

Preface

The primary object of study in this book is small-amplitude periodic solutions of two-dimensional autonomous systems of ordinary differential equations,

$$\dot{x} = P(x, y), \quad \dot{y} = Q(x, y),$$

for which the right-hand sides are polynomials. Such systems are called polynomial systems. If the origin is an isolated singularity of a polynomial (or real analytic) system, and if there does not exist an orbit that tends to the singularity, in either forward or reverse time, with a definite limiting tangent direction, then the singularity must be either a *center*, in which case there is a neighborhood of the origin in which every orbit except the origin is periodic, or a *focus*, in which case there is a neighborhood of the origin in which every orbit spirals towards or away from the origin. The problem of distinguishing between a center and a focus for a given polynomial system or a family of such systems is known as the Poincaré center problem or the center-focus problem. Although it dates from the end of the 19th century, it is completely solved only for linear and quadratic systems ($\max\{\deg(P), \deg(Q)\}$ equal to 1 or 2, respectively) and a few particular cases in families of higher degree.

Relatively simple analysis shows that when the matrix of the linearization of the system at the singular point has eigenvalues with nonzero real parts, the singular point is a focus. If, however, the real parts of the eigenvalues are zero then the type of the singular point depends on the nonlinear terms of polynomials in a nontrivial way. A general method due to Poincaré and Lyapunov reduces the problem to that of solving an infinite system of polynomial equations whose variables are parameters of the system of differential equations. That is, the center-focus problem is reduced to the problem of finding the variety of the ideal generated by a collection of polynomials, called the focus quantities of the system.

A second problem, called the cyclicity problem, is to estimate the number of limit cycles, that is, isolated periodic solutions, that can bifurcate from a center or focus when the coefficients of the system of differential equations are perturbed by an arbitrarily small amount, but in such a way as to remain in a particular family of systems, for example in the family of all quadratic polynomial systems if the

original system was quadratic. This problem is a part of the still unresolved 16th Hilbert problem and is often called the local 16th Hilbert problem. In fact, in order to find an upper bound for the cyclicity of a center or focus in a polynomial system it is sufficient to obtain a basis for the above-mentioned ideal of focus quantities. Thus the study of these two famous problems in the qualitative theory of differential equations can be carried out through the study of polynomial ideals, that is, through the study of an object of commutative algebra.

Recent decades have seen a surge of interest in the center and cyclicity problems. Certainly an important reason for this is that the resolution of these problems involves extremely laborious computations, which nowadays can be carried out using powerful computational facilities. Applications of concepts that could not be utilized even 30 years ago are now feasible, often even on a personal computer, because of advances in the mathematical theory, in the computer software of computational algebra, and in computer technology. This book is intended to give the reader a thorough grounding in the theory, and explains and illustrates methods of computational algebra, as a means of approaching the center-focus and cyclicity problems.

The methods we present can be most effectively exploited if the original real system of differential equations is properly complexified; hence, the idea of complexifying a real system, and more generally working in a complex setting, is one of the central ideas of the text. Although the idea of extracting information about a real system of ordinary differential equations from its complexification goes back to Lyapunov, it is still relatively scantily used. Our belief that it deserves exposition at the level of a textbook has been a primary motivation for this work. In addition to that, it has appeared to us that by and large specialists in the qualitative theory of differential equations are not well versed in these new methods of computational algebra, and conversely that there appears to be a general lack of knowledge on the part of specialists in computational algebra about the possibility of an algebraic treatment of these problems of differential equations. We have written this work with the intention of trying to help to draw together these two mathematical communities.

Thus, the readers we have had in mind in writing this work have been graduate students and researchers in nonlinear differential equations and computational algebra, and in fields outside mathematics in which the investigation of nonlinear oscillation is relevant. The book is designed to be suitable for use as a primary textbook in an advanced graduate course or as a supplementary source for beginning graduate courses. Among other things, this has meant motivating and illustrating the material with many examples, and including a great many exercises, arranged in the order in which the topics they cover appear in the text. It has also meant that we have given complete proofs of a number of theorems that are not readily available in the current literature and that we have given much more detailed versions of proofs that were written for specialists. All in all, researchers working in the theory of limit cycles of polynomial systems should find it a valuable reference resource, and because it is self-contained and written to be accessible to nonspecialists, researchers in other fields should find it an understandable and helpful introduction to the tools

they need to study the onset of stable periodic motion, such as ideals in polynomial rings and Gröbner bases.

The first two chapters introduce the primary technical tools for this approach to the center and cyclicity problems, as well as questions of linearizability and isochronicity that are naturally investigated in the same manner. The first chapter lays the groundwork of computational algebra. We give the main properties of ideals in polynomial rings and their affine varieties, explain the concept of Gröbner bases, a key component of various algorithms of computational algebra, and provide explicit algorithms for elimination and implicitization problems and for basic operations on ideals in polynomial rings and on their varieties. The second chapter begins with the main theorems of Lyapunov's second method, theorems that are aimed at the investigation of the stability of singularities (in this context often termed equilibrium points) by means of Lyapunov functions. We then cover the basics of the theory of normal forms of ordinary differential equations, including an algorithm for the normalization procedure and a criterion for convergence of normalization transformations and normal forms.

Chapter 3 is devoted to the center problem. We describe how the concept of a center can be generalized to complex systems, in order to take advantage of working over the algebraically closed field \mathbb{C} in place of \mathbb{R} . This leads to the study of the variety, in the space of parameters of the system, that corresponds to systems with a center, which is called the center variety. We present an efficient computational algorithm for computing the focus quantities, which are the polynomials that define the center variety. Then we describe two main mechanisms for proving the existence of a center in a polynomial system, Darboux integrability and time-reversibility, thereby completing the description of all the tools needed for this method of approach to the center-focus problem. This program and its efficiency are demonstrated by applying it to resolve the center problem for the full family of quadratic systems and for one particular family of cubic systems. In a final section, as a complement to the rest of the chapter, particularly aspects of symmetry, the important special case of Liénard systems is presented.

If all solutions in a neighborhood of a singular point are periodic, then a question that arises naturally is whether all solutions have the same period. This is the so-called isochronicity problem that has attracted study from the time of Huygens and the Bernoullis. In Chapter 4 we present a natural generalization of the concept of isochronicity to complex systems of differential equations, the idea of linearizability. We then introduce and develop methods for investigating linearizability in the complex setting.

As indicated above, one possible mechanism for the existence of a center is time-reversibility of the system. Chapter 5 presents an algorithm for computing all time-reversible systems within a given polynomial family. This takes on additional importance because in all known cases the set of time-reversible systems forms exactly one component of the center variety. The algorithm is derived using the study of invariants of the rotation group of the system and is a nice application of that theory and the algebraic theory developed in Chapter 1.

The last chapter is devoted to the cyclicity problem. We describe Bautin's method, which reduces the study of cyclicity to finding a basis of the ideal of focus quantities, and then show how to obtain the solution for the cyclicity problem in the case that the ideal of focus quantities is radical. In the case that the ideal generated by the first few focus quantities is not radical, the problem becomes much more difficult; at present there is no algorithmic approach for its treatment. Nevertheless we present a particular family of cubic systems for which it is possible, using Gröbner basis calculations, to obtain a bound on cyclicity. Finally, as a further illustration of the applicability of the ideas developed in the text, we investigate the problem of the maximum number of cycles that can maintain the original period of an isochronous center in \mathbb{R}^2 when it is perturbed slightly within the collection of centers, the so-called problem of bifurcation of critical periods.

Specialists perusing the table of contents and the bibliography will surely miss some of their favorite topics and references. For example, we have not mentioned methods that approach the center and cyclicity problems based on the theory of resultants and triangular decomposition, and have not treated the cyclicity problem specifically in the important special case of Liénard systems, such as we did for the center problem. We are well aware that there is much more that could be included, but one has to draw the line somewhere, and we can only say that we have made choices of what to include and what to omit based on what seemed best to us, always with an eye to what we hoped would be most valuable to the readers of this book.

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