Preface

As an undergraduate student at a good engineering school, I had never heard of stochastic processes or Lie groups (even though I double majored in Mathematics). As a faculty member in engineering I encountered many problems where the recurring themes were "noise" and "geometry." When I went to read up on both topics I found fairly little at this intersection. Now, to be certain, there are many wonderful texts on one of these subjects or the other. And to be fair, there are several advanced treatments on their intersection. However, for the engineer or scientist who has the modest goal of modeling a stochastic (i.e., time-evolving and random) mechanical system with equations with an eye towards numerically simulating the system's behavior rather than proving theorems, very few books are out there. This is because mechanical systems (such as robots, biological macromolecules, spinning tops, satellites, automobiles, etc.) move in multiple spatial dimensions, and the configuration space that describes allowable motions of objects made up of rigid components does not fit into the usual framework of linear systems theory. Rather, the configuration space manifold is usually either a Lie group or a homogeneous space[1].

My mission then became clear: write a book on stochastic modeling of (possibly complicated) mechanical systems that a well-motivated first-year graduate student or undergraduate at the senior level in engineering or the physical sciences could pick up and read cover-to-cover without having to carry around twenty other books. The key point that I tried to keep in mind when writing this book was that the art of mathematical modeling is very different than the art of proving theorems. The emphasis here is on "how to calculate" quantities (mostly analytically by hand and occasionally numerically by computer) rather than "how to prove." Therefore, some topics that are treated at great detail in mathematics books are covered at a superficial level here, and some concrete analytical calculations that are glossed over in mathematics books are explained in detail here. In other words the goal here is not to expand the frontiers of mathematics, but rather to translate known results to a broader audience.

The following quotes from Felix Klein[2] in regard to the modern mathematics of his day came to mind often during the writing process:

> The exposition, intended for a few specialized colleagues, refrains from indicating any connection with more general questions. Hence it is barely accessible to colleagues in neighboring fields and totally inaccessible to a larger circle. . .

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[1]The reader is not expected to know what these concepts mean at this point.

[2]F. Klein, *Development of Mathematics in the 19th Century*, translated by M. Ackerman as part of *Lie Groups: History, Frontiers and Applications*, Vol. IX, Math Sci Press, 1979.

> In fact, the physicist can use little, and the engineer none at all, of these theories in his tasks.

The later of these was also referenced in Arnol'd's classic book[3] as an example of how work that is initially viewed as esoteric can become central to applied fields.

In order to emphasize the point that this book is for practitioners, as I present results they generally are not in "definition–proof–theorem" format. Rather, results and derivations are presented in a flowing style. Section headings punctuate results so that the presentation (hopefully) does not ramble on too much.

Another difference between this book and one on pure mathematics is that while pathological examples can be viewed as the fundamental motivation for many mathematical concepts (e.g., the behavior of $\sin\frac{1}{x}$ as $x \to 0$), in most applications most functions and the domains on which they are defined do not exhibit pathologies. And so practitioners can afford to be less precise than pure mathematicians.

A final major difference between this presentation and those written by mathematicians is that rather than the usual "top-down" approach in which examples follow definitions and theorems, the approach here is "bottom-up" in the sense that examples are used to motivate concepts throughout this book and the companion volume. Then after the reader gains familiarity with the concepts, definitions are provided to capture the essence of the examples.

To help with the issue of motivation and to illustrate the art of mathematical modeling, case studies from a variety of different engineering and scientific fields are presented. In fact, so much material is covered that this book has been split into two volumes. Volume 1 (which is what you are reading now) focuses on basic stochastic theory and geometric methods. The usefulness of some of these methods may not be clear until the second volume. For example, some results pertaining to differential forms and differential geometry that are presented in Volume 1 are not applied to stochastic models until they find applications in Volume 2 in the form of integral geometry (also called geometric probability) and in multivariate statistical analysis. Volume 2 serves as an in-depth (but accessible) treatment of Lie groups, and the extension of statistical and information-theoretic techniques to that domain.

I have organized Volume 1 into the following 9 chapters and an appendix: Chapter 1 provides an introduction and overview of the kinds of the problems that can be addressed using the mathematical modeling methods of this book. Chapter 2 reviews every aspect of the Gaussian distribution, and uses this as the quintessential example of a probability density function. Chapter 3 discusses probability and information theory and introduces notation that will be used throughout these volumes. Chapter 4 is an overview of white noise, stochastic differential equations (SDEs), and Fokker–Planck equations on the real line and in Euclidean space. The relationship between Itô and Stratonovich SDEs is explained and examples illustrate the conversions between these forms on multi-dimensional examples in Cartesian and curvilinear coordinate systems. Chapter 5 provides an introduction to geometry including elementary projective, algebraic, and differential geometry of curves and surfaces. That chapter begins with some concrete examples that are described in detail. Chapter 6 introduces differential forms and the generalized Stokes' theorem. Chapter 7 generalizes the treatment of surfaces and polyhedra to manifolds and polytopes. Geometry is first described using a coordinate-dependent presentation that some differential geometers may find old fashioned, but it

[3]See Arnol'd, VI, *Mathematical Methods of Classical Mechanics*, Springer-Verlag, Berlin, 1978.

is nonetheless fully rigorous and general, and far more accessible to the engineer and scientist than the elegant and powerful (but cryptic) coordinate-free descriptions. Chapter 8 discusses stochastic processes in manifolds and related probability flows. Chapter 9 summarizes the current volume and introduces Volume 2. The appendix provides a comprehensive review of concepts from linear algebra, multivariate calculus, and systems of first-order ordinary differential equations. To the engineering or physical science student at the senior level or higher, some of this material will be known already. But for those who have not seen it before, it is presented in a self-contained manner. In addition, exercises at the end of each chapter in Volume 1 reinforce the main points. There are more than 150 exercises in Volume 1. Volume 2 also has many exercises. Over time I plan to build up a full solution set that will be uploaded to the publisher's webpage, and will be accessible to instructors. This will provide many more worked examples than space limits allow within the volumes.

Volume 1 can be used as a textbook in several ways. Chapters 2–4 together with the appendix can serve as a one-semester course on continuous-time stochastic processes. Chapters 5–8 can serve as a one-semester course on elementary differential geometry. Or, if chapters are read sequentially, the whole book can be used for self-study. Each chapter is meant to be relatively self-contained, with its own references to the literature. Altogether there are approximately 250 references that can be used to facilitate further study.

The stochastic models addressed here are equations of motion for physical systems that are forced by noise. The time-evolving statistical properties of these models are studied extensively. Information theory is concerned with communicating and extracting content in the presence of noise. Lie groups either can be thought of as continuous sets of symmetry operations, or as smooth high-dimensional surfaces which have an associated operator. That is, the same mathematical object can be viewed from either a more algebraic or more geometric perspective.

Whereas the emphasis of Volume 1 is on basic theory of continuous-time stochastic processes and differential geometric methods, Volume 2 provides an in-depth introduction to matrix Lie groups, stochastic processes that evolve on Lie groups, and information-theoretic inequalities involving groups. Volume 1 only has a smattering of information theory and Lie groups. Volume 2 emphasizes information theory and Lie groups to a much larger degree.

Information theory consists of several branches. The branch originating from Shannon's mathematical theory of communication is covered in numerous engineering textbooks with minor variants on the titles "Information Theory" or "Communications Theory." A second branch of information theory, due to Wiener, is concerned with filtering of noisy data and extracting a signal (such as in radar detection of flying objects). The third branch originated from the field of mathematical statistics in which people like Fisher, de Bruijn, Cramér, and Rao developed concepts in statistical estimation. It is primarily this third branch that is addressed in Volume 1, and so very little of the classical engineering information theory is found here. However, Shannon's theory is reviewed in detail in Volume 2, where connections between many aspects of information and group theory are explored. And Wiener's filtering ideas (which have a strong connection with Fourier analysis) find natural applications in the context of deconvolving functions on Lie groups (an advanced topic that is also deferred to Volume 2).

Volume 2 is a more formal and more advanced presentation that builds on the basics covered in Volume 1. It is composed of three parts. Part 1 begins with a detailed treatment of Lie groups including elementary algebraic, differential geometric, and functional analytic properties. Classical variational calculus techniques are reviewed, and the coordinate-free extension of these concepts to Lie groups (in the form of the Euler–Poincaré equation) are derived and used in examples. In addition, the basic concepts of group representation theory are reviewed along with the concepts of convolution of functions and Fourier expansions on Lie groups. Connections with multivariate statistical analysis and integral geometry are also explored. Part 2 of Volume 2 is concerned with the connections between information theory and group theory. An extension of the de Bruijn inequality to the context of Lie groups is examined. Classical communication-theory problems are reviewed, and information inequalities that have parallels in group theory are explained. Geometric and algebraic problems in coding theory are also examined. A number of connections to problems in engineering and biology are provided. For example, it is shown how a spherical optical encoder developed by the author and coworkers[4] can be viewed as a decoding problem on the rotation group, $SO(3)$. Also, the problem of noise in coherent optical communication systems is formulated and the resulting Fokker–Planck equation is shown to be quite similar to that of the stochastic Kinematic cart that is described in the introductory chapter of Volume 1. This leads to Part 3 of Volume 2, which brings the discussion back to issues close to those in Volume 1. Namely, stochastic differential equations and Fokker–Planck equations are revisited. In Volume 2 all of these equations evolve on Lie groups (particularly the rotation and rigid-body-motion groups). The differential geometric techniques that are presented in Volume 1 are applied heavily in this setting. Several closely related (though not identical) concepts of "mean" and "covariance" of probability densities on Lie groups are reviewed, and their propagation under iterated convolutions is studied. As far as the descriptions of probability densities on Lie groups are concerned, closed-form Gaussian-like approximations are possible in some contexts, and Fourier-based solutions are more convenient in others. The coordinate-based tools needed for realizing these expressions as concrete quantities (which can in principle be implemented numerically) are provided in Volume 2.

During a lecture I attended while writing this book, an executive from a famous computer manufacturer said that traditionally technical people have been trained to be "I-shaped," meaning an education that is very deep in one area, but not broad. The executive went on to say that he now hires people who are "T-shaped," meaning that they have a broad but generally shallow background that allows them to communicate with others, but in addition have depth in one area. From this viewpoint, the present book and its companion volume are "Π-shaped," with a broad discussion of geometry that is used to investigate three areas of knowledge relatively deeply: stochastic models, information theory, and Lie groups.

It has been a joy to write these books. It has clarified many issues in my own mind. And I hope that you find them both interesting and useful. And while I have worked hard to eliminate errors, there will no doubt be some that escaped my attention. Therefore I welcome any comments/corrections and plan to keep an updated online erratum page which can be found by searching for my name on the Web.

There are so many people without whom this book would not have been completed. First, I must thank John J. Benedetto for inviting me to contribute to this series that he is editing, and Tom Grasso at Birkhäuser for making the process flow smoothly.

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[4]Stein, D., Scheinerman, E.R., Chirikjian, G.S., "Mathematical models of binary spherical-motion encoders," *IEEE-ASME Trans. Mechatron.*, 8, pp. 234–244, 2003.

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Last but not least, I would like to thank my family. Writing a single-author book can be a solitary experience. And so it is important to have surroundings that are “fuuuun.”

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