Fields of Extremals and Sufficient Conditions for a Class of Variational Games

Dean Carlson
Mathematical Reviews
American Mathematical Society
Ann Arbor, MI, USA
dac@ams.org

Abstract

In this paper, using G. Leitmann's direct sufficiency method, extended for differential games, we present sufficient conditions for open loop strategies for a class of continuous time variational games. The result presented here is an extension of the classical Weierstrass sufficiency results for free problems in the calculus of variations.

Key words. Differential games, fields of extremals, calculus of variations.

AMS Subject Classifications. Primary 91A23; Secondary 49N70, 49K10.

1 Introduction

The study of calculus of variations is mainly focused on "single-player" games and has not concerned itself, for the most part, with the case of n-player games. The first exception to this case is the papers of C. F. Roos in the 1920's, which through a series of papers explored what we refer to here as variational games. In particular, we mention his 1925 paper in which he considers the mathematical theory of competition in the context of a dynamic economic growth model [13]. The culmination of these works appears to be his 1927 paper "Generalized Lagrange Problems in the Calculus of Variations" that appeared in the Transactions of the American Mathematical Society [14]. In this latter paper, he considers a general two-player game in the form of two coupled Lagrange problems in the calculus of variations. He presents a rather complete theory for what we now call a Nash equilibrium (some 25 years before Nash's theorem!) providing analogues of the classical necessary conditions, including the Euler-Lagrange equations, Weierstrass necessary conditions, Legendre conditions, as well as transversality conditions. The paper concludes with analogues of the classical sufficient conditions in the spirit of Weierstrass and Legendre for weak and strong local minimizers. Unfortunately, he includes no examples illustrating the application of his work. Moreover, with the advent of optimal control and differential games, much of this classical

P. Bernhard et al. (eds.), *Advances in Dynamic Games and Their Applications*,
Annals of the International Society of Dynamic Games 10, DOI: 10.1007/978-0-8176-4834-3_2,

theory has been neglected. Indeed, the only reference I am aware of regarding this early work is a reference to the 1925 paper in the textbook by Kamien and Schwartz [9], although I am sure there are others.

In the work presented here, we consider a class of variational games with the idea of presenting a version of Weierstrass's sufficiency theorem and the classical field theory. Our approach is to present, via the direct method of G. Leitmann, a theorem which allows us to conclude that when a solution of the Euler-Lagrange equations can be embedded in a family of extremals then it is indeed an open loop Nash equilibrium. The direct method of G. Leitmann is a non-variational method which utilizes coordinate transformations to construct an "equivalent variational problem" for which the solution may be obtained easily (often by inspection). Once the equivalent problem is solved, the inverse coordinate transformation gives a solution to the original problem. The original formulation by Leitmann has been generalized to dynamic games and additionally has incorporated the notion of an equivalent problem arising in the work of C. Carathéodory for single-player games.

The plan of our paper is as follows. In Sec. 2 we introduce the class of variational games we are concerned with. Section 3 introduces the direct method presenting the fundamental lemma. In Sec. 4 we introduce the notion of a field of extremals and present our sufficient conditions for an open loop Nash equilibrium. Section 5 we give a presentation of the "classical presentation" of field theory illustrating the connections between the results of Sec. 4 with those of the classical variational theory. Finally, in Sec. 6 we conclude with two elementary examples illustrating our result.

2 The Class of Games Considered

We consider an N-person game in which the state of player $j = 1, 2, \ldots N$ is a real-valued function $x_j(\cdot) : [a, b] \to \mathbb{R}$ with fixed initial value $x_j(a) = x_{aj}$ and fixed terminal value $x_j(b) = x_{bj}$. The objective of each player is to minimize a Lagrange type functional,

$$I_j(\mathbf{x}(\cdot)) = \int_a^b L_j(t, \mathbf{x}(t), \dot{x}_j(t))\, dt, \tag{1}$$

over all of his possible admissible strategies (see below), $\dot{x}_j(\cdot)$. The notation used here is that $\mathbf{x} = (x_1, x_2, \ldots x_N) \in \prod_{j=1}^{N} \mathbb{R} \doteq \mathbb{R}^N$. We assume that $L_j : A_j \to \mathbb{R}$ is a continuous function defined on the open set $A_j \subset \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}$ with the additional properties that $L_j(t, \cdot, \cdot)$ is twice continuously differentiable on $A_j(t) \doteq \{(\mathbf{x}, p_j) : \quad (t, \mathbf{x}, p_j) \in A_j\}$.

Clearly, the strategies of the other players influences the decision of the j-th player and so each player is unable to minimize independently of the other players. As a consequence, the players seek to play a Nash equilibrium instead.

To introduce this concept we first introduce the following notation. For each fixed $j = 1, 2, \ldots, N$, $\mathbf{x} = (x_1, x_2, \ldots, x_N) \in \mathbb{R}^N$, and $y_j \in \mathbb{R}$ we use the notation,

$$[\mathbf{x}^j, y_j] \doteq (x_1, x_2, \ldots, x_{j-1}, y_j, x_{j+1}, \ldots x_N).$$

With this notation we have the following definitions.

Definition 2.1. We say a function $\mathbf{x}(\cdot) = (x_1(\cdot), x_2(\cdot), \ldots, x_N(\cdot)) : [a, b] \to \mathbb{R}^N$ is admissible if it is continuous, has a piecewise continuous first derivative, satisfies the fixed initial and terminal conditions

$$x_j(a) = x_{aj} \quad \text{and} \quad x_j(b) = x_{bj}, \quad j = 1, 2, \ldots N, \tag{2}$$

and $(t, x_j(t), \dot{x}_j(t)) \in A_j$ for all $t \in [a, b]$.

Definition 2.2. Given an admissible function $\mathbf{x}(\cdot)$ we say a function $y_j(\cdot) : [a, b] \to \mathbb{R}$ is admissible for player j relative to $\mathbf{x}(\cdot)$ if the function $[\mathbf{x}^j, y_j](\cdot)$ is admissible.

With these definitions we can now give the definition of a Nash equilibrium.

Definition 2.3. An admissible function $\mathbf{x}^*(\cdot) : [a, b] \to \mathbb{R}^N$ is a Nash equilibrium if for each player $j = 1, 2, \ldots N$ and each function $y_j(\cdot) : [a, b] \to \mathbb{R}$ that is admissible for player j relative to $\mathbf{x}^*(\cdot)$ one has:

$$\begin{aligned} I_j(\mathbf{x}^*(\cdot)) &= \int_a^b L_j(t, \mathbf{x}^*(t), \dot{x}_j^*(t))\, dt \\ &\leq \int_a^b L_j(t, [\mathbf{x}^*(t)^j, y_j(t)], \dot{y}_j(t))\, dt \\ &= I_j([\mathbf{x}^{*j}, y_j](\cdot)). \end{aligned} \tag{3}$$

Remark 2.1. From the above definition it is clear that when all of the players "play" a Nash equilibrium, then each player's strategy is his best response to that of the other players. In other words, if player j applies any other strategy, than the equilibrium strategy, his cost functional will not decrease.

Remark 2.2. The above dynamic game clearly is not the most general structure one can imagine, even in a variational framework. In particular, the cost functionals are coupled only through their state variables and not through their strategies. While not the most general, one can argue that this form is general enough to cover many cases of interest since in a "real-world setting," an individual player will not know the strategies of the other players (see e.g., Dockner and Leitmann [8]).

The similarity of the above dynamic game to a free problem in the calculus of variations begs the question as to how much of the classical variational theory can be extended to this setting. It is indeed one aspect of this question that partly motivates this paper. We first recall the classical first-order necessary condition for this problem, giving the following theorem.

Theorem 2.1. *If $\mathbf{x}^*(\cdot)$ is a Nash equilibrium, then the following system of Euler-Lagrange equations are satisfied:*

$$\frac{d}{dt}\frac{\partial L_j}{\partial \dot{x}_j}\left(t, \mathbf{x}^*(t), \dot{x}_j^*(t)\right) = \frac{\partial L_j}{\partial x_j}\left(t, \mathbf{x}^*(t), \dot{x}_j^*(t)\right), \quad t \in [a,b]. \tag{4}$$

Proof. The proof follows directly from the classical theory of the calculus of variations upon the recognition that for each $j = 1, 2 \ldots N$ the trajectory $x_j^*(\cdot)$ minimizes the functional,

$$I_j[\mathbf{x}^{*j}, y_j](\cdot)) = \int_a^b L_j(t, [\mathbf{x}^{*j}(t), y_j(t)], \dot{y}_j(t))\, dt$$

over all continuous functions with piecewise continuous derivatives, $y_j(\cdot) : [a,b] \to \mathbb{R}$ satisfying the fixed end conditions given by (2). $\square$

Clearly, this is a set of standard necessary conditions and as such only provides candidates (i.e., the usual suspects!) for a Nash equilibrium. To insure that the candidate is indeed a Nash equilibrium, one must show that it satisfies some additional conditions. In the classical variational theory, these conditions typically mean that one has to show that the candidate can be embedded in a field of extremals and that some additional convexity conditions are also satisfied. As we shall see shortly, an analogous theory can also be developed here.

3 Leitmann's Direct Method

In this section we briefly outline a coordinate transformation method originally developed for single-player games in [10] and [11] (see also [2]), and further extended to N-player games in [8] and [3] which will enable us to derive our results. In particular, in Leitmann [12] (see also Carlson and Leitmann [3]) we have the following theorem.

Lemma 3.1. *For $j = 1, 2, \ldots, N$ let $x_j = z_j(t, \tilde{x}_j)$ be a transformation of class C^1 having a unique inverse $\tilde{x}_j = \tilde{z}_j(t, x_j)$ for all $t \in [a,b]$ such that there is a one-to-one correspondence $\mathbf{x}(t) \Leftrightarrow \tilde{\mathbf{x}}(t)$, for all admissible trajectories $\mathbf{x}(\cdot)$ satisfying the boundary conditions* (2) *and for all $\tilde{\mathbf{x}}(\cdot)$ satisfying:*

$$\tilde{x}_j(a) = \tilde{z}_j(a, x_{aj}) \quad \text{and} \quad \tilde{x}_j(b) = \tilde{z}_j(b, x_{bj})$$

for all $j = 1, 2, \ldots, N$. Further for each $j = 1, 2, \ldots, N$, let $\tilde{L}_j(\cdot,\cdot,\cdot) : [a,b] \times \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ be a given integrand having the same properties as $L_j(\cdot,\cdot,\cdot)$. For a given admissible $\mathbf{x}^*(\cdot) : [a,b] \to \mathbb{R}^N$, suppose the transformations $x_j = z_j(t, \tilde{x}_j)$ are such that there exist C^1 functions $G_j(\cdot,\cdot) : [a,b] \times \mathbb{R} \to \mathbb{R}$ so that the functional identities

$$\begin{aligned} L_j(t, [\mathbf{x}^*(t)^j, x_j(t)], \dot{x}_j(t)) \; &- \; \tilde{L}_j(t, [\mathbf{x}^*(t)^j, \tilde{x}_j(t)], \dot{\tilde{x}}_j(t)) \\ &= \frac{d}{dt} G_j(t, \tilde{x}_j(t)) \end{aligned} \tag{5}$$

hold on $[a, b]$. *If* $\tilde{x}_j^*(\cdot)$ *yields an extremum of* $\tilde{I}_j([\mathbf{x}^{*j}(\cdot), \cdot])$ *with* $\tilde{x}_j^*(\cdot)$ *satisfying the transformed boundary conditions, then* $x_j^*(\cdot)$ *with* $x_j^*(t) = z_j(t, \tilde{x}^*(t))$ *yields an extremum for* $I_j([\mathbf{x}^{*j}(\cdot), \cdot])$ *with the boundary conditions* (2).

Moreover, the function $\mathbf{x}^*(\cdot)$ *is an open-loop Nash equilibrium for the variational game.*

Proof. Leitmann [12]. □

Remark 3.1. This result has been successfully applied to a number of examples having applications in mathematical economics and we refer the reader to the references. Additionally, this lemma has been extended to include various classes of control systems (e.g., affine in the strategies) [6], infinite horizon models [5], as well as multiple integral problems [4].

Two immediate and useful corollaries are as follows.

Corollary 3.1. *The existence of* $G_j(\cdot, \cdot)$, $j = 1, 2 \ldots, N$, *in* (5) *imply that the following identities hold for* $(t, \tilde{x}_j) \in (a, b) \times \mathbb{R}$ *and* $\tilde{q}_j \in \mathbb{R}$ *for* $j = 1, 2, \ldots, N$*:*

$$L_j\left(t, [\mathbf{x}^{*j}(t), z_j(t, \tilde{x}_j)], \frac{\partial z_j(t, \tilde{x}_j)}{\partial t} + \frac{\partial z_j(t, \tilde{x}_j)}{\partial \tilde{x}_j}\tilde{q}_j\right) - \tilde{L}_j(t, [\mathbf{x}^{*j}(t), \tilde{x}_j], \tilde{q}_j) \equiv \frac{\partial G_j(t, \tilde{x}_j)}{\partial t} + \frac{\partial G_j(t, \tilde{x}_j)}{\partial \tilde{x}_j}\tilde{q}_j. \qquad (6)$$

Corollary 3.2. *For each* $j = 1, 2, \ldots, N$ *the left-hand side of the identity,* (6) *is linear in* $\tilde{q}_j$*, that is, it is of the form,*

$$\theta_j(t, \tilde{x}_j) + \psi_j(t, \tilde{x}_j)\tilde{q}_j$$

and,

$$\frac{\partial G_j(t, \tilde{x}_j)}{\partial t} = \theta_j(t, \tilde{x}_j) \quad \text{and} \quad \frac{\partial G_j(t, \tilde{x}_j)}{\partial \tilde{x}_j} = \psi(t, \tilde{x}_j)$$

on $[a, b] \times \mathbb{R}$.

The utility of the above lemma rests in being able to choose not only the transformation $\mathbf{z}(\cdot, \cdot)$ but also the integrand $\tilde{L}(\cdot, \cdot, \cdot)$. It is this flexibility that will enable us to extend the classical calculus of variations theory to the class of dynamic games considered here.

4 A Direct Method Proof of a Classical Sufficiency Condition

In this section we demonstrate how the direct method described above can be used to extend a classical result from the calculus of variations to the class of variational games considered here. Moreover, the approach presented here further provides an elementary proof of this result in the single-player case. We begin by extending the notion of a field of extremals beginning with the following definition.

Definition 4.1. For $j = 1, 2, \ldots N$, let $\xi_j(\cdot,\cdot) : [a,b] \times \mathbb{R} \to \mathbb{R}$ be a set of twice continuously differentiable functions and put $\xi(\cdot,\cdot) = (\xi_1(\cdot,\cdot), \xi_2(\cdot,\cdot), \ldots, \xi_N(\cdot,\cdot)) : [a,b] \times \mathbb{R}^N \to \mathbb{R}^N$. We say that $\xi(\cdot,\cdot)$ is a family of extremals for the differential game if for each constant vector $\beta = (\beta_1, \beta_2, \ldots, \beta_N) \in \mathbb{R}^N$ one has that the functions $t \to \xi_j(t, \beta_j)$ satisfies the Euler-Lagrange equations

$$\frac{d}{dt}\frac{\partial L_j}{\partial \dot{x}_j}(t, \xi(t,\beta), \frac{\partial \xi_j(t,\beta_j)}{\partial t}) = \frac{\partial L_j}{\partial x_j}(t, \xi(t,\beta), \frac{\partial \xi_j(t,\beta_j)}{\partial t}), \quad t \in [a,b], \quad (7)$$

and that $(t, \xi_j(t,\beta_j), \frac{\partial \xi_j(t,\beta_j)}{\partial t}) \in A_j$ for $t \in [a,b]$ and $j = 1, 2 \ldots N$.

When given a game to solve, one naturally makes an effort to solve the corresponding necessary conditions to obtain candidates of optimality. For the case considered here, these necessary conditions are the system of Euler-Lagrange equations (4). Consequently, suppose that we have a candidate for a Nash equilibrium, say $\mathbf{x}^*(\cdot)$ that satisfies the Euler-Lagrange equations as well as the fixed end conditions (2). The following theorem gives additional conditions that allow us to determine when this candidate is an open-loop Nash equilibrium

Theorem 4.1. *In addition to the hypotheses given in Sec. 1, assume that for each* $j = 1, 2, \ldots N$ *the functions* $p_j \to L_j(t, \mathbf{x}, p_j)$ *are convex on* $A_j(t, \mathbf{x}) = \{p_j \in \mathbb{R} : (\mathbf{x}, p_j) \in A_j(t)\}$. *Furthermore, let* $\mathbf{x}^*(\cdot)$ *be an admissible function for the dynamic game which satisfies the Euler-Lagrange equation* (4) *and assume that there exists a family of extremals,* $\xi(\cdot,\cdot)$ *having the following properties.*

(1) The transformations $x_j = \xi_j(t, \beta_j)$ *are of class* C^2 *with a unique inverse* $\beta_j = \tilde{\xi}_j(t, x_j)$.
(2) There exists a constant vector $\beta^* \in \mathbb{R}^N$ *such that* $x_j^*(t) = \xi_j(t, \beta_j^*)$ *for* $t \in [a,b]$.
(3) The partial derivatives $\frac{\partial}{\partial \beta_j}\xi_j(t,\beta_j)$ *are nonzero for all* (t, β_j).

Then $\mathbf{x}^*(\cdot)$ *is an open-loop Nash equilibrium for the dynamic game.*

Remark 4.1. To put the above theorem into a classical context, in the one-player game the family of extremals, assumed to exist in the above theorem, is referred to as a "field of extremals" in the classic book by Bolza [1] and in this case we have a direct comparison to the classical sufficiency condition due to Weierstrass and Hilbert. In the next section we will explore this relationship further by presenting these results in a more classical manner.

Remark 4.2. A family of extremals satisfying the conditions indicated above defines a one-to-one correspondence between the admissible trajectories of the dynamic game considered, say $\mathbf{x}(\cdot)$, and a set of functions $\tilde{\mathbf{x}}(\cdot)$ satisfying the boundary conditions:

$$\tilde{x}_j(a) = \tilde{\xi}_j(a, x_{aj}) = \beta_j^* \quad \text{and} \quad \tilde{x}_j(b) = \tilde{\xi}(b, x_{bj}) = \beta_j^*.$$

To see this we notice that if $\mathbf{x}(\cdot)$ is admissible for the original problem then we have the function $\tilde{\mathbf{x}}(\cdot) : [a,b] \to \mathbb{R}^N$ defined component-wise by $\tilde{x}_j(t) = \tilde{\xi}_j(t, x_j(t))$, for $j = 1, 2, \dots N$, satisfies the fixed-end conditions

$$\tilde{x}_j(a) = \tilde{\xi}_j(a, x_{aj}) \quad \text{and} \quad \tilde{x}_j(b) = \tilde{\xi}_j(b, x_{bj}).$$

Further, since $\beta_j^* = \xi(t, x_j^*(t))$ for all $t \in [a,b]$, it follows immediately that $\beta_j^* = \tilde{\xi}(a, x_{aj}) = \tilde{\xi}(b, x_{bj})$ as desired.

Proof of Theorem 4.1. To begin, we note that by Taylor's theorem, for each $t \in [a,b]$, $\beta \in \mathbb{R}^N$, $\tilde{\mathbf{p}} = (\tilde{p}_1, \tilde{p}_2, \dots \tilde{p}_N) \in \mathbb{R}^N$ and $j = 1, 2, \dots, N$ there exists $\gamma_j(t, \beta, \tilde{p}_j) \in [0,1]$ such that:

$$\begin{aligned} & L_j\left(t, \xi(t,\beta), \frac{\partial \xi_j}{\partial t}(t,\beta_j) + \frac{\partial \xi_j}{\partial \beta_j}(t,\beta_j)\tilde{p}_j\right) = \\ & L_j\left(t, \xi(t,\beta), \frac{\partial \xi_j}{\partial t}(t,\beta_j)\right) + \frac{\partial L_j}{\partial p_j}\left(t, \xi(t,\beta), \frac{\partial \xi_j}{\partial t}(t,\beta_j)\right)\frac{\partial \xi_j}{\partial \beta_j}(t,\beta_j)\tilde{p}_j \\ & + \frac{1}{2}\frac{\partial^2 L_j}{\partial p_j^2}\Bigg|_{\left(t,\xi(t,\beta), \frac{\partial \xi_j}{\partial t}(t,\beta_j) + \gamma_j(t,\beta,\tilde{p}_j)\frac{\partial \xi_j}{\partial \beta_j}(t,\beta_j)\tilde{p}_j\right)} \left(\frac{\partial \xi_j}{\partial \beta_j}(t,\beta_j)\right)^2 \tilde{p}_j{}^2. \end{aligned} \tag{8}$$

Now define $\tilde{L}_j(\cdot,\cdot,\cdot) : [a,b] \times \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ by the formula,

$$\tilde{L}_j(t,\beta,\tilde{p}_j) = \frac{1}{2}\frac{\partial^2 L_j}{\partial p_j^2}\left(\frac{\partial \xi_j}{\partial \beta_j}(t,\beta_j)\right)^2 \tilde{p}_j{}^2,$$

in which the second partial derivative of L_j in the above is evaluated at

$$\left(t, \xi(t,\beta), \frac{\partial \xi_j}{\partial t}(t,\beta_j) + \gamma_j(t,\beta,\tilde{p}_j)\frac{\partial \xi_j}{\partial \beta_j}(t,\beta_j)\tilde{p}_j\right),$$

and let $\theta(\cdot,\cdot) : [a,b] \times \mathbb{R}^N \to \mathbb{R}$ and $\psi(\cdot,\cdot) : [a,b] \times \mathbb{R}^N \to \mathbb{R}$ be given by

$$\theta(t,\beta) = L_j\left(t, \xi(t,\beta), \frac{\partial \xi_j}{\partial t}(t,\beta_j)\right)$$

and

$$\psi(t,\beta) = \frac{\partial}{\partial p_j}L_j\left(t, \xi(t,\beta), \frac{\partial \xi_j}{\partial t}(t,\beta_j)\right)\frac{\partial \xi_j}{\partial \beta_j}(t,\beta_j).$$

Observe that (8) may now be compactly written as:

$$\begin{aligned} L_j(t, \xi(t,\beta), \frac{\partial \xi_j(t,\beta_j)}{\partial t} + \frac{\partial \xi_j(t,\beta_j)}{\partial \beta_j}\tilde{p}_j) - \tilde{L}_j(t,\beta,\tilde{p}_j) \\ = \theta(t, [\beta^j, \beta_j]) + \psi(t, [\beta^j, \beta_j])\tilde{p}_j. \end{aligned} \tag{9}$$

To apply the direct method we must show that the right-hand side in (9) is exact, when viewed as a function of the variables (t, β_j). To do this it is sufficient to show that we have:

$$\frac{\partial}{\partial \beta_j}\theta(t,[\beta^j,\beta_j]) = \frac{\partial}{\partial t}\psi(t,[\beta^j,\beta_j]).$$

To this end we observe that since $\xi_j(\cdot,\cdot)$, $j = 1, 2, \dots N$, is a family of extremals we have:

$$\begin{aligned}
\frac{\partial}{\partial \beta_j}\theta(t,[\beta^j,\beta_j]) &= \frac{\partial L_j}{\partial x_j}(t,\xi(t,\beta),\frac{\partial \xi_j}{\partial t}(t,\beta_j))\frac{\partial \xi_j}{\partial \beta_j}(t,\beta_j) \\
&\quad + \frac{\partial L_j}{\partial p_j}(t,\xi(t,\beta),\frac{\partial \xi_j}{\partial t}(t,\beta_j))\frac{\partial^2 \xi_j}{\partial \beta_j \partial t}(t,\beta_j) \\
&= \frac{\partial}{\partial t}\left[\frac{\partial L_j}{\partial p_j}(t,\xi(t,\beta),\frac{\partial \xi_j}{\partial t}(t,\beta_j))\right]\frac{\partial \xi_j}{\partial \beta_j}(t,\beta_j) \\
&\quad + \frac{\partial L_j}{\partial \xi_j}(t,\xi(t,\beta),\frac{\partial \xi_j}{\partial t}(t,\beta_j))\frac{\partial^2 \xi_j}{\partial \beta_j \partial t}(t,\beta_j) \\
&= \frac{\partial}{\partial t}\left[\frac{\partial}{\partial \xi_j}L_j(t,\xi(t,\beta),\frac{\partial \xi_j}{\partial t}(t,\beta_j))\right]\frac{\partial \xi_j}{\partial \beta_j}(t,\beta_j) \\
&\quad + \frac{\partial}{\partial \xi_j}L_j\left(t,\xi(t,\beta),\frac{\partial \xi_j}{\partial t}(t,\beta_j)\right)\frac{\partial^2 \xi_j}{\partial t \partial \beta_j}(t,\beta_j) \\
&= \frac{\partial}{\partial t}\psi(t,[\beta^j,\beta_j]),
\end{aligned}$$

where we have used the fact that the family of extremals is twice continuously differentiable. From this we can conclude that there exists a function $G_j(\cdot,\cdot)$: $[a,b] \times \mathbb{R}^N \to \mathbb{R}$ such that:

$$\begin{aligned}
L_j(t,\xi(t,\beta),\frac{\partial \xi_j}{\partial t}(t,\tilde{x}_j) &+ \frac{\partial \xi_j}{\partial x_j}(t,\tilde{x}_j)\tilde{p}_j) - \tilde{L}_j(t,\beta,\tilde{p}_j) \\
&= \frac{\partial G_j}{\partial t}(t,[\beta^j,\tilde{x}_j]) + \frac{\partial G_j}{\partial \tilde{x}_j}(t,[\beta^j,\tilde{x}_j])\tilde{p}_j.
\end{aligned}$$

Thus, for the trajectory $\mathbf{x}^*(\cdot) = \xi(\cdot,\beta^*)$ and any function $x_j(\cdot)$ that is admissible for player j relative to $\mathbf{x}^*(\cdot)$ for player j, we have:

$$L_j(t,[\mathbf{x}(t)^{*j},x_j(t)],\dot{x}_j(t)) - \tilde{L}_j(t,[\beta^{*j},\tilde{x}_j(t)],\dot{\tilde{x}}_j(t)) = \frac{d}{dt}G(t,[\beta^{*j},\tilde{x}_j(t)]),$$

in which $\tilde{x}_j(\cdot) : [a,b] \to \mathbb{R}$ is defined by $\tilde{x}_j(\cdot) = \tilde{\xi}_j(\cdot, x_j(\cdot))$. This is precisely Eq. (5) so that we can apply the direct method. To this end we consider the following N auxiliary problems of minimizing the functionals:

$$\tilde{I}_j(\tilde{x}_j(\cdot)) = \int_a^b \tilde{L}_j(t,[\beta^{*j},\tilde{x}_j(t)],\dot{\tilde{x}}_j(t))\,dt \tag{10}$$

subject to the fixed (constant) end conditions

$$\tilde{x}_j(a) = \tilde{x}_j(b) = \beta_j^*. \tag{11}$$

Observe that since L_j is convex in its last argument we have $(t, \tilde{x}_j, \tilde{p}_j) \to \tilde{L}_j(t, [\beta^{*j}, \tilde{x}_j], \tilde{p}_j)$ is non-negative and assumes the value zero whenever $\tilde{p}_j = 0$. This means that minimizers for each of the N auxiliary problems are given by $\tilde{x}_j^*(t) \equiv \beta_j^*$ for $j = 1, 2, \ldots N$. Thus an application of the direct method gives us that the trajectory $\mathbf{x}^*(\cdot)$ is an open-loop Nash equilibrium for the original dynamic game. $\square$.

5 The Classical Approach to Sufficient Conditions

As indicated earlier, Bolza [1] referred to a family of extremals satisfying the hypotheses of Theorem 2 as a field of extremals. In other treatments, the notion of a field of extremals (see e.g., Cesari [7]) is defined in terms of a "slope function." To see how this concept arises, suppose that we have a candidate for a Nash equilibrium, say $\mathbf{x}^*(\cdot) = (x_1^*(\cdot), \ldots, x_N^*(\cdot))$, and a family of extremals $\{\xi_j(\cdot, \cdot)\}_{j=1}^N$ satisfying the following properties.

(i) There is a vector $\beta^* = (\beta_1^*, \ldots \beta_N^*) \in \mathbb{R}^N$ such that for each $j = 1, 2, \ldots N$ and all for $t \in [a, b]$ one has $x_j^*(t) = \xi_j(t, \beta_j^*)$.

(ii) For each $j = 1, 2 \ldots N$, there exists a set S_j that is a subset of a set of the form $\{(t, x_j) : t \in [a, b],\ x_j^*(t) - k < x_j < x_j^*(t) + k,\ k \in \mathbb{R}_+\}$ such that the equation $x_j = \xi_j(t, \tilde{x}_j)$ implicitly defines a function $\tilde{x}_j : S_j \to \mathbb{R}$ having continuous first partial derivatives on S (i.e. $\tilde{x}_j = \tilde{\xi}_j(t, x_j)$).

(iii) For each $j = 1, 2, \ldots N$ the functions $\xi_j(\cdot, \cdot)$ are twice continuously differentiable in an interval of the form $[a, b] \times [A_j, B_j]$ such that β_j^* is an interior point of $[A_j, B_j]$.

For each j, we view the function $\xi_j(\cdot, \beta_j)$ as a set of curves that cover S_j with exactly one curve through each point of S_j.

Now define a family of slope functions $\pi_j : S_j \to \mathbb{R}$, $j = 1, 2, \ldots N$ by the equations:

$$\pi_j(t, x_j) = \frac{\partial \xi_j}{\partial t}(t, \tilde{\xi}_j(t, x_j)), \quad (t, x_j) \in S_j, \quad j = 1, 2, \ldots N. \tag{12}$$

Observe that at a point $(t, x_j) \in S_j$, $\pi_j(t, x_j)$ is a tangent vector of the unique curve $\xi_j(\cdot, \beta_j)$ that passes through the point (t, x_j). Hence, one has

$$\pi_j(t, \xi_j(t, \beta_j)) = \frac{\partial \xi_j}{\partial t}(t, \beta_j), \tag{13}$$

which, as a consequence of (iii), implies that π_j has continuous first partial derivatives. Differentiating (13) yields:

$$\frac{\partial \pi_j}{\partial t}(t, \xi_j(t, \beta_j)) + \frac{\partial \pi_j}{\partial x_j}(t, \xi_j(t, \beta_j)) \frac{\partial \xi_j}{\partial t}(t, \beta_j) = \frac{\partial^2 \xi_j}{\partial t^2}(t, \xi_j(t, \beta_j)) \tag{14}$$

which becomes, via (13),

$$\frac{\partial \pi_j}{\partial t}(t, x_j) + \pi_j(t, x_j)\frac{\partial \pi_j}{\partial x_j}(t, x_j) = \frac{\partial^2 \xi_j}{\partial t^2}(t, x_j). \tag{15}$$

Letting $\mathbf{S} = S_1 \times \cdots S_N$ and $\pi(\cdot,\cdot) = (\pi_1(\cdot,\cdot), \ldots \pi_N(\cdot,\cdot))$ (viewed as a function from $[a,b] \times \mathbf{S}$ into $\mathbb{R}^N$) we refer to the pair $(\mathbf{S}, \pi)$ as a field $\mathcal{F}$ about $\mathbf{x}^*(\cdot)$ and say that the trajectory $\mathbf{x}^*(\cdot)$ is embedded in the field $\mathcal{F}$. To continue this further, and to recapture Hilbert's theory we assume that the integrands L_j are smooth enough so that all the partial derivatives taken below exist and are continuous. Let $\mathbf{x}^*(\cdot)$ be a twice continuously differentiable function that is embedded in field $\mathcal{F}$ and consider the functionals, for $j = 1, 2, \ldots N$, $\mathcal{J}_j$ defined on $\mathcal{Y}_j = \{y \in C^2([a,b];\mathbb{R}) : \ (t, [\mathbf{x}^{*j}(t), y(t)], \dot{y}(t)) \in A_j, \ a \leq t \leq b\}$ by

$$\begin{aligned}
\mathcal{J}_j(y(\cdot)) &= \int_a^b \Big\{ L_j(t, [\mathbf{x}^*(t)^j, y(t)], \pi_j(t, y(t))) \\
&\qquad + [\dot{y}(t) - \pi_j(t, y(t))] \frac{\partial L_j}{\partial p_j}(t, [\mathbf{x}^*(t)^j, y(t)], \pi_j(t, y(t))) \Big\}\, dt \\
&= \int_\sigma \Big\{ L_j(t, [\mathbf{x}^*(t)^j, y], \pi_j(t, y)) \\
&\qquad - \pi_j(t, y) \frac{\partial L_j}{\partial p_j}(t, [\mathbf{x}^*(t)^j, y], \pi_j(t, y)) \Big\}\, dt \\
&\qquad + \frac{\partial L_j}{\partial p_j}(t, [\mathbf{x}^*(t)^j, y], \pi_j(t, y))\, dy
\end{aligned} \tag{16}$$

in which $\sigma = \{(t, y(t)) : a \leq t \leq b\} \subset \mathbb{R}^2$ is viewed as a curve in the plane and the last integral is interpreted as a line integral. From classical calculus, this line integral is path independent (i.e., depends only endpoints $(a, y(a)), (b, y(b)) \in \mathbb{R}^2$ if and only if one has:

$$\begin{aligned}
\frac{\partial}{\partial x_j}\Big\{ L_j(t, [\mathbf{x}^*(t)^j, y], \pi_j(t, y)) \ &- \ \pi_j(t, y) \frac{\partial L_j}{\partial p_j}(t, [\mathbf{x}^*(t)^j, y], \pi_j(t, y)) \Big\} \\
&= \ \frac{\partial}{\partial t}\Big\{ \frac{\partial L_j}{\partial p_j}(t, [\mathbf{x}^*(t)^j, y], \pi_j(t, y)) \Big\}.
\end{aligned} \tag{17}$$

Expanding the left side of the above gives us:

$$\begin{aligned}
\frac{\partial}{\partial x_j}\Big\{ L_j(t, [\mathbf{x}^*(t)^j, y], \pi_j(t, y)) \ &- \ \pi_j(t, y) \frac{\partial L_j}{\partial p_j}(t, [\mathbf{x}^*(t)^j, y], \pi_j(t, y)) \Big\} \\
&= \ \frac{\partial L_j}{\partial x_j} - \pi_j \left[\frac{\partial^2 L_j}{\partial x_j \partial p_j} + \frac{\partial^2 L_j}{\partial p_j^2} \frac{\partial \pi_j}{\partial x_j} \right]
\end{aligned} \tag{18}$$

in which $(t, [\mathbf{x}^*(t)^j, y], \pi_j(t, y))$ are the arguments of L_j and its partial derivatives (t, y) are the arguments π_j and its partial derivatives. Similarly, expanding the right side gives us:

$$\frac{\partial}{\partial t}\Big\{\frac{\partial L_j}{\partial p_j}(t,[\mathbf{x}^*(t)^j,y],\pi_j(t,y))\Big\} = \frac{\partial^2 L_j}{\partial t \partial p_j} + \sum_{i\neq j}\frac{\partial^2 L_j}{\partial x_i \partial p_j}\dot{x}_j^*(t) + \frac{\partial^2 L_j}{\partial p_j^2}\frac{\partial \pi_j}{\partial t}, \tag{19}$$

with the same convention regarding arguments as above. In view of the computations regarding the field, for each $(t,y)\in S_j$ there is a parameter β so that $y=\xi_j(t,\beta)$ and $\pi_j(t,y)=\frac{\partial \xi_j}{\partial t}(t,\beta)$. Moreover, we also have for each $i\neq j$ that $\dot{x}_i^*(t)=\pi_i(t,x*_i(t))$. Thus, using these facts, by equating (18) and (19) and arranging terms we arrive at the fact that (17) is equivalent to having,

$$\begin{aligned}\frac{\partial L_j}{\partial x_j} &= \frac{\partial^2 L_j}{\partial t\partial p_j} + \sum_{i\neq j}\frac{\partial^2 L_j}{\partial x_i\partial p_j}\frac{\partial \xi_i}{\partial t}(t,\beta_i^*) + \frac{\partial^2 L_j}{\partial x_j\partial p_j}\frac{\partial \xi_j}{\partial t}(t,\beta)\\ &\qquad + \frac{\partial^2 L_j}{\partial p_j^2}\frac{\partial^2 \xi_j}{\partial t^2}(t,\beta)\\ &= \frac{d}{dt}\frac{\partial L_j}{\partial p_j}(t,[\mathbf{x}^*(t)^j,\xi_j(t,\beta)],\frac{\partial \xi_j}{\partial t}(t,\beta)),\end{aligned} \tag{20}$$

which holds since we are assuming that $\xi(\cdot,\cdot)$ is a field of extremals. In the above computations we have used Eq. (15). Thus, in this setting we have shown the following version of the Hilbert invariant integral theorem.

Theorem 5.1. *Let $\mathbf{x}^*(\cdot)$ be any trajectory that can be embedded in a field $\mathcal{F}$, and suppose that for $j=1,2,\ldots$, fixed we have two piecewise smooth functions $y_1(\cdot)$ and $y_2(\cdot)$ such that the plane curves $(t,y_i(t))\in S_j$ for $i=1,2$ have common endpoints. Then it is the case that $\mathcal{J}_j(y_1(\cdot))=\mathcal{J}_j(y_2(\cdot))$.*

Now define, for a fixed trajectory $\mathbf{x}^*(\cdot)$ embedded in a field $\mathcal{F}$, the "Weierstrass excess functions" $E_j(\cdot,\cdot,\cdot,\cdot): S_j\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ by the formula:

$$\begin{aligned}E_j(t,y,q,p) = L_j(t,[\mathbf{x}^*(t)^j,y],p) - L_j(t,[\mathbf{x}^*(t)^j,y],q)&\\ -(p-q)\frac{\partial L_j}{\partial p_j}(t,[\mathbf{x}^*(t)^j,y],q).&\end{aligned} \tag{21}$$

Using this notation one can easily show the following result.

Theorem 5.2. *Let $\mathbf{x}^*(\cdot)$ be a smooth admissible trajectory embedded in a field $\mathcal{F}=(\mathbf{S},\pi)$ and for each $j=1,2,\ldots,N$ let $y_j(\cdot)$ be any function defined on $[a,b]$ such that $(t,y_j(t))\in S_j$ for all $t\in[a,b]$ and such that $y_j(a)=x_j^*(a)$ and $y_j(b)=x_j^*(b)$ (i.e., $y_j(\cdot)$ is admissible for player j relative to $\mathbf{x}^*(\cdot)$). Then one has:*

$$I_j([\mathbf{x}^*(\cdot)^j,y_j(\cdot)]) - I_j(\mathbf{x}^*(\cdot)) = \int_a^b E_j(t,y_j(t),\pi_j(t,y_j(t)),\dot{y}_j(t))\,dt.$$

Proof. We first observe that from the definition of the Weierstrass excess functions we immediately have for each $j = 1, 2, \ldots N$ that $\mathcal{J}_j(x_j^*(\cdot)) = I_j(\mathbf{x}^*(\cdot))$ since, $\dot{x}_j^*(\cdot) = \pi_j(\cdot, x_j^*(\cdot))$. Further, from the above theorem we also have that $\mathcal{J}_j(y_j(\cdot)) = \mathcal{J}_j(x_j^*(\cdot))$. Thus we have:

$$
\begin{aligned}
I_j([\mathbf{x}^*(\cdot)^j\,,\, y_j(t)]) - I_j(\mathbf{x}^*(\cdot)) &= I_j([\mathbf{x}^*(\cdot)^j, y_j(t)]) - \mathcal{J}_j(x_j^*(\cdot)) \\
&= I_j([\mathbf{x}^*(\cdot)^j, y_j(t)]) - \mathcal{J}_j(y_j(\cdot)) \\
&= \int_a^b L_j(t, [\mathbf{x}^*(t)^j, y_j(t)], \dot{y}_j(t))\, dt - \\
&\quad \int_a^b \Big\{ L_j(t, [\mathbf{x}^*(t)^j, y(t)], \pi_j(t, y(t))) \\
&+ [\dot{y}(t) - \pi_j(t, y(t))] \frac{\partial L_j}{\partial p_j}(t, [\mathbf{x}^*(t)^j, y(t)], \pi_j(t, y(t))) \Big\}\, dt \\
&= \int_a^b E_j(t, y_j(t), \pi_j(t, y_j(t)), \dot{y}_j(t))\, dt,
\end{aligned}
$$

as desired. □

From this last theorem we immediately have the following game-theoretic version of the classical Weierstrass-Hilbert sufficiency theorem.

Theorem 5.3. *If $\mathbf{x}^*(\cdot)$ is a smooth admissible trajectory of the variational game (1)–(2) which can be embedded in a field $\mathcal{F} = (\mathbf{S}, \mathbf{p})$, and if for each $j = 1, 2, \ldots, N$ and $y_j : [a, b] \to \mathbb{R}$ that is admissible for player j relative to $\mathbf{x}^*(\cdot)$ one has*

$$E_j(t, y_j(t), \pi_j(t, y_j(t)), \dot{y}_j(t)) \geq 0 \quad \textit{for all} \quad t \in [a, b],$$

then $I_j(\mathbf{x}^*(\cdot)) \leq I_j([\mathbf{x}^*(\cdot)^j, y_j(t)])$, which implies that $\mathbf{x}^*(\cdot)$ is an open-loop Nash equilibrium for the game (1)–(2).

Proof. The proof follows immediately from the previous theorem. □

Remark 5.1. To compare the Weierstrass-Hilbert sufficiency theorem to our result in the previous section we observe that Eq. (8) may be rewritten, for $\beta_i = \beta_i^*$ for $i \neq j$ as:

$$
\begin{aligned}
&E_j\left(t, \xi_j(t, \beta_j), \frac{\partial \xi_j}{\partial t}(t, \beta_j), \frac{\partial \xi_j}{\partial t}(t, \beta_j) + \frac{\partial \xi_j}{\partial \beta_j}(t, \beta_j)\tilde{p}_j\right) \\
&= \frac{1}{2} \frac{\partial^2 L_j}{\partial p_j^2}\bigg|_{\left(t, \xi(t,\beta), \frac{\partial \xi_j}{\partial t}(t,\beta_j) + \gamma_j(t,\beta) \frac{\partial \xi_j}{\partial \beta_j}(t,\beta_j)\tilde{p}_j\right)} \left(\frac{\partial \xi_j}{\partial \beta_j}(t, \beta_j)\right)^2 \tilde{p}_j^2 \\
&\geq 0
\end{aligned}
$$

due to our convexity assumption on the integrand $p_j \to L_j(t, \mathbf{x}, p_j)$.

Remark 5.2. Generally speaking, embedding a candidate for a Nash equilibrium, say $\mathbf{x}^*(\cdot)$, satisfying the Euler-Lagrange equations into a field $\mathcal{F}$ can only be guaranteed locally. As a consequence, the resulting sufficient condition only yields a local open-loop Nash equilibrium. In practice, for elementary examples it is possible to construct "global fields" which allows one to conclude that the candidate is a global Nash equilibrium. Sufficient conditions that insure the existence of an appropriate field for open-loop dynamic games are presently unavailable. We conjecture that theorems similar to those from the classical variational calculus can be appropriately modified to provide such conditions. One concept which will need to be addressed in this regard will be the notion of conjugate points. We leave these ideas for future research.

6 Examples

In this section we present several examples illustrating the above, theory beginning first with a one-player game (i.e., a calculus of variations problem).

6.1 Example

Consider the problem:

$$\text{minimize } J(x(\cdot)) = \int_a^b \dot{x}(t)^4 \, dt \tag{22}$$

over all piecewise smooth functions $x(\cdot) : [a,b] \to \mathbb{R}$ satisfying the fixed-end conditions:

$$x(a) = x_a \quad \text{and} \quad x(b) = x_b. \tag{23}$$

The Euler-Lagrange equation in this case is:

$$\frac{d}{dt} 4\dot{x}(t)^3 = 0$$

which gives rise to the extremals $x(t) = \alpha t + \beta$ for constants $\alpha, \beta \in \mathbb{R}$. The candidate for optimality is given as $x^*(t) = \alpha^* t + \beta^*$ in which α^* and β^* are chosen so that $x^*(\cdot)$ satisfies the end conditions (22). Explicitly, it is easy to check that $\alpha^* = (x_a - x_b)/(a-b)$ and that $\beta^* = x_a - \alpha^* a = b(x_b - x_a)/(a-b) = -b\alpha^*$. From this two-parameter family of extremals we must choose a one-parameter family. The one to choose is to let $\xi(t,\beta) = \alpha^* t + \beta$. Geometrically, the family of equations $x = \alpha^* t + \beta$ is a family of parallel lines that cover the tx-plane with a unique line (i.e., extremal) passing through each point of the plane. Furthermore, this gives rise to the desired one-to-one correspondence for the trajectories of the original problem (i.e., those that satisfy the end conditions (23)) and the piecewise trajectories $\tilde{x}(\cdot) : [a,b] \to \mathbb{R}$ that satisfy the fixed-end conditions:

$$\tilde{x}(a) = \beta^* \quad \text{and} \quad \tilde{x}(b) = \beta^*. \tag{24}$$

To see, this observe that if $x(\cdot)$ is a trajectory for the original problem then define $\tilde{x}(\cdot)$ by the formula

$$\tilde{x}(t) = x(t) - \alpha^* t, \quad t \in [a, b],$$

and observe that we have $\tilde{x}(a) = x_a - \alpha^* a = \beta^*$ and $\tilde{x}(b) = x_b - \alpha^* b = \beta^*$. Conversely, if $\tilde{x}(\cdot)$ satisfies (24) then we have that:

$$x(t) = \alpha^* t + \tilde{x}(t)$$

satisfies $x(a) = \alpha^* a + \tilde{x}(b) = \alpha^* a + \beta^* = x_a$ and $x(b) = \alpha^* b + \tilde{x}(b) = \alpha^* b + \beta^* = x_b$ as desired.

Now the Weierstrass sufficiency theorem tells us that $x^*(\cdot)$ is an optimal solution. What is new here is our proof of that fact, which is based on the direct method. To see how this works out in the example, observe that for $L(t, x, p) = p^4$ and $\xi(t, \beta) = \alpha^* t + \beta$ we have by Taylor's formula:

$$\begin{aligned} L(t, \xi(t,\beta), \xi_t(t,\beta) + \xi_\beta(t,\beta)\tilde{p}) = L(t, z(t,\beta), \xi_t(t,\beta)) \\ + \ \frac{\partial L}{\partial z}(t, \xi(t,\beta), \xi_t(t,\beta))\xi_\beta(t,\beta)\tilde{p} \\ + \ \frac{1}{2}\frac{\partial^2 L}{\partial z^2}(t, \xi(t,\beta), \xi_t(t,\beta) + \gamma(t,\beta,\tilde{p})\xi_\beta(t,\beta)\tilde{p})\xi_\beta(t,\beta)^2\tilde{p}^2, \end{aligned} \tag{25}$$

where $\gamma(t, \beta, \tilde{p}) \in [0, 1]$. Here the subscripts $\xi_t(\cdot, \cdot)$ and $\xi_\beta(\cdot, \cdot)$ are partial derivatives. From the above we want to choose $\tilde{L}(\cdot, \cdot, \cdot)$ to be

$$\tilde{L}(t, \beta, \tilde{p}) = \frac{1}{2}\frac{\partial^2 L}{\partial p^2}(t, \xi(t,\beta), \xi_t(t,\beta) + \gamma(t,\beta,\tilde{p})\xi_\beta(t,\beta)\tilde{p})\xi_\beta(t,\beta)^2\tilde{p}^2$$

so that the auxiliary problem becomes:

$$\text{minimize } \tilde{J}(\tilde{x}(\cdot)) = \int_a^b \tilde{L}(t, \tilde{x}(t), \dot{\tilde{x}}(t))\, dt$$

over all trajectories $\tilde{x}(\cdot) : [a, b] \to \mathbb{R}$ satisfying the end conditions (24). Further from the above we have the following:

$$\begin{aligned} L(t, \xi(t,\beta), \xi_t(t,\beta) + \xi_\beta(t,\beta)\tilde{p}) \ - \ \tilde{L}(t, \beta, \tilde{p}) = \\ L(t, \xi(t,\beta), \xi_t(t,\beta)) + \frac{\partial L}{\partial p}(t, \xi(t,\beta), \xi_t(t,\beta))\xi_\beta(t,\beta)\tilde{p}. \end{aligned}$$

To apply the direct method we must show that there exists a function $G(t, \beta)$ such that

$$G_t(t, \beta) = L(t, \xi(t,\beta), \xi_t(t,\beta))$$

and

$$G_\beta(t, \beta) = \frac{\partial L}{\partial p}(t, \xi(t,\beta), \xi_t(t,\beta))\xi_\beta(t,\beta)$$

both hold. A necessary and sufficient condition for this to hold is that $G_{t\beta}(t,\beta) = G_{\beta t}(t,\beta)$ or equivalently that

$$\frac{\partial}{\partial \beta} L(t, \xi(t,\beta), \xi_t(t,\beta)) = \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial p}(t, \xi(t,\beta), \xi_t(t,\beta))\xi_\beta(t,\beta)\right).$$

To see that this is the case we observe that (where we have suppressed the (t,β) arguments for brevity):

$$\begin{aligned}
\frac{\partial}{\partial \beta} L(t, \xi(t,\beta), \xi_t(t,\beta)) &= \frac{\partial L}{\partial x}(t,\xi,\xi_t)\cdot \xi_\beta + \frac{\partial L}{\partial p}(t,\xi,\xi_t)\cdot \xi_{t\beta} \\
&= \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial x}(t,\xi,\xi_t)\right)\xi_\beta + \frac{\partial L}{\partial p}(t,\xi,\xi_t)\cdot \xi_{t\beta} \\
&= \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \xi}(t, \xi(t,\beta), \xi_t(t,\beta))\xi_\beta(t,\beta)\right).
\end{aligned}$$

The second line in the above is a consequence of the fact that $t \to \xi(t,\beta)$ solves the Euler-Lagrange equation for each β and the last line is just the product rule. This means we do indeed have the desired function $G(\cdot,\cdot)$ so that we can write

$$L(t, \xi(t,\beta), \xi_t(t,\beta) + \xi_\beta(t,\beta)\tilde{q}) - \tilde{L}(t,\beta,\tilde{p}) = G_t(t,\beta) + G_\beta(t,\beta)\tilde{p}$$

which gives the fundamental identity

$$\begin{aligned}
&L(t, x(t), \dot{x}(t)) + \tilde{L}(t, \tilde{x}(t), \dot{\tilde{x}}(t)) \\
&\qquad = G_t(t,\tilde{x}(t)) + G_\beta(t,\tilde{x}(t))\dot{\tilde{x}}(t) = \frac{d}{dt}G(t,\tilde{x}(t)),
\end{aligned}$$

for each pair of trajectories $\{x(\cdot), \tilde{x}(\cdot)\}$ related through $x(t) = \xi(t,\tilde{x}(t))$. For the simple example considered here we can explicitly determine all of these functions. Indeed, since $\xi_t = \alpha^*$, $\xi_\beta = 1$, $L_p = 4p^3$, and $L_{pp} = 12p^2$ we have that (25) becomes

$$(\alpha^* + \tilde{p})^4 = (\alpha^*)^4 + 4(\alpha^*)^3\tilde{p} + \frac{1}{2}12(\alpha^* + \gamma(t,\beta,\tilde{p})\tilde{p})^2\tilde{p}^2.$$

Expanding this last statement and collecting terms it is easy to see that $\gamma = \gamma(t,\beta,\tilde{p})$ satisfies the quadratic equation

$$6\tilde{p}\gamma^2 + 12\alpha^*\tilde{p}\gamma - \tilde{p}^2 - 4\alpha^*\tilde{p} = 0$$

from which γ is easily determined by the quadratic formula. In addition, we also must have $G(\cdot,\cdot)$ satisfy $G_t(t,\beta) = (\alpha^*)^4$ and $G_\beta(t,\beta) = 4(\alpha^*)^3$ so that from the first we must have $G(t,\beta) = (\alpha^*)^4 t + g(\beta)$ so that $g'(\beta) = G_\beta(t,\beta) = 4(\alpha^*)^3$ which implies that $g(\beta) = 4(\alpha^*)^3\beta + C$ where C is an arbitrary constant. Thus, we have $G(t, \beta) = (\alpha^*)^4 t + 4(\alpha^*)^3 \beta + C$ for any constant C. Furthermore, we have that the auxiliary problem consists of minimizing

$$\tilde{I}(\tilde{x}(\cdot)) = \int_a^b 6(\alpha^* + \gamma(t, \tilde{x}(t), \dot{\tilde{x}}(t))\dot{\tilde{x}}(t))^2 \dot{\tilde{x}}(t)^2 \, dt$$

over all piecewise smooth trajectories $\tilde{x}(\cdot) : [a, b] \to \mathbb{R}$ satisfying the fixed-end conditions

$$\tilde{x}(a) = \beta^* = \tilde{x}(b).$$

Clearly, the integrand is non-negative for all choices of $\tilde{x}(\cdot)$ and, moreover, is identically zero when $\dot{\tilde{x}}(t) \equiv 0$. Thus, $\tilde{x}^*(t) \equiv \beta^*$ is an optimal solution and by the direct method we have $x^*(t) = \alpha^* t + \beta^*$ is an optimal solution of the original problem.

6.2 Example

In this example we consider a two-player game in which the objective of player $j = 1, 2$ is to maximize the objective functional

$$I_j(x_1(\cdot), x_2(\cdot)) = \int_0^1 \sqrt{\dot{x}_j(t) - x_1(t) - x_2(t)} \, dt$$

with the fixed-end conditions

$$x_j(0) = x_{j0} \quad \text{and} \quad x_j(1) = x_{j1}.$$

We first observe that for each $j = 1, 2$ we have $p_j \to L_j(x_1, x_2, p_j) = \sqrt{p_j - x_1 - x_2}$ is a concave function for fixed (x_1, x_2) since we have

$$\frac{\partial^2 L_j}{\partial p_j^2} = \frac{-1}{4(p_j - x_1 - x_2)^{\frac{3}{2}}} < 0,$$

so that our hypotheses apply. In addition, the Euler-Lagrange equations take the form

$$\frac{d}{dt} \frac{1}{\sqrt{\dot{x}_j(t) - x_1(t) - x_2(t)}} = -\frac{1}{\sqrt{\dot{x}_j(t) - x_1(t) - x_2(t)}}, \quad j = 1, 2.$$

To solve this pair of coupled equations we observe that if we let $v_j(t) = (\dot{x}_j(t) - x_1(t) - x_2(t))^{-1/2}$ we see that $\dot{v}_j(t) = -v_j(t)$ which gives us that $v_j(t) = A_j e^{-t}$ for each $j = 1, 2$. Thus, we have

$$\dot{x}_j(t) - x_1(t) - x_2(t) = \alpha_j e^{2t}, \quad j = 1, 2,$$

where $\alpha_j \geq 0$ is otherwise an arbitrary constant. Adding the two equation together gives us:

$$\dot{x}_1(t) + \dot{x}_2(t) - 2(x_1(t) + x_2(t)) = (\alpha_1 + \alpha_2)e^{2t}$$

which has the general solution

$$x_1(t) + x_2(t) = (\alpha_1 + \alpha_2)te^{2t} + Ce^{2t},$$

where C is a constant of integration. From this it follows that for each $j = 1, 2$ we have:

$$\dot{x}_j(t) = (\alpha_j + C)e^{2t} + (\alpha_1 + \alpha_2)te^{2t},$$

giving us

$$x_j(t) = \left[\frac{1}{2}(\alpha_j + C) + \frac{\alpha_1 + \alpha_2}{2}t - \frac{\alpha_1 + \alpha_2}{4}\right] e^{2t} + \beta_j,$$

in which β_j is an arbitrary constant of integration. Choosing $C = \frac{\alpha_1 + \alpha_2}{2}$ we obtain

$$x_j(t) = \frac{1}{2}\left[\alpha_j + (\alpha_1 + \alpha_2)t\right] e^{2t} + \beta_j.$$

To get a candidate for optimality we need to choose α_j and β_j so that the fixed-end conditions are satisfied. It is easy to see that this gives a linear system of equations for the four unknown constants $\alpha_1,\ \alpha_2,\ \beta_1,\ \beta_2$ which is uniquely solvable for any set of boundary conditions. This means we can find $\alpha_1^*,\ \alpha_2^*,\ \beta_1^*, \beta_2^*$ to give us a candidate for the optimal solution given by:

$$x_j^*(t) = \frac{1}{2}\left[\alpha_j^* + (\alpha_1^* + \alpha_2^*)t\right] e^{2t} + \beta_j^*.$$

From this it is easy to see that if we choose

$$\xi_j(t, \beta_j) = \frac{1}{2}\left[\alpha_j^* + (\alpha_1^* + \alpha_2^*)t\right] e^{2t} + \beta_j$$

we obtain a one-parameter family of extremals for the dynamic game which allows us to conclude that $\mathbf{x}^*(\cdot) = (x_1^*(\cdot), x_2^*(\cdot))$ is an open loop Nash equilibrium for this dynamic game.

References

[1] O. Bolza, *Calculus of variations*, 3rd ed., Chelsea Publishing Company, New York, 1973.

[2] D.A. Carlson, *An observation on two methods of obtaining solutions to variational problems*, Journal of Optimization Theory and Applications **114** (2002), 345–362.

[3] D.A. Carlson and G. Leitmann, *An extension of the coordinate transformation method for open-loop Nash equilibria*, Journal of Optimization Theory and its Applications **123** (2004), no. 1, 27–47.

[4] ______, *A coordinate transformation method for the extremization of multiple integrals*, Journal Optimization Theory and Applications **127** (2005), 523–533.

[5] ______, *The direct method for a class of infinite horizon dynamic games*, Optimal Control and Dynamic Games, Applications in Finance, Management Science and Economics (New York) (Christophe Deissenberg and Richard F. Hartl, eds.), Springer, 2005.

[6] ______, *A direct method for open-loop dynamic games for affine control systems*, Dynamic Games: Theory and Applications (New York) (Alain Haurie and Georges Zaccour, eds.), Springer, 2005, pp. 37–55.

[7] L. Cesari, *Optimization-theory and applications: Problems with ordinary differential equations*, Applications of Applied Mathematics, vol. 17, Springer-Verlag, New York, 1983.

[8] E.J. Dockner and G. Leitmann, *Coordinate transformation and derivation of open-loop Nash equilibrium*, Journal of Optimization Theory and Applications **110** (2001), no. 1, 1–16.

[9] M.I. Kamien and N. L. Schwartz, *Dynamic optimization: The calculus of variations and optimal control in economics and management sciences*, second ed., Advanced Textbooks in Economics, vol. 31, North-Holland, Amsterdam, 1991.

[10] G. Leitmann, *A note on absolute extrema of certain integrals*, International Journal of Non-Linear Mechanics **2** (1967), 55–59.

[11] ______, *On a class of direct optimization problems*, Journal Optimization Theory and Applications **108** (2001), no. 3, 467–481.

[12] ______, *A direct method of optimization and its applications to a class of differential games*, Dynamics of Continuous, Discrete and Impulsive Systems, Series A **11** (2004), 191–204.

[13] C. F. Roos, *A mathematical theory of competition*, American Journal of Mathematics **46** (1925), 163–175.

[14] ______, *Generalized Lagrange problems in the calculus of variations*, Transactions of the American Mathematical Society **30** (1928), no. 2, 360–384.

Springer
http://www.springer.com/978-0-8176-4833-6

Advances in Dynamic Games and Their Applications
Analytical and Numerical Developments
Bernhard, P.; Gaitsgory, V.; Pourtallier, O. (Eds.)
2009, XIV, 462 p. 94 illus., 9 illus. in color., Hardcover
ISBN: 978-0-8176-4833-6
A product of Birkhäuser Basel