Preface

Harmonic analysis is a venerable part of modern mathematics. Its roots began, perhaps, with late eighteenth-century discussions of the wave equation. Using the method of separation of variables, it was realized that the equation could be solved with a data function of the form $\varphi(x) = \sin jx$ for $j \in \mathbb{Z}$. It was natural to ask, using the philosophy of superposition, whether the equation could then be solved with data on the interval $[0, \pi]$ consisting of a finite linear combination of the $\sin jx$. With an affirmative answer to that question, one is led to ask about *infinite* linear combinations.

This was an interesting venue in which physical reasoning interacted with mathematical reasoning. Physical intuition certainly suggests that *any* continuous function φ can be a data function for the wave equation. So one is led to ask whether *any* continuous φ can be expressed as an (infinite) superposition of sine functions. Thus was born the fundamental question of Fourier series.

No less an *eminence gris* than Leonhard Euler argued against the proposition. He pointed out that some continuous functions, such as

$$\varphi(x) = \begin{cases} \sin(x - \pi) & \text{if } \ 0 \le x < \pi/2, \\[1ex] \dfrac{2(x - \pi)}{\pi} & \text{if } \ \pi/2 \le x \le \pi, \end{cases}$$

are actually not one function, but the juxtaposition of *two* functions. How, Euler asked, could the juxtaposition of two functions be written as the sum of single functions (such as $\sin jx$)? Part of the problem, as we can see, is that mathematics was nearly 150 years away from a proper and rigorous definition of function.[1] We were also more than 25 years away from a rigorous definition (to be later supplied by Cauchy and Dirichlet) of what it means for a series to converge.

[1] It was Goursat, in 1923, who gave a fairly modern definition of function. Not too many years before, no less a figure than H. Poincaré lamented the sorry state of the function concept. He pointed out that each new generation created bizarre "functions" only to show that the preceding generation did not know what it was talking about.

Fourier[2] in 1822 finally provided a means for producing the (formal) Fourier series for virtually any given function. His reasoning was less than airtight, but it was calculationally compelling and it seemed to work.

Fourier series live on the interval $[0, 2\pi)$, or even more naturally on the circle group $\mathbb{T}$. The Fourier analysis of the real line (i.e., the Fourier transform) was introduced at about the same time as Fourier series. But it was not until the mid-twentieth century that Fourier analysis on $\mathbb{R}^N$ came to fruition (see [BOC2], [STW]). Meanwhile, abstract harmonic analysis (i.e., the harmonic analysis of locally compact abelian groups) had developed a life of its own. And the theory of Lie group representations provided a natural crucible for noncommutative harmonic analysis.

The point here is that the subject of harmonic analysis is a point of view and a collection of tools, and harmonic analysts continually seek new venues in which to ply their wares. In the 1970s, E.M. Stein and his school introduced the idea of studying classical harmonic analysis—fractional integrals and singular integrals—on the Heisenberg group. This turned out to be a powerful device for developing sharp estimates for the integral operators (the Bergman projection, the Szegő projection, etc.) that arise naturally in the several complex variables setting. It also gave sharp subelliptic estimates for the $\overline{\partial}_b$ problem.

It is arguable that modern harmonic analysis (at least *linear harmonic analysis*) is the study of integral operators. Zygmund and Stein have pioneered this point of view, and Stein's introduction of Heisenberg group analysis validated it and illustrated it in a vital context. Certainly the integral operators of several complex variables are quite different from those that arise in the classical setting of one complex variable. And it is not just the well-worn differences between one-variable analysis and several-variable analysis. It is the nonisotropic nature of the operators of several complex variables. There is also a certain noncommutativity arising from the behavior of certain key vector fields. In appropriate contexts, the structure of the Heisenberg group very naturally models the structure of the canonical operators of several complex variables, and provides the means for obtaining sharp estimates thereof.

The purpose of the present book is to exposit this rich circle of ideas. And we intend to do so in a context for students. The harmonic analysis of several complex variables builds on copious background material. We provide the necessary background in classical Fourier series, leading up to the Hilbert transform. That will be our entree into singular integrals. Passing to several real-variables, we shall meet the Riesz fractional integrals and the Calderón–Zygmund singular integrals. The aggregate of all the integral operators encountered thus far will provide motivation (in Appendix 3) for considering pseudodifferential operators.

The material on Euclidean integral operators that has been described up to this point is a self-contained course in its own right. But for us it serves as an introduction to analysis on the Heisenberg group. In this new arena, we must first

[2] In his book *The Analytical Theory of Heat* [FOU].

provide suitable background material on the function theory of several complex variables. This includes analyticity, the Cauchy–Riemann equations, pseudoconvexity, and the Levi problem. All of this is a prelude to the generalized Cayley transform and an analysis of the automorphism group of the Siegel upper half-space. From this venue the Heisenberg group arises in a complex-analytically natural fashion.

Just to put the material presented here into context: We develop the ideas of integral operators up through pseudodifferential operators *not* because we are going to use pseudodifferential operators as such. Rather, they are the natural climax for this study. For us these ideas are of particular interest because they put into context, and explain, the idea of "order" of an integral operator (and of an error term). This material appears in Appendix 3. In addition, when we later make statements about asymptotic expansions for the Bergman kernel, the pseudodifferential ideas will help students to put the ideas into context. The pseudodifferential operator ideas are also lurking in the background when we discuss subelliptic estimates for the $\overline{\partial}$ problem in the last section of the book.

In addition, we present some of the ideas from the real-variable theory of Hardy spaces *not* because we are going to use them in the context of the Heisenberg group. Rather, they are the natural culmination of a study of integral operators in the context of harmonic analysis. Thus Chapters 1–5 of this book constitute a basic instroduction to harmonic analysis. Chapters 6–8 provide a bridge between harmonic analysis and complex function theory. And Chapters 9 and 10 are dessert: They introduce students to some of the cutting-edge ideas about the Siegel upper half-space and the Heisenberg group.

Analysis on the Heisenberg group still smacks of Euclidean space. But now we are working in a step-one nilpotent Lie group. So dilations, translations, convolutions, and many other artifacts of harmonic analysis take a new form. Even such a fundamental idea as fractional integration must be rethought. Certainly one of the profound new ideas is that the critical dimension for integrability is no longer the topological dimension. Now we have a new idea of *homogeneous dimension*, which is actually one greater than the topological dimension. And there are powerful analytic reasons why this must be so.

We develop the analysis of the Heisenberg group in some detail, so that we may define and calculate bounds on both fractional and singular integrals in this new setting. We provide applications to the study of the Szegő and Poisson–Szegő integrals. The book concludes with a treatment of domains of finite type—which is the next development in this chain of ideas, and is the focus of current research.

We provide considerable background here for the punch line, which is analysis on the Heisenberg group. Much of this basic material in Fourier and harmonic analysis has been covered in other venues (some by this author), but it would be a disservice to the reader were we to send him or her running off to a number of ancillary sources. We want this book to be as self-containted as possible.

We do not, however, wish the book to be boring for the experienced reader. So we put the most basic material on Fourier series in an appendix. Even there, proofs are isolated so that the reader may review this material quickly and easily. The first chapter of the book is background and history, and may be read quickly. Chapters 2 and 3 provide basic material on Fourier analysis, particularly the Fourier transform (although the ideas about singular integrals in Chapter 2 are seminal and should be absorbed carefully). In these two chapters we have also exploited the device of having proofs isolated at the end of the chapter. Many readers will have seen some of this material in a graduate real-variables course. They will want to move on expeditiously to the more exciting ideas that pertain to the task at hand. We have made every effort to aid in this task.

We introduce in this graduate text a few didactic tools to make the reading stimulating and engaging for students:

1. Each chapter begins with a *Prologue*, introducing students to the key ideas that will unfold in the text that follows.
2. Each section begins with a *Capsule*, giving a quick preview of that unit of material.
3. Each key theorem or proposition is preceded by a *Prelude*, putting the result in context and providing motivation.
4. At key junctures we include an *Exercise for the Reader* to encourage the neophyte to pick up a pencil, do some calculations, and get involved with the material.

We hope that these devices will break up the usual dry exposition of a research monograph and make this text more like an invitation to the subject.

I have taught versions of this material over the years, most recently in spring of 2006 at Washington University in St. Louis. I thank the students in the course for their attention and for helping me to locate many mistakes and misstatements. Lina Lee, in particular, took wonderful notes from the course and prepared them as TEX files. Her notes are the basis for much of the second part of this book. I thank the American Institute of Mathematics for hospitality and support during some of the writing.

In total, this is an ambitious introduction to a particular direction in modern harmonic analysis. It presents harmonic analysis *in vitro*—in a context in which it is actually applied: complex variables and partial differential equations. This will make the learning experience more meaningful for graduate students who are just beginning to forge a path of research. We expect the readers of this book to be ready to take a number of different directions in exploring the research literature and beginning his or her own investigations.

— *Steven G. Krantz*

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