

The locus of a perturbed relay system (LPRS) theory

2.1 Introduction to the LPRS

As we considered in the previous chapter, the motions in relay servo systems are normally analyzed as motions in two separate dynamic subsystems: the “slow” subsystem and the “fast” subsystem. The “fast” subsystem pertains to self-excited oscillations or periodic motions. The “slow” subsystem deals with forced motions caused by an input signal or by a disturbance, a non-zero initial conditions component of the motion, and usually pertains to the averaged (over the period of the self-excited oscillation) motion. The two dynamic subsystems interact with each other via a set of parameters: the results of the solution of the “fast” subsystem are used by the “slow” subsystem. This decomposition of the dynamics is possible if the external input is much slower than the self-excited oscillations, which is normally the case. Exactly as in the DF method, we shall proceed from the assumption that the external signals applied to the system are slow in comparison to the oscillations.

Consider again the harmonic balance equation (1.19). Using the formulas for the negative reciprocal of the DF (1.20) and the equivalent gain of the relay (1.23), we can rewrite formula (1.19) as follows:

$$W_I(j\Omega) = -\frac{1}{2} \frac{1}{k_{n(DF)}} + j \frac{\pi}{4c} y_{(DF)}(0). \quad (2.1)$$

In the imaginary part of (2.1), we view the condition of the switch of the relay from minus to plus (defined as zero time) as the equality of the system output to the negative half hysteresis ($-b$): $y_{(DF)}(t = 0) = -b$. It follows from (1.21), (1.23), and (2.1) that the frequency of the oscillations and the equivalent gain in the system (1.16) can be varied by changing the hysteresis value $2b$ of the relay. Therefore, the following two mappings can be considered: $M_1 : b \rightarrow \Omega$, $M_2 : b \rightarrow k_n$. Assume that M_1 has an inverse mapping (it follows from (1.21), (1.23), and (2.1) for the DF analysis and is proved below by deriving an analytical formula) $M_1^{-1} : \Omega \rightarrow b$. Applying

the chain rule, consider the mapping $M_2(M_1^{-1}) : \Omega \rightarrow b \rightarrow k_n$. Now let us define a certain function J as the expression on the right-hand side of formula (2.1) with the additional requirement that the values of the equivalent gain and the output at zero time should be exact values. Applying the mapping $M_2(M_1^{-1}) : \omega \rightarrow b \rightarrow k_n, \omega \in [0; \infty)$, in which we treat the frequency ω as an independent parameter, we get the following for J :

$$J(\omega) = -\frac{1}{2} \frac{1}{k_n} + j \frac{\pi}{4c} y(t)|_{t=0} \quad (2.2)$$

where $k_n = M_2(M_1^{-1}(\omega))$, $y(t)|_{t=0} = M_1^{-1}(\omega)$, $t = 0$ is the time of the switch of the relay from “ $-c$ ” to “ $+c$.” Thus, $J(\omega)$ comprises the two mappings and is defined as a characteristic of the response of the linear part to the unequally spaced pulse input $u(t)$, subject to $f_0 \rightarrow 0$ as the frequency ω varies. The real part of $J(\omega)$ contains information about gain k_n , and the imaginary part of $J(\omega)$ comprises the condition of the switching of the relay and, consequently, contains information about the frequency of the oscillations. By deriving the function that satisfies the above requirements, we can obtain exact values of the frequency of the oscillations and the *equivalent gain*.

We call the function $J(\omega)$ defined above, along with its plot on the complex plane (with the frequency ω varied), the locus of a perturbed relay system (LPRS). Suppose we have computed the LPRS of a given system. Then (as in the DF analysis) we can determine the frequency of the oscillations (as well as the amplitude) and the equivalent gain k_n (Fig. 2.1). The point of intersection of the LPRS and the straight line, which lies at the distance $\pi b/(4c)$ below (if $b > 0$) or above (if $b < 0$) the horizontal axis and parallel to it (line “ $-\pi b/4c$ ”), offers computing the frequency of the oscillations and the equivalent gain k_n of the relay.

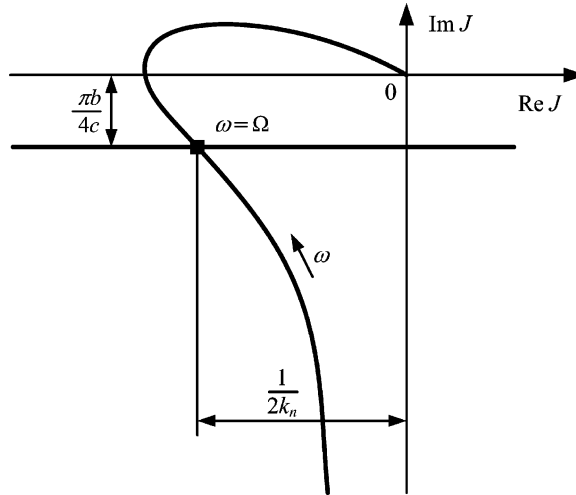


Fig. 2.1. The LPRS and oscillation analysis

According to (2.2), the frequency Ω of the oscillations can be computed by solving the equation:

$$\operatorname{Im} J(\Omega) = -\frac{\pi b}{4c}, \quad (2.3)$$

(i.e., $y(0) = -b$ is the condition of the relay switch) and the gain k_n can be computed as:

$$k_n = -\frac{1}{2\operatorname{Re} J(\Omega)}. \quad (2.4)$$

Formula (2.3) provides a periodic solution and is, therefore, a *necessary condition for the existence of a periodic motion* in the system.¹ Formula (2.2) is only a definition and not intended for the purpose of computing the LPRS $J(\omega)$. It is shown below that although $J(\omega)$ is defined through the parameters of the oscillations in a closed-loop system, it can be easily derived from the parameters of the linear part without employing the variables of formula (2.2).

2.2 Computing the LPRS for a non-integrating plant

2.2.1 Matrix state-space description approach

Deriving the computing formula of the LPRS involves only the parameters of the linear part for the case of the non-integrating (self-regulating) linear part given by the matrix differential equations. Let the system be described by the equations (1.16), where $\mathbf{A}$ is nonsingular.

Let us find the periodic solution of system (1.16) at the unequally spaced relay switching caused by a non-zero constant input signal f_0 . A common way to find a periodic solution is to use a Poincaré map. Because the control switches are unequally spaced and the oscillations are not symmetric, a Poincaré return map must be considered. Suppose that an asymmetric periodic process of the period T exists in the system. Then, considering the solution for the constant control u ,

$$\mathbf{x}(t) = e^{At}\mathbf{x}(0) + \mathbf{A}^{-1}(e^{At} - \mathbf{I})\mathbf{B}u,$$

the periodic solution of system (1.16) for the control $u = \pm 1$ (it will be shown below that the LPRS is a characteristic of the linear part only and we can assume without loss of generality $c = 1$) can be written as

$$\eta = e^{\mathbf{A}\theta_1}\rho + \mathbf{A}^{-1}(e^{\mathbf{A}\theta_1} - \mathbf{I})\mathbf{B}, \quad (2.5)$$

$$\rho = e^{\mathbf{A}\theta_2}\eta - \mathbf{A}^{-1}(e^{\mathbf{A}\theta_2} - \mathbf{I})\mathbf{B}, \quad (2.6)$$

¹ The actual existence of a periodic motion depends on a number of other factors, too, including orbital stability of the obtained periodic solution and initial conditions.

where $\rho = \mathbf{x}(0) = \mathbf{x}(T)$, $\eta = \mathbf{x}(\theta_1)$, for the periodic solution, and θ_1, θ_2 are the positive and the negative pulse durations of the periodic control $u(t)$. Formulas (2.5) and (2.6) are Poincaré return maps for the system (sequential numbers of switches are not shown). The periodic solution of system (1.16) can be obtained through finding a fixed point of the Poincaré return map (solution of (2.5) and (2.6)), which is given as follows:

$$\rho = (\mathbf{I} - e^{\mathbf{A}T})^{-1} \mathbf{A}^{-1} [e^{\mathbf{A}T} - 2e^{\mathbf{A}\theta_2} + \mathbf{I}] \mathbf{B}, \quad (2.7)$$

$$\eta = (\mathbf{I} - e^{\mathbf{A}T})^{-1} \mathbf{A}^{-1} [2e^{\mathbf{A}\theta_1} - e^{\mathbf{A}T} - \mathbf{I}] \mathbf{B}. \quad (2.8)$$

We now need to consider the periodic solution (2.7) and (2.8) as a result of the feedback action. The conditions of the switches of the relay can be written as

$$\begin{aligned} f_0 - y(0) &= b \\ f_0 - y(\theta_1) &= -b. \end{aligned} \quad (2.9)$$

Having solved the set of equations (2.9) for f_0 , we obtain: $f_0 = (y(0) + y(\theta_1))/2$. Hence, the constant term of $\sigma(t)$ is

$$\sigma_0 = f_0 - y_0 = (y(0) + y(\theta_1))/2 - y_0, \quad (2.10)$$

and the real part of the LPRS definition formula can be transformed into

$$\operatorname{Re} J(\omega) = -0.5 \lim_{\gamma \rightarrow \frac{1}{2}} \frac{0.5[y(0) + y(\theta_1)] - y_0}{u_0}, \quad (2.11)$$

where $\gamma = \frac{\theta_1}{\theta_1 + \theta_2} = \frac{\theta_1}{T}$. Then $\theta_1 = \gamma T$, $\theta_2 = (1 - \gamma)T$, $u_0 = 2\gamma - 1$, and (2.11) can be written as

$$\operatorname{Re} J(\omega) = -0.5 \lim_{\gamma \rightarrow \frac{1}{2}} \frac{0.5\mathbf{C}[\rho + \eta] - y_0}{2\gamma - 1},$$

where ρ and η are given by (2.7) and (2.8), respectively. The imaginary part of the definition formula of $J(\omega)$ can be transformed into:

$$\operatorname{Im} J(\omega) = \frac{\pi}{4} \mathbf{C} \lim_{\gamma \rightarrow \frac{1}{2}} \rho.$$

Finally, the state-space description-based formula of the LPRS can be derived on the basis of the previous two formulas and (2.7), (2.8) as follows:

$$\begin{aligned} J(\omega) &= -0.5\mathbf{C}[A^{-1} + \frac{2\pi}{\omega}(\mathbf{I} - e^{\frac{2\pi}{\omega}\mathbf{A}})^{-1}e^{\frac{\pi}{\omega}\mathbf{A}}]\mathbf{B} \\ &\quad + j\frac{\pi}{4}\mathbf{C}(\mathbf{I} + e^{\frac{\pi}{\omega}\mathbf{A}})^{-1}(\mathbf{I} - e^{\frac{\pi}{\omega}\mathbf{A}})\mathbf{A}^{-1}\mathbf{B}. \end{aligned} \quad (2.12)$$

Therefore, if the system is given in the state-space form (1.16), then formula (2.12) can be used to compute the LPRS. The LPRS computed as (2.12) comprises all possible periodic solutions and equivalent gain values

for a given linear part. For that reason, the LPRS is a relatively universal frequency-domain characteristic of the linear part of a relay servo system. An actual periodic solution for a given linear part and parameters of the relay can be found from equation (2.3). A detailed derivation of the LPRS is given in the Appendix.

The subroutine “lprsmatr” (see Appendix) can be used for the LPRS computing per formula (2.12).

2.2.2 Partial fraction expansion technique

We now derive the LPRS formula when the description of the linear part is given in the form of the transfer function expanded into partial fractions. We first prove the additivity property of the LPRS $J(\omega)$.

Theorem 2.1. (*additivity property*). *If the transfer function $W_l(s)$ of the linear part is a sum of n transfer functions $W_l(s) = W_1(s) + W_2(s) + \dots + W_n(s)$, then the LPRS $J(\omega)$ can be calculated as a sum of n LPRS: $J(\omega) = J_1(\omega) + J_2(\omega) + \dots + J_n(\omega)$, where $J_i(\omega)$ ($i = 1, \dots, n$) is the LPRS of the relay system with the transfer function of the linear part being $W_i(s)$.*

Proof. We prove the property for $n = 2$: if the property is true for $n = 2$, it is true for any n . Consider the steady asymmetric oscillations in the system when $f(t) \equiv f_0 \neq 0$. Assume that a unimodal asymmetric limit cycle occurs (Fig. 1.4). Suppose that the frequency Ω of the oscillations is known, and the control amplitude c , as well as the pulse duration (θ_1 and θ_2) of the periodic control $u(t)$, are given. If $W_l(s) = W_1(s) + W_2(s)$, then the output is $y(t) = y_1(t) + y_2(t)$, where $y_i(t), i = \overline{1, 2}$ is the output of the linear part, which has the transfer function $W_i(s), i = \overline{1, 2}$ with its input $u(t)$ as above. Substitute $y_1(t) + y_2(t)$ for $y(t)$ in (2.10) and obtain $\sigma_0 = \sigma_{01} + \sigma_{02}$, where $\sigma_{01} = (y_1(0) + y_1(\theta_1))/2 - y_{01}$, $\sigma_{02} = (y_2(0) + y_2(\theta_1))/2 - y_{02}$, y_{01} and y_{02} are the constant terms of $y_1(t)$ and $y_2(t)$, respectively. Thus, when the parameters of $u(t)$ are as specified above, the constant term of $\sigma(t)$ is equal to the sum of the constant terms of $\sigma_1(t)$ and $\sigma_2(t)$ where $\sigma_1(t)$ and $\sigma_2(t)$ are the errors in two different relay systems with the transfer functions $W_1(s)$ and $W_2(s)$, respectively. Because the additivity property is true for σ_0 , it is also true for σ_0/u_0 because u_0 is constant and, consequently, this is true for $\lim(\sigma_0/u_0)$. It is also obvious that $y(0) = y_1(0) + y_2(0)$. Thus, according to (2.2): $J(\omega) = J_1(\omega) + J_2(\omega)$. ■

The additivity property offers a way of computing the LPRS $J(\omega)$ via expanding $W_l(s)$ into the sum of first- and second-order dynamics (partial fractions), calculating the component LPRS $J_i(\omega)$ for each of them, and summing the LPRS $J_i(\omega)$. Analytical formulas for $J(\omega)$ of first- and second-order dynamics are derived in this chapter below and presented in Table 2.1. The respective MATLAB functions are given in the Appendix (functions “lprslord,” “lprshint,” “lprs2ord1,” “lprs2ord2,” “lprs2ord3,” “lprs2ord4,” and “lprsfopdt”).

Table 2.1. Formulas of the LPRS $J(\omega)$

Tr. fun. $W(s)$	The LPRS $J(\omega)$
$\frac{K}{s}$	$0 - j\frac{\pi^2 K}{8\omega}$
$\frac{K}{Ts+1}$	$\frac{K}{2}(1 - \alpha \operatorname{csch} \alpha) - j\frac{\pi K}{4} \tanh(\alpha/2),$ $\alpha = \pi/(T\omega)$
$\frac{K}{(T_1 s+1)(T_2 s+1)}$	$\frac{K}{2}[1 - T_1/(T_1 - T_2)\alpha_1 \operatorname{csch} \alpha_1 - T_2/(T_2 - T_1)\alpha_2 \operatorname{csch} \alpha_2]$ $- j\frac{\pi K}{4}/(T_1 - T_2)[T_1 \tanh(\alpha_1/2) - T_2 \tanh(\alpha_2/2)],$ $\alpha_1 = \pi/(T_1 \omega), \quad \alpha_2 = \pi/(T_2 \omega)$
$\frac{K}{s^2+2\xi s+1}$	$\frac{K}{2}[(1 - (B + \gamma C)/(\sin^2 \beta + \sinh^2 \alpha)]$ $- j\frac{\pi K}{4}(\sinh \alpha - \gamma \sin \beta)/(\cosh \alpha + \cos \beta),$ $\alpha = \pi\xi/\omega, \quad \beta = \pi(1 - \xi^2)^{1/2}/\omega, \quad \gamma = \alpha/\beta$ $B = \alpha \cos \beta \sinh \alpha + \beta \sin \beta \cosh \alpha,$ $C = \alpha \sin \beta \cosh \alpha - \beta \cos \beta \sinh \alpha$
$\frac{Ks}{s^2+2\xi s+1}$	$\frac{K}{2}[\xi(B + \gamma C) - \pi/\omega \cos \beta \sinh \alpha]/(\sin^2 \beta + \sinh^2 \alpha)]$ $- j\frac{\pi K}{4}(1 - \xi^2)^{-1/2} \sin \beta/(\cosh \alpha + \cos \beta),$ $\alpha = \pi\xi/\omega, \quad \beta = \pi(1 - \xi^2)^{1/2}/\omega, \quad \gamma = \alpha/\beta$ $B = \alpha \cos \beta \sinh \alpha + \beta \sin \beta \cosh \alpha,$ $C = \alpha \sin \beta \cosh \alpha - \beta \cos \beta \sinh \alpha$
$\frac{Ks}{(s+1)^2}$	$\frac{K}{2}[\alpha(-\sinh \alpha + \alpha \cosh \alpha)/\sinh^2 \alpha - j0.25\pi\alpha/(1 + \cosh \alpha)],$ $\alpha = \pi/\omega$
$\frac{Ks}{(T_1 s+1)(T_2 s+1)}$	$\frac{K}{2}/(T_2 - T_1)[\alpha_2 \operatorname{csch} \alpha_2 - \alpha_1 \operatorname{csch} \alpha_1]$ $- j\frac{\pi K}{4}/(T_2 - T_1)[\tanh(\alpha_1/2) - \tanh(\alpha_2/2)],$ $\alpha_1 = \pi/(T_1 \omega), \quad \alpha_2 = \pi/(T_2 \omega)$
$\frac{Ke^{-\tau s}}{Ts+1}$	$\frac{K}{2}(1 - \alpha e^\gamma \operatorname{csch} \alpha) + j\frac{\pi K}{4}\left(\frac{2e^{-\alpha}e^\gamma}{1+e^{-\alpha}} - 1\right),$ $\alpha = \frac{\pi}{T\omega}, \quad \gamma = \frac{\tau}{T}$

2.2.3 Transfer function description approach

Another formula for $J(w)$ can now be derived for the case of the linear part given by a transfer function. Suppose the linear part does not have integrators. We write the Fourier series expansion of the signal $u(t)$ (Fig. 1.5)

$$u(t) = u_0 + 4c/\pi \sum_{k=1}^{\infty} \sin(\pi k \theta_1/(\theta_1 + \theta_2))/k \times \{\cos(k\omega\theta_1/2) \cos(k\omega t) + \sin(k\omega\theta_1/2) \sin(k\omega t)\},$$

where $u_0 = c(\theta_1 - \theta_2)/(\theta_1 + \theta_2)$, $\omega = 2\pi/(\theta_1 + \theta_2)$. Therefore, $y(t)$ as a response of the linear part with the transfer function $W_l(s)$ can be written as

$$\begin{aligned} y(t) = & y_0 + 4c/\pi \sum_{k=1}^{\infty} \sin(\pi k \theta_1/(\theta_1 + \theta_2))/k \\ & \times \{\cos(k\omega\theta_1/2) \cos[k\omega t + \varphi_l(k\omega)] \\ & + \sin(k\omega\theta_1/2) \sin[k\omega t + \varphi_l(k\omega)]\} A_l(k\omega), \end{aligned} \quad (2.13)$$

where $\varphi_l(k\omega) = \arg W_l(jk\omega)$, $A_l(k\omega) = |W_l(jk\omega)|$, $y_0 = u_0|W_l(j0)|$. The conditions of the switches of the relay have the form of equations (2.9) where $y(0)$ and $y(\theta_1)$ can be obtained from (2.13) if we set $t = 0$ and $t = \theta_1$, respectively:

$$y(0) = y_0 + 4c/\pi \sum_{k=1}^{\infty} [0.5 \sin(2\pi k\theta_1/(\theta_1 + \theta_2)) \operatorname{Re} W_l(jk\omega) + \sin^2(\pi k\theta_1/(\theta_1 + \theta_2)) \operatorname{Im} W_l(jk\omega)]/k, \quad (2.14)$$

$$y(\theta_1) = y_0 + 4c/\pi \sum_{k=1}^{\infty} [0.5 \sin(2\pi k\theta_1/(\theta_1 + \theta_2)) \operatorname{Re} W_l(jk\omega) - \sin^2(\pi k\theta_1/(\theta_1 + \theta_2)) \operatorname{Im} W_l(jk\omega)]/k. \quad (2.15)$$

Differentiating (2.9) with respect to f_0 (and taking into account (2.14) and (2.15)), we obtain the formulas containing the derivatives at the point $\theta_1 = \theta_2 = \theta = \pi/\omega$. Solving those equations for $d(\theta_1 - \theta_2)/df_0$ and $d(\theta_1 + \theta_2)/df_0$, we obtain: $d(\theta_1 + \theta_2)/df_0|_{f_0=0} = 0$, which corresponds to the derivative of the frequency of the oscillations, and:

$$\frac{d(\theta_1 - \theta_2)}{df_0}|_{f_0=0} = 2\theta/[c(|W_l(0)| + 2 \sum_{k=1}^{\infty} \cos(\pi k) \operatorname{Re} W_l(\omega k))]. \quad (2.16)$$

Considering the formula of the closed-loop system transfer function, we can write:

$$\frac{d(\theta_1 - \theta_2)}{df_0}|_{f_0=0} = k_n/(1 + k_n|A_l(0)|)2\theta/c. \quad (2.17)$$

Solving equations (2.16) and (2.17) together for k_n , we obtain the following expression:

$$k_n = 0.5 / \sum_{k=1}^{\infty} (-1)^k \operatorname{Re} W_l(k\pi/\theta). \quad (2.18)$$

Taking into account formula (2.18), the identity $\omega = \pi/\theta$, and the definition of the LPRS (2.2), we obtain the final form of expression for $\operatorname{Re} J(\omega)$. Similarly, solving the set of equations (2.9), where $\theta_1 = \theta_2 = \theta$ and $y(0)$ and $y(\theta_1)$ have the form (2.14) and (2.15), respectively, we obtain the final formula of $\operatorname{Im} J(\omega)$. Putting the real and the imaginary parts together, we obtain the final formula of the LPRS $J(\omega)$ for relay systems with non-integrating plants:

$$J(\omega) = \sum_{k=1}^{\infty} (-1)^{k+1} \operatorname{Re} W_l(k\omega) + j \sum_{k=1}^{\infty} \frac{1}{2k-1} \operatorname{Im} W_l[(2k-1)\omega]. \quad (2.19)$$

The subroutine “lprsser200” (see Appendix) can be used for the LPRS computing per formula (2.19), which takes the sum of 200 terms of the series.

2.2.4 Orbital stability of relay systems

The stability of periodic orbits (limit cycles) is usually referred to as orbital stability. The notion of orbital stability is different from the notion of stability of an equilibrium point: for an orbitally stable motion, the difference between the perturbed and unperturbed motions does not necessarily vanish. What is important is that the perturbed motion in an orbitally stable system converges to the orbit of the unperturbed system. More details about this type of stability are provided in [62]. In relay feedback systems, analysis of orbital stability can be reduced to the analysis of certain equivalent discrete-time systems with time instants corresponding to the switches of the relay, which can be obtained from the original system by the Poincaré map of the motion with an initial perturbation. The stability condition based on this approach was proposed in [2]. If we assume that the initial state is $\mathbf{x}(0) = \rho + \delta\rho$, where $\delta\rho$ is the initial perturbation, and find the mapping $\delta\rho \rightarrow \delta\eta$, we can make a conclusion about orbital stability of the system by considering the Jacobian matrix of this mapping. A detailed derivation of the Jacobian matrix that relates the perturbations at switching times is given in the Appendix.

Therefore, the stability criterion can be formulated as follows.

Theorem 2.2. *The relay feedback system (1.16) is locally orbitally asymptotically stable if and only if all eigenvalues of the matrix*

$$\Phi_0 = \left[\mathbf{I} - \frac{\mathbf{v}(\frac{T}{2}-)\mathbf{C}}{\mathbf{C}\mathbf{v}(\frac{T}{2}-)} \right] e^{\mathbf{A}\frac{T}{2}}, \quad (2.20)$$

where $T = \frac{2\pi}{\Omega}$ is the period of the oscillations, $\mathbf{v}$ is the velocity matrix,

$$\mathbf{v}\left(\frac{T}{2}-\right) = 2(\mathbf{I} - e^{\mathbf{A}T})^{-1} \left(e^{\mathbf{A}\frac{T}{2}} - e^{\mathbf{A}T} \right) \mathbf{B} = 2(\mathbf{I} + e^{\mathbf{A}T/2})^{-1} e^{\mathbf{A}T/2} \mathbf{B},$$

have magnitudes less than one.

In addition to the stability analysis, the direction of the relay switch must be verified, too [94]. This condition is formulated as the following inequality,

$$\dot{y}\left(\frac{T}{2}-\right) = \mathbf{C}\mathbf{v}\left(\frac{T}{2}-\right) > 0,$$

where $\mathbf{v}\left(\frac{T}{2}-\right)$ is given by the previous formula.

2.3 Computing the LPRS for an integrating plant

2.3.1 Matrix state-space description approach

For an integrating linear part, the formulas derived above cannot be used without certain modifications. Despite the fact that the solution $\mathbf{x}(t)$ of the

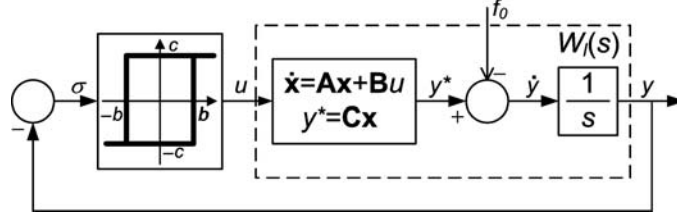


Fig. 2.2. Relay servo system with integrating linear part

system is well-defined even if the matrix $\mathbf{A}$ does not have an inverse, the above results are not applicable to an integrating linear part. In the case of unequally spaced switches, a system with a conventional description, strictly speaking, cannot have a periodic process even if a ramp signal is applied to the input of the system in Fig. 1.3. The motion occurring in such a system is a combination of a periodic and a ramp motion — due to unlimited integration. To enable the system to have an asymmetric periodic motion, we must transpose the constant input signal to the integrator input (Fig. 2.2). The balance of the constant terms of the signals in the various points of the system must be achieved for periodic motion to occur.

Similarly, we derive the formulas of $J(\omega)$ for the case of an integrating linear part. The state-space description of the system (Fig. 2.2) has the following form,

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u, \quad (2.21)$$

$$\dot{y} = \mathbf{C}\mathbf{x} - f_0, \quad (2.22)$$

$$u = \begin{cases} +c & \text{if } \sigma = -y \geq b \text{ or } \sigma > -b, u(t-0) = c \\ -c & \text{if } \sigma = -y \leq -b \text{ or } \sigma < b, u(t-0) = -c \end{cases}$$

where $\mathbf{A} \in R^{(n-1) \times (n-1)}$, $\mathbf{B} \in R^{(n-1) \times 1}$, $\mathbf{C} \in R^{1 \times (n-1)}$, $\mathbf{A}$ is nonsingular, f_0 is a constant input to the system, σ is the error signal, and $u(t-0)$ is the control value at the time immediately preceding the current time. Note that formula (2.22) defines not the output y but its derivative, which adds an integrator to the linear part. A separate consideration of the variable $y(t)$ from the other state variables is possible due to the integrating property of the linear part. This allows us at first to find a periodic solution for $\mathbf{x}(t)$ (for a given unequally spaced switching), and after that to determine a periodic solution for the system output. The periodic solution for $\mathbf{x}(t)$ is given above (formulas (2.7) and (2.8)). The periodic output $y(t)$ can be obtained by integrating equation (2.22) from the initial states determined by formulas (2.7) and (2.8). As a result, for the control amplitude $c = 1$, the system output can be written as

$$y_1(t) = y_1(0) - \mathbf{C}\mathbf{A}^{-1}\mathbf{B}t - f_0t + \mathbf{C}\mathbf{A}^{-1}[(e^{\mathbf{A}t} - \mathbf{I})\rho + \mathbf{A}^{-1}(e^{\mathbf{A}t} - \mathbf{I})\mathbf{B}], \quad (2.23)$$

$$y_2(t) = y_1(\theta_1) + \mathbf{C}\mathbf{A}^{-1}\mathbf{B}t - f_0t + \mathbf{C}\mathbf{A}^{-1}[(e^{\mathbf{A}t} - \mathbf{I})\eta - \mathbf{A}^{-1}(e^{\mathbf{A}t} - \mathbf{I})\mathbf{B}], \quad (2.24)$$

where $y_1(t) = y(t)$, $y_2(t) = y(t + \theta_1)$.

The time t in formulas (2.23) and (2.24) is independent, and $t = 0$ in formula (2.23) is the time of the switch from minus to plus, and in formula (2.24) $t = 0$ is the time of the switch from plus to minus. For periodic motion, the following equations hold, which represents a Poincaré return map:

$$y(\theta_1) = y(0) - (\mathbf{CA}^{-1}\mathbf{B} - f_0)\theta_1 + \mathbf{CA}^{-1}[(e^{\mathbf{A}\theta_1} - \mathbf{I})\rho + \mathbf{A}^{-1}(e^{\mathbf{A}\theta_1} - \mathbf{I})\mathbf{B}], \quad (2.25)$$

$$y(0) = y(\theta_1) + (\mathbf{CA}^{-1}\mathbf{B} - f_0)\theta_2 + \mathbf{CA}^{-1}[(e^{\mathbf{A}\theta_2} - \mathbf{I})\eta - \mathbf{A}^{-1}(e^{\mathbf{A}\theta_2} - \mathbf{I})\mathbf{B}]. \quad (2.26)$$

Analysis of equations (2.25) and (2.26) shows that the set of equations has a solution if and only if

$$f_0 = -\mathbf{CA}^{-1}\mathbf{B}(2\gamma - 1), \quad (2.27)$$

$$\gamma = \frac{\theta_1}{\theta_1 + \theta_2} = \frac{\theta_1}{T},$$

which corresponds to the situation when the constant term of the signal $y^*(t)$ is equal to f_0 and, therefore, the constant term at the integrator input is zero — the only possibility for the system to have a periodic process. Furthermore, equations (2.25) and (2.26) are equivalent and have an infinite number of solutions. To understand why, note that if a periodic signal with zero constant term is applied to the integrator input, its output signal is not uniquely determined, but, depending on the initial value, can represent an infinite number of biased periodic signals. To define a unique solution, introduce an additional condition:

$$y(\theta_1) = -y(0). \quad (2.28)$$

The solution of equations (2.25) and (2.28) results in

$$y(0) = \mathbf{CA}^{-1}\mathbf{B}\gamma(1 - \gamma)T + \frac{1}{4}\mathbf{CA}^{-2}\{(\mathbf{I} - e^{\mathbf{A}T})^{-1}[6e^{\mathbf{A}T} - 3(e^{\mathbf{A}\theta_1} + e^{\mathbf{A}\theta_2}) - e^{\mathbf{A}T}(e^{\mathbf{A}\theta_1} + e^{\mathbf{A}\theta_2}) + 2\mathbf{I}] - (e^{\mathbf{A}\theta_1} + e^{\mathbf{A}\theta_2}) + 2\mathbf{I}\}\mathbf{B}. \quad (2.29)$$

The output at $t = \theta_1$ is a negative value of the same formula. Thus, we find the periodic solution of system (2.21), (2.22). The LPRS formula can be derived from the analysis of the closed-loop system with an unequally spaced switching control having an infinitesimally small asymmetry. The constant term y_0 of the output $y(t)$ is determined as the sum of integrals of functions (2.23) and (2.24) divided by the period T

$$y_0 = \frac{1}{T}\left\{\int_0^{\theta_1} y_1(\tau)d\tau + \int_0^{\theta_2} y_2(\tau)d\tau\right\}, \quad (2.30)$$

where $y_1(\tau)$ is given by (2.23) and $y_2(\tau)$ is given by (2.24). The formula for the real part of $J(\omega)$ can be transformed into

$$\operatorname{Re} J(\omega) = \lim_{\gamma \rightarrow \frac{1}{2}} \frac{y_0}{2\gamma - 1}, \quad (2.31)$$

where expression (2.30) can be used for computing y_0 . The formula of the imaginary part of $J(\omega)$ is determined by (2.29) with a coefficient, which follows from the LPRS definition. Finally, the LPRS for the case of an integrating linear part can be expressed with the following formula

$$\begin{aligned} J(\omega) = & \frac{1}{4} \mathbf{C} \mathbf{A}^{-2} \{ (\mathbf{I} - \mathbf{D}^2)^{-1} [\mathbf{D}^2 - (\mathbf{I} + \frac{4\pi}{\omega} \mathbf{A}) \mathbf{D} + \mathbf{D}^3 - \mathbf{I}] + \mathbf{D} - \mathbf{I} \} \mathbf{B} \\ & + j \frac{\pi}{8} \mathbf{C} \mathbf{A}^{-1} \{ \frac{\pi}{\omega} + \mathbf{A}^{-1} [(\mathbf{I} - \mathbf{D}^2)^{-1} (3\mathbf{D}^2 - 3\mathbf{D} - \mathbf{D}^3 + \mathbf{I}) - \mathbf{D} - \mathbf{I}] \} \mathbf{B}, \end{aligned} \quad (2.32)$$

where $\mathbf{D} = e^{\frac{\pi}{\omega} \mathbf{A}}$. Therefore, the state-space description-based LPRS formula for the case of an integrating linear part has been derived above.

The subroutine “lprsmatrnt” (see Appendix) can be used for the LPRS computing per formula (2.32).

2.3.2 Transfer function description approach

We derive the LPRS formula for the case of an integrating linear part given by a transfer function. The model suitable for the following analysis is given in Fig. 2.2. One can notice that the periodic terms of the signals of the system Fig. 2.2 are the same as the periodic terms of respective signal of the system Fig. 1.3. For that reason, we can use some results of the above analysis for the case of a non-integrating linear part. The constant input f_0 causes an asymmetry in the periodic motion. In a steady periodic motion, the constant term of the input signal to the integrator is zero. Therefore, the constant input is compensated for by the constant term of the signal $y^*(t)$, and the output of the system can again be written as in formula (2.13). However, the value of y_0 in (2.13) is different. Now it does not directly depend on u_0 . The values of $y(0)$ and $y(\theta_1)$ are given by formulas (2.14) and (2.15), as before. In other words, the input $\sigma(t)$ to the relay has two terms: the constant term σ_0 and the periodic term $\sigma_p(t)$. The periodic term $\sigma_p(t)$ coincides with that of formula (2.13) (the negative value of the latter), and the constant term is $\sigma_0 = -y_0$. Because the input to the relay does not include the external input f_0 , the following equation holds:

$$y(0) + y(\theta_1) = 0.$$

Solving this equation, we find that $\sigma_0 = -y_0$, and

$$\sigma_0 = \frac{2c}{\pi} \sum_{k=1}^{\infty} \sin\left(\frac{2\pi k \theta_1}{\theta_1 + \theta_2}\right) \operatorname{Re} W_I(jk\omega).$$

The equivalent gain k_n can be obtained as a reciprocal of the derivative $d\sigma_0/du_0$ at $\theta_1 = \theta_2 = \pi/\omega$. We compute the following limit

$$\lim_{\gamma \rightarrow \frac{1}{2}} \frac{\sigma_0}{u_0} = 2 \sum_{k=1}^{\infty} (-1)^k \cdot \operatorname{Re} W_l(jk\omega).$$

The real part of the LPRS is given by $\operatorname{Re} J(\omega) = -0.5/k_n$, where the equivalent gain k_n is the reciprocal of the above limit. The imaginary part of the LPRS remains the same for the case of an integrating linear part. And finally, a formula for the LPRS can be constructed on the basis of the definition (2.2) and the above analysis. The final formula of the LPRS is given as follows:

$$J(\omega) = \sum_{k=1}^{\infty} (-1)^{k+1} \operatorname{Re} W_l(k\omega) + j \sum_{k=1}^{\infty} \frac{1}{2k-1} \operatorname{Im} W_l[(2k-1)\omega]. \quad (2.33)$$

One can see that formula (2.33) coincides with formula (2.19). Therefore, despite the different model and different mechanism of generation of the constant term in the error signal, the LPRS formula expressed in terms of the frequency response of the linear part remains the same. For an accurate calculation of a point of $J(\omega)$, the few first terms of the series (2.33) are enough as a rule. It can be shown that the series (2.33) always converges for strictly proper transfer functions. Formula (2.33) can also be used for the LPRS calculation from a frequency response characteristic (Bode plot, Nyquist plot) of the linear part.

2.3.3 Orbital stability of relay systems

An integrating plant provides significantly different dynamics in comparison with a non-integrating plant. Therefore, the stability conditions in [2] cannot be directly used for stability analysis of the systems with integrating plants. The formal reason is that the matrix $\mathbf{A}$ is not invertible. However, with the plant description as in (2.21), (2.22), the matrix $\mathbf{A}$ refers only to the non-integrating part of the plant and, thus, has an inverse. Again, if we assume that the initial state is $\mathbf{x}(0) = \rho + \delta\rho$, where $\delta\rho$ is the initial perturbation, and find the mapping $\delta\rho \rightarrow \delta\eta$, we can make a conclusion about the orbital stability of the system by considering the Jacobian matrix of this mapping. A detailed derivation of the Jacobian matrix that relates the perturbations at switching times is given in the Appendix.

Therefore, the stability criterion can be formulated as follows.

Theorem 2.3. *The relay feedback system (2.21), (2.22) is locally orbitally asymptotically stable if and only if all the eigenvalues of the matrix*

$$\Phi_0 = -\frac{\mathbf{v} \left(\frac{T}{2} - \right) \mathbf{C} \mathbf{A}^{-1} (e^{\mathbf{A} \frac{T}{2}} - \mathbf{I})}{\dot{y}_p \left(\frac{T}{2} \right)} + e^{\mathbf{A} \frac{T}{2}}, \quad (2.34)$$

where $T = \frac{2\pi}{\Omega}$ is the period of the oscillations, $\mathbf{v}$ is the velocity matrix,

$$\mathbf{v}\left(\frac{T}{2}-\right) = 2\left(\mathbf{I} + e^{\mathbf{A}T/2}\right)^{-1} e^{\mathbf{A}T/2}\mathbf{B},$$

and

$$\dot{y}_p\left(\frac{T}{2}\right) = \mathbf{C}\mathbf{A}^{-1}\mathbf{B} - 2\mathbf{C}\mathbf{A}^{-1}\left(\mathbf{I} + e^{\mathbf{A}T/2}\right)^{-1}\mathbf{B}$$

have magnitudes less than one.

In addition to the stability analysis, the direction of the relay switch must be verified, too. This condition is formulated as the following inequality:

$$\dot{y}_p\left(\frac{T}{2}\right) > 0,$$

where $\dot{y}_p\left(\frac{T}{2}\right)$ is given by the previous formula.

2.4 Computing the LPRS for a plant with a time delay

2.4.1 Matrix state-space description approach

Consider now the linear part with a time delay. Let the plant be

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}u \\ y &= \mathbf{C}\mathbf{x}\end{aligned}$$

and the control

$$u = \begin{cases} +1 & \text{if } \sigma(t - \tau) = f_0 - y(t - \tau) \geq b \quad \text{or} \quad \sigma(t - \tau) > -b, \quad u(t-) = 1 \\ -1 & \text{if } \sigma(t - \tau) = f_0 - y(t - \tau) \leq -b \quad \text{or} \quad \sigma(t - \tau) < b, \quad u(t-) = -1 \end{cases}$$

where $\mathbf{A} \in R^{n \times n}$, $\mathbf{B} \in R^{n \times 1}$, $\mathbf{C} \in R^{1 \times n}$ are matrices, and $\mathbf{A}$ is nonsingular. We note that $t = 0$ corresponds to the time that the error signal reaches the hysteresis values $\sigma = b$, $\dot{\sigma} > 0$. The control $u(t)$ switches from -1 to $+1$ not at time $t = 0$ but at time $t = \tau$. The solution for the constant control $u = \pm 1$ is

$$\mathbf{x}(t) = e^{\mathbf{A}(t-\tau)}\mathbf{x}(\tau) \pm \mathbf{A}^{-1}(e^{\mathbf{A}(t-\tau)} - \mathbf{I})\mathbf{B}, \quad t > \tau.$$

Therefore, also

$$\mathbf{x}(\tau) = e^{\mathbf{A}\tau}\mathbf{x}(0) - \mathbf{A}^{-1}(e^{\mathbf{A}\tau} - \mathbf{I})\mathbf{B}.$$

Denoting $\rho_p = \mathbf{x}(\tau) = \mathbf{x}(T + \tau)$, $\eta_p = \mathbf{x}(\theta_1 + \tau)$, where θ_1 is the length of the positive pulse of control, we can partly use the results obtained above for the linear part without time delay. A detailed derivation of the LPRS

for the case being considered is given in the Appendix. The final state-space description-based formula of the LPRS can be written as follows:

$$J(\omega) = -0.5\mathbf{C} \left[\mathbf{A}^{-1} + \frac{2\pi}{\omega} \left(\mathbf{I} - e^{\frac{2\pi}{\omega}\mathbf{A}} \right)^{-1} e^{\left(\frac{\pi}{\omega}-\tau\right)\mathbf{A}} \right] \mathbf{B} \\ + j\frac{\pi}{4}\mathbf{C} \left(\mathbf{I} + e^{\frac{\pi}{\omega}\mathbf{A}} \right)^{-1} \left(\mathbf{I} + e^{\frac{\pi}{\omega}\mathbf{A}} - 2e^{\left(\frac{\pi}{\omega}-\tau\right)\mathbf{A}} \right) \mathbf{A}^{-1}\mathbf{B}. \quad (2.35)$$

The subroutine “lprsmatrdel” (see Appendix) can be used for the LPRS computing per formula (2.35).

2.4.2 Orbital asymptotic stability

Let us extend the above methodology to the case of a plant with a time delay. Consider only the case of symmetric oscillations and apply a simplified approach. Again we need to find the mapping of the initial perturbation (at time $t = 0$) into the perturbation at the time corresponding to the condition $\sigma = -b, \dot{\sigma} < 0$. Denoting the initial perturbation $\delta\mathbf{x}(0)$, we can write the mapping $\delta\mathbf{x}(0) \rightarrow \delta\rho$ as

$$\delta\rho = e^{\mathbf{A}\tau}\delta\mathbf{x}(0),$$

and the mapping $\delta\mathbf{x}(0) \rightarrow \delta\rho \rightarrow \delta\mathbf{x}(T/2)$ as

$$\delta\mathbf{x}(T/2) = e^{\mathbf{A}(T/2-\tau)}\delta\rho = e^{\mathbf{A}(T/2-\tau)}e^{\mathbf{A}\tau}\delta\mathbf{x}(0) = e^{\mathbf{A}T/2}\delta\mathbf{x}(0).$$

We note that the stability condition (2.20) is the product of two multipliers. The first one is a mapping due to the change of the switching instant when the initial perturbation is present, and the second one is the mapping $\delta\mathbf{x}(0) \rightarrow \delta\mathbf{x}(T/2)$. The second multiplier comes from above, and the first multiplier stays the same, subject to the formula for the velocity matrix below. Therefore, the stability criterion can be formulated as follows.

Theorem 2.4. *The relay feedback system with a time-delay plant is locally orbitally asymptotically stable if and only if all the eigenvalues of the matrix*

$$\Phi_0 = \left[\mathbf{I} - \frac{\mathbf{v}(\frac{T}{2}-)\mathbf{C}}{\mathbf{C}\mathbf{v}(\frac{T}{2}-)} \right] e^{\mathbf{A}\frac{T}{2}},$$

where $T = \frac{2\pi}{\Omega}$ is the period of the oscillations, $\mathbf{v}$ is the velocity matrix,

$$\begin{aligned} \mathbf{v}\left(\frac{T}{2}-\right) &= \dot{\mathbf{x}}\left(\frac{T}{2}-\right) = \mathbf{A}\mathbf{x}\left(\frac{T}{2}\right) + \mathbf{B} = -\mathbf{A}\mathbf{x}(0) + \mathbf{B} \\ &= -\mathbf{A}(\mathbf{I} + e^{\mathbf{A}T/2})^{-1}\mathbf{A}^{-1} [\mathbf{I} + e^{\mathbf{A}T/2} - 2e^{\mathbf{A}(T/2-\tau)}] \mathbf{B} + \mathbf{B} \\ &= -(\mathbf{I} + e^{\mathbf{A}T/2})^{-1} [\mathbf{I} + e^{\mathbf{A}T/2} - 2e^{\mathbf{A}(T/2-\tau)}] \mathbf{B} + \mathbf{B} \\ &= 2(\mathbf{I} + e^{\mathbf{A}T/2})^{-1} e^{\mathbf{A}(T/2-\tau)} \mathbf{B} \end{aligned}$$

have magnitudes less than one.

In addition to the stability analysis, the direction of the relay switch must be verified, too. This condition is formulated as the following inequality

$$\dot{y}\left(\frac{T}{2}-\right) = \mathbf{C}\mathbf{v}\left(\frac{T}{2}-\right) > 0,$$

where $\mathbf{v}\left(\frac{T}{2}-\right)$ is given by the previous formula.

2.5 LPRS of first-order dynamics

As mentioned above, one of the possible techniques of LPRS computing is to represent the transfer function as partial fractions, compute the LPRS of the component transfer functions (partial fractions), and add those partial LPRS together in accordance with Theorem 2.1. To apply this technique, we have to know the formulas of the LPRS for first- and second-order dynamics. These are of similar meaning and importance as the characteristics of first- and second-order dynamics in linear system analysis.

The knowledge of the LPRS of the low-order dynamics is important for other reasons, too. Some features of the LPRS of low-order dynamics can be extended to higher-order systems. Those features are considered in Chapter 4.

Let us find the formula of the LPRS for the first-order dynamics given by the transfer function $W(s) = K/(Ts + 1)$.

We derive an analytical formula for $J(\omega)$, $\omega \in [0; \infty)$. There exist non-symmetrical oscillations in the system (Fig. 2.3) if $f(t) \equiv f_0 \neq 0$. The system model can be written as the following set of equations:

$$\begin{cases} y(\theta_1) = y(0)\exp(-\theta_1/T) + cK(1 - \exp(-\theta_1/T)) \\ y(\theta_1 + \theta_2) = y(\theta_1)\exp(-\theta_2/T) - cK(1 - \exp(-\theta_2/T)) \\ y(\theta_1 + \theta_2) = y(0) \\ f_0 - y(0) = b \\ f_0 - y(\theta_1) = -b. \end{cases} \quad (2.36)$$

Solving (2.36), we obtain θ_1 and θ_2 :

$$\theta_1 = -T \ln(2b/(f_0 - b - cK) + 1), \quad (2.37)$$

$$\theta_2 = -T \ln(1 - 2b/(f_0 + b + cK)). \quad (2.38)$$

Consider the limit $\lim_{f_0 \rightarrow 0} (u_0/f_0)$: on the one hand, it can be derived from (2.37), (2.38) (taking into account that $u_0 = c(\theta_1 - \theta_2)/(\theta_1 + \theta_2)$)

$$\lim_{f_0 \rightarrow 0} (u_0/f_0) = -2bcT/(\theta(b - cK)(b + cK)) \quad (2.39)$$

where $\theta = \lim_{f_0 \rightarrow 0} \theta_{1,2} = -T \ln(1 - 2b/(cK + b))$; on the other hand, it can be related to the gain k_n by the formula of a closed-loop system

$$\lim_{f_0 \rightarrow 0} (u_0/f_0) = k_n/(1 + k_n K). \quad (2.40)$$

A formula for k_n can easily be found from (2.39) and (2.40)

$$k_n = -T(\exp(-2\theta/T) - 1)/[K(2\theta \exp(-\theta/T) + T(\exp(-2\theta/T) - 1))]$$

from which a formula for $\text{Re}J(\omega)$ (where $\omega = \pi/\theta$) can be obtained. An expression for $\text{Im}J(\omega)$ can be found by solving the set of equations (2.36) with $f_0 = 0$. Finally we obtain

$$J(\omega) = \frac{K}{2} \left(1 - \frac{\pi}{T\omega} \text{csch} \frac{\pi}{T\omega}\right) - j \frac{\pi K}{4} \tanh \frac{\pi}{2\omega T}, \quad (2.41)$$

where $\text{csch } x$ and $\tanh x$ are hyperbolic cosecant and tangent, respectively. The subroutine “lprs1ord” (see Appendix) can be used for the LPRS computing per formula (2.41).

The plot of the LPRS for $K = 1$, $T = 1$ is given in Fig. 2.3. The whole plot is totally located in the 4th quadrant. The point $(0.5K; -j\frac{\pi}{4}K)$ corresponds to the frequency $\omega = 0$, and the point $(0; j0)$ corresponds to the frequency $\omega = \infty$. The high-frequency segment of the LPRS has the imaginary axis as an asymptote.

With the formula for the LPRS available, we can easily find the frequency of periodic motions in the relay servo system with the linear part being the first-order dynamics. The LPRS is a continuous function of the frequency, and for every hysteresis value from the range $b \in [0; cK]$, there exists a periodic solution of the frequency that can be determined from (2.3), (2.41), which is

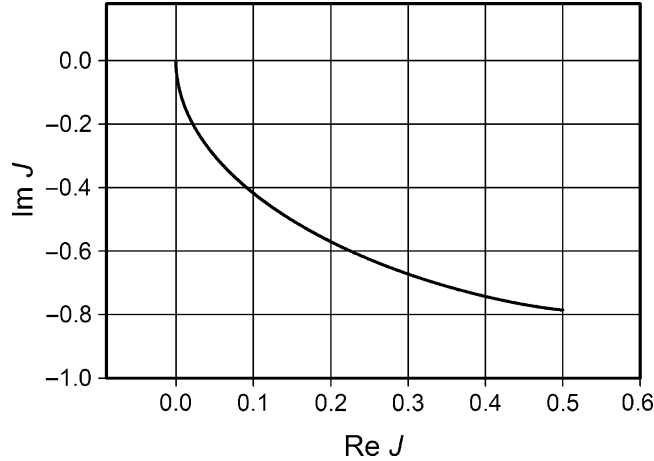


Fig. 2.3. The LPRS of first-order dynamics

$$\Omega = \frac{\pi}{2T} \tanh^{-1} \left(\frac{b}{cK} \right). \quad (2.42)$$

It is easy to show that when the hysteresis value b tends to zero, then the frequency of the periodic solution tends to infinity

$$\lim_{b \rightarrow 0} \Omega = \infty,$$

and when the hysteresis value b tends to cK , then the frequency of the periodic solution tends to zero

$$\lim_{b \rightarrow cK} \Omega = 0.$$

From (2.41), we can also see that the imaginary part of the LPRS is a monotone function of the frequency. Therefore, the condition of the existence of a finite frequency periodic solution holds for any non-zero hysteresis value from the specified range, and the limit for $b \rightarrow 0$ exists and corresponds to infinite frequency.

It is easy to show that the oscillations are always orbitally stable. The stability of a periodic solution is usually verified by finding eigenvalues of the Jacobian of the corresponding Poincaré map [62]. For the first-order system, the only eigenvalue of this Jacobian will always be zero, as there is only one system variable, which also determines the condition of the switch of the relay.

2.6 LPRS of second-order dynamics

Now we carry out a similar analysis for second-order dynamics. Let the matrix $\mathbf{A}$ of (1.16) be $\mathbf{A} = [0 \ 1; -a_1 \ -a_2]$. Here, consider a few cases, all with $a_1 > 0, a_2 > 0$.

A. Let $a_2^2 - 4a_1 < 0$. Then the plant transfer function can be written as:

$$W(s) = K/(T^2 s^2 + 2\xi Ts + 1). \quad (2.43)$$

The LPRS formula can be found, for example, by expanding the above transfer function into partial fractions and applying formula (2.41) obtained for the first-order dynamics. However, the coefficients of those partial fractions will be complex numbers, and this circumstance must be considered. The formula of the LPRS for the second-order dynamics given by transfer function (2.43) can be written as follows:

$$J(\omega) = \frac{K}{2} \left(1 - \frac{g + \gamma h}{\sin^2 \beta + \sinh^2 \alpha} \right) - j \frac{\pi K}{4} \frac{\sinh \alpha - \gamma \sin \beta}{\cosh \alpha + \cos \beta} \quad (2.44)$$

where

$$\begin{aligned}\alpha &= \frac{\pi\xi}{\omega T}, \\ \beta &= \frac{\pi\sqrt{1-\xi^2}}{\omega T}, \\ \gamma &= \alpha/\beta \\ g &= \alpha \cos \beta \sinh \alpha + \beta \sin \beta \cosh \alpha, \\ h &= \alpha \sin \beta \cosh \alpha + \beta \cos \beta \sinh \alpha.\end{aligned}$$

The subroutine “lprs2ord1” (see Appendix) can be used for the LPRS computing per formula (2.44).

The plots of the LPRS for $K = 1$, $T = 1$ and different values of damping factor ξ are given in Fig. 2.4 (#1 – $\xi = 1$, #2 – $\xi = 0.85$, #3 – $\xi = 0.7$, #4 – $\xi = 0.55$, #5 – $\xi = 0.4$). The high-frequency segment of the LPRS of the second-order plant approaches the real axis.

Now, with the LPRS formula available, we analyze possible existence of the periodic solution in the relay feedback system with the plant being the second-order dynamics. Consider two limits of $J(\omega)$ that can be obtained from (2.44):

$$\lim_{\omega \rightarrow \infty} J(\omega) = (0; j0); \quad \lim_{\omega \rightarrow 0} J(\omega) = (0.5K; -j\frac{\pi}{4}K).$$

They give the two boundary points of the LPRS corresponding to zero frequency and infinite frequency. Analysis of function (2.44) shows that it does not have intersections with the real axis except at the origin. Because $J(\omega)$ is a continuous function of the frequency ω (this follows from formula (2.44)), a solution of equation (2.3) exists for any $b \in (0; cK)$. This means that a periodic solution of finite frequency exists for the second-order system for every value

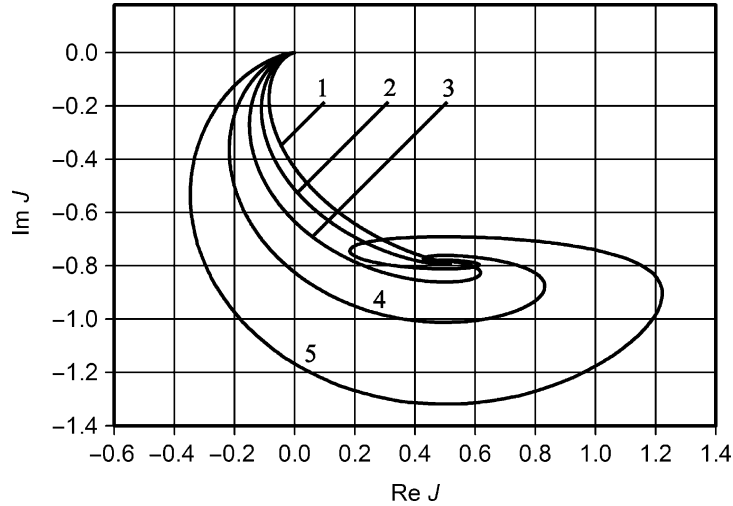


Fig. 2.4. The LPRS of second-order dynamics

of b within the specified range, and there is a periodic solution of infinite frequency for $b = 0$.

Now we analyze the stability of those periodic solutions. We write the Jacobian of the Poincaré map of the relay system:

$$\Phi = \left[\mathbf{I} - \frac{v\mathbf{C}}{\mathbf{C}v} \right] e^{\mathbf{A}\pi/\omega}, \quad (2.45)$$

where $v = 2(\mathbf{I} + e^{\mathbf{A}\pi/\omega})^{-1}e^{\mathbf{A}\pi/\omega}\mathbf{B}$. If all the eigenvalues of the matrix Φ have magnitudes smaller than *one*, the periodic motion is orbitally asymptotically stable. For the second-order system, we obtain analytical formulas of the matrix Φ eigenvalues

$$\begin{aligned} \lambda_1 &= 0, \\ \lambda_2 &= -a_1\alpha_1^2\pi^2/\omega^2 + \alpha_0(a_2\alpha_1\pi/\omega - \alpha_0), \end{aligned} \quad (2.46)$$

where

$$\begin{aligned} \alpha_0 &= \frac{\lambda_{1A} \exp(\lambda_{2A}\pi/\omega) - \lambda_{2A} \exp(\lambda_{1A}\pi/\omega)}{\lambda_{1A} - \lambda_{2A}} \\ \alpha_1 &= \frac{\exp(\lambda_{1A}\pi/\omega) - \exp(\lambda_{2A}\pi/\omega)}{\lambda_{1A} - \lambda_{2A}} \frac{\omega}{\pi}. \end{aligned}$$

λ_{1A} and λ_{2A} are eigenvalues of the matrix A ,

$$\lambda_{1A} = 0.5(-a_2 + \sqrt{a_2^2 - 4a_1}), \lambda_{2A} = 0.5(-a_2 - \sqrt{a_2^2 - 4a_1}).$$

Therefore, if $|\lambda_2| < 1$, then the periodic solution is stable. From (2.46), we can also find the limit corresponding to the oscillations of infinite frequency, which is of much interest in sliding mode control theory: $\lim_{\omega \rightarrow \infty} \lambda_2 = 0$. Therefore, the periodic solution of infinite frequency is stable.

B. Consider the case when $a_2^2 - 4a_1 = 0$. To obtain the LPRS formula, we use formula (2.44) and find the limit for $\xi \rightarrow 1$. The LPRS for this case is given in Fig. 2.4 (#1). All subsequent analysis and conclusions are the same as in case A.

C. Assume that $a_2^2 - 4a_1 > 0$. Then the transfer function can be expanded into two partial fractions, and according to Theorem 2.1, the LPRS can be computed as a sum of the two components. The subsequent analysis is similar to the previous one.

D. Assume that $a_1 = 0$. Then the transfer function is $W(s) = K/[s(Ts + 1)]$. For this plant, the LPRS is given by the following formula, which can be obtained via partial fraction expansion of the transfer function expression and application of the LPRS formulas of the first-order dynamics:

$$J(\omega) = \frac{K}{2} \left(\frac{\pi}{T\omega} \operatorname{csch} \frac{\pi}{T\omega} - 1 \right) + j \frac{\pi K}{4} \left(\tanh \frac{\pi}{2\omega T} + \frac{\pi}{2\omega} \right). \quad (2.47)$$

The plot of the LPRS for $K = 1$, $T = 1$ is given in Fig. 2.5. The whole plot is totally located in the 3rd quadrant. The point $(0.5K; -j\infty)$ corresponds

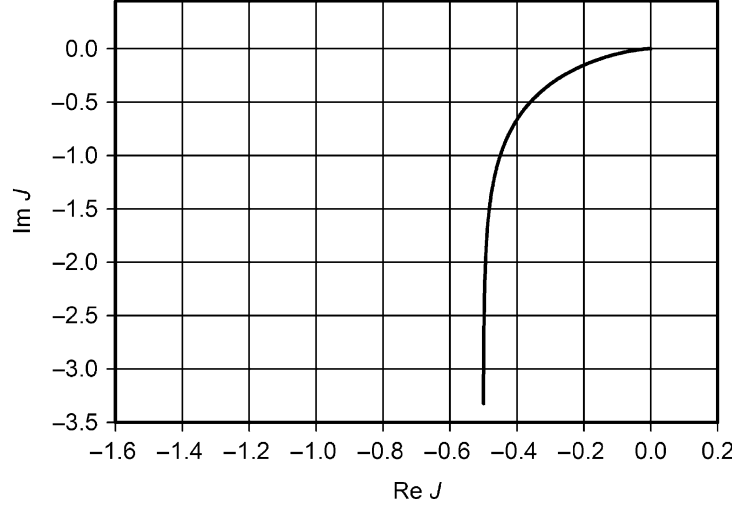


Fig. 2.5. The LPRS of integrating second-order plant

to the frequency $\omega = 0$, and the point $(0; j0)$ corresponds to the frequency $\omega = \infty$. The high-frequency segment of the LPRS has the real axis as an asymptote.

Again, applying the LPRS formula and the same approach, we can prove that the periodic solution of the relay feedback system, with the plant being the second-order dynamics, exists; that in the case of the ideal relay it is the oscillations of infinite frequency; and that the periodic solution is orbitally asymptotically stable.

2.7 LPRS of first-order plus dead-time dynamics

Many industrial processes can be adequately approximated by the first-order plus time-delay transfer function

$$W(s) = \frac{K e^{-\tau s}}{Ts + 1} \quad (2.48)$$

where K is the process gain, T is a time constant, and τ is a time delay (dead time). This factor results in the particular importance of the analysis of these dynamics. To apply the above idea to the process (2.48), we need to obtain the formula of the LPRS for the transfer function (2.48).

Consider the equation of the periodic process with unequally spaced switching in the relay feedback system (Fig. 1.5) with the plant being the transfer function (2.48). At first, for an auxiliary purpose, we find a response of the first-order plant without time delay to the steady periodic pulse control of the amplitude c , with positive pulse length θ_1 and negative

pulse length θ_2 . The steady periodic response of such a plant can be described by the following expressions:

$$y^*(\theta_1) = y^*(0) \cdot e^{-\theta_1/T} + cK(1 - e^{-\theta_1/T}) \quad (2.49)$$

$$y^*(0) = y^*(\theta_1) \cdot e^{-\theta_2/T} + cK(1 - e^{-\theta_2/T}). \quad (2.50)$$

Formulas (2.49) and (2.50) are a Poincaré return map for the feedback relay system with the plant being a first-order transfer function. Solution of (2.49) and (2.50) provides the following result:

$$y_{\min} = y^*(0) = cK \frac{2e^{-\theta_2/T} - e^{-(\theta_1+\theta_2)/T} - 1}{1 - e^{-(\theta_1+\theta_2)/T}} \quad (2.51)$$

$$y_{\max} = y^*(\theta_1) = cK \frac{1 + e^{-(\theta_1+\theta_2)/T} - 2e^{-\theta_2/T}}{1 - e^{-(\theta_1+\theta_2)/T}}. \quad (2.52)$$

Denote the values of the output at the switching instants $y_{\min}$ and $y_{\max}$ (2.51) and (2.52). With $y_{\min}$ and $y_{\max}$ available, we can now write the equations of the asymmetric periodic process in the system with the first-order plus dead-time plant:

$$y(\theta_1) = y_{\min} \cdot e^{-(\theta_1-\tau)/T} + cK(1 - e^{-(\theta_1-\tau)/T}) \quad (2.53)$$

$$y(0) = y_{\max} \cdot e^{-(\theta_2-\tau)/T} - cK(1 - e^{-(\theta_2-\tau)/T}) \quad (2.54)$$

$$f_0 - y(0) = b \quad (2.55)$$

$$f_0 - y(\theta_1) = -b. \quad (2.56)$$

First, we derive the formula of the imaginary part of the LPRS for the given plant. According to the definition, the imaginary part of the LPRS is the value of the system output at the time of the switch from “−” to “+.” Because the input f_0 tends to zero, to derive the formula of the imaginary part we consider the symmetric oscillations. In that case $y(\theta_1) = -y(0)$, and the solution of equations (2.53)–(2.56) is fairly straightforward:

$$\lim_{f_0 \rightarrow 0} y(0) = cK \left(\frac{2e^{-\alpha} \cdot e^{\gamma}}{1 + e^{-\alpha}} - 1 \right) \quad (2.57)$$

where

$$\alpha = \frac{\theta}{T} = \frac{\pi}{T\omega} \text{ and } \gamma = \frac{\tau}{T}.$$

Now we derive the formula of the real part of the LPRS for the given plant. We solve equations (2.53)–(2.56) for θ_1 and θ_2 :

$$\theta_1 = -T \ln \frac{f_0 + b - cK}{f_0 - b + cK - 2cK e^{\gamma}} \quad (2.58)$$

$$\theta_2 = -T \ln \frac{f_0 - b + cK}{2cKe^\gamma + f_0 + b - cK}. \quad (2.59)$$

We find the limiting value of the positive and negative pulse length for $f_0 \rightarrow 0$:

$$\lim_{f_0 \rightarrow 0} \theta_1 = \lim_{f_0 \rightarrow 0} \theta_2 = \theta = T \cdot \ln \frac{cK(2e^\gamma - 1) + b}{cK - b}. \quad (2.60)$$

In formula (2.60), θ is half of the period of the symmetric oscillations. Consequently, the frequency of the oscillations is: $\Omega = \pi/\theta$.

Now we derive a formula of $\lim_{f_0 \rightarrow 0} \frac{\theta_1 - \theta_2}{f_0}$. It can be derived from (2.58) and (2.59) but it must not contain b or f_0 on the right-hand side. For that reason, formula (2.60) is helpful. After a number of transformations, we obtain:

$$\lim_{f_0 \rightarrow 0} \frac{\theta_1 - \theta_2}{f_0} = \frac{T(1 + e^{-\alpha}) \cdot (1 - e^{-\alpha})}{cKe^\gamma \cdot e^{-\alpha}}. \quad (2.61)$$

Formula (2.61) does not contain b or f_0 in the right-hand side.

Taking into account the relation between θ_1, θ_2 and u_0 , we obtain the following limit:

$$\lim_{f_0 \rightarrow 0} \frac{u_0}{f_0} = \frac{c}{2\theta} \cdot \lim_{f_0 \rightarrow 0} \frac{\theta_1 - \theta_2}{f_0} = \frac{(1 + e^{-\alpha}) \cdot (1 - e^{-\alpha})}{2\alpha K e^\gamma \cdot e^{-\alpha}}. \quad (2.62)$$

Another expression for the same limit is the formula of the closed-loop system that uses the equivalent gain of the relay k_n :

$$\lim_{f_0 \rightarrow 0} \frac{u_0}{f_0} = \frac{k_n}{1 + k_n K}. \quad (2.63)$$

Equating the right-hand sides of (2.62) and (2.63), we obtain the equation for the equivalent gain k_n . After solving it, we obtain the formula of the real part of the LPRS (taking into account the fact that the real part is the reciprocal of the equivalent gain with the coefficient -0.5). Finally, we put together the real and the imaginary parts and obtain the formula for the LPRS for the first-order plus dead-time transfer function as follows:

$$J(\omega) = \frac{K}{2} (1 - \alpha e^\gamma \operatorname{csch} \alpha) + j \frac{\pi}{4} K \left(\frac{2e^{-\alpha} e^\gamma}{1 + e^{-\alpha}} - 1 \right). \quad (2.64)$$

The subroutine “lprsfopdt” (see Appendix) can be used for the LPRS computing per formula (2.64).

Let us compute the LPRS and plot it for various values of γ . The plots of the LPRS for $\gamma = 0$ (#1), $\gamma = 0.2$ (#2), $\gamma = 0.5$ (#3), $\gamma = 1.0$ (#4), and $\gamma = 1.5$ (#5) are depicted in Fig. 2.6. All the plots begin at the point $(0.5, -j\pi/4)$ that corresponds to the frequency $\omega = 0$. Plot number 1 (which corresponds to zero dead-time) comes to the origin, which corresponds to

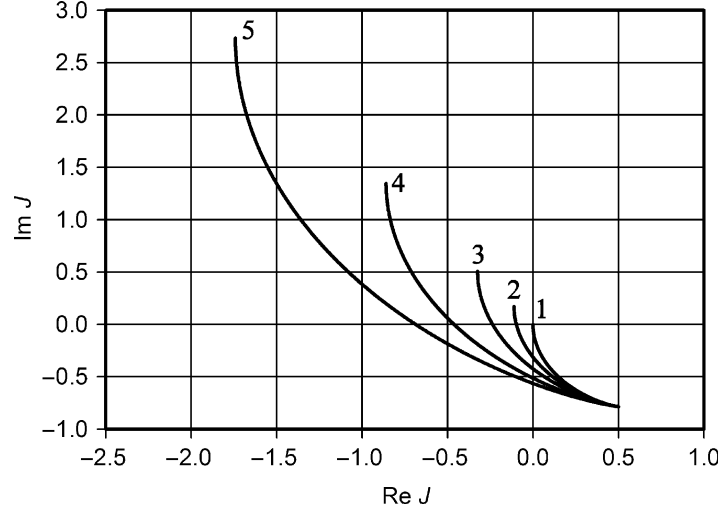


Fig. 2.6. The LPRS of first-order plus dead-time dynamics

infinite frequency. Other plots are defined only for frequencies less than those corresponding to half of the period. Therefore, they do not reach the origin.

Formula (2.64) can be validated by computing the LPRS for the same values of γ as in Fig. 2.6, with the use of the series expression (2.19). An application of formula (2.19) to the transfer function (2.48) provides the same results as formula (2.64).

2.8 Some properties of the LPRS

The knowledge of certain properties of the LPRS is computationally helpful, especially for the design of linear compensators with the use of the LPRS method. One of these properties, probably the most important, was formulated in Theorem 2.1: it is the additivity property. A few other properties relating to the boundary points corresponding to zero frequency and infinite frequency are considered below.

Consider a non-integrating linear part of the relay servo system given by equations (1.16). We find the coordinates of the initial point of the LPRS corresponding to zero frequency. For that purpose, let us find the limit of function $J(\omega)$ for ω tending to zero. Using formula (2.12), we can write:

$$\lim_{\omega \rightarrow 0} J(\omega) = \mathbf{C} \lim_{\omega \rightarrow 0} \left\{ -0.5[\mathbf{A}^{-1} + \frac{2\pi}{\omega}(\mathbf{I} - e^{\frac{2\pi}{\omega}\mathbf{A}})^{-1}e^{\frac{\pi}{\omega}\mathbf{A}}] + j\frac{\pi}{4}(\mathbf{I} + e^{\frac{\pi}{\omega}\mathbf{A}})^{-1}(\mathbf{I} - e^{\frac{\pi}{\omega}\mathbf{A}})\mathbf{A}^{-1} \right\} \mathbf{B}.$$

We evaluate the following two limits:

$$\begin{aligned}\lim_{\omega \rightarrow 0} \left[\frac{2\pi}{\omega} ((\mathbf{I} - e^{\frac{2\pi}{\omega} \mathbf{A}})^{-1} e^{\frac{\pi}{\omega} \mathbf{A}}) \right] &= \lim_{\omega \rightarrow 0} \left[\frac{2\pi}{\omega} e^{-\frac{2\pi}{\omega} \mathbf{A}} e^{\frac{\pi}{\omega} \mathbf{A}} \right] = \lim_{\omega \rightarrow 0} \left[\frac{2\pi}{\omega} e^{-\frac{\pi}{\omega} \mathbf{A}} \right] = \mathbf{0} \\ \lim_{\omega \rightarrow 0} [(\mathbf{I} + e^{\frac{\pi}{\omega} \mathbf{A}})^{-1} (\mathbf{I} - e^{\frac{\pi}{\omega} \mathbf{A}})] &= \lim_{\omega \rightarrow 0} [e^{-\frac{\pi}{\omega} \mathbf{A}} e^{\frac{\pi}{\omega} \mathbf{A}}] = \mathbf{I}.\end{aligned}$$

With these two limits, we can write the limit for the LPRS as follows:

$$\lim_{\omega \rightarrow 0} J(\omega) = \left[-0.5 + j\frac{\pi}{4} \right] \mathbf{CA}^{-1}\mathbf{B}. \quad (2.65)$$

The product of matrices $\mathbf{CA}^{-1}\mathbf{B}$ in (2.65) is the negative value of the gain of the plant transfer function. We have thus proved that for a *non-integrating linear part of the relay servo system, the initial point of the corresponding LPRS is $(0.5K; -j\pi/4K)$, where K is the static gain of the linear part.* This coincides with the above analysis of the LPRS of the first- and second-order dynamics: see, for example, Fig. 2.3 and Fig. 2.4.

To find the limit of $J(\omega)$ as ω tends to infinity, consider the following power series expansion of the exponential function.

$$\lim_{\omega \rightarrow \infty} \exp\left(\frac{\pi}{\omega} \mathbf{A}\right) = \lim_{\omega \rightarrow \infty} \sum_{n=0}^{\infty} \frac{(\pi/\omega)^n}{n!} \mathbf{A}^n = \mathbf{I} + \lim_{\omega \rightarrow \infty} \sum_{n=1}^{\infty} \frac{(\pi/\omega)^n}{n!} \mathbf{A}^n = \mathbf{I},$$

and another limit:

$$\begin{aligned}\lim_{\omega \rightarrow \infty} \left\{ \frac{2\pi}{\omega} [\mathbf{I} - \exp\left(\frac{2\pi}{\omega} \mathbf{A}\right)]^{-1} \right\} &= \lim_{\lambda \rightarrow 0} \{ \lambda [\mathbf{I} - \exp(\lambda \mathbf{A})]^{-1} \} \\ &= \lim_{\lambda \rightarrow 0} \left\{ \left(\frac{\partial \lambda}{\partial \lambda} \right) \left[\frac{\partial (\mathbf{I} - \exp(\lambda \mathbf{A}))}{\partial \lambda} \right]^{-1} \right\} = -\mathbf{A}^{-1}.\end{aligned}$$

Finally, taking account of the above two limits, we can prove that the final point of the LPRS for the non-integrating linear part is the origin:

$$\lim_{\omega \rightarrow \infty} J(\omega) = 0 + j0. \quad (2.66)$$

Reasoning along the same lines, we can obtain the initial and final points of the LPRS for integrating linear parts, which are as follows:

$$\lim_{\omega \rightarrow 0} J(\omega) = 0.5\mathbf{CA}^{-1}\mathbf{B} - j\infty \quad (2.67)$$

$$\lim_{\omega \rightarrow \infty} J(\omega) = 0 + j0. \quad (2.68)$$

Some further investigation of asymptotic behavior of the LPRS is done in Chapter 4, which is devoted to analysis of sliding mode control systems. It is shown there that the location of the high-frequency segment of the LPRS determines whether chattering or ideal sliding mode occurs in the system. Here we only consider a few rules that may be helpful for LPRS computing and plotting, as well as verifying calculations of the LPRS.

2.9 LPRS of nonlinear plants

2.9.1 Additivity property

In all the previous sections, we considered relay servo systems with linear plants only. This is a limitation of the aforementioned method. However, the LPRS is a characteristic of the relay servo system that remains meaningful and useful even if the plant is nonlinear. Of course, the same methods of computing cannot be used for nonlinear plants. However, other techniques of computing can be developed on the basis of some properties considered below. The application of this approach is demonstrated in the chapter devoted to analysis and design of the pneumatic servomechanism.

Consider the relay feedback system depicted in Fig. 2.7. Let the system be given by the following equations,

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{g}(\mathbf{x}, u), \\ y &= h(\mathbf{x}),\end{aligned}\tag{2.69}$$

$$u = \begin{cases} +c & \text{if } \sigma = f_0 - y \geq b \text{ or } \sigma > -b, u(t-0) = c \\ -c & \text{if } \sigma = f_0 - y \leq -b \text{ or } \sigma < b, u(t-0) = -c, \end{cases}$$

where $\mathbf{g}$ and h are nonlinear functions. We limit our analysis to static symmetric nonlinearities. Assume as before that in the autonomous mode, a symmetric periodic motion exists in the system, and if a non-zero external input is applied to the system, then a periodic motion with unequally spaced switches of the relay occurs (see Fig. 1.5). We use the same definition of the LPRS that was introduced above — except now we have the system with a nonlinear plant

$$J(\omega) = -\frac{1}{2} \lim_{f_0 \rightarrow 0} \frac{\sigma_0}{u_0} + j \frac{\pi}{4c} \lim_{f_0 \rightarrow 0} y(t)|_{t=0},\tag{2.70}$$

where $t = 0$ is the time of the switch of the relay from “−c” to “+c.” The definition in formula (2.70) does not require that the plant necessarily be linear. Therefore, if via some technique, we compute and plot the LPRS for this nonlinear plant that satisfies the given definition, then we can calculate the frequency of possible periodic motions and the equivalent gain of the relay

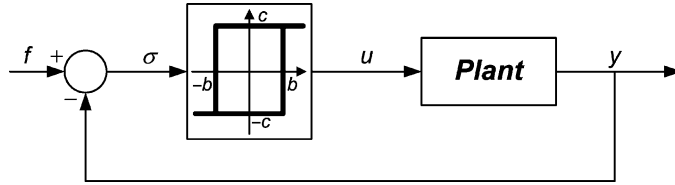


Fig. 2.7. Relay servo system with a nonlinear plant

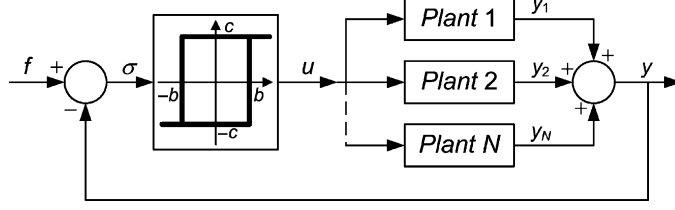


Fig. 2.8. Multichannel plant system

for input-output analysis. Nonlinear plants in the general case do not provide any means for the LPRS computing other than a direct application of formula (2.70). Yet, some features of the LPRS and the system under consideration allow for a simpler approach. The problem is, therefore, to develop a technique or techniques that use those features and do not directly involve the variables of formula (2.70), but rather use the parameters of the plant. Consider the main feature that allows us to simplify the task of LPRS computing.

Assume that the plant can be represented as a number of parallel channels (plant components) as depicted in Fig. 2.8 with each component satisfying the above requirements for the plant. Then the following property is valid.

Theorem 2.5. (*Additivity property of the LPRS*). *If the plant of the relay servo system can be represented as a sum of N nonlinear (in a general case) plants as depicted in Fig. 2.8,*

$$\begin{aligned}\dot{\mathbf{x}}_i &= \mathbf{g}_i(\mathbf{x}_i, u), \\ y_i &= h_i(\mathbf{x}_i), \quad i = \overline{1, N}, \\ y &= \sum_{i=1}^N y_i,\end{aligned}$$

each satisfying the assumptions of equations (2.69), then the LPRS of this system is equal to the sum of N LPRS, each of which corresponds to the relay servo system (Fig. 2.7) with the plant being the plant component of the original system given by i -th equation (Plant 1, Plant 2, Plant N in Fig. 2.8)

$$J(\omega) = J_1(\omega) + J_2(\omega) + \dots + J_N(\omega), \quad (2.71)$$

where $J_i(\omega)$ is the LPRS for i -th plant.

Proof. Suppose the system in Fig. 2.8 is a type 0 servo system (has a non-integrating plant), a constant input $f(t) \equiv f_0$ is applied, and a limit cycle of frequency Ω does occur. Then the output $y(t)$ at the switching time is $y(0) = y^+$ (switching from “−” to “+”) and $y(\theta_1) = y^-$ (switching from “−” to “+”).

Each variable $y_i(t)$ ($i = \overline{1, N}$) at the switching time is equal to $y_i(0) = y_i^+$ and $y_i(\theta_1) = y_i^-$. The signals in each system have the following constant terms (mean values): $\sigma_0, u_0, y_0, y_{0i}$ ($i = \overline{1, N}$). The following identities obviously hold:

$$\begin{cases} \sum_{i=1}^N y_i^+ = y^+, \\ \sum_{i=1}^N y_i^- = y^-, \\ \sum_{i=1}^N y_{0i} = y_0. \end{cases} \quad (2.72)$$

The periodic solution of the system in Fig. 2.8 (the switching conditions) can be described as follows:

$$\begin{cases} f_0 - y^+ = b, \\ f_0 - y^- = -b \end{cases} \quad (2.73)$$

Because each signal $y_i(t)$ ($i = \overline{1, N}$) is a periodic function, the periodic solution for each plant component in the relay system in Fig. 2.7 (the plant is supposed to be i -th component of the original plant) exists if the input f_0 and hysteresis value b are equal to

$$f_{0i} = \frac{1}{2}(y_i^+ + y_i^-) \quad (2.74)$$

$$b_{0i} = \frac{1}{2}(y_i^- + y_i^+) \quad (2.75)$$

respectively, which is a solution of system (2.73) for i -th component. Note that the constant inputs and the hysteresis values are different for each of the systems with an i -th plant. Therefore, a periodic solution for the system in Fig. 2.7 with the plant being a plant component from the system in Fig. 2.8 exists, and the output of this system coincides with the component output $y_i(t)$ in the system in Fig. 2.8. The relay controls are identical,

$$u_i(t) = u_2(t) = \dots = u_N(t) = u(t),$$

which is also true with respect to the constant terms (mean values):

$$u_{01} = u_{02} = \dots = u_{0N} = u_0. \quad (2.76)$$

Then the following equality holds:

$$\begin{aligned} \sum_{i=1}^N \frac{\sigma_{0i}}{u_{0i}} &= \frac{1}{u_0} \sum_{i=1}^N \sigma_{0i} = \frac{1}{u_0} \sum_{i=1}^N (f_{0i} - y_{0i}) \\ &= \frac{1}{u_0} \sum_{i=1}^N \left[\frac{1}{2}(y_i^+ + y_i^-) - y_{0i} \right] \\ &= \frac{1}{u_0} \left[\frac{1}{2}(y^+ + y^-) - y_0 \right] \\ &= \frac{f_0 - y_0}{u_0} = \frac{\sigma_0}{u_0}. \end{aligned} \quad (2.77)$$

Formula (2.77) holds for any given frequency $\omega = \Omega$. This means that (2.71) holds with respect to the real part of the LPRS. It is also valid with respect to the imaginary part of the LPRS that directly follows from (2.72). ■

It should be noted that (2.72) and (2.77) provide an additivity property of the LPRS that is valid not only for infinitesimally small constant terms but also for any finite values. This is very important as we are going to use this property for numerical computing of the LPRS for nonlinear plants.

The proved property suggests some techniques for LPRS computing. The above definition implies existence of the LPRS only within the frequency range where periodic solution is possible. Sometimes it is necessary to calculate the LPRS beyond the frequency range where a periodic solution exists. A typical example of this occurs when the desired frequency of the oscillations is beyond the range of possible frequencies of the oscillations in a non-compensated system.

The main property of the LPRS provides a solution to this problem. If the LPRS definition does not allow for LPRS calculation at the frequency of interest Ω , then additional dynamics with known LPRS can be connected in parallel with the given plant to allow oscillations of frequency Ω to exist in the system. The LPRS of the original plant can be calculated as the LPRS of the latter system minus the LPRS of the known dynamics (at frequency Ω).

Therefore, if the LPRS cannot be calculated at a certain frequency of interest Ω with the use of the LPRS definition, and if by adding components in parallel with the plant (or deleting parallel components) the frequency of interest Ω can be generated, then the LPRS of the given plant at frequency Ω can be calculated as a difference (sum) of the LPRS of the consolidated plant and the LPRS of the components connected in parallel with the given plant; this follows from the additivity property. In the case of linear dynamics connected in parallel with the plant, the LPRS of the linear dynamics can be calculated through the formulas presented in Table 2.1. Thus, the calculation of the LPRS of nonlinear plants depends on how well the auxiliary parallel components are chosen to obtain a necessary frequency of the oscillations.

A typical application of this property is the calculation of the LPRS of a nonlinear plant with two integrators. A periodic solution in the relay system with such a plant doesn't exist (despite the formula of $J(\omega)$ for corresponding linear plant). On the other hand, if second-order dynamics are connected in parallel with such a plant, a periodic solution may exist and the LPRS can be calculated (the switching frequency can be varied by changing the hysteresis value and the parameters of the parallel component).

2.9.2 The LPRS extended definition and open-loop LPRS computing

The LPRS is a plant response to the asymmetric square-wave pulse input signal. It can also be easily seen from formulas (2.12), (2.19), and (2.32) that the LPRS is a function of the plant parameters only. This means that it is

possible to construct a certain definition (different from the original definition of the LPRS) that would involve an open-loop consideration of the plant.

Suppose the plant is a type 0 servo system (non-integrating) and the control is a square-wave pulse signal of frequency ω , relative pulse duration γ , and amplitude c . Then, the coefficients of the Fourier series of the steady periodic output of the plant can be computed. Relative pulse duration γ is a quotient of positive pulse duration θ_1 and the period of oscillations. Under those assumptions, the plant output is

$$y(t) = y_0 + \sum_{k=1}^{\infty} \{a_k \cdot \cos(k\omega t) + b_k \cdot \sin(k\omega t)\}, \quad (2.78)$$

where y_0, a_k, b_k are the coefficients of the Fourier series, and $t = 0$ is the time of the control switching from “-c” to “+c.”

It follows from (2.78) that at the switching times, the following equalities hold:

$$y^+ = y_0 + \sum_{k=1}^{\infty} a_k, \quad (2.79)$$

$$y^- = y_0 + \sum_{k=1}^{\infty} \{a_k \cdot \cos(2\pi k\gamma) + b_k \cdot \sin(2\pi k\gamma)\}. \quad (2.80)$$

Consider the following hypothetical experiment. Suppose that the plant is closed by the feedback (see the system in Fig. 2.7) but the error signal link is disconnected from the relay input and a periodic asymmetric signal from an external generator is introduced to the relay input instead. This results in a square-wave pulse signal of frequency ω , relative pulse duration γ , and amplitude c from the relay. At a certain time, we instantaneously disconnect the relay input from the external generator and connect it to the adder output. To allow the oscillations to remain in the closed-loop system, the system input f_0 and relay hysteresis b must satisfy equations (2.73), from which f_0 and b values can be obtained (formulas (2.74) and (2.75)). This shows the existence of a certain equivalence between the open-loop and closed-loop generation of a periodic motion.

Simultaneous consideration of (2.70), (2.74), and (2.75) results in the formula for the LPRS based on the plant output spectrum in the open-loop experiment

$$J(\omega) = - \lim_{\gamma \rightarrow \frac{1}{2}} \frac{0.5[y^+ + y^-] - y_0}{2u_0} + j \frac{\pi}{4c} \lim_{\gamma \rightarrow \frac{1}{2}} \frac{y^+ - y^-}{2} \quad (2.81)$$

where $u_0 = c(2\gamma - 1)$, y^+ , and y^- are defined by (2.79) and (2.80), respectively.

In practice, the values of γ close to 0.5 can be used for LPRS calculation as per (2.81). If the system is of non-zero type (integrating plant), the adjustments stated above should be made.

We call formula (2.81) the *extended or open-loop definition of the LPRS* because it allows for the LPRS to be defined at frequencies that may not be the frequencies of the oscillations in the closed-loop system (Fig. 2.7). As per (2.81), the LPRS of a given plant can be computed at any frequency.

We call the set of coefficients of the Fourier expansion of plant output y_0, a_k, b_k ($k = 1, \dots, m$) at a given frequency ω the *spectral characteristic of the plant at frequency ω* . We have shown above (see the sections devoted to obtaining the LPRS formula from the plant transfer function) that the spectral characteristic at any given frequency can be transformed into the LPRS at the same frequency. In some cases, it is very convenient to calculate and memorize the spectral characteristic. If, for example, a linear compensator is connected in series with the plant (in the error signal), the resulting spectral characteristic is easily calculated as a propagation of the plant output spectrum through the linear compensator. This technique is convenient for the design of a linear compensator. In the chapter devoted to pneumatic servomechanism analysis and design, this technique is further investigated.

2.10 Application of periodic signal mapping to computing the LPRS of some special nonlinear plants

Periodic signal mapping was introduced in Chapter 1, and the relation between this mapping and Poincaré mapping was reviewed above. Now, once the open-loop definition of the LPRS is available, we can apply the periodic signal mapping technique to the open-loop definition and design the methodology for computing the LPRS for nonlinear plants. However, this type of analysis differs from the LPRS analysis in the case of a linear plant. Moreover, according to formula (2.81), the oscillation (the control signal) must be asymmetric to enable us to compute the LPRS. Therefore, we have to consider asymmetric signals and the Fourier series with non-zero constant term, which is a more complex problem compared to the one solved above. Yet, for the purpose of computing the LPRS, the frequency of the control signal can be considered a known value. Also, we can assume a certain small (but sufficient for computing the real part of the LPRS) asymmetry of the control signal. Therefore, we can write the following spectral representation of the control:

$$\mathbf{Q}_u = [q_{u0} \ q_{u1} \ q_{u2} \ q_{u3} \ \dots] \quad (2.82)$$

where $q_{u0} = u_0 = c(\theta_1 - \theta_2)/(\theta_1 + \theta_2)$, $\omega = 2\pi/(\theta_1 + \theta_2)$,

$$q_{uk} = \frac{4c}{\pi} \sin(\pi k \theta_1 / (\theta_1 + \theta_2)) / k \times \{ \cos(k\omega\theta_1/2) \cos(k\omega t) + \sin(k\omega\theta_1/2) \sin(k\omega t) \}, k = \overline{1, \infty}.$$

In comparison to formula (1.8), formula (2.82) has a constant term. The mapping via linear dynamics given by the transferfunction $W(s)$ is given by the

following element-by-element multiplication formula,

$$\mathbf{Q}_y = \mathbf{Q}_u \bullet \mathbf{S} \quad (2.83)$$

where y is the output of these linear dynamics (which can be an internal variable of the plant), and

$$\mathbf{S} = [W(j0) \ W(j\omega) \ W(j2\omega) \ W(j3\omega) \ \dots] \quad (2.84)$$

assuming that $W(s)$ is non-integrating.

We consider a few nonlinear models of the plant: the Hammerstein model, the Wiener model, and a nonlinearity preceded and followed by linear dynamics.

The Hammerstein model is given by a cascade connection of a single-valued memoryless nonlinearity followed by linear dynamics (Fig. 2.9). We limit our discussion to single-valued symmetric nonlinearities.

Because the control $u(t)$ is either “+ c ” or “− c ,” the input nonlinearity of the Hammerstein model changes only the value of the effective control amplitude. Instead of “ $\pm c$,” we have to use “ $\pm c_1$,” where $c_1 = g(c)$, where the input nonlinearity of the Hammerstein model is $g(u)$. Therefore, the LPRS is not affected by the presence of the input single-valued symmetric nonlinearity of the plant. Yet, when determining the frequency of the periodic motion and the equivalent gain value, the modified value of the relay amplitude must be used instead of the original one.

In the same way, we analyze the effect of the output nonlinearity of the Wiener model. The Wiener model is given by a cascade connection of linear dynamics followed by a single-valued memoryless nonlinearity (Fig. 2.10). Again let us consider only symmetric nonlinearities.

We derive the formula of the LPRS for the plant given by the Wiener model (Fig. 2.10). We rewrite (2.70) in the following form using the notation for the figure,

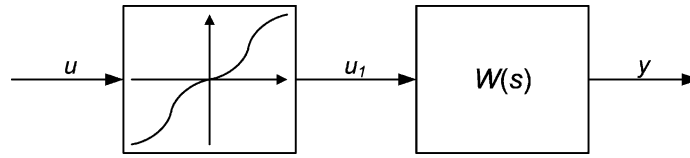


Fig. 2.9. Hammerstein system

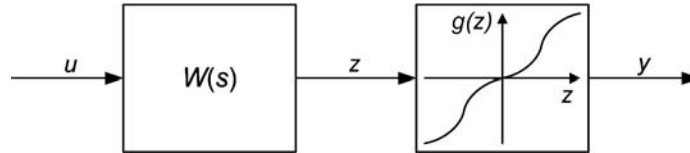


Fig. 2.10. Wiener system

$$\begin{aligned}
J(\omega) &= -\frac{1}{2} \lim_{f_0 \rightarrow 0} \frac{f_0 - y_0}{u_0} + j \frac{\pi}{4c} \lim_{f_0 \rightarrow 0} y(t) \Big|_{t=0} \\
&= -\frac{1}{2} \left(\frac{df_0}{du_0} - \frac{dy_0}{du_0} \right) + j \frac{\pi}{4c} y(t) \Big|_{t=0} \\
&= -\frac{1}{2} \left(\frac{df_0}{du_0} - \frac{dg}{dz} \Big|_{z=0} \frac{dz_0}{du_0} \right) + j \frac{\pi}{4c} g(z(t) \Big|_{t=0}),
\end{aligned} \tag{2.85}$$

where $\dots|_{z=0}$ denotes the derivative at the point $z = 0$. It follows from formula (2.85) that the LPRS of the Wiener system plant (Fig. 2.10) can be computed as follows,

$$J(\omega) = \frac{dg}{dz} \Big|_{z=0} \text{Re} J_l(\omega) + j \text{Im} J_l(\omega) \tag{2.86}$$

where $J_l(\omega)$ is the LPRS computed for the linear part only (as per (2.12), (2.32), (2.19)), and the hysteresis value is modified as follows,

$$b_1 = g^{-1}(b), \tag{2.87}$$

where $g^{-1}(y)$ is the inverse function with respect to the function $g(z)$.

The Hammerstein and Wiener models provide examples of simple nonlinear plants to which the LPRS method can be applied with minimal modifications. However, in real applications, plants dynamics often feature nonlinearities preceded by and followed by linear dynamics as shown in Fig. 2.11. We note that other types of connections between the plant nonlinearity and linear dynamics can be brought to the configuration in Fig. 2.11. We can write the following model of control signal propagation through the nonlinear plant. Assume that the control is given by (2.82). Then the following holds,

$$\mathbf{Q}_{y1} = \mathbf{Q}_u \bullet \mathbf{S}_1, \tag{2.88}$$

where y_1 is the output of the linear dynamics with the transfer function $W_1(s)$, and $\mathbf{Q}_{y1}$ is the spectrum matrix of $y_1(t)$,

$$\mathbf{S}_1 = [W_1(j0) \ W_1(j\omega) \ W_1(j2\omega) \ W_1(j3\omega) \ \dots]. \tag{2.89}$$

The time-domain form of y_1 is obtained as the inverse Fourier transform of its spectral representation:

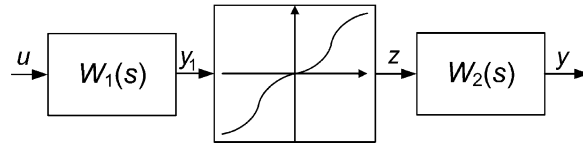


Fig. 2.11. Nonlinearity preceded and followed by linear dynamics

$$y_1(t) = q_{y10} + 2 \sum_{k=1}^{\infty} |q_{y1k}| \cos(\omega t + \arg q_{y1k}). \quad (2.90)$$

As a result, the output of the nonlinearity in the time domain is

$$z(t) = g(y_1(t)). \quad (2.91)$$

In the spectral domain it is as follows:

$$\mathbf{Q}_z = [q_{z0} \ q_{z1} \ q_{z3} \ q_{z5} \ \dots] \quad (2.92)$$

where

$$q_{zk} = \frac{1}{T} \int_{-T/2}^{T/2} z(t) [\cos(2k-1)\omega t - j \sin(2k-1)\omega t] dt, k = \overline{1, \infty}, \quad (2.93)$$

$$q_{z0} = \frac{1}{T} \int_{-T/2}^{T/2} z(t) dt. \quad (2.94)$$

Subsequently, the output of the plant is given by the following spectral representation,

$$\mathbf{Q}_y = \mathbf{Q}_z \bullet \mathbf{S}_2, \quad (2.95)$$

where

$$\mathbf{S}_2 = [W_2(j0) \ W_2(j\omega) \ W_2(j3\omega) \ W_2(j5\omega) \ \dots] \quad (2.96)$$

and in the time domain

$$y(t) = q_{y0} + 2 \sum_{k=1}^{\infty} |q_{yk}| \cos(\omega t + \arg q_{yk}), \quad (2.97)$$

where q_{yk} are Fourier coefficients that comprise matrix $\mathbf{Q}_y$. Therefore, utilizing the open-loop computing formula of the LPRS (2.81), the following approximate finite difference schema can be designed for computing the LPRS of the nonlinear plant Fig. 2.11.

$$J(\omega) \approx -\frac{0.5[y^+ + y^-] - y_0}{2u_0} + j \frac{\pi}{4c} \frac{y^+ - y^-}{2}, \quad (2.98)$$

where $y^+ = y(0)$, $y^- = y(\theta_1)$, which is given by (2.97), $y_0 = q_{y0}$, and u_0 is selected to be small enough for the finite difference estimate of the LPRS (2.98) to be precise. Values of u_0 that are too high result in errors in the evaluation of the equivalent gain, and values that are too small may not provide a sufficient resolution of the finite difference approach. As a rule of thumb, u_0 should correspond to the range of expected relative pulse duration values of the control pulses in the system under an external input.

2.11 Comparison of the LPRS with other methods of analysis of relay systems

The LPRS method deals with a classic problem that is examined in many sources. Naturally, the LPRS method should overlap with other existing methods and produce the same results under certain circumstances. Thus it is interesting to compare the LPRS and other methods. The closest two methods are the describing function method and Tsypkin's method.

The describing function method ([8, 50]): Because the DF method is based upon the filtering hypothesis, one might expect that the LPRS method provides the same result under this hypothesis. Indeed, if only the first terms of the series (2.19) of the real and imaginary parts are used (in accordance with the filtering hypothesis), this formula coincides with that of the DF method. The LPRS method, therefore, provides a more precise model of the oscillations and of the input-output properties of a relay system compared to the DF method. In particular, it takes into account the non-sinusoidal shape of the output signal, and the precision enhancement is due to that. If the actual shape of the output signal is close to sinusoidal, both methods provide similar results. Another difference is that the LPRS method does not require harmonic balance conditions to be fulfilled in the closed-loop system; it can handle systems where this condition is not fulfilled (i.e., a system consisting of a hysteretic relay and a first-order linear part or sliding mode control systems).

Tsypkin's method ([94]). The main similarity between Tsypkin's method and the LPRS is in the imaginary parts of the two loci. The imaginary part of the Tsypkin locus is defined as the output value in a periodic motion at the time of the relay switch from minus to plus. The imaginary part of the LPRS is essentially the same: the difference is only in the coefficient. However, the real part of the Tsypkin locus is defined as the derivative of the output at the time of the switch ($t = 0-$) and is intended for verifying the condition of the proper direction of the switch. The real part of the LPRS is defined as a ratio of the two infinitesimally small constant terms of the signals caused by the infinitesimally small asymmetry of switching in a closed-loop system. As a result, the Tsypkin locus is a method of analysis of possible periodic motions only; whereas the LPRS is intended for more complex analysis: the solution of the periodic problem and input-output analysis (disturbance rejection and external signal propagation).

A brief example demonstrates some aspects of the above comparison. Let the plant be the first-order plus dead-time transfer function: $W_l(s) = 0.5 \exp(-0.5s)/(1.5s+1)$. The frequency of the periodic solution found via application of the LPRS and of the Tsypkin locus is $\Omega_{LPRS} = \Omega_{Ts} = 3.593s^{-1}$ (exact value); the same frequency found via application of the DF method is $\Omega_{DF} = 3.516s^{-1}$ (the error between the two values is 2.1%). The *equivalent gain* values found via application of the LPRS and the DF methods are

$k_{nLPRS} = 6.258$ and $k_{nDF} = 5.371$, respectively. The error between these two values is 14.1%. The true value of the equivalent gain is the same as the one found via the LPRS application. From those results, we can see that even if the DF method may seem precise in terms of the frequency of a periodic solution, the error of the input-output properties may be much larger.

2.12 An example of analysis of oscillations and transfer properties

Let us find a periodic solution and analyze the transfer properties of the relay feedback system with an integrating linear part, given by equations (2.21), (2.22) with the following parameters:

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -0.4 & -0.5 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \mathbf{C} = [1 \quad 0],$$

$c = 1$, $b = 0.1$. Let an external harmonic signal of frequency 0.01 Hz ($\omega_{in} = 0.0628\text{s}^{-1}$) and amplitude $a_{in} = 20$ be applied to the closed-loop system. We find the frequency and the amplitude of the self-excited oscillation and analyze the system response to the specified external signal.

We compute the LPRS as per formula (2.32) and plot it (Fig. 2.12). For computing the LPRS, the subroutine “lprsmatrnt” (see Appendix) can be used. The solution of equation (2.3) corresponding to the point of intersection of the LPRS and the line parallel to the real axis drawn at a distance of $\pi b/(4c) = 0.0785$ below provides the frequency of the oscillations: $\Omega = 0.625\text{s}^{-1}$. The amplitude of the self-excited oscillation is $a = 6.52$.

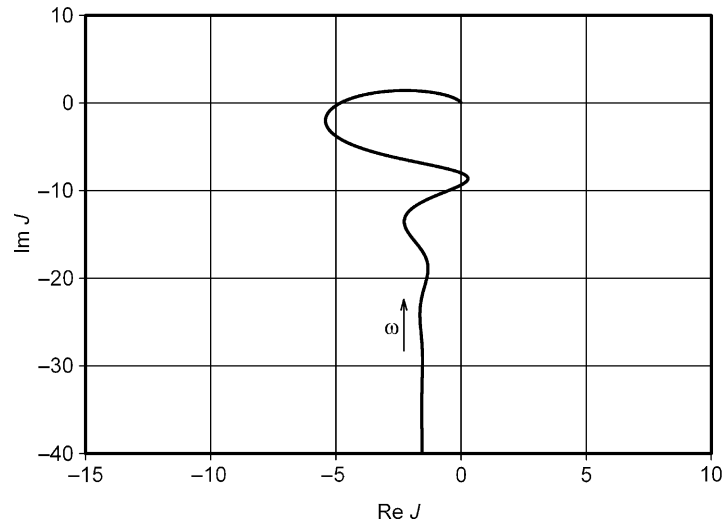


Fig. 2.12. LPRS for example of Section 2.12

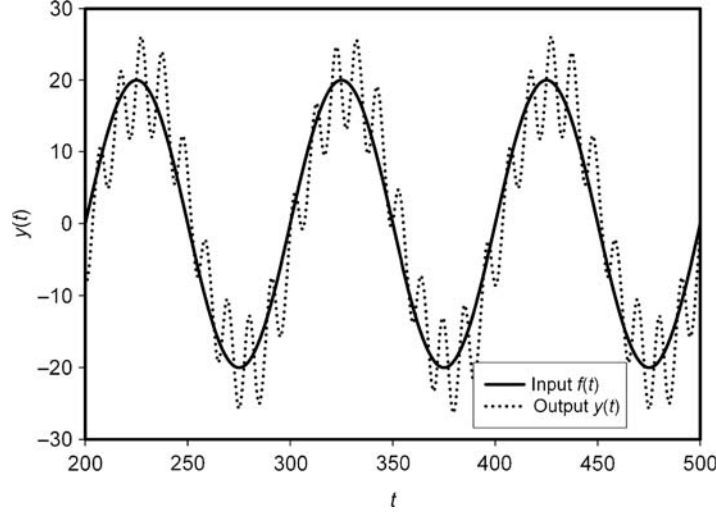


Fig. 2.13. System response to external harmonic excitation (simulations)

We assess the orbital stability of the system. The eigenvalues of the matrix Φ_0 computed per (2.34) are $\lambda_1 = -0.319$ and $\lambda_2 = 0.062$, which have magnitudes smaller than one. Therefore, the system is orbitally asymptotically stable. According to formula (2.4), we calculate the equivalent gain: $k_n = 0.103$. We assess the tracking quality of the input signal by the system using the linearized model, which is obtained via substitution of the gain k_n for the relay characteristic. Employing the methods of linear systems analysis, we obtain the component of the motion of frequency ω_{in} in the output signal. Note: The output signal also contains the self-excited periodic component $y_p(t)$ of frequency $\Omega = 0.625\text{s}^{-1}$. Therefore, the output is $y(t) = 19.8 \sin(0.0628t - 0.242) + y_p(t)$, where $y_p(t)$ is the periodic component of the motion of frequency $\Omega = 0.625\text{s}^{-1}$ (note that $y_p(t)$ is not a harmonic signal). This result matches well the results obtained through simulations (Fig. 2.13).

2.13 Conclusions

The frequency domain methodology of analysis is based on the notion of the LPRS and an approach that involves substitution of the relay element with the *equivalent gain*. The LPRS comprises both the oscillatory and the transfer properties of a relay system, and succeeds even if the filtering hypothesis fails; therefore, it is a relatively universal characterization of a discontinuous control system. We prove that despite the fact that the LPRS is defined through the parameters of the periodic motion in the closed-loop system, it is actually a

characterization of the linear part only. We derive three different techniques for computing the LPRS for both non-integrating and integrating linear parts, and consider certain properties of the LPRS. Finally, we demonstrate that the LPRS concept can be extended to nonlinear plants.



<http://www.springer.com/978-0-8176-4752-0>

Discontinuous Control Systems
Frequency-Domain Analysis and Design

Boiko, I.

2009, XIV, 212 p., Hardcover

ISBN: 978-0-8176-4752-0

A product of Birkhäuser Basel