimes_{\Bbbk} F(\mathcal{O}_X) \xrightarrow{F(\pi_{2*}(d'_0))} F(\mathbf{R}\pi_{2*}(\mathcal{L}^{sk}\otimes\mathbf{c}_{(m)}))$$

is a convolution of (2.25). Moreover for $m\geq 2\dim X$, (2.16) implies that

$$F(\mathbf{R}\pi_{2*}(\mathcal{L}^{sk}\otimes\mathbf{c}_{(m)}))\simeq F(\mathcal{L}^{sk})\oplus F(\mathbf{R}\pi_{2*}(\pi_1^*\mathcal{L}^{sk}\otimes T_m))[m]\,. \tag{2.26}$$

Let us fix $p<z$ and choose $m>\max\{z+n-p, 2\dim X\}$.

Lemma 2.26. *Take $s>m$ such that $R\pi^i_{2,*}(\pi_1^*\mathcal{L}^{sk}\otimes T_m)=0$ for $i>0$ and $k\geq 1$. Then $\mathcal{H}^i(F(\mathbf{R}\pi_{2*}(\mathcal{L}^{sk}\otimes\mathbf{c}_{(m)})))=0$ unless $i\in[z-m,z+n-m]$ or $i\in[z,z+n]$. Moreover $F(\mathbf{R}\pi_{2*}(\mathcal{L}^{sk}\otimes\mathbf{c}_{(m)}))\to F(\mathcal{L}^{sk})$ induces an isomorphism*

$$\beta_k^F: F(\mathbf{R}\pi_{2*}(\mathcal{L}^{sk}\otimes\mathbf{c}_{(m)}))_{\geq p}\simeq F(\mathcal{L}^{sk})$$

in $D^b(Y)$.

Proof. One has

$$\begin{aligned} &\mathcal{H}^i(F(\mathbf{R}\pi_{2*}(\mathcal{L}^{sk}\otimes\mathbf{c}_{(m)})))\simeq \mathcal{H}^i(F(\pi_{2*}(\pi_1^*\mathcal{L}^{sk}\otimes T_m))) \\ &\qquad\qquad \text{for } i\in[z-m,z+n-m]\,, \\ &\mathcal{H}^i(F(\mathbf{R}\pi_{2*}(\mathcal{L}^{sk}\otimes\mathbf{c}_{(m)})))\simeq \mathcal{H}^i(F(\mathcal{L}^{sk}))\quad \text{for } i\in[z,z+n]\,, \\ &\mathcal{H}^i(F(\mathbf{R}\pi_{2*}(\mathcal{L}^{sk}\otimes\mathbf{c}_{(m)})))=0 \\ &\qquad\qquad \text{for the remaining values of } i. \end{aligned}$$

The result follows. ∎

Let us take $m>n+2(1+\dim X+\dim Y)$ so that we can apply Lemma 2.26. We consider the truncated complex $\mathcal{C}^\bullet{}_{(m)}$ given by (2.15). Since F is fully faithful, we have an isomorphism

$$\begin{aligned} &\operatorname{Hom}_{D^b(X)}(\mathcal{R}_p[r],\Gamma(X,\mathcal{L}^{p-q})\otimes_{\Bbbk}\mathcal{R}_q)\xrightarrow{F} \\ &\qquad\qquad \operatorname{Hom}_{D^b(Y)}(F(\mathcal{R}_p)[r],F(\Gamma(X,\mathcal{L}^{p-q})\otimes_{\Bbbk}\mathcal{R}_q))\,. \end{aligned}$$

Then (2.20) remains true if we replace Φ by F, and one obtains, as in Section 2.2.2, an isomorphism

$$
\begin{aligned}
F\colon \operatorname{Hom}_{D^b(X\times Y)}(\pi_1^*\mathcal{L}^{-p}\otimes\pi_2^*\mathcal{R}_p[r],\pi_1^*\mathcal{L}^{-q}\otimes\pi_2^*\mathcal{R}_q) &\simeq \\
\operatorname{Hom}_{D^b(X\times Y)}(\pi_1^*\mathcal{L}^{-p}\otimes\pi_2^*F(\mathcal{R}_p)[r],\pi_1^*\mathcal{L}^{-q}\otimes\pi_2^*F(\mathcal{R}_q))\,. &\quad (2.27)
\end{aligned}
$$

Taking in particular $q=p-1$ and $r=0$, we see that d_p induces a morphism $F(d_p)\colon \pi_1^*\mathcal{L}^{-p}\otimes\pi_2^*F(\mathcal{R}_p)\to\pi_1^*\mathcal{L}^{-(p-1)}\otimes\pi_2^*F(\mathcal{R}_{p-1})$ and we have a complex

$$
\begin{aligned}
(1\otimes F)(\mathcal{C}^\bullet{}_{(m)})\equiv\pi_1^*\mathcal{L}^{-m}\otimes\pi_2^*F(\mathcal{R}_m)\xrightarrow{F(d_m)}\cdots\to \\
\pi_1^*\mathcal{L}^{-1}\otimes\pi_2^*F(\mathcal{R}_1)\xrightarrow{F(d_1)}\pi_1^*\mathcal{O}_X\otimes\pi_2^*F(\mathcal{O}_X) \quad (2.28)
\end{aligned}
$$

of objects of $D^b(X\times Y)$. Furthermore, (2.27) implies that there exists a convolution $\pi_1^*\mathcal{O}_X\otimes\pi_2^*F(\mathcal{O}_X)\xrightarrow{F(d_0)}\mathbf{Fc}_{(m)}$ of $(1\otimes F))(\mathcal{C}^\bullet{}_{(m)})$.

Analogously for any s and k, one has a complex

$$
\begin{aligned}
\pi_1^*\mathcal{L}^{sk}\otimes(1\otimes F)(\mathcal{C}^\bullet{}_{(m)})\equiv\pi_1^*\mathcal{L}^{sk-m}\otimes\pi_2^*F(\mathcal{R}_m)\xrightarrow{F(d_m)}\cdots\to \\
\pi_1^*\mathcal{L}^{sk-1}\otimes\pi_2^*F(\mathcal{R}_1)\xrightarrow{F(d_1)}\pi_1^*\mathcal{L}^{sk}\otimes\pi_2^*F(\mathcal{O}_X) \quad (2.29)
\end{aligned}
$$

of objects of $D^b(X\times Y)$, and $\pi_1^*\mathcal{L}^{sk}\otimes\pi_2^*F(\mathcal{O}_X)\xrightarrow{1\otimes F(d_0)}\pi_1^*\mathcal{L}^{sk}\otimes\mathbf{Fc}_{(m)}$ is a convolution of $\pi_1^*\mathcal{L}^{sk}\otimes(1\otimes F)(\mathcal{C}^\bullet{}_{(m)})$.

Lemma 2.27. *1. There is a (not uniquely determined) isomorphism*

$$\eta_k\colon \Phi_{X\to Y}^{\mathbf{Fc}_{(m)}}(\mathcal{L}^{sk})=\mathbf{R}\pi_{2*}(\pi_1^*\mathcal{L}^{sk}\otimes\mathbf{Fc}_{(m)})\xrightarrow{\sim}F(\mathbf{R}\pi_{2*}(\pi_1^*\mathcal{L}^{sk}\otimes\mathbf{Fc}_{(m)}))$$

in the derived category $D^b(Y)$ which makes the diagram

$$
\begin{tikzcd}
\pi_{2*}(\pi_1^*\mathcal{L}^{sk}\otimes\pi_2^*F(\mathcal{O}_X)) \arrow[r, "\mathbf{R}\pi_{2*}(1\otimes F(d_0))"] \arrow[d, "\simeq"] & \mathbf{R}\pi_{2*}(\pi_1^*\mathcal{L}^{sk}\otimes\mathbf{Fc}_{(m)}) = \Phi_{X\to Y}^{\mathbf{Fc}_{(m)}}(\mathcal{L}^{sk}) \arrow[d, "\eta_k"', "\simeq"] \\
\Gamma(X,\mathcal{L}^{sk})\otimes_{\Bbbk}F(\mathcal{O}_X) \arrow[r, "F(\pi_{2*}(d_0'))"] & F(\mathbf{R}\pi_{2*}(\pi_1^*\mathcal{L}^{sk}\otimes\mathbf{Fc}_{(m)}))
\end{tikzcd}
\quad (2.30)
$$

commutative.

2. $\mathcal{H}^i(\mathbf{Fc}_{(m)})=0$ *unless* $i\in[z-m,z+n-m]$ *or* $i\in[z,z+n]$*. Moreover* $\mathbf{Fc}_{(m)}$ *can be expressed as a biproduct*

$$\mathbf{Fc}_{(m)}\simeq(\mathbf{Fc}_{(m)})_{\geq p}\oplus(\mathbf{Fc}_{(m)})_{\leq p}\,.$$

Proof. 1. By applying $\mathbf{R}\pi_{2*}$ to the complex $\pi_1^*\mathcal{L}^{sk}\otimes(1\otimes F)(\mathcal{C}^\bullet{}_{(m)})$ one obtains the complex $F(\mathbf{R}\pi_{2*}(\mathcal{L}^{sk}\otimes\mathcal{C}^\bullet{}_{(m)}))$ described in (2.25). Then, by Remark 2.14, we have an isomorphism $\eta_k\colon \mathbf{R}\pi_{2*}(\pi_1^*\mathcal{L}^{sk}\otimes\mathbf{Fc}_{(m)})\xrightarrow{\sim}F(\mathbf{R}\pi_{2*}(\pi_1^*\mathcal{L}^{sk}\otimes\mathbf{Fc}_{(m)}))$ between the convolutions which makes the diagram (2.30) commutative.

2. Assume that there is an integer i not in the prescribed rank such that $\mathcal{H}^i(\mathbf{Fc}_{(m)}))\neq 0$. Then for $k\gg 0$ one has $0\neq\pi_{2*}(\pi_1^*\mathcal{L}^{sk}\otimes\mathcal{H}^i(\mathbf{Fc}_{(m)}))\simeq\pi_{2*}(\mathcal{H}^i(\pi_1^*\mathcal{L}^{sk}\otimes\mathbf{Fc}_{(m)}))$ and $R^j\pi_{2*}(\pi_1^*\mathcal{L}^{sk}\otimes\mathcal{H}^{i-1}(\mathbf{Fc}_{(m)}))\simeq R^j\pi_{2*}(\mathcal{H}^{i-1}(\pi_1^*\mathcal{L}^{sk}\otimes\mathbf{Fc}_{(m)}))=0$ for every $j>0$. There is a spectral sequence $E_2^{p,q}=R^p\pi_{2*}(\mathcal{H}^q(\pi_1^*\mathcal{L}^{sk}\otimes\mathbf{Fc}_{(m)}))$ converging to $E_\infty^{p+q}=\mathcal{H}^{p+q}(\mathbf{R}\pi_{2*}(\pi_1^*\mathcal{L}^{sk}\otimes\mathbf{Fc}_{(m)}))$. Since $E^{-2,i+1}=E_2^{2,i-1}=0$, every nonzero element in $E_2^{0,i}$ is a cycle that survives to infinity. Then $E_2^{0,i}\neq 0$ implies that $\mathcal{H}^i(\mathbf{R}\pi_{2*}(\pi_1^*\mathcal{L}^{sk}\otimes\mathbf{Fc}_{(m)}))\neq 0$. If we also take the preceding part 1 into account, this contradicts Lemma 2.26.

Now, since $m-n-2>2(\dim X+\dim Y)$ and $X\times Y$ is smooth, Lemma 2.22 gives the decomposition $\mathbf{Fc}_{(m)}\simeq(\mathbf{Fc}_{(m)})_{\geq p}\oplus(\mathbf{Fc}_{(m)})_{\leq p}$. □

Note that if $F=\Phi^{\mathcal{K}^\bullet}_{X\to Y}$ then $\mathcal{K}^\bullet\simeq(\mathbf{Fc}_{(m)})_{\geq p}$ as we saw in Lemma 2.23. It is therefore convenient in our general situation to define the following objects of $D^b(X\times Y)$

$$\mathcal{K}^\bullet\simeq(\mathbf{Fc}_{(m)})_{\geq p}\,,\quad \mathcal{Q}^\bullet\simeq(\mathbf{Fc}_{(m)})_{\leq p} \tag{2.31}$$

(see Corollary A.35). Then we have an isomorphism of functors

$$\Phi^{\mathbf{Fc}_{(m)}}_{X\to Y}\simeq\Phi^{\mathcal{K}^\bullet}_{X\to Y}\oplus\Phi^{\mathcal{Q}^\bullet}_{X\to Y}\,.$$

Lemma 2.28. *The natural morphism $\Phi^{\mathbf{Fc}_{(m)}}_{X\to Y}(\mathcal{L}^{sk})\to\Phi^{\mathcal{K}^\bullet}_{X\to Y}(\mathcal{L}^{sk})$ induces for every k an isomorphism*

$$\beta_k^\Phi\colon(\Phi^{\mathbf{Fc}_{(m)}}_{X\to Y}(\mathcal{L}^{sk}))_{\geq p}\simeq\Phi^{\mathcal{K}^\bullet}_{X\to Y}(\mathcal{L}^{sk})\,.$$

Proof. By Lemma 2.27, one has $\mathcal{H}^q(\mathcal{Q}^\bullet)=0$ for $q\notin[z-m,z+n-m]$. Then $R^p\pi_{Y*}(\mathcal{H}^q(\pi_X^*\mathcal{L}^{sk}\otimes\mathcal{Q}^\bullet))=0$ for $q\notin[z-m,z+n-m+\dim X]$. Since $m>z+n-p+\dim X$, one has $\mathcal{H}^i(\Phi^{\mathcal{Q}^\bullet}_{X\to Y}(\mathcal{L}^{sk}))=0$ for $i\geq p$ and the result follows. □

As a consequence, (2.30) can be completed to a commutative diagram

$$\begin{tikzcd}
\pi_{2*}(\pi_1^*\mathcal{L}^{sk}\otimes\pi_2^*F(\mathcal{O}_X)) \arrow[r, "\mathbf{R}\pi_{2*}(1\otimes F(d_0))"] \arrow[d, "\simeq"] & \Phi^{\mathbf{Fc}_{(m)}}_{X\to Y}(\mathcal{L}^{sk}) \arrow[r, "\beta_k^\Phi"] \arrow[d, "\eta_k"', "\simeq"] \arrow[dr, "f_k"] & \Phi^{\mathcal{K}^\bullet}_{X\to Y}(\mathcal{L}^{sk}) \arrow[d, "\gamma_k"', "\simeq"] \\
\Gamma(X,\mathcal{L}^{sk})\otimes_{\Bbbk}F(\mathcal{O}_X) \arrow[r, "F(\pi_{2*}(d_0'))"] & F(\mathbf{R}\pi_{2*}(\pi_1^*\mathcal{L}^{sk}\otimes\mathbf{Fc}_{(m)})) \arrow[r, "\beta_k^F"] & F(\mathcal{L}^{sk})
\end{tikzcd} \tag{2.32}$$

where β_k^F is the isomorphism given by Lemma 2.26 and

$$f_k = \gamma_k \circ \beta_k^\Phi = \beta_k^F \circ \eta_k \colon \Phi_{X\to Y}^{\mathbf{Fc}_{(m)}}(\mathcal{L}^{sk}) \to F(\mathcal{L}^{sk})\,.$$

Even if η_k is not unique, the morphism γ_k is, in view of the following lemma.

Lemma 2.29. *The isomorphism $\gamma_k \colon \Phi_{X\to Y}^{\mathcal{K}^\bullet}(\mathcal{L}^{sk}) \simeq F(\mathcal{L}^{sk})$ in the derived category $D^b(Y)$ is uniquely determined by the commutativity of the diagram* (2.32)*. Thus the morphisms γ_k are functorial, in the sense that, for every morphism $\alpha\colon \mathcal{L}^{sk} \to \mathcal{L}^{s\ell}$, there is a commutative diagram in $D^b(Y)$*

$$\begin{tikzcd} \Phi_{X\to Y}^{\mathcal{K}^\bullet}(\mathcal{L}^{sk}) \arrow[r, "\Phi_{X\to Y}^{\mathcal{K}^\bullet}(\alpha)"] \arrow[d, "\gamma_k"', "\simeq"] & \Phi_{X\to Y}^{\mathcal{K}^\bullet}(\mathcal{L}^{s\ell}) \arrow[d, "\gamma_\ell"', "\simeq"] \\ F(\mathcal{L}^{sk}) \arrow[r, "F(\alpha)"] & F(\mathcal{L}^{s\ell})\,. \end{tikzcd}$$

Proof. Since

$$\begin{aligned} &\mathrm{Hom}_{D^b(Y)}(\pi_{2*}(\pi_1^*\mathcal{L}^{sk-p} \otimes \pi_2^* F(R_p))[r], F(\mathcal{L}^{sk})) \\ &\qquad \simeq \mathrm{Hom}_{D^b(Y)}(\Gamma(Y, \mathcal{L}^{sk-p}) \otimes_{\Bbbk} F(R_p))[r], F(\mathcal{L}^{sk})) \\ &\qquad \simeq \mathrm{Hom}_{D^b(X)}(\Gamma(Y, \mathcal{L}^{sk-p}) \otimes_{\Bbbk} R_p[r], \mathcal{L}^{sk}) = 0 \end{aligned}$$

as F is fully faithful, Lemma 2.12 implies that f_k is the unique morphism which makes the diagram (2.32) commutative. Under this condition the morphism γ_k is unique as well. Functoriality follows straightforwardly. □

Since X and Y are smooth, the categories $D^b(X)$ and $D^b(Y)$ have Serre functors. Moreover $\{\mathcal{L}^{sk}\}_{k\in\mathbb{Z}}$ is an ample sequence in $D^b(X)$, and then we can apply Proposition 2.11 to obtain an existence result. The uniqueness of the kernel is given by Theorem 2.25.

Proposition 2.30. *Let X, Y be smooth projective varieties and let $F\colon D^b(X) \to D^b(Y)$ be an exact fully faithful functor. If F is bounded, then it is an integral functor and its kernel is uniquely determined in $D^b(X \times Y)$ up to isomorphism.*

Orlov's representability theorem 2.15 follows from Proposition 2.30 due to the following result:

Proposition 2.31. *If X, Y are projective varieties and X is smooth, every exact functor $F\colon D^b(X) \to D^b(Y)$ is bounded.*

Proof. As we have already noticed, the functor F admits a left adjoint $G\colon D^b(Y) \to D^b(X)$, cf. Remark 2.16. Take an embedding $Y \hookrightarrow \mathbb{P}^N$ induced by a very ample line bundle $\mathcal{L}$. Pulling back to Y the right resolution (2.11), we have a right resolution

$$0 \to \mathcal{L}^{-j} \to V_0^j \otimes_{\Bbbk} \mathcal{O}_X \to V_1^j \otimes_{\Bbbk} \mathcal{L} \to \cdots \to \\ V_{N-1}^j \otimes_{\Bbbk} \mathcal{L}^{N-1} \to V_N^j \otimes_{\Bbbk} \mathcal{L}^N \to 0 \quad (2.33)$$

for every integer $j > 0$. Then, for every $j > 0$ there is a spectral sequence $E_2^{p,q} = \mathcal{H}^p(V_q^j \otimes_{\Bbbk} G(\mathcal{L}^q)) = V_q^j \otimes_{\Bbbk} G(\mathcal{L}^q))$ converging to $E_\infty^{p+q} = \mathcal{H}^{p+q}(G(\mathcal{L}^{-j}))$. Since each $G(\mathcal{L}^q)$ is bounded, there exist integers $p_0 \leq p_1$ such that $\mathcal{H}^p(G(\mathcal{L}^q)) = 0$ for every $q \in [0, N]$ if $p \notin [p_0, p_1]$. This implies that $\mathcal{H}^k(G(\mathcal{L}^{-j})) = 0$ for $k \notin [p_0, p_1 + N]$ and for every $j > 0$. A similar spectral sequence argument implies that for every sheaf $\mathcal{F}$ on X one has

$$\operatorname{Hom}^i_{D^b(Y)}(\mathcal{L}^{-j}, F(\mathcal{F})) = \operatorname{Hom}^i_{D^b(X)}(G(\mathcal{L}^{-j}), \mathcal{F}) = 0 \qquad (2.34)$$

for $i \notin [-p_1 - N, -p_0 + \dim X]$ and for every $j > 0$, since X is smooth and then $\operatorname{Hom}^i_{D^b(X)}(\mathcal{G}, \mathcal{F}) = 0$ for any sheaf $\mathcal{G}$ on X and any $i > \dim X$. Again, for every value of j there is a spectral sequence with $E_2^{p,q} = \operatorname{Ext}^p_Y(\mathcal{L}^{-j}, \mathcal{H}^q(F(\mathcal{F})))$ converging to $E_\infty^{p+q} = \operatorname{Hom}^{p+q}_{D^b(Y)}(\mathcal{L}^{-j}, F(\mathcal{F}))$. Since $F(\mathcal{F})$ is bounded, for $j \gg 0$ one has $E_2^{p,q} = 0$ for all values of q; then the spectral sequence degenerates yielding an isomorphism $\operatorname{Hom}_Y(\mathcal{L}^{-j}, \mathcal{H}^q(F(\mathcal{F}))) \simeq \operatorname{Hom}^q_{D^b(Y)}(\mathcal{L}^{-j}, F(\mathcal{F}))$. If $\mathcal{H}^q(F(\mathcal{F})) \neq 0$, we can find $j \gg 0$ such that $\operatorname{Hom}_Y(\mathcal{L}^{-j}, \mathcal{H}^q(F(\mathcal{F}))) \neq 0$. Thus, (2.34) implies that $\mathcal{H}^q(F(\mathcal{F})) = 0$ unless $q \in [-p_1 - N, -p_0 + \dim X]$. $\square$

Remark 2.32. Given two kernels $\mathcal{K}^\bullet$, $\mathcal{G}^\bullet$ in $D^b(X \times Y)$, any morphism $f\colon \mathcal{K}^\bullet \to \mathcal{G}^\bullet$ in the derived category induces a morphism of functors $\Phi^f_{X\to Y}\colon \Phi^{\mathcal{K}^\bullet}_{X\to Y} \to \Phi^{\mathcal{G}^\bullet}_{X\to Y}$ which in turn induces another morphism $\bar{f}\colon \mathcal{K}^\bullet \to \mathcal{G}^\bullet$. This morphism may fail to coincide with f. Moreover, the groups $\operatorname{Hom}_{D^b(X\times Y)}(\mathcal{K}^\bullet, \mathcal{G}^\bullet)$ and $\operatorname{Hom}(\Phi^{\mathcal{K}^\bullet}_{X\to Y}, \Phi^{\mathcal{G}^\bullet}_{X\to Y})$ may not be isomorphic. In other words, the functor that maps the kernel $\mathcal{K}^\bullet$ to the integral functor $\Phi^{\mathcal{K}^\bullet}_{X\to Y}$ is not fully faithful in general.

Take for instance for $X = Y$ an elliptic curve, $\mathcal{K}^\bullet = \mathcal{O}_\Delta$ and $\mathcal{G}^\bullet = \mathcal{O}_\Delta[2]$. The Serre functor of $X \times X$ consists in the shift by 2, so that

$$\operatorname{Hom}_{D^b(X\times X)}(\mathcal{O}_\Delta, \mathcal{O}_\Delta[2]) \simeq \operatorname{Hom}_{D^b(X\times X)}(\mathcal{O}_\Delta, \mathcal{O}_\Delta)^* \simeq \Bbbk\,.$$

On the other hand, $\Phi^{\mathcal{K}^\bullet}_{X\to Y}$ is the identity functor $\operatorname{Id}_{D^b(X)}$ and $\Phi^{\mathcal{G}^\bullet}_{X\to Y} \simeq \operatorname{Id}_{D^b(X)}[2]$. If $g\colon \operatorname{Id}_{D^b(X)} \to \operatorname{Id}_{D^b(X)}[2]$ is a functor morphism, then for every sheaf $\mathcal{F}$ on X, the induced morphism $g(\mathcal{F})\colon \mathcal{F} \to \mathcal{F}[2]$ is zero since $\operatorname{Hom}_{D^b(X)}(\mathcal{F}, \mathcal{F}[2]) = \operatorname{Ext}^2_X(\mathcal{F}, \mathcal{F}) = 0$ because X is a curve. One easily proves that the morphism $g(\mathcal{F}^\bullet)$ is also zero for every bounded complex $\mathcal{F}^\bullet$; thus $g = 0$, that is,

$$\operatorname{Hom}(\Phi^{\mathcal{K}^\bullet}_{X\to Y}, \Phi^{\mathcal{G}^\bullet}_{X\to Y}) \simeq \operatorname{Hom}(\operatorname{Id}_{D^b(X)}, \operatorname{Id}_{D^b(X)}[2]) = 0\,.$$

The fact that the functor mapping kernels to integral functors may fail to be fully faithful can be regarded as an intrinsic limitation of the triangulated structure of the derived category. Actually, if we pass to the setting of dg-categories, the corresponding functor (suitably defined) turns out to be an equivalence. See Theorem A.57 and the comments in "Notes and further reading."

△

2.3 Fourier-Mukai functors

The following definition introduces the objects that will be our main concern in this book.

Definition 2.33. An integral functor $\Phi^{\mathcal{K}^\bullet}_{X\to Y}\colon D^b(X) \to D^b(Y)$ is called a Fourier-Mukai functor if it is an exact equivalence of derived categories. If in addition the kernel is a concentrated complex, the functor will be said to be a Fourier-Mukai transform. △

We give now some basic properties of the Fourier-Mukai functors. Later on we shall describe some geometric applications of Fourier-Mukai functors and shall establish a criterion for testing whether an integral functor is a Fourier-Mukai functor.

The composition of two Fourier-Mukai functors is a Fourier-Mukai functor as well (and, as we know, the kernel of the composition is the convolution of the two kernels, cf. Proposition 1.3). However, the composition of two Fourier-Mukai transforms may fail to be a Fourier-Mukai transform, because its kernel may not be a concentrated complex, as we shall see in Example 2.59.

Fourier-Mukai functors behave well with respect to the WIT condition. Let $\Phi = \Phi^{\mathcal{K}^\bullet}_{X\to Y}\colon D^b(X) \to D^b(Y)$ be a Fourier-Mukai functor; the functor i-th cohomology sheaf $\Phi^i(\bullet) = \mathcal{H}^i(\Phi(\bullet))$ will be called the i-th Fourier-Mukai functor. Given a quasi-inverse $\widehat{\Phi}\colon D^b(Y) \to D^b(X)$ of Φ, we have an isomorphism

$$\widehat{\Phi}(\Phi(\mathcal{E}^\bullet)) \simeq \mathcal{E}^\bullet$$

in the derived category. When $\mathcal{E}^\bullet$ is a sheaf $\mathcal{E}$ in degree zero, the above isomorphism means that there is a convergent spectral sequence

$$E_2^{p,q} = \widehat{\Phi}^p(\Phi^q(\mathcal{E})) \implies \begin{cases} \mathcal{E} & \text{if } p+q=0 \\ 0 & \text{otherwise.} \end{cases} \tag{2.35}$$

Proposition 2.34. *If Φ is a Fourier-Mukai functor and $\mathcal{E}$ is a WIT$_i$ sheaf, then the unique nonzero Fourier-Mukai sheaf $\widehat{\mathcal{E}}$ is a WIT$_{-i}$ sheaf. Moreover $\widehat{\widehat{\mathcal{E}}} = \widehat{\Phi}^{-i}(\widehat{\mathcal{E}}) \simeq \mathcal{E}$.*

We shall also need the following result.

Proposition 2.35. *Let $\Phi\colon D^b(X) \to D^b(Y)$ be a Fourier-Mukai functor and assume that X is smooth of dimension n. For every (closed) point $x \in X$ the following inequality holds true:*

$$\sum_i \dim \operatorname{Hom}^1_{D(Y)}(\Phi^i(\mathcal{O}_x), \Phi^i(\mathcal{O}_x)) \le n\,.$$

Proof. There is a spectral sequence $E_2^{p,q} = \bigoplus_i \operatorname{Hom}^p_{D(Y)}(\Phi^i(\mathcal{O}_x), \Phi^{i+q}(\mathcal{O}_x))$ converging to $E_\infty^{p+q} = \operatorname{Hom}^{p+q}_{D(Y)}(\Phi(\mathcal{O}_x), \Phi(\mathcal{O}_x))$. The exact sequence of the lower terms yields $0 \to E_2^{1,0} \to E_\infty^1$. By the Parseval formula (Proposition 1.34), one has $\operatorname{Hom}^1_{D(Y)}(\Phi(\mathcal{O}_x), \Phi(\mathcal{O}_x)) \simeq \operatorname{Hom}^1_{D(X)}(\mathcal{O}_x, \mathcal{O}_x) \simeq \Bbbk^n$. □

2.3.1 Some geometric applications of Fourier-Mukai functors

The existence of a Fourier-Mukai functor between the derived categories of two smooth algebraic varieties (or the equivalent conditions that the two algebraic varieties have equivalent derived categories, cf. Theorem 2.15) imposes strong constraints on their geometry. A first manifestation of this fact is Theorem 2.38 which states that the two (smooth projective) varieties X and Y have the same dimension, and that their canonical bundles satisfy some stringent conditions. Corollary 2.40 will establish that the rational Chow rings of X and Y are isomorphic (as $\mathbb{Q}$-vector spaces). According to Theorem 2.49, under a condition on the Kodaira dimension of X, the varieties X and Y are birational. If the hypotheses are strengthened by assuming that X has an ample canonical bundle, then X and Y are isomorphic (Theorem 2.51).

The following definition is commonly adopted for varieties with equivalent bounded derived categories.

Definition 2.36. Two projective varieties X and Y are Fourier-Mukai partners if there is an exact equivalence of triangulated categories $F\colon D^b(X) \xrightarrow{\sim} D^b(Y)$. △

Note that we do not impose that any of the varieties is smooth. However, if one is smooth, the other is smooth as well.

Lemma 2.37. *Let X be a smooth projective variety.*

1. *Every Fourier-Mukai partner of X is smooth.*
2. *A projective variety Y is a Fourier-Mukai partner of X if and only if there is Fourier-Mukai functor $\Phi^{\mathcal{K}^\bullet}_{X\to Y}\colon D^b(X) \xrightarrow{\sim} D^b(Y)$.*

Proof. Assume that X is a smooth variety and that there is an equivalence of categories $F\colon D^b(Y) \to D^b(X)$. For every (closed) point $y \in X$ and every sheaf $\mathcal{F}$ on Y, one has $\operatorname{Hom}^i(\mathcal{O}_y, \mathcal{F}) \simeq \operatorname{Hom}^i(F(\mathcal{O}_y), F(\mathcal{F}))$. Since X is smooth and $F(\mathcal{O}_y)$ and $F(\mathcal{F})$ are bounded, there is only a finite number of indexes i with $\operatorname{Hom}^i(\mathcal{O}_y, \mathcal{F}) \neq 0$. Then $\mathcal{O}_y$ is of finite homological dimension, and hence Y is smooth at y by Serre's Theorem [266], [215, 19.2]. This proves the first part. The second follows from Orlov's representability theorem 2.15. □

We briefly recall the notion of determinant bundle for an object $\mathcal{M}^\bullet$ of $D^b(X)$ where X is a smooth projective variety. This is defined by

$$\det(\mathcal{M}^\bullet) = \bigotimes_i (\det(\mathcal{E}^i))^{(-1)^i}$$

where $\mathcal{E}^\bullet$ is any bounded complex of locally free sheaves isomorphic to $\mathcal{M}^\bullet$ in the derived category and $\det(\mathcal{E}^i)$ is the highest exterior power of $\mathcal{E}^i$.

A direct computation shows that

$$\det(\mathcal{M}^\bullet \otimes \mathcal{L}) = \det(\mathcal{M}^\bullet) \otimes \mathcal{L}^{\operatorname{rk}(\mathcal{M}^\bullet)} \tag{2.36}$$

for every line bundle $\mathcal{L}$.

Theorem 2.38. *Let X, Y be smooth projective varieties that are Fourier-Mukai partners, so that there is a Fourier-Mukai functor $\Phi^{\mathcal{K}^\bullet}_{X\to Y}\colon D^b(X) \to D^b(Y)$.*

1. *The right and left adjoints to $\Phi^{\mathcal{K}^\bullet}_{X\to Y}$ are functorially isomorphic*

$$\Phi^{\mathcal{K}^{\bullet\vee}\otimes\pi_X^*\omega_X[m]}_{Y\to X} \simeq \Phi^{\mathcal{K}^{\bullet\vee}\otimes\pi_Y^*\omega_Y[n]}_{Y\to X}$$

   (here $m = \dim X$ and $n = \dim Y$) and they are both quasi-inverses to $\Phi^{\mathcal{K}^\bullet}_{X\to Y}$.

2. *X and Y have the same dimension, $m = n$.*

3. *ω_X and ω_Y have the same order, that is, ω_X^k is trivial if and only if ω_Y^k is trivial. Thus, ω_X is trivial if and only if ω_Y is trivial and in this case the functor $\Phi^{\mathcal{K}^{\bullet\vee}[n]}_{Y\to X}$ is a quasi-inverse to $\Phi^{\mathcal{K}^\bullet}_{X\to Y}$.*

4. *$\omega_X^r \simeq \mathcal{O}_X$ and $\omega_Y^r \simeq \mathcal{O}_Y$ with $r = \operatorname{rk}(\mathcal{K}^\bullet)$.*

Proof. 1. Since a quasi-inverse is both a right and a left adjoint, the uniqueness of adjoints together with Proposition 1.13 yields the statement.

2. Applying the above functorial isomorphism to the skyscraper sheaf $\mathcal{O}_y$ we obtain $\mathbf{L}j_y^*\mathcal{K}^{\bullet\vee} \otimes \omega_X[m] \simeq \mathbf{L}j_y^*\mathcal{K}^{\bullet\vee}[n]$. Since the functors we have applied are equivalences of categories, both objects are nonzero in $D^b(X)$. Then there is an

integer q_0 which is the minimum of the q's with $\mathcal{H}^q(\mathbf{L}j_y^*\mathcal{K}^{\bullet\vee}) \neq 0$ and one gets $q_0 + m = q_0 + n$ so that $m = n$.

3. Assume for instance that ω_X^k is trivial; the other case is proved analogously. By Corollary 1.18, one has $S_Y^k \simeq \Phi_{X\to Y}^{\mathcal{K}^\bullet} \circ S_X^k \circ (\Phi_{X\to Y}^{\mathcal{K}^\bullet})^{-1}$, where $(\Phi_{X\to Y}^{\mathcal{K}^\bullet})^{-1}$ is a quasi-inverse to $\Phi_{X\to Y}^{\mathcal{K}^\bullet}$. Since ω_X^k is trivial, $S_X^k(\mathcal{F}^\bullet) \simeq \mathcal{F}^\bullet[kn]$ and then $S_Y^k \simeq \Phi_{X\to Y}^{\mathcal{K}^\bullet} \circ [kn] \circ (\Phi_{X\to Y}^{\mathcal{K}^\bullet})^{-1} \simeq [kn]$. Therefore $\omega_Y^k \simeq \mathcal{O}_Y$.

4. Taking determinant bundles in the expression $\mathbf{L}j_y^*\mathcal{K}^{\bullet\vee} \otimes \omega_X \simeq \mathbf{L}j_y^*\mathcal{K}^{\bullet\vee}$, we have $\det(\mathbf{L}j_y^*\mathcal{K}^{\bullet\vee}) \simeq \det(\mathbf{L}j_y^*\mathcal{K}^{\bullet\vee}) \otimes \omega_X^{r_y}$ with $r_y = \operatorname{rk}(\mathbf{L}j_y^*\mathcal{K}^{\bullet\vee})$ by Equation (2.36), and therefore $\omega_X^{r_y} \simeq \mathcal{O}_X$. Now the functoriality of the Chern classes gives $r_y = \operatorname{rk}(\mathcal{K}^{\bullet\vee}) = r$. The proof of the second formula is analogous. □

As we recalled at the beginning of this chapter, when $\mathcal{K}^\bullet$ is a sheaf $\mathcal{Q}$ concentrated in degree zero the dual complex may be different from the concentrated complex given by the dual sheaf $\mathcal{Q}^*$ in degree zero. This point deserves a comment.

Example 2.39. Take $\mathcal{K}^\bullet = \mathcal{O}_\Delta$, the structure sheaf of the diagonal $\Delta \subset X \times X$. The integral functor $\Phi_{X\to X}^{\mathcal{O}_\Delta}$ is isomorphic to the identity functor as we have seen in Example 1.2. We have that $\mathcal{O}_\Delta^* = 0$ because $\mathcal{O}_\Delta$ is a torsion sheaf, so that $\Phi_{X\to X}^{\mathcal{O}_\Delta^*\otimes\pi_1^*\omega_X[n]} = 0$ and this cannot be a quasi-inverse to $\Phi_{X\to X}^{\mathcal{O}_\Delta}$. The identity functor is of course a quasi-inverse to itself, and according to Theorem 2.38 it must coincide with $\Phi_{X\to X}^{\mathcal{O}_\Delta^\vee\otimes\pi_i^*\omega_X[n]}$ for $i = 1, 2$. Let us check that this is indeed the case. Since X is smooth, the diagonal is a regular embedding and then a standard local computation using the Koszul complex yields the formulas

$$\mathcal{E}xt^i_{\mathcal{O}_{X\times X}}(\mathcal{O}_\Delta, \mathcal{O}_{X\times Y}) \simeq \begin{cases} 0 & \text{for } q \neq n \\ \delta_*(\omega_X^*) & \text{for } q = n. \end{cases} \tag{2.37}$$

Thus, $\mathcal{O}_\Delta^\vee \simeq \delta_*(\omega_X)[-n]$ in the derived category, and therefore $\mathcal{O}_\Delta^\vee \otimes \pi_i^*\omega_X[n] \simeq \mathcal{O}_\Delta$ so that $\Phi_{X\to X}^{\mathcal{O}_\Delta^\vee\otimes\pi_i^*\omega_X[n]} \simeq \Phi_{X\to X}^{\mathcal{O}_\Delta}$ is the identity.

Looking at things the other way round, one should say that Theorem 2.38 together with the uniqueness of the kernel (Theorem 2.25) proves $\mathcal{O}_\Delta^\vee \otimes \pi_i^*\omega_X[n] \simeq \mathcal{O}_\Delta$ and therefore yields the formula 2.37 without resorting to the Koszul complex. △

Theorem 2.38 allows us to prove an important property of the map $f^{\mathcal{K}^\bullet}$ defined in Equation (1.12).

Corollary 2.40. *Let X, Y be smooth projective varieties and let $\Phi_{X\to Y}^{\mathcal{K}^\bullet} \colon D^b(X) \to D^b(Y)$ be a Fourier-Mukai functor. The induced map $f^{\mathcal{K}^\bullet} \colon A^\bullet(X)\otimes\mathbb{Q} \to A^\bullet(Y)\otimes\mathbb{Q}$ is an isomorphism of $\mathbb{Q}$-vector spaces. Moreover, if $\Bbbk = \mathbb{C}$, the induced f-map in cohomology $f^{\mathcal{K}^\bullet} \colon H^\bullet(X,\mathbb{Q}) \to H^\bullet(Y,\mathbb{Q})$ is also an isomorphism of $\mathbb{Q}$-vector spaces* *which induces an isomorphism of vector spaces between the even cohomology rings.*

Proof. By Theorem 2.38, $\Phi_{Y\to X}^{\mathcal{K}^{\bullet\vee}\otimes\pi_X^*\omega_X[m]}$ is a quasi-inverse to $\Phi_{X\to Y}^{\mathcal{K}^\bullet}$, so that the convolution $\mathcal{K}^\bullet * (\mathcal{K}^{\bullet\vee}\otimes\pi_X^*\omega_X[m])$ is isomorphic to $\mathcal{O}_\Delta$ in the derived category because of the uniqueness of the kernel (Theorem 2.25). The functoriality of the map f (cf. Eq. (1.13)) yields $f^{\mathcal{K}^\bullet}\circ f^{\mathcal{K}^{\bullet\vee}\otimes\pi_X^*\omega_X[m]} = f^{\mathcal{O}_\Delta}$. Since $v(\mathcal{O}_\Delta) = \delta_*(1)$ by Grothendieck-Riemann-Roch for the diagonal immersion δ, we have $f^{\mathcal{O}_\Delta} = \mathrm{Id}$ (here v is the Mukai vector defined in Eq. (1.1)). One analogously proves that $f^{\mathcal{K}^{\bullet\vee}\otimes\pi_X^*\omega_X[m]}\circ f^{\mathcal{K}^\bullet} = \mathrm{Id}$. The cohomology statement is proved in a similar way. □

Part 4 of Theorem 2.38 implies that whenever the kernel $\mathcal{K}^\bullet$ is not of rank zero, a certain power ω_X^r of the canonical bundle of X has to be trivial, with $r\neq 0$. This is a strong geometric constraint: if X is a curve, it has to be elliptic (and then $\omega_X\simeq\mathcal{O}_X$); if X is a surface, it has to be Abelian, K3 (in which cases $\omega_X\simeq\mathcal{O}_X$), Enriques (for which $\omega_X^2\simeq\mathcal{O}_X$) or bielliptic (for which $\omega_X^{12}\simeq\mathcal{O}_X$) (cf. [141, Thm. 6.3]). In dimension 3 the most important example is provided by Calabi-Yau varieties (for which, by definition, $\omega_X\simeq\mathcal{O}_X$). This is the reason why the Fourier-Mukai transform has been mostly studied for this kind of variety.

However, this by no means implies that if all powers ω_X^r (with nonzero exponent) are nontrivial, then Fourier-Mukai functors $\Phi_{X\to Y}^{\mathcal{K}^\bullet}\colon D^b(X)\to D^b(Y)$ cannot exist. Rather they do exist, but *the kernel $\mathcal{K}^\bullet$ must be of rank zero*, as in the case of the structure sheaf of the diagonal.

We shall deal with the case of rank zero kernels when in Chapter 6 we shall study integral transforms for families, or *relative* integral transforms. Given two families of varieties $X\to S$ and $Y\to S$, we shall define an integral functor $\Phi_{X\to Y}^{\mathcal{K}^\bullet}\colon D^b(X)\to D^b(Y)$ by means of a *relative* kernel $\mathcal{K}^\bullet$ in the derived category $D^b(X\times_S Y)$ of the fiber product. That transform will be defined as the ordinary integral functor with kernel $i_*\mathcal{K}^\bullet$, where $i\colon X\times_S Y\hookrightarrow X\times Y$ is the natural immersion. Even when the integral functor $\Phi_{X\to Y}^{\mathcal{K}^\bullet}$ is an equivalence of categories we cannot use Theorem 2.38 to get information about ω_X, because as a complex in $D^b(X\times Y)$ we have $\mathrm{rk}(i_*\mathcal{K}^\bullet)=0$. Here one needs a relative version of Theorem 2.38 where the relative canonical sheaves $\omega_{X/S}$ and $\omega_{Y/S}$ replace the absolute canonical sheaves.

Let X, Y be smooth projective varieties and $\Phi_{X\to Y}^{\mathcal{K}^\bullet}$ a Fourier-Mukai functor. By Theorem 2.38, a quasi-inverse is given by $\Phi_{Y\to X}^{\mathcal{K}^{\bullet\vee}\otimes\pi_X^*\omega_X} = \Phi_{Y\to X}^{\mathcal{K}^{\bullet\vee}\otimes\pi_Y^*\omega_Y}$. Let W and $W^\vee$ be the supports of $\mathcal{K}^\bullet$ and $\mathcal{K}^{\bullet\vee}$, respectively (see Definition A.90).

Proposition 2.41. *One has $W = W^\vee$ and the two projections $\pi_{X|W}\colon W\to X$, $\pi_{Y|W}\colon W\to Y$ are surjective.*

Proof. There is a convergent spectral sequence with $E_2^{p,q} = \mathcal{E}xt^p_{\mathcal{O}_{X\times Y}}(\mathcal{H}^p(\mathcal{K}^\bullet), \mathcal{O})$ and $E_\infty^{p+q} = \mathcal{H}^{p+q}(\mathcal{K}^{\bullet\vee})$. This proves that $W^\vee \subseteq W$. Reversing the roles of $\mathcal{K}^\bullet$ and $\mathcal{K}^{\bullet\vee}$ we get that $W \subseteq W^\vee$. Now, since $\Phi^{\mathcal{K}^\bullet}_{Y\to X}$ is an equivalence, $\Phi^{\mathcal{K}^\bullet}_{Y\to X}(\mathcal{O}_x) = Lj_x^*\mathcal{K}^\bullet \neq 0$ for every $x \in X$; this proves that the morphism $\pi_{X|W}: W \to X$ is surjective. The surjectivity of $\pi_{Y|W}: W \to Y$ is proved analogously by using the fact that $\Phi^{\mathcal{K}^{\bullet\vee}}_{Y\to X}$ is also a Fourier-Mukai functor by Theorem 2.38. $\square$

The existence of a Fourier-Mukai functor $\Phi^{\mathcal{K}^\bullet}_{X\to Y}: D^b(X) \to D^b(Y)$ (which is equivalent to the existence of an exact equivalence by Theorem 2.15) has interesting effects on the geometry of X and Y. Our next aim is to prove some strong results in that direction due to Orlov and Kawamata

We begin by recalling some standard definitions. If X is a smooth projective variety, and $\mathcal{L}$ is a line bundle on X, then for $n \gg 0$ the dimension of the global sections $\Gamma(X, \mathcal{L}^n)$ of $\mathcal{L}^n$ is a polynomial in n of a certain degree $d \leq \dim X$; the degree of the null polynomial is $-\infty$ by decree. The degree of such polynomial is called the *Kodaira dimension of* $\mathcal{L}$ and it is denoted by $\kappa(X, \mathcal{L})$. So one knows that $\kappa(X, \mathcal{L}) \leq \dim X$.

In particular, the *Kodaira dimension of* X is defined as $\kappa(X) = \kappa(X, \omega_X)$. One can also define the Kodaira dimension in terms of the projective rational maps defined by $\mathcal{L}^s$ ($s \geq 0$), assuming that they exist. In that case (which corresponds to $\kappa(X, \mathcal{L}) > -\infty$), $\kappa(X, \mathcal{L})$ is the maximum of the dimensions of the images of those maps, and it is also the transcendence degree of the graded ring $R(X, \mathcal{L}) = \oplus_{s\geq 0}\Gamma(X, \mathcal{L}^s)$ minus 1.

Lemma 2.42. (Kodaira's Lemma) *If $\kappa(X) = \dim X$ (resp. $\kappa(X, \omega_X^*) = \dim X$), there exist an ample divisor H and an integer s_0 such that for any integer $s \geq s_0$ there is an effective divisor D_s such that $\omega_X^s \simeq \mathcal{O}_X(H) \otimes \mathcal{O}_X(D_s)$ (resp. $(\omega_X^*)^s \simeq \mathcal{O}_X(H) \otimes \mathcal{O}_X(D_s)$).*

Proof. Assume that $\kappa(X) = \dim X$. Let $H \hookrightarrow X$ be a smooth ample divisor, which exists by Bertini's theorem [141, 8.18], and consider the exact sequence

$$0 \to \omega_X^s(-H) \to \omega_X^s \to {\omega_X}^s_{|H} \to 0\,.$$

Since $\chi(X, \omega_X^s)$ is a polynomial in s of degree $\kappa(X) = \dim X$, and $\chi(H, {\omega_X}^s_{|H})$ is a polynomial in s of degree $\kappa(H, \omega_{X|H}) \leq \dim H = \dim X - 1$, we see that for $s \gg 0$ the line bundle $\omega_X^s(-H)$ has a section. Thus, $\omega_X^s(-H) \simeq \mathcal{O}_X(D_s)$ for some effective divisor D_s. The other case is analogous. $\square$

Recall that a line bundle $\mathcal{L}$ on a projective variety is *numerically effective* or *nef* if for any morphism $\phi: C \to X$ where C is a projective curve, one has $\deg \phi^*\mathcal{L} \geq 0$. We can consider only closed immersions $C \hookrightarrow X$, because we can always replace $\phi: C \to X$ by its image.

Lemma 2.43. *Let $f\colon Y \to X$ be a projective morphism.*

1. *If $\mathcal{L}$ is a nef line bundle on X, then $f^*\mathcal{L}$ is nef on Y.*
2. *If f is surjective, then a line bundle $\mathcal{L}$ on X is nef if and only if $f^*\mathcal{L}$ is nef on Y.*

Proof. The first claim is obvious. For the second, let C be a projective curve, $\phi\colon C \to X$ a morphism and $\mathcal{N}$ a very ample line bundle for the projective morphism $f_C\colon C \times_Y X \to C$. For $n \gg 0$, $\mathcal{N}^n$ has a section which defines a divisor $H \hookrightarrow C \times_Y X$. The curve $\tilde{C} = H^r$ ($r = \dim Y - \dim X$) intersects every fiber in a finite number of points, so that the projection $\pi\colon \tilde{C} \to C$ is a finite morphism. Moreover the composition $\phi \circ \pi\colon \tilde{C} \to X$ factors as $f \circ \rho$, where $\rho\colon \tilde{C} \to Y$ is the induced morphism. Since $f^*\mathcal{L}$ is nef, $\deg \rho^* f^* \mathcal{L} \geq 0$, then $\deg \phi^*\mathcal{L} \geq 0$ as well. □

We can define the *numerical Kodaira dimension* of a line bundle $\mathcal{L}$ on a projective variety as the maximum $\nu(X, \mathcal{L})$ of the integer numbers m such that there is a proper morphism $\varphi\colon T \to X$ from a variety T of dimension m with the property $\varphi^*(c_1(\mathcal{L}))^m \cdot T \neq 0$. The intersection numbers $\varphi^*(c_1(\mathcal{L}))^m \cdot T$ can be defined in terms of the Snapper polynomial. To this end, let us recall that for any line bundle $\mathcal{N}$ on a m-dimensional projective variety T, the Euler characteristic $\chi(T, \mathcal{N}^n)$ of $\mathcal{L}^n$ is a polynomial in n of a certain degree $d \leq m$, called the Snapper polynomial [119, Ex. 18.3.6], and that

$$\chi(T, \mathcal{N}^n) = \frac{1}{m!} c_1(\mathcal{N})^m \cdot T\, n^m + \text{terms of lower degree.}$$

It is clear that we can define the numerical Kodaira dimension of $\mathcal{L}$ by considering only closed immersions $\varphi\colon T \hookrightarrow X$. Moreover, the numerical Kodaira dimension of any power of a line bundle $\mathcal{L}$ equals that of $\mathcal{L}$, namely, $\nu(X, \mathcal{L}) = \nu(X, \mathcal{L}^s)$ for any $s \neq 0$.

When $\mathcal{L}$ is nef, the numerical Kodaira dimension is the maximum of the integers m such that $c_1(\mathcal{L})^m$ is not numerically trivial. In this case, the numerical Kodaira dimension is bounded by the Kodaira dimension, $\nu(X, \mathcal{L}) \leq \kappa(X, \mathcal{L})$.

If X is a projective Gorenstein variety, the *numerical Kodaira dimension of* X is defined as $\nu(X) = \nu(X, \omega_X)$.

Lemma 2.44. *Let $f\colon Y \to X$ be a projective morphism and $\mathcal{L}$ a line bundle on X. Then $\nu(Y, f^*\mathcal{L}) \leq \nu(X, \mathcal{L})$. Moreover, if f is surjective, one has $\nu(Y, f^*\mathcal{L}) = \nu(X, \mathcal{L})$.*

Proof. The first claim is obvious. The proof of the second is similar to that of Lemma 2.43. Let $\varphi\colon T \to X$ be a proper morphism such that $\varphi^*(\mathcal{L}^m) \cdot T \neq 0$ with

$m = \dim T$, and let $\mathcal{N}$ be a very relatively ample line bundle for the projective morphism $f_T\colon T\times_Y X \to T$. For $n \gg 0$, $\mathcal{N}^n$ has a section which defines a divisor $H \hookrightarrow T\times_Y X$. Then $\tilde{T} = H^r$ ($r = \dim Y - \dim X$) intersects every fiber in a finite number of points, so that the projection $\pi\colon \tilde{T} \to T$ is a finite morphism. Moreover the composition $\varphi\circ\pi\colon \tilde{T}\to X$ factors as $f\circ\rho$, where $\rho\colon \tilde{T}\to Y$ is the induced morphism. It follows that $\rho^*(f^*\mathcal{L}^m)\cdot\tilde{T} \simeq \pi^*(\varphi^*\mathcal{L}^m)\cdot\tilde{T} \neq 0$, and then $\nu(X,\mathcal{L}) \le \nu(Y, f^*\mathcal{L})$, so that $\nu(Y,f^*\mathcal{L}) = \nu(X,\mathcal{L})$ as claimed. □

Finally, we need a technical result whose proof we include although it is standard.

Lemma 2.45. *Let Z be a normal variety and $\mathcal{F}$ a rank r coherent sheaf on Z. If $\mathcal{L}_1$ and $\mathcal{L}_2$ are line bundles on Z such that $\mathcal{F}\otimes\mathcal{L}_1 \simeq \mathcal{F}\otimes\mathcal{L}_2$, then $\mathcal{L}_1^r \simeq \mathcal{L}_2^r$.*

Proof. Modding the torsion out we can assume that $\mathcal{F}$ is torsion-free. Since Z is normal there is a codimension two closed subset Z' such that $\mathcal{F}$ is locally free of rank r on $U = Z - Z'$. By the theorem on generic smoothness [141, III.10.7], U can be assumed to be smooth. Taking determinants, we get $\det(\mathcal{F}_{|U})\otimes\mathcal{L}^r_{1|U} \simeq \det(\mathcal{F}_{|U})\otimes\mathcal{L}^r_{2|U}$ and thus $\mathcal{L}^r_{1|U} \simeq \mathcal{L}^r_{2|U}$. Since Z is normal and Z' has codimension 2, this isomorphism can be extended to an isomorphism $\mathcal{L}_1^r \simeq \mathcal{L}_2^r$ (cf. [140, Theorem 3.8]). □

Let X and Y be smooth projective varieties and $\Phi^{\mathcal{K}^\bullet}_{X\to Y}$ a Fourier-Mukai functor. For every irreducible component Z of the support W of $\mathcal{K}^\bullet$, we denote by $\widetilde{Z}\to Z$ its normalization and by $\tilde{p}_X\colon \widetilde{Z}\to X$, $\tilde{p}_Y\colon \tilde{Z}\to Y$ the induced maps.

Lemma 2.46. *One has $\tilde{p}_X^*\omega_X^r \simeq \tilde{p}_Y^*\omega_Y^r$ for some $r>0$. In particular $\tilde{p}_X^*K_X$ and $\tilde{p}_Y^*K_Y$ are $\mathbb{Q}$-linearly equivalent. Moreover, we can chose an irreducible component $Z_X(\mathcal{K}^\bullet)$ of W such that $p_X = \pi_{X|Z_X(\mathcal{K}^\bullet)}\colon Z_X(\mathcal{K}^\bullet)\to X$ and then also $\tilde{p}_X\colon \widetilde{Z}_X(\mathcal{K}^\bullet)\to X$, are dominant.*

Proof. By Theorem 2.38, one has $\dim Y = \dim X$ and if we denote by n this dimension, a quasi-inverse to $\Phi^{\mathcal{K}^\bullet}_{X\to Y}$ is given by $\Phi^{\mathcal{K}^{\bullet\vee}\otimes\pi_X^*\omega_X[n]}_{Y\to X} \simeq \Phi^{\mathcal{K}^{\bullet\vee}\otimes\pi_Y^*\omega_Y[n]}_{Y\to X}$. The uniqueness of the kernel (Theorem 2.25) implies that $\mathcal{K}^{\bullet\vee}\otimes\pi_X^*\omega_X \simeq \mathcal{K}^{\bullet\vee}\otimes\pi_Y^*\omega_Y$, so that $\mathcal{H}^i(\mathcal{K}^{\bullet\vee})\otimes\pi_X^*\omega_X \simeq \mathcal{H}^i(\mathcal{K}^{\bullet\vee})\otimes\pi_Y^*\omega_Y$ for every i. If $\rho\colon \widetilde{Z}\to X\times Y$ is the composition of $\widetilde{Z}\to Z$ and the immersion $Z\hookrightarrow X\times Y$, we see that $\rho^*(\mathcal{H}^i(\mathcal{K}^{\bullet\vee})\otimes p_X^*\omega_X) \simeq \rho^*(\mathcal{H}^i(\mathcal{K}^{\bullet\vee})\otimes p_Y^*\omega_Y)$. By Lemma 2.45, $\tilde{p}_X^*\omega_X^r \simeq \tilde{p}_Y^*\omega_Y^r$ where r is the rank of $\mathcal{H}^i(\mathcal{K}^{\bullet\vee})_{|Z}$, which is not zero.

By Proposition 2.41, $\pi_{X|W}\colon W\to X$ is surjective. Then we can choose an irreducible component $Z_X(\mathcal{K}^\bullet)$ of W which dominates X. □

The following Proposition 2.48 and Theorem 2.49 express a result due to Kawamata, usually known as "D-equivalence implies K-equivalence." We now give the precise definition of K-equivalent algebraic varieties.

Definition 2.47. Two smooth projective algebraic varieties X and Y are K-equivalent if there are a normal variety $\widetilde{Z}$ and projective birational morphisms $\tilde{p}_X\colon \widetilde{Z} \to X$, $\tilde{p}_Y\colon \widetilde{Z} \to Y$ such that $\tilde{p}_X^* K_X$ and $\tilde{p}_Y^* K_Y$ are $\mathbb{Q}$-linearly equivalent, that is, $r\tilde{p}_X^* K_X$ and $r\tilde{p}_Y^* K_Y$ are linearly equivalent for some $r \neq 0$. $\triangle$

Notice that if $\mathcal{F}^\bullet$ is a simple object of $D^b(Y)$, i.e., $\operatorname{Hom}_{D(Y)}(\mathcal{F}^\bullet, \mathcal{F}^\bullet) = \Bbbk$, the support of $\mathcal{F}^\bullet$ is connected. This can be seen as follows: assume that $\operatorname{Supp}\mathcal{F}^\bullet = Y_1 \coprod Y_2$; if we represent $\mathcal{F}^\bullet$ as a finite complex $\mathcal{E}^\bullet$ of locally free sheaves of finite rank, the natural map $\mathcal{E}^\bullet \to j_{1*}j_1^*\mathcal{E}^\bullet \oplus j_{2*}j_2^*\mathcal{E}^\bullet$ is a quasi-isomorphism. Then $\mathcal{F}^\bullet \simeq j_{1*}j_1^*\mathcal{E}^\bullet \oplus j_{2*}j_2^*\mathcal{E}^\bullet$ in the derived category, so that $\mathcal{F}^\bullet$ is not simple.

Proposition 2.48. *Let $Z_X(\mathcal{K}^\bullet)$ be an irreducible component of $W = \operatorname{Supp}\mathcal{K}^\bullet$ such that $p_X\colon Z_X(\mathcal{K}^\bullet) \to X$ is dominant.*

1. *There is a nonempty open subset U of X such that $p_X^{-1}(x) = \pi_X^{-1}(x) \cap W \simeq \operatorname{Supp}\Phi^{\mathcal{K}^\bullet}_{X\to Y}(\mathcal{O}_x)$ for every point $x \in U$.*
2. *If $\dim Z_X(\mathcal{K}^\bullet) = \dim X$, then $\tilde{p}_X\colon \widetilde{Z}_X(\mathcal{K}^\bullet) \to X$ and $\tilde{p}_Y\colon \widetilde{Z}_X(\mathcal{K}^\bullet) \to Y$ are birational and $\tilde{p}_X^* K_X$ and $\tilde{p}_Y^* K_Y$ are $\mathbb{Q}$-linearly equivalent. That is, X and Y are K-equivalent.*

Proof. If $Z_1, \dots, Z_s$ are the irreducible components of W other than $Z = Z_X(\mathcal{K}^\bullet)$, and $T = \cup_i(Z \cap Z_i)$, we take $U = X - p_X(T)$. Since $\Phi^{\mathcal{K}^\bullet}_{X\to Y}(\mathcal{O}_x)$ is simple, its support $\operatorname{Supp}\Phi^{\mathcal{K}^\bullet}_{X\to Y}(\mathcal{O}_x) \simeq \pi_X^{-1}(x) \cap W$ is connected as we have just seen. Then $\operatorname{Supp}\Phi^{\mathcal{K}^\bullet}_{X\to Y}(\mathcal{O}_x)$ has to be contained in $\pi_X^{-1}(x) \cap Z = p_X^{-1}(x)$ if $x \in U$; this proves the first part.

We now prove the second part. We have to prove that both p_X and p_Y are birational. Consider first the projection $p_X\colon Z \to X$. Since $\dim Z = \dim X$, p_X is generically finite. By Zariski's main theorem [141, 11.4], to prove that it is birational, we need only to check that it is generically injective. By the first part, $p_X^{-1}(x) = \pi_X^{-1}(x) \cap W$ for generic $x \in X$, and then $\pi_X^{-1}(x) = \{(x, y_1), \dots, (x, y_s)\}$ is finite. Thus, $Lj_x^*\mathcal{K}^\bullet = \Phi^{\mathcal{K}^\bullet}_{X\to Y}(\mathcal{O}_x)$ is supported at the points $y_1, \dots, y_s$, and $\operatorname{Hom}_{D^b(Y)}(\Phi^{\mathcal{K}^\bullet}_{X\to Y}(\mathcal{O}_x), \Phi^{\mathcal{K}^\bullet}_{X\to Y}(\mathcal{O}_x))$ is of dimension s. By the Parseval formula (cf. Proposition 1.34), one has

$$\operatorname{Hom}_{D^b(Y)}(\Phi^{\mathcal{K}^\bullet}_{X\to Y}(\mathcal{O}_x), \Phi^{\mathcal{K}^\bullet}_{X\to Y}(\mathcal{O}_x)) \simeq \operatorname{Hom}_{D^b(X)}(\mathcal{O}_x, \mathcal{O}_x) = \Bbbk\,,$$

so that $s = 1$. This proves that p_X is generically injective.

Our last step is to show that $p_Y\colon Z \to Y$ is also birational. If we prove that it is dominant, taking into account that Z is also an irreducible component of $W^\vee =$

$\operatorname{Supp}\mathcal{K}^{\bullet\vee}$ (cf. Proposition 2.41), we deduce as above that p_Y is birational. Since p_X is birational, there is an open subset $U' \subseteq U$ such that π_X induces an isomorphism between $p_X^{-1}(U') = \pi_X^{-1}(U') \cap W$ and U'. If we assume that p_Y is not dominant, then $\dim p_Y(Z) < \dim Y$ and there exist distinct points x_1, x_2 of U' such that $y = \pi_Y(\pi_X^{-1}(x_1)\cap W) = \pi_Y(\pi_X^{-1}(x_2)\cap W)$. Thus, the point y is the support of both $\Phi^{\mathcal{K}^\bullet}_{X\to Y}(\mathcal{O}_{x_1})$ and $\Phi^{\mathcal{K}^\bullet}_{X\to Y}(\mathcal{O}_{x_2})$, so that $\operatorname{Hom}^i_{D^b(Y)}(\Phi^{\mathcal{K}^\bullet}_{X\to Y}(\mathcal{O}_{x_1}), \Phi^{\mathcal{K}^\bullet}_{X\to Y}(\mathcal{O}_{x_2})) \neq 0$ for some integer i. By the Parseval formula, this implies that $\operatorname{Hom}^i_{D^b(X)}(\mathcal{O}_{x_1}, \mathcal{O}_{x_2}) \neq 0$, which is absurd.

Finally, since $\tilde{p}_X^*\omega_X^r = \tilde{p}_Y^*\omega_Y^r$, we eventually obtain that $\tilde{p}_X^*K_X$ and $\tilde{p}_Y^*K_Y$ are $\mathbb{Q}$-linearly equivalent. □

Theorem 2.49. [175, 176] *Let X, Y be smooth Fourier-Mukai partners.*

1. *The line bundle ω_X (resp. ω_X^*) is nef if and only if ω_Y (resp. ω_Y^*) is nef.*
2. *X and Y have the same numerical Kodaira dimension, $\nu(X) = \nu(Y)$.*
3. *If the Kodaira dimension $\kappa(X)$ is equal to $\dim X$ (or if $\kappa(X, \omega_X^*) = \dim X$), then X and Y are K-equivalent.*

Proof. By Lemma 2.37, there is a Fourier-Mukai functor $F = \Phi^{\mathcal{K}^\bullet}_{X\to Y}$. Let $Z_X(\mathcal{K}^\bullet)$ be an irreducible component of the support W of $\mathcal{K}^\bullet$ which dominates X (cf. Lemma 2.46; we use the same notation as in this lemma).

1. If ω_Y is nef so is ω_Y^r, then $\tilde{p}_Y^*\omega_Y^r$ is nef by Lemma 2.43 and since $\tilde{p}_X$ is surjective and $\tilde{p}_X^*\omega_X^r \simeq \tilde{p}_Y^*\omega_Y^r$, the same lemma implies that ω_X^r, and hence ω_X, is nef. The case when ω_Y^* is nef is proved analogously.

2. By Lemma 2.44, $\nu(X, \omega_X) = \nu(Z_X(\mathcal{K}^\bullet), p_X^*\omega_X) = \nu(Z_X(\mathcal{K}^\bullet), p_Y^*\omega_Y) \leq \nu(Y, \omega_Y)$. Reversing the roles of X and Y one proves the converse statement.

3. Assume first that $\kappa(X) = \dim X$. By Kodaira's Lemma 2.42, one can take $m > 0$ such that the two isomorphisms $\tilde{p}_X^*\omega_X^m \simeq \tilde{p}_Y^*\omega_Y^m$ and $\omega_X^m \simeq \mathcal{O}_X(H) \otimes \mathcal{O}_X(D)$, where H is an ample divisor and D is effective, hold true.

Let us see that $\tilde{p}_Y\colon \widetilde{Z}_X(\mathcal{K}^\bullet) \to Y$ is quasi-finite (i.e., it has finite fibers) outside $\tilde{p}_X^{-1}(D)$. First note that since p_X and p_Y are the restrictions to $Z_X(\mathcal{K}^\bullet)$ of the projections of $X \times Y$ onto its factors, no curve can be contracted by both of them; since normalization is a finite morphism, the same happens for $\tilde{p}_X$ and $\tilde{p}_Y$. Assume now that there is a curve C contained in a fiber $\tilde{p}_Y^{-1}(y)$ and not entirely contained in $\tilde{p}_X^{-1}(D)$; then we have $\tilde{p}_Y^{-1}K_Y \cdot C = 0$ and

$$m\tilde{p}_Y^{-1}K_Y \cdot C = m\tilde{p}_X^{-1}K_X \cdot C = \tilde{p}_X^{-1}H \cdot C + \tilde{p}_X^{-1}D \cdot C \geq \tilde{p}_X^{-1}H \cdot C\,.$$

Since C cannot be contracted by $\tilde{p}_X$ and H is ample, we get $m\tilde{p}_Y^{-1}K_Y \cdot C > 0$, which is a contradiction. Thus $\dim Z_X(\mathcal{K}^\bullet) = \dim \widetilde{Z}_X(\mathcal{K}^\bullet) \leq n = \dim Y$ so that

$\dim Z_X(\mathcal{K}^\bullet) = n$, and we conclude by Proposition 2.48. The case $\kappa(X, \omega_X^*) = \dim X$ is analogous. □

Remark 2.50. The variety $\widetilde{Z}_X(\mathcal{K}^\bullet)$ in Lemma 2.46 may be assumed to be *smooth* possibly by replacing it with a resolution of its singularities. △

A consequence of Kawamata's Theorem 2.49 is a celebrated "reconstruction theorem" due to Bondal and Orlov [49].

Theorem 2.51. *Let X, Y be smooth Fourier-Mukai partners. If either ω_X or ω_Y is ample or anti-ample, there is an isomorphism $X \simeq Y$.*

Proof. Assume that ω_X is ample or anti-ample. If $C \hookrightarrow \widetilde{Z}_X(\mathcal{K}^\bullet)$ is a curve contracted by $\tilde{p}_Y$, then $\tilde{p}_X^*\omega_X^m{}_{|C} \simeq \tilde{p}_Y^*\omega_Y^m{}_{|C} \simeq \mathcal{O}_C$, which is impossible because ω_X is ample or anti-ample and C is not contracted by $\tilde{p}_X$ as we have seen in the proof of Theorem 2.49. Hence $\tilde{p}_Y$ is an isomorphism and we have a birational morphism $f\colon Y \to X$ of smooth varieties such that $f^*\omega_X^m \simeq \omega_Y^m$ with $m > 0$ when ω_X is ample. Let us now prove that f is actually an isomorphism. In the exact sequence of differentials

$$f^*\Omega_X \xrightarrow{df} \Omega_Y \to \Omega_{Y/X} \to 0$$

the morphism df is injective (because it is injective at the generic point and $f^*\Omega_X$ is locally free); then $\Omega_{Y/X}$ is a torsion sheaf supported by a closed subscheme $Y' \neq Y$ which coincides with the zeroes of the determinant of df. This determinant is a section of $\omega_Y \otimes f^*\omega_X^{-1}$. Then $(\det(df))^m$ is a section of $(\omega_Y \otimes f^*\omega_X^{-1})^m \simeq \mathcal{O}_Y$ vanishing on Y'. Thus Y' is empty, so that $\Omega_{Y/X} = 0$ and f is smooth of relative dimension zero, and being also birational, it is an isomorphism. □

We have chosen to give a detailed account of this proof of this important theorem because the techniques introduced and its underlying ideas will be useful elsewhere in this book, in particular when we shall study the Fourier-Mukai partners of a variety. However, the original proof by Bondal and Orlov in [49] is more direct and does not make use of Orlov's representability theorem 2.15. It also enlightens how the derived category of a variety encodes information about its points and the line bundles on it. For this reason we offer here a brief sketch of that proof.

One starts by defining the *point objects* and *invertible objects* in a triangulated category. One shows that when the latter is the derived category $D^b(X)$ of coherent sheaves of a smooth algebraic variety with ample canonical or anticanonical sheaf, the point objects are the complexes of the form $\mathcal{O}_x[i]$ with $i \in \mathbb{Z}$. On the other hand, when point objects have this form, one shows that the invertible objects are the objects of the form $\mathcal{L}[i]$ with $i \in \mathbb{Z}$, where $\mathcal{L}$ is a line bundle.

Assume now that X is a smooth projective algebraic variety with ample or anti-ample canonical bundle, and Y is a Fourier-Mukai partner of X, i.e., that there is an exact equivalence of triangulated categories $F\colon D^b(X) \to D^b(Y)$. One proves that F maps point objects to point objects, and this in turn implies that the point objects in Y are again exactly the shifted skyscraper sheaves. Along the same lines, one proves that F maps invertible objects to invertible objects. The result about point objects implies now that line bundles are mapped into shifted line bundles. By suitably redefining the functor F, one can obtain that skyscraper sheaves are mapped to skyscrapers, thus providing a set-theoretic identification of X with Y, and that line bundles are mapped to line bundles. The latter property implies that X and Y are homeomorphic. After redefining F again, we can assume that $F(\mathcal{O}_X) \simeq \mathcal{O}_Y$, and since F commutes with the Serre functor, one has $F(\omega_X^i) \simeq \omega_Y^i$ for all i. Since ω_X is ample or anti-ample, the topology of X has a basis formed by open sets U_α labeled by $\alpha \in \operatorname{Hom}_X(\omega_X^i, \omega_X^j)$, $i, j \in \mathbb{Z}$; by definition, U_α is the set of points in X where α does not vanish. The properties of F that one has so far proved imply that the open sets $V_\alpha \subset Y$ defined in the same way in terms of ω_Y are also a basis of the topology of Y. A theorem by Illusie [158] implies that ω_Y is ample or anti-ample, respectively.

Moreover, the equivalence F induces an isomorphism between the graded canonical algebras $\oplus_{i\geq 0}\operatorname{Hom}_X(\mathcal{O}_X, \omega_X^i)$ and $\oplus_{i\geq 0}\operatorname{Hom}_Y(\mathcal{O}_Y, \omega_Y^i)$. When ω_X and ω_Y are ample, this gives rise to an algebraic isomorphism

$$X \simeq \operatorname{Proj}(\oplus_{i\geq 0}\operatorname{Hom}_X(\mathcal{O}_X, \omega_X^i)) \xrightarrow{\sim} \operatorname{Proj}(\oplus_{i\geq 0}\operatorname{Hom}_Y(\mathcal{O}_Y, \omega_Y^i)) \simeq Y\,.$$

When ω_X and ω_Y are anti-ample, one proceeds in a similar way with the algebras $\oplus_{i\geq 0}\operatorname{Hom}_X(\mathcal{O}_X, \omega_X^{-i})$ and $\oplus_{i\geq 0}\operatorname{Hom}_Y(\mathcal{O}_Y, \omega_Y^{-i})$. It is worth saying that this proof does not makes use of Orlov's representability theorem 2.15.

2.3.2 Characterization of Fourier-Mukai functors

We have already seen that a spanning class may be used to test if an exact fully faithful functor is an equivalence of categories. But one can also find a suitable spanning class for the derived category $D^b(X)$ and use it to state conditions for an integral functor to be a Fourier-Mukai functor.

The results in the first part of this section are valid in arbitrary characteristic, while starting from Proposition 2.56 we need to assume that the characteristic is zero.

The following two propositions are taken from [61].

Proposition 2.52. *On a smooth proper variety X the skyscraper sheaves $\mathcal{O}_x$ form a spanning class for the derived category $D^b(X)$.*

Proof. For every $\mathcal{M}^\bullet \in D^b(X)$ and every point $x \in X$ there is a spectral sequence $E_2^{p,q} = \operatorname{Ext}^p_{\mathcal{O}_X}(\mathcal{H}^{-q}(\mathcal{M}^\bullet), \mathcal{O}_x) \implies E_\infty^{p+q} = \operatorname{Hom}^{p+q}_{D(X)}(\mathcal{M}^\bullet, \mathcal{O}_x)$. If $\mathcal{M}^\bullet \neq 0$, then there exists an integer q such that $\mathcal{H}^q(\mathcal{M}^\bullet) \neq 0$. Let q_0 be the maximum of such q's and let x be a point in the support of $\mathcal{H}^{q_0}(\mathcal{M}^\bullet)$. Then we have a nonzero element $e \in E_2^{0,-q_0} = \operatorname{Hom}_{D(X)}(\mathcal{H}^{q_0}(\mathcal{M}^\bullet), \mathcal{O}_x)$, and that element survives to give a nonzero element of $E_\infty^{-q_0} = \operatorname{Hom}^{-q_0}_{D(X)}(\mathcal{M}^\bullet, \mathcal{O}_x)$. This implies that the skyscraper sheaves $\mathcal{O}_x$ form a spanning class on the right for the derived category $D^b(X)$, that is, they satisfy Condition 2 in Definition 2.1. Note that this does not require X to be smooth, while this is necessary to prove Condition 1. Take then $\mathcal{M}^\bullet \neq 0$. Since X is smooth, ω_X is a line bundle, so that $\mathcal{N}^\bullet = \mathcal{M}^\bullet \otimes \omega_X \neq 0$ and we can apply the above argument to $\mathcal{N}^\bullet$ and find $\bar{q}_0$ and $\bar{x}$ such that $\operatorname{Ext}^{-\bar{q}_0}_{\mathcal{O}_X}(\mathcal{N}^\bullet, \mathcal{O}_{\bar{x}}) = \operatorname{Hom}^{-\bar{q}_0}_{D(X)}(\mathcal{N}^\bullet, \mathcal{O}_{\bar{x}}) \neq 0$. By Serre duality $\operatorname{Hom}^{n+\bar{q}_0}_{D(X)}(\mathcal{O}_x, \mathcal{N}^\bullet) \simeq \operatorname{Ext}^{-\bar{q}_0}_{\mathcal{O}_X}(\mathcal{N}^\bullet, \mathcal{O}_{\bar{x}})^* \neq 0$, thus finishing the proof. □

We also need to ascertain when the derived category $D^b(X)$ is indecomposable.

Proposition 2.53. *Let X be a smooth proper variety. Then $D^b(X)$ is indecomposable if and only if X is connected.*

Proof. If X is not connected, write $X = X_1 \coprod X_2$, and then $D^b(X) \simeq D^b(X_1) \oplus D^b(X_2)$. Assume now that X is connected and that there exist full nontrivial subcategories $\mathfrak{A}_1$, $\mathfrak{A}_2$ with $D^b(X) \simeq \mathfrak{A}_1 \oplus \mathfrak{A}_2$. For any integral closed subvariety $Y \hookrightarrow X$, the sheaf $\mathcal{O}_Y$ is indecomposable, so that it is isomorphic to an object either in $\mathfrak{A}_1$ or $\mathfrak{A}_2$. Moreover, for every point $y \in Y$, the sheaf $\mathcal{O}_y$ is isomorphic to an object in the same $\mathfrak{A}_j$ as $\mathcal{O}_Y$, because otherwise $\operatorname{Hom}_{D(X)}(\mathcal{O}_Y, \mathcal{O}_y) = 0$ and this is not true. Let X_j be the union of all integral subvarieties Y such that $\mathcal{O}_Y$ is isomorphic to an object of $\mathfrak{A}_2$. Then X_1, X_2 are closed subsets and $X = X_1 \coprod X_2$, because if $y \in X_1 \cap X_2$, then $\mathcal{O}_y$ is isomorphic both to an object of $\mathfrak{A}_1$ and an object of $\mathfrak{A}_2$, and this is absurd. Since X is connected, one of the X_j's, say X_2, is empty. Then for every object $\mathcal{K}^\bullet$ in $D^b(X_2)$ one has

$$\operatorname{Hom}^i_{D(X)}(\mathcal{K}^\bullet, \mathcal{O}_x) = 0\,, \quad \text{for any } i \in \mathbb{Z},\, x \in X$$

and therefore $\mathcal{K}^\bullet \simeq 0$ because the skyscraper sheaves form a spanning class by Proposition 2.52. □

Let X be a smooth projective variety.

Definition 2.54. A sheaf $\mathcal{F}$ on X is special if $\mathcal{F} \otimes \omega_X \simeq \mathcal{F}$. An object $\mathcal{F}^\bullet$ of $D^b(X)$ is special if $\mathcal{F}^\bullet \otimes \omega_X \simeq \mathcal{F}^\bullet$ in $D^b(X)$. △

Then, when the canonical bundle ω_X is trivial, every object of $D^b(X)$ is special. An object $\mathcal{F}^\bullet$ of $D^b(X)$ is special if and only if its cohomology sheaves $\mathcal{H}^i(\mathcal{F}^\bullet)$ are special sheaves, as the following proposition shows.

Proposition 2.55. *Let $\mathcal{F}^\bullet$ be a complex in $D^b(X)$ such that all its cohomology sheaves $\mathcal{H}^i(\mathcal{F}^\bullet)$ are special sheaves. If $f^i\colon \mathcal{H}^i(\mathcal{F}^\bullet) \to \mathcal{H}^i(\mathcal{F}^\bullet)\otimes\omega_X$ are isomorphisms of sheaves, there is an isomorphism $f\colon \mathcal{F}^\bullet \to \mathcal{F}^\bullet\otimes\omega_X$ in the derived category such that $\mathcal{H}^i(f)=f^i$ for every i.*

Proof. We proceed by induction on the number of nonzero cohomology sheaves. If $\mathcal{H}^n(\mathcal{F}^\bullet)$ is the highest nonzero cohomology sheaf, we can assume that $\mathcal{F}^m=0$ for $m>n$. Assume that $\mathcal{F}^\bullet$ has only a nonzero cohomology sheaf. Then $\mathcal{F}^\bullet \simeq \mathcal{H}^m(\mathcal{F}^\bullet)[m]$ in $D^b(X)$ and we set $f=f^m[m]\colon \mathcal{F}^\bullet\to\mathcal{F}^\bullet\otimes\omega_X$. In the general case, we can assume by induction that there is an isomorphism $\tilde{f}\colon {\mathcal{F}^\bullet}_{\le(n-1)} \to {\mathcal{F}^\bullet}_{\le(n-1)}\otimes\omega_X$ in the derived category inducing in cohomology the morphisms f^i for $i\le n-1$. Consider the exact triangle in $K(\mathfrak{Qco}(X))$

$$ {\mathcal{F}^\bullet}_{\le(n-1)} \xrightarrow{i} \mathcal{F}^\bullet \to \operatorname{Cone}(i) \xrightarrow{\beta} {\mathcal{F}^\bullet}_{\le(n-1)}[1]\,. $$

Note that β is homotopic to zero, so $\beta=0$ in $K(\mathfrak{Qco}(X))$ and then also in the derived category. Since $\operatorname{Cone}(i)\simeq\mathcal{H}^m(\mathcal{F}^\bullet)[m]$ in the derived category, we have a commutative diagram in $D^b(X)$ whose arrows are exact triangles

$$ \begin{array}{ccccccc} {\mathcal{F}^\bullet}_{\le(n-1)} & \xrightarrow{i} & \mathcal{F}^\bullet & \longrightarrow & \operatorname{Cone}(i) & \xrightarrow{\beta=0} & {\mathcal{F}^\bullet}_{\le(n-1)}[1] \\ \simeq\downarrow f' & & & & \simeq\downarrow f^m[m] & & \simeq\downarrow f'[1] \\ {\mathcal{F}^\bullet}_{\le(n-1)} & \xrightarrow{i} & \mathcal{F}^\bullet & \longrightarrow & \operatorname{Cone}(i) & \xrightarrow{\beta=0} & {\mathcal{F}^\bullet}_{\le(n-1)}[1]\,. \end{array} $$

Then, there is an isomorphism $f\colon\mathcal{F}^\bullet\to\mathcal{F}^\bullet\otimes\omega_X$ in $D^b(X)$ which completes the diagram. ∎

From this point on, we assume that the base field $\Bbbk$ has characteristic zero.

Proposition 2.56. [202, 61] *Let X and Y be smooth projective varieties of the same dimension n, and let $\mathcal{K}^\bullet$ be a kernel in $D^b(X\times Y)$. The following conditions are equivalent:*

1. *$\Phi^{\mathcal{K}^\bullet}_{X\to Y}$ is a Fourier-Mukai functor;*
2. *$\Phi^{\mathcal{K}^\bullet}_{X\to Y}$ is fully faithful and $\mathbf{L}j_x^*\mathcal{K}^\bullet$ is a special object of $D^b(Y)$ for all $x\in X$.*

In particular, if $\mathcal{Q}$ is a sheaf on $X\times Y$ strongly simple over X, then $\Phi^{\mathcal{Q}}_{X\to Y}$ is a Fourier-Mukai transform if and only if $\mathcal{Q}_x$ is special for all $x\in X$.

Proof. We can assume that Y is connected so that $D^b(Y)$ is indecomposable by Proposition 2.53. To prove that 2 implies 1, in view of Propositions 2.52 and 2.5 we need to show that $\Phi^{\mathcal{K}^\bullet}_{X\to Y} S_X(\mathcal{O}_x) \simeq S_Y \Phi^{\mathcal{K}^\bullet}_{X\to Y}(\mathcal{O}_x)$ for every closed point x. Indeed, by the speciality of the complexes $\mathbf{L}j_x^*\mathcal{K}^\bullet$, we have

$$\Phi^{\mathcal{K}^\bullet}_{X\to Y} S_X(\mathcal{O}_x) \simeq \mathbf{L}j_x^*\mathcal{K}^\bullet[n] \simeq \mathbf{L}j_x^*\mathcal{K}^\bullet\omega_Y[n] \simeq S_Y \Phi^{\mathcal{K}^\bullet}_{X\to Y}(\mathcal{O}_x)\,.$$

The fact that 1 implies 2 is proved similarly. □

The results of the previous proposition will be mostly used in the following form.

Proposition 2.57. *Assume that X and Y are smooth projective varieties of the same dimension with trivial canonical bundles and that $\mathcal{K}^\bullet$ is an object in $D^b(X\times Y)$ strongly simple over X. Then the functor $\Phi^{\mathcal{K}^\bullet}_{X\to Y}$ is a Fourier-Mukai functor and the functor $\Phi^{\mathcal{K}^{\bullet\vee}[n]}_{Y\to X}$ is a quasi-inverse to $\Phi^{\mathcal{K}^\bullet}_{X\to Y}$. In particular, if $\mathcal{Q}$ is a locally free sheaf on $X\times Y$ strongly simple over X, the functor $\Phi^{\mathcal{Q}^*[n]}_{Y\to X}$ is a quasi-inverse for $\Phi^{\mathcal{Q}}_{X\to Y}$.*

Corollary 2.58. *Let X, Y be as in Proposition 2.57 and let $\mathcal{K}^\bullet$ be an object in $D^b(X\times Y)$ which is strongly simple over X. Then $\mathcal{K}^\bullet$ is strongly simple over Y as well and $\Phi^{\mathcal{K}^\bullet}_{Y\to X}$ is a Fourier-Mukai functor with inverse $\Phi^{\mathcal{K}^{\bullet\vee}[n]}_{X\to Y}$.*

Proof. By Proposition 2.57, $\Phi^{\mathcal{K}^{\bullet\vee}[n]}_{Y\to X}$ is an exact equivalence, and then $\mathcal{K}^{\bullet\vee}$ is strongly simple over Y by Theorem 1.27. By Remark 1.26, $\mathcal{K}^\bullet$ is strongly simple over Y, so that the statement follows again from Proposition 2.57. □

We are now in a position to show that the composition of two Fourier-Mukai transforms may fail to be a Fourier-Mukai transform.

Example 2.59. Let X be a K3 surface. We shall give an introduction to the geometry of K3 surfaces in Chapter 4; what we shall need here is that $\omega_X \simeq \mathcal{O}_X$ and $H^1(X,\mathcal{O}_X)=0$. Let us consider the integral functor $\Phi = \Phi^{\mathcal{I}_\Delta}_{X\to X}\colon D^b(X)\to D^b(X)$, where $\mathcal{I}_\Delta$ is the ideal sheaf of the diagonal in $X\times X$. One easily checks that $\mathcal{I}_\Delta$ is strongly simple, so that Φ is a Fourier-Mukai functor by Corollary 2.58. Again a straightforward computation shows that $\Phi(\mathcal{O}_X)\simeq \mathcal{O}_X[-2]$. Moreover, if $\mathcal{L}$ is a line bundle on X which has no cohomology in every degree, one has $\Phi(\mathcal{L})\simeq\mathcal{L}[-1]$. Comparing the two results, we see that for k big enough, the kernel of iterated composition Φ^k is not a shifted sheaf. △

Let X and Y be smooth projective varieties and $\mathcal{K}^\bullet$ a kernel in $D^b(X\times Y)$. We consider another pair $\tilde X$, $\tilde Y$ of smooth projective varieties and another kernel $\tilde{\mathcal{K}}^\bullet$ in $D^b(\tilde X\times\tilde Y)$. We then have a kernel $\mathcal{K}^\bullet \overset{\mathbf{L}}{\boxtimes} \tilde{\mathcal{K}}^\bullet$ in $D^b(X\times\tilde X\times Y\times\tilde Y)$.

Lemma 1.28 about the product of integral functors can be strengthened in the case of Fourier-Mukai functors.

Corollary 2.60. *If the integral functors $\Phi^{\mathcal{K}^\bullet}_{X\to Y}$ and $\Phi^{\tilde{\mathcal{K}}^\bullet}_{\tilde{X}\to\tilde{Y}}$ are Fourier-Mukai functors, then the functor $\Phi^{\mathcal{K}^\bullet \overset{\mathbf{L}}{\boxtimes} \tilde{\mathcal{K}}^\bullet}_{X\times\tilde{X}\to Y\times\tilde{Y}}$ is a Fourier-Mukai functor as well.*

Proof. In view of Lemma 1.28 and Proposition 2.56 we need only to show that for evey closed point $x,\tilde{x})\in X\times\tilde{X}$ the restriction $\mathbf{L}j^*_{(x,\tilde{x})}(\mathcal{K}^\bullet \overset{\mathbf{L}}{\boxtimes} \tilde{\mathcal{K}}^\bullet)$ is a special object. From one side, one has the isomorphism $\mathbf{L}j^*_{(x,\tilde{x})}(\mathcal{K}^\bullet \overset{\mathbf{L}}{\boxtimes} \tilde{\mathcal{K}}^\bullet)\simeq \mathbf{L}j^*_x\mathcal{K}^\bullet \overset{\mathbf{L}}{\boxtimes} \mathbf{L}j^*_{\tilde{x}}\tilde{\mathcal{K}}^\bullet$. From the other, as $\omega_{Y\times\tilde{Y}}\simeq\omega_Y\boxtimes\omega_{\tilde{Y}}$, we have

$$\mathbf{L}j^*_{(x,\tilde{x})}(\mathcal{K}^\bullet \overset{\mathbf{L}}{\boxtimes} \tilde{\mathcal{K}}^\bullet)\otimes\omega_{Y\times\tilde{Y}}\simeq(\mathbf{L}j^*_x\mathcal{K}^\bullet\otimes\omega_Y)\overset{\mathbf{L}}{\boxtimes}(\mathbf{L}j^*_{\tilde{x}}\tilde{\mathcal{K}}^\bullet\otimes\omega_{\tilde{Y}})\,.$$

□

In a number of important examples that will be thoroughly investigated in the next chapters, the hypotheses of Proposition 2.56 are met when Y is a connected component of the moduli spaces of simple sheaves on X, X is a connected component of the moduli spaces of simple sheaves on Y, and $\mathcal{Q}$ is the corresponding bi-universal family (provided it exists). In the case of surfaces, there are particular results that will be very useful. We describe here some of them.

Let Y be a smooth projective surface and X a fine moduli space of *special* stable sheaves on Y with fixed Mukai vector v (cf. Eq. (1.1)). Let $\mathcal{Q}$ be a universal sheaf on $X\times Y$ for the corresponding moduli problem, so that $\mathcal{Q}$ is flat over X and $\mathcal{Q}_x$ is a stable special sheaf on Y with Mukai vector v. Given closed points x and z in X, one has $\chi(\mathcal{Q}_x,\mathcal{Q}_z)=-v^2$ by Equation (1.7).

The following result can be found in [70].

Proposition 2.61. *Assume that X is a projective surface.*

1. *X is smooth if and only if $v^2=0$.*
2. *In this case, the integral functor $\Phi^{\mathcal{Q}}_{X\to Y}\colon D^b(X)\to D^b(Y)$ is a Fourier-Mukai functor.*

Proof. Let x be a closed point of X. Since the $\operatorname{Hom}^1_X(\mathcal{Q}_x,\mathcal{Q}_x)$ is the tangent space at x to the moduli space X of the sheaves $\{\mathcal{Q}_x\}$ on Y, one has that X is smooth at x if and only if $\dim\operatorname{Hom}^1_X(\mathcal{Q}_x,\mathcal{Q}_x)=2$. To compute this dimension, we note that the sheaf $\mathcal{Q}_x$ is stable and special, so that $\operatorname{Hom}_Y(\mathcal{Q}_x,\mathcal{Q}_z)\simeq\Bbbk$ by the stability and $\operatorname{Hom}^2_X(\mathcal{Q}_x,\mathcal{Q}_x)\simeq\Bbbk$ by Serre duality. It follows that $-v^2=\chi(\mathcal{Q}_x,\mathcal{Q}_x)=2-\dim\operatorname{Hom}^1_X(\mathcal{Q}_x,\mathcal{Q}_x)$ which proves the first claim.

Assume now that $v^2=0$. If x and z are different closed points of X, one has that $\operatorname{Hom}_Y(\mathcal{Q}_x,\mathcal{Q}_z)=0$ because $\mathcal{Q}_x$ and $\mathcal{Q}_z$ are nonisomorphic stable sheaves with the same Chern characters. Since the sheaves $\mathcal{Q}_X$ are special, Serre duality gives $\operatorname{Hom}^2_X(\mathcal{Q}_x, \mathcal{Q}_z) = 0$. Finally, from $\chi(\mathcal{Q}_x, \mathcal{Q}_z) = -v^2 = 0$ we deduce that also $\operatorname{Hom}^1_X(\mathcal{Q}_x, \mathcal{Q}_z) = 0$, so that $\mathcal{Q}$ is strongly simple over X. The claim follows now from Proposition 2.56. □

Remark 2.62. Proposition 2.61 holds also true for pure stable sheaves in the sense of Simpson as described in Section C.2, because pure stable sheaves have the properties of torsion-free stable sheaves we have used in its proof. △

2.3.3 Fourier-Mukai functors between moduli spaces

We would like to show that in many cases, integral transforms define in a natural way algebraic morphisms between moduli spaces.

Let $\Phi\colon D^b(X) \to D^b(Y)$ be an integral functor, where X and Y are smooth projective varieties. Let $\mathbf{M}_{X,P}$ be the functor associating to any variety T the set of equivalence classes of all coherent sheaves $\mathcal{E}$ on $T \times X$, flat over T and whose restrictions $\mathcal{E}_t = j_t^*\mathcal{E}$ to the fibers $X_t \simeq X$ of $\pi_T\colon T \times X \to T$ have Hilbert polynomial P. Here, two sheaves $\mathcal{E}$ and $\mathcal{E}'$ are considered to be equivalent if $\mathcal{E} \simeq \mathcal{E}' \otimes \pi_T^*\mathcal{L}$ for a line bundle $\mathcal{L}$ on T. Furthermore, let $\mathbf{M}_X$ be a subfunctor of $\mathbf{M}_{X,P}$ parameterizing WIT_i sheaves for a certain index i. Corollary 1.9 implies that if $\mathcal{E}$ is in $\mathbf{M}_X(T)$, the sheaves $\widehat{\mathcal{E}} = \Phi^i_T(\mathcal{E})$ are flat over T, so that for a fixed i the fibers $(\widehat{\mathcal{E}})_t \simeq \widehat{\mathcal{E}}_t$ have the same Hilbert polynomial $\hat{P}$. Moreover $\Phi^i_T(\mathcal{E} \otimes \pi_T^*\mathcal{L}) \simeq \Phi^i_T(\mathcal{E}) \otimes \pi_T^*\mathcal{L}$. Thus, Φ^i_T maps $\mathbf{M}_X(T)$ to $\mathbf{M}_{Y,\hat{P}}(T)$. The polynomial $\hat{P}$ can be computed by the Grothendieck-Riemann-Roch formula in terms of P, the Chern classes of the kernel $\mathcal{K}^\bullet$ of Φ and the Todd class of X.

By compatibility of integral functors with base change (Proposition 1.8), Φ^i induces a morphism of functors

$$\Phi^i_{\mathbf{M}}\colon \mathbf{M}_X \to \mathbf{M}_{Y,\hat{P}}\,.$$

Proposition 2.63. *Assume that $\mathbf{M}_X$ has a coarse moduli scheme M_X.*

1. *If there is a subfunctor $\mathbf{M}_Y \subset \mathbf{M}_{Y,\hat{P}}$ containing the image of $\Phi^i_{\mathbf{M}}$ that is also coarsely representable by a moduli scheme M_Y, then the integral functor Φ gives rise to an* algebraic morphism *of schemes*
$$\Phi^i_M\colon M_X \to M_Y\,.$$
2. *If Φ is a Fourier-Mukai functor, then the image functor $\mathbf{M}_Y = \Phi^i(\mathbf{M}_X)$ is coarsely representable by a moduli scheme M_Y, and Φ induces a* scheme isomorphism
$$\Phi^i_M\colon M_X \xrightarrow{\sim} M_Y\,.$$
Moreover M_X is a fine moduli scheme (that is, it represents the moduli functor $\mathbf{M}_X$) if and only if M_Y is a fine moduli scheme.

Proof. 1. Since $\mathbf{M}_Y$ is coarsely represented by M_Y, there exists a morphism of functors $\mathbf{M}_Y \to \mathrm{Hom}(-, M_Y)$, where the latter is the functor of points of M_Y. The composition with Φ^i is a morphism of functors $\mathbf{M}_X \to \mathrm{Hom}(-, M_Y)$ which, by the definition of coarse moduli, factors in a unique way through a morphism of functors $\mathrm{Hom}(-, M_X) \to \mathrm{Hom}(-, M_Y)$. This corresponds to a scheme morphism $\Phi^i \colon M_X \to M_Y$. Part 2 is straightforward, due to the uniqueness of the coarse moduli of a functor. □

An important example is given by the moduli functor of skyscraper sheaves $\mathcal{O}_x$ on X. Assume that the skyscraper sheaves are all WIT_i; this happens for instance if the kernel of Φ is a concentrated complex, in which case they are WIT_0. Then we have:

Corollary 2.64. *If Φ is a Fourier-Mukai functor, then X is a fine moduli space for the moduli functor $\mathbf{M}_Y$ of the sheaves $\Phi^i(\mathcal{O}_x)$ over Y.*

To illustrate another example, let X and Y be polarized smooth projective varieties (see Section C.2), and let $\mathbf{M}^{ss}_{X,P}$, $\mathbf{M}^{ss}_{Y,\hat{P}}$ be the corresponding moduli functors of (Gieseker) semistable sheaves. Assume that all semistable sheaves $\mathcal{F}$ in $\mathbf{M}^{ss}_{X,P}$ are WIT_i and that their images $\Phi^i(\mathcal{F})$ are semistable. We have:

Corollary 2.65. *1. Φ^i induces a morphism of schemes $\Phi^i_M \colon M^{ss}_{X,P} \to M^{ss}_{Y,\hat{P}}$.*

2. If Φ is a Fourier-Mukai functor and $\Phi^i(\mathbf{M}^{ss}_{X,P}) = \mathbf{M}^{ss}_{Y,\hat{P}}$, the induced morphism is an isomorphism of schemes $\Phi^i_M \colon M^{ss}_{X,P} \xrightarrow{\sim} M^{ss}_{Y,\hat{P}}$.

Corollary 2.65 implies that Φ^i transforms S-equivalent semistable sheaves on X to S-equivalent sheaves on Y (for the notion of S-equivalence, see Section C.2). However, Φ^i may transform non-S-equivalent semistable sheaves on X to S-equivalent sheaves on Y, even if Φ is a Fourier-Mukai functor. Thus, in general the morphism $\Phi^i_M \colon M^{ss}_{X,P} \to M^{ss}_{Y,\hat{P}}$ induced by a a Fourier-Mukai functor may fail to be injective or surjective. There is however a partial result.

Corollary 2.66. *If Φ is a Fourier-Mukai functor and there is a stable sheaf $\mathcal{F}$ in $M^s_{X,P}$ such that $\Phi^i(\mathcal{F})$ is stable, the functor Φ^i induces a surjective birational morphism*

$$\Phi^i_M \colon \tilde{M}_X \to \tilde{M}_Y\,,$$

where $\tilde{M}_X$ and $\tilde{M}_Y$ are the irreducible components of $M^{ss}_{X,P}$ and $M^{ss}_{Y,\hat{P}}$ which contain $[\mathcal{F}]$ and $[\Phi^i(\mathcal{F})]$, respectively.

Proof. If $\mathcal{G}$ is a semistable sheaf on X and $\Phi^i_M([\mathcal{G}]) = [\Phi^i(\mathcal{F})]$, then $\Phi^i(\mathcal{G})$ is S-equivalent to $\mathcal{F}$, and thus $\Phi^i(\mathcal{G}) \simeq \Phi^i(\mathcal{F})$ because $\Phi^i(\mathcal{F})$ is stable. Hence, $\mathcal{G} \simeq \mathcal{F}$ by the invertibility of Φ, and the fiber of Φ^i_M over the point $[\Phi^i(\mathcal{F})]$ is a single point. Since $M^{ss}_{X,P}$ and $M^{ss}_{Y,\hat{P}}$ are projective (see Theorem C.6), by Zariski's main theorem [136, 4.4.3] there exist open neighborhoods V of $[\Phi^i(\mathcal{F})]$ and $U = (\Phi^i_M)^{-1}(V)$ of $[\mathcal{F}]$ such that $\Phi^i_M \colon U \xrightarrow{\sim} V$. Moreover, $\Phi^i_M(\tilde{M}) = \tilde{M}_Y$ because $\Phi^i_M(\tilde{M})$ is irreducible and contains $[\Phi^i(\mathcal{F})]$. Then $\Phi^i_M \colon \tilde{M}_X \to \tilde{M}_Y$ is birational. □

2.4 Notes and further reading

Historical remarks. As we already mentioned, the first instance of a Fourier-Mukai transform is contained in Mukai's 1981 paper [224]. A Fourier-Mukai transform on K3 surfaces was first constructed by the authors of this book in 1994 [24] (see also [26]). Later a similar construction was done by Mukai [228]. We shall study Fourier-Mukai transforms on K3 surfaces in Chapter 4. Other examples of Fourier-Mukai transforms will be described in Chapter 6 as relative integral functors.

The talk by Bondal and Orlov at the 2002 International Congress of Mathematicians [50] is a nice review of work done by them and others on equivalences between derived categories of coherent sheaves.

Spherical objects. Fourier-Mukai functors can be constructed by using the so-called spherical objects, first introduced by Bondal and Polishchuk [51]. A complex $\mathcal{E}^\bullet$ in $D^b(X)$ is spherical if: (i) $\operatorname{Hom}^i_{D^b(X)}(\mathcal{E}^\bullet, \mathcal{E}^\bullet)$ is equal to $\Bbbk$ for $i = 0, \dim X$ and to zero otherwise; (ii) $\mathcal{E}^\bullet$ is special, i.e., $\mathcal{E}^\bullet \otimes \omega_X \xrightarrow{\sim} \mathcal{E}^\bullet$. For instance, the structure sheaf of a Calabi-Yau variety is a spherical object. We exploited this property in Example 2.59, which indeed generalizes to any Calabi-Yau variety.

For any object $\mathcal{E}^\bullet$ of $D^b(X)$, one defines the twist functor $T_{\mathcal{E}^\bullet}$ as the integral functor whose kernel is the cone of the evaluation morphism $\mathcal{E}^{\bullet\vee} \overset{\mathbf{L}}{\boxtimes} \mathcal{E}^\bullet \to \mathcal{O}_\Delta$. Whenever $\mathcal{E}^\bullet$ is spherical, the twist functor $T_{\mathcal{E}^\bullet}$ is a Fourier-Mukai functor, as proved by Seidel and Thomas [265].

Results for singular varieties. Kawamata proved a generalization of Orlov's representability theorem 2.15 to stacks associated with normal varieties with quotient singularities [176]. A characterization of Fourier-Mukai functors on Cohen-Macaulay varieties was given in [144, 143].

An alternative setting: differential graded categories. Though they are a powerful tool in algebraic geometry as well as in algebraic analysis, representation theory and several other branches of mathematics, derived categories suffer from a number of drawbacks. In particular, the underlying triangulated structure appears too poor to allow for an entirely satisfactory description of functors between these categories and of natural algebraic or homotopical operations. Bondal and Kapranov [46] proposed the idea of using differential graded categories as an "enhancement" of derived categories in order to provide a more flexible and rich environment.

Differential graded categories — whose first appearance in the literature dates back to the 1960s [179] — can be thought of as "differential graded algebras with many objects," pretty much in the same vein as additive categories can be thought of as "rings with many objects." The basics of differential graded categories are briefly presented in Section A.4.4. For a more detailed overview the reader is referred to Keller's beautiful exposition [178]. Here we shall limit ourselves to focus attention on a few issues that appear to be more relevant to our purposes.

The category $\mathrm{dgcat}_{\Bbbk}$ of small differential graded $\Bbbk$-categories (Definition A.50) admits a structure of model category, whose weak equivalences are the quasi-equivalences (Theorem A.51). One denotes by $Ho(\mathrm{dgcat}_{\Bbbk})$ the localization of $\mathrm{dgcat}_{\Bbbk}$ with respect to quasi-equivalences.

As proved in [105, 284], the monoidal category $(Ho(\mathrm{dgcat}_{\Bbbk}), \overset{\mathbf{L}}{\otimes})$ admits an internal Hom-functor $\mathbf{R}\mathcal{H}om$ (see Theorem A.56). Within this framework, Toën [284] has recently worked out a version of derived Morita theory, where the morphisms between dg-categories of modules over two dg-categories $\mathfrak{E}$, $\mathfrak{F}$ are described as the dg-category of $(\mathfrak{E}$-$\mathfrak{F})$-bimodules.

As an application of his theory, Toën proved some results that can be viewed as a strengthening of Orlov's representability theorem 2.15. Let us consider the Abelian category $\mathfrak{Qco}(X)$ of quasi-coherent sheaves on an algebraic variety X. We shall denote by $D_{dg}(X)$ the dg-derived category $D_{dg}(\mathbf{C}_{dg}(\mathfrak{Qco}(X)))$ (see Definition A.54); one has that the homotopy category $H^0(D_{dg}(X))$ is equivalent to $D(X)$. As Theorem A.57 shows, given two algebraic varieties X, Y, there is natural isomorphism in $Ho(\mathrm{dgcat}_{\Bbbk})$

$$D_{dg}(X \times Y) \simeq \mathbf{R}\mathcal{H}om_c(D_{dg}(X), D_{dg}(Y)),$$

where $\mathbf{R}\mathcal{H}om_c$ denotes the full subcategory of $\mathbf{R}\mathcal{H}om$ consisting of coproduct preserving quasi-functors. In particular, when X and Y are smooth and projective, it turns out (Equation A.10) that

$$\mathrm{parf}_{dg}(X \times_{\Bbbk} Y) \simeq \mathbf{R}\mathcal{H}om(\mathrm{parf}_{dg}(X), \mathrm{parf}_{dg}(Y)),$$

where $\mathrm{parf}_{dg}(X)$ is the full sub-dg-category of $D_{dg}(X)$ whose objects are the perfect complexes. We can rephrase this equivalence by saying that, in the dg environment, all functors are integral functors.

One should compare these results with the representability theorem [47, 6.8] proved by Bondal, Larsen, and Lunts. Actually, the dg-category $\mathrm{parf}_{dg}(X)$ can be viewed as a standard enhancement of the derived category $D(X)$ in the sense of Definition [47, 5.1].

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