Chapter 2

Closed Convex Processes

Introduction

Naturally, the first question which arises is: *What are the set-valued analogues of continuous linear operators?*

Since the graph of a continuous linear operator $A \in \mathcal{L}(X, Y)$ is a (closed) vector subspace of $X \times Y$, it is quite natural to regard set-valued maps, with closed convex cones as their graphs, as these set-valued analogues. Such set-valued maps are called *closed convex processes*[1] and the maps the graph of which are vector subspaces are called *linear processes*.

The main class of examples of closed convex processes is provided by derivatives of set-valued maps which are introduced and studied exhaustively in Chapter 5.

We shall prove that closed convex processes enjoy (almost) all properties of continuous linear operators, including Banach's Open Mapping and Closed Graph Theorems (Section 2) and the Uniform Boundedness Theorem (Section 3.)

As continuous linear operators, closed convex processes can be *transposed* and the Bipolar Theorem can be adapted to closed convex processes. They thus enjoy the benefits of a duality theory exposed in Section 5. For instance, a duality criterion of invariance by a

[1]The term "process" has been coined by R.T. Rockafellar for denoting maps the graph of which are cones in a study of economic "processes" (with constant return to scale.)

closed convex process is given in this section for linear processes and in Chapter 4 (Section 2) in the general case.

We shall also provide in Chapter 3 a theorem on existence of eigenvectors of closed convex processes (Theorem 3.6.2.)

For proving these results, we recall in Section 4 the Bipolar Theorem for continuous linear operators, the Closed Range Theorem and the properties of *support functions*. The latter allow characterization of closed convex subsets by an (infinite) family of linear inequalities thanks to a version of the Hahn-Banach Separation Theorem. It also enables one to deal with the class of upper semicontinuous convex positively homogeneous functions instead of handling closed convex subsets.

We conclude the chapter with a short section on upper *hemicontinuous set-valued maps*, characterized by the upper semicontinuity of the support functions of their values, a more familiar property. This characterization is mainly useful for set-valued maps with closed convex images.

2.1 Definitions

Let us introduce the set-valued analogues of continuous linear operators, which are the closed convex processes.

Definition 2.1.1 (Closed Convex Process) *Let $F : X \rightsquigarrow Y$ be a set-valued map from a normed space X to a normed space Y. We shall say that F is*

— convex *if its graph is convex*

— closed *if its graph is closed*

— *a* process *(or* positively homogeneous*) if its graph is a cone*

— *a* linear process *if its graph is a vector subspace.*

Hence a closed convex process *is a set-valued map whose graph is a closed convex cone.*

We shall see that most of the properties of continuous linear operators are enjoyed by closed convex processes.

Let us begin with the following obvious statements.

Lemma 2.1.2 *A set-valued map F is convex if and only if*

$$\begin{cases} \forall\, x_1,\, x_2 \in \mathrm{Dom}(F), \;\; \forall\, \lambda \in [0,1], \\ \lambda F(x_1) + (1-\lambda)F(x_2) \subset F(\lambda x_1 + (1-\lambda)x_2) \end{cases}$$

It is a process if and only if

$$\forall\, x \in X,\, \forall\, \lambda > 0,\; \lambda F(x) = F(\lambda x) \;\; \textit{and} \;\; 0 \in F(0)$$

and a convex process if and only if it is a process satisfying

$$\forall\, x_1,\, x_2 \in X,\;\; F(x_1) + F(x_2) \subset F(x_1 + x_2)$$

We observe that the domain and the image of a closed convex process are convex cones (not necessarily closed.)

The main examples of closed processes are provided by contingent derivatives of set-valued maps that we shall introduce in Chapter 5.

We associate with a closed convex process its *norm* defined in the following way.

Definition 2.1.3 (Norm of a Closed Convex Process) *Let $F : X \rightsquigarrow Y$ be a closed convex process. Its* norm *$\|F\|$ is equal to*

$$\begin{cases} \|F\| := \sup_{x \in \mathrm{Dom}(F)} d(0, F(x))/\|x\| \\ \quad = \sup_{x \in \mathrm{Dom}(F)} \inf_{v \in F(x)} \|v\|/\|x\| \\ \quad = \sup_{x \in \mathrm{Dom}(F) \cap B} \inf_{v \in F(x)} \|v\| \end{cases} \tag{2.1}$$

2.2 Open Mapping and Closed Graph Theorems

The Banach Open Mapping Theorem can be extended to closed convex processes:

Theorem 2.2.1 (Open Mapping) *Let X, Y be Banach spaces. Assume that a closed convex process $F : X \rightsquigarrow Y$ is* surjective *(in the sense that* $\mathrm{Im}(F) = Y$*). Then F^{-1} is Lipschitz:*

There exists a constant $l > 0$ such that, for all $x_1 \in F^{-1}(y_1)$ and for any $y_2 \in Y$, we can find a solution $x_2 \in F^{-1}(y_2)$ satisfying:

$$\|x_1 - x_2\| \;\leq\; l\|y_1 - y_2\|$$

Actually, this theorem holds true for closed convex maps:

Theorem 2.2.2 (Robinson-Ursescu) *Let X, Y be Banach spaces, $F : X \leadsto Y$ be a closed convex set-valued map. Suppose that y_0 belongs to the interior of the image of F and let $x_0 \in F^{-1}(y_0)$.*

Then there exist positive constants l and γ such that for any $y \in y_0 + \gamma B$, there exists a solution x to the inclusion $F(x) \ni y$ satisfying

$$\|x - x_0\| \;\leq\; l\|y - y_0\|$$

Proof — For simplicity, we prove this theorem only when the Banach space X is *reflexive*[2].

Let us introduce the function ρ defined by

$$\rho(y) \;:=\; \inf_{x \in F^{-1}(y)} \|x - x_0\| \;=\; d(x_0, F^{-1}(y)) \tag{2.2}$$

(It takes the value $+\infty$ when y does not belong to $\mathrm{Im}(F)$.)

Since the set-valued map F is convex, this function is obviously convex. Assume for a while it is lower semicontinuous.

Then, Baire's Theorem implies that ρ is continuous[3] on the interior of $\mathrm{Im}(F)$, which is not empty by assumption. Since continuous convex functions are locally Lipschitz, there exists a ball of radius $\gamma > 0$ centered at y_0 and a constant $l' > 0$ such that for all y in this ball,

$$\|\rho(y)\| \;=\; \|\rho(y) - \rho(y_0)\| \;\leq\; l'\|y - y_0\|$$

[2] See for instance [35, Theorem 3.3.1] for the nonreflexive case and the original papers of Robinson and Ursescu.

[3] Let us recall this result: The domain of ρ is the union of the sections

$$S_n \;:=\; \{y \mid \rho(y) \leq n\}$$

which are closed because ρ is lower semicontinuous. Since the interior of the domain of ρ is not empty, the interior of one of these sections is not empty. So, the convex function ρ being bounded on an open subset $\mathrm{Int}(S_n)$ for some n, it is continuous (and even, locally Lipschitz) on the interior of its domain.

because $\rho(y_0) = 0$. Therefore

$$d\left(x_0, F^{-1}(y)\right) \leq l'\|y - y_0\|$$

By setting $l = 2l'$ we end the proof. It remains to check that the function ρ is lower semicontinuous. $\square$

Lemma 2.2.3 *Let us consider a reflexive Banach space X, a Banach space Y, a closed convex process $F : X \rightsquigarrow Y$.*

Then the function ρ defined by (2.2) is lower semicontinuous.

Proof — It is enough to prove that nonempty sections

$$\{y \mid \rho(y) \leq \lambda\}$$

are closed. Let us consider a sequence of elements y_n of such a section converging to some y. Then, by reflexivity of X, there exists $x_n \in F^{-1}(y_n)$ satisfying $\|x_n - x_0\| = \rho(y_n) \leq \lambda$. Hence the elements x_n remain in the ball $x_0 + \lambda B$. Since X is reflexive, there exists a (weak) cluster point x of the sequence x_n. Then (x, y) is a weak cluster point of the sequence (x_n, y_n), which thus belongs to the graph of F because, being closed and convex, it is closed in $X \times Y$ when X is supplied with the weak topology and Y with the norm topology. Since the elements x_n belong to the ball $x_0 + \lambda B$, which is weakly closed, the cluster point x belongs also to this ball, so that

$$\rho(y) \leq \|x - x_0\| \leq \lambda \; \square$$

Proof of Theorem 2.2.1 — We take $x_0 = 0$, $y_0 = 0$ in Theorem 2.2.2. To say that 0 belongs to the interior of the cone $\mathrm{Im}(F)$ amounts to saying that $\mathrm{Im}(F)$ is equal to Y, i.e., that F is surjective. By Theorem 2.2.2 for a constant $l > 0$ we have:

$$\forall\, y \in Y, \exists\, x \in F^{-1}(y) \text{ such that } \|x\| \leq l\|y\|$$

Fix y_1, $y_2 \in Y$ and let us choose any solution $x_1 \in F^{-1}(y_1)$ and $e \in F^{-1}(y_2 - y_1)$ satisfying $\|e\| \leq l\|y_1 - y_2\|$. Then $x_2 := x_1 + e$ belongs to $F^{-1}(y_2)$, since F is a convex process and satisfies the estimate

$$\|x_1 - x_2\| = \|e\| \leq l\|y_1 - y_2\| \; \square$$

Corollary 2.2.4 *Let us consider Banach spaces X, Y, a continuous linear operator $A \in \mathcal{L}(X,Y)$ and a closed convex subset $K \subset X$. Let us assume that there exists $x_0 \in K$ such that $Ax_0 \in \mathrm{Int}(A(K))$.*

Then there exist positive constants l and γ such that for any $y \in A(x_0) + \gamma B$, there exists a solution $x \in K$ to the equation $Ax = y$ satisfying $\|x - x_0\| \leq l\|y - A(x_0)\|$.

Remark — Actually, Corollary 2.2.4 is equivalent to Theorem 2.2.2; we apply it with $K := \mathrm{Graph}(F)$ and $A := \pi_Y$. □

Since the restriction $F := A|_K$ of a continuous linear operator $A \in \mathcal{L}(X,Y)$ to a closed convex cone is a closed convex process, we obtain the following consequence:

Corollary 2.2.5 *Let us consider Banach spaces X, Y, a continuous linear operator $A \in \mathcal{L}(X,Y)$ and a closed convex cone $K \subset X$ such that $A(K) = Y$. Then the set-valued map $y \rightsquigarrow A^{-1}(y) \cap K$ is Lipschitz: there exists a positive constant l such that,*

$$\forall\, y_1,\, y_2 \,\in\, Y,\;\; A^{-1}(y_1) \cap K \;\subset\; A^{-1}(y_2) \cap K \;+\; l\|y_1 - y_2\|B$$

As in the case of continuous linear operators, the Open Mapping Theorem is equivalent to the Closed Graph Theorem, which can be stated as follows.

Theorem 2.2.6 (Closed Graph Theorem) *A closed convex process F from a Banach space X to another Y whose domain is the whole space is Lipschitz: there exists a (Lipschitz) constant $l > 0$ such that*

$$\forall\, x_1,\, x_2 \,\in X,\;\; F(x_1) \;\subset\; F(x_2) \;+\; l\|x_1 - x_2\|B \tag{2.3}$$

Thus, the norm of F is finite whenever $\mathrm{Dom}(F) = X$.

Proof — It is sufficient to apply the Open Mapping Theorem 2.2.1 to the closed convex process F^{-1}. □

2.3 Uniform Boundedness Theorem

We can now adapt the Uniform Boundedness Theorem to the case of closed convex processes.

Theorem 2.3.1 (Uniform Boundedness) *Let X and Y be Banach spaces and F_h be a family of closed convex processes from X to Y, "pointwise bounded" in the sense that*

$$\forall\, x \in X,\ \exists\, y_h \in F_h(x) \text{ such that } \sup_h \|y_h\| < +\infty \tag{2.4}$$

Then this family is "uniformly bounded" in the sense that

$$\sup_h \|F_h\| < +\infty$$

Hence we can speak of *bounded* families of closed convex processes, without specifying whether it is pointwise or uniform.

Proof — Let us consider the positively homogeneous convex lower semicontinuous functions ρ_h defined by

$$\rho_h(x) := \inf_{y \in F_h(x)} \|y\| = d(0, F_h(x))$$

(which are lower semicontinuous because the closed convex processes F_h are Lipschitz) and the function ρ defined by

$$\forall\, x \in X,\ \rho(x) := \sup_h \rho_h(x)$$

Assumption (2.4) implies that this function ρ is finite. Since it is also positively homogeneous, convex and lower semicontinuous (being the supremum of such functions), it is continuous at 0. Hence there exists a constant l such that $\sup_h d(0, F_h(x)) = \rho(x) \le l\|x\|$, i.e., $\|F_h\| \le l < \infty$. $\square$

The following consequence of Theorem 2.3.1 extends to closed convex processes the following useful convergence result.

Theorem 2.3.2 (Crossed Convergence) *Consider a metric space U, Banach spaces X, Y and a set-valued map associating to each*

$u \in U$ a closed convex process $F(u) : X \rightsquigarrow Y$. *Let us assume that the family of closed convex processes $F(u)$ is pointwise bounded.*

Then the following conditions are equivalent:

$$\left\{\begin{array}{ll} i) & \textit{the map } u \rightsquigarrow \mathrm{Graph}(F(u)) \textit{ is lower semicontinuous} \\ \\ ii) & \textit{the map } (u,x) \rightsquigarrow F(u)(x) \textit{ is lower semicontinuous} \end{array}\right.$$

Proof — For proving that *i)* implies *ii)*, let us consider a sequence of elements (u_n, x_n) converging to (u, x) and an element $y \in F(u)(x)$. We have to approximate it by elements $y_n \in F(u_n)(x_n)$.

Since $u \rightsquigarrow \mathrm{Graph}(F(u))$ is lower semicontinuous, we can approximate (x, y) by elements $(\widehat{x}_n, \widehat{y}_n) \in \mathrm{Graph}(F(u_n))$. By the pointwise boundedness assumption and Theorem 2.3.1, there exist $l > 0$ and solutions $f_n \in F(u_n)(x_n - \widehat{x}_n)$ satisfying

$$\|f_n\| \ \leq \ l\|x_n - \widehat{x}_n\|$$

The right hand side of the above inequality converges to zero when n goes to infinity. Because $F(u_n)$ is a convex process, the element $y_n := \widehat{y}_n + f_n$ does belong to $F(u_n)(x_n)$. Consequently, y_n converging to y, we deduce that the set-valued map $(u, x) \rightsquigarrow F(u)(x)$ is lower semicontinuous at (u, x).

The converse is obviously true even when the family $(F(u))_{u \in U}$ is unbounded. □

2.4 The Bipolar Theorem

Definition 2.4.1 *Let K be a nonempty subset of a Banach space X. We associate with any continuous linear form $p \in X^\star$*

$$\sigma_K(p) \ := \ \sigma(K, p) \ := \ \sup_{x \in K} < p, x > \in \ \mathrm{R} \cup \{+\infty\}$$

The function $\sigma_K : X^\star \mapsto \mathrm{R} \cup \{+\infty\}$ is called the support function of K. *Its domain is a convex cone called the* barrier cone *denoted by*

$$b(K) \ := \ \mathrm{Dom}(\sigma_K) \ := \ \{p \ \in \ X^\star \mid \sigma_K(p) < \infty\} \qquad (2.5)$$

We say that the subsets of $X^\star$ defined by

$$\left\{\begin{array}{lllll} i) & K^\circ & := & \{p \in X^\star \mid \sigma_K(p) \leq 1\} \\ ii) & K^- & := & \{p \in X^\star \mid \sigma_K(p) \leq 0\} \\ iii) & K^+ & := & -K^- \\ iv) & K^\perp & := & \{p \in X^\star \mid \forall x \in K, \ < p, x > = 0\} \end{array}\right.$$

are the polar set, (negative) polar cone, positive polar cone *and* orthogonal *of K respectively.*

When $L \subset X^\star$, we define the polar set $L^\circ \subset X$ as the subset of elements $x \in X$ (and not $X^{\star\star}$) satisfying $< p, x > \leq 1$ for all $p \in L$. The polar cone $L^- \subset X$ and the orthogonal $L^\perp \subset X$ of L are defined in the same way. The subsets

$$K^{\circ\circ} := (K^\circ)^\circ \subset X \quad \& \quad K^{--} := (K^-)^- \subset X$$

are called respectively the *bipolar set* and *bipolar cone* of a subset $K \subset X$ and the subspace $K^{\perp\perp} := (K^\perp)^\perp \subset X$ the *biorthogonal* of K.

It is clear that K° is a *closed convex subset containing* 0, that K^- is a *closed convex cone*, that $K^\perp$ is a *closed subspace* of $X^\star$ and that

$$K^\perp = K^- \cap K^+ \subset K^- \subset K^\circ \subset b(K)$$

Examples

- When $K = \{x\}$, then $\sigma_K(p) = < p, x >$
- When $K = B_X$, then $\sigma_{B_X}(p) = \|p\|_\star$
- If K is a *cone*, then

$$\sigma_K(p) = \left\{\begin{array}{lll} 0 & \text{if} & p \in K^- \\ +\infty & \text{if} & p \notin K^- \end{array}\right. \square$$

When $K = \emptyset$, we set $\sigma_\emptyset(p) = -\infty$ for every $p \in X^\star$.
The Separation Theorem can be stated in the following way:

Theorem 2.4.2 (Separation theorem) *Let K be a nonempty subset of a Banach space X. Its* closed convex hull *is characterized by linear constraint inequalities in the following way:*

$$\overline{co}(K) = \{x \in X \mid \forall p \in X^\star, \; < p, x > \le \sigma_K(p)\}$$

Furthermore, there is a bijective correspondance between *nonempty closed convex subsets of X* and *nontrivial lower semicontinuous positively homogeneous convex functions on $X^\star$.*

Remark — The Separation Theorem holds true not only in Banach spaces, but in any Hausdorff locally convex topological vector-space. In particular, we can use it when X is supplied with the weakened topology. The geometrical interpretation can be stated as follows: the closed convex hull of a nonempty subset is the intersection of all closed half-spaces containing it. □

We observe that a subset K is *bounded if and only if its support function is finite.*

We mention the following consequence, known as the *Bipolar theorem.*

Theorem 2.4.3 (Bipolar Theorem) *Let X, Y be Banach spaces and $K \subset X$. The* bipolar *cone K^{--} is the closed convex cone spanned by K.*

If $A \in \mathcal{L}(X,Y)$ is a continuous linear operator from X to Y and K is a subset of X, then

$$(A(K))^- = A^{\star^{-1}}(K^-)$$

where $A^\star$ denotes the transpose of A.

Thus the closed cone spanned by $A(K)$ is equal to $\left(A^{\star^{-1}}(K^-)\right)^-$

We provide now a simple criterion which implies that the image of a closed subset is closed.

Theorem 2.4.4 (Closed Range Theorem) *Let X be a Banach space, Y be a reflexive space, $K \subset X$ be a* weakly closed *subset and $A \in \mathcal{L}(X,Y)$ a continuous linear operator satisfying*

$$\mathrm{Im}(A^\star) + b(K) = X^\star \tag{2.6}$$

Then the image $A(K)$ is closed. In particular, if K is a closed convex cone *and if*

$$\mathrm{Im}(A^\star) + K^- = X^\star$$

then

$$A(K) = \left(A^{\star^{-1}}(K^-)\right)^-$$

Proof — Let us consider a sequence of elements $x_n \in K$ such that $A(x_n)$ converges to some y in Y. We shall check that this sequence is weakly bounded, and thus, weakly relatively compact. Let us take for that purpose any $p \in X^\star$, which can, by assumption (2.6), be written $p := A^\star q + r$, where $q \in Y^\star$, $r \in b(K)$. Therefore,

$$\begin{cases} \sup_n < p, x_n > = \sup_n(< q, Ax_n > + < r, x_n >) \\ \leq \sup_n(\|q\|\|Ax_n\| + \sigma_K(r)) < +\infty \end{cases}$$

since the converging sequence $(Ax_n)_{n\in\mathbf{N}}$ is bounded.

Therefore the sequence $(x_n)_{n\in\mathbf{N}}$ has a weak cluster point x which belongs to K since it is weakly closed. □

Remark — Actually, arguments of the above proof imply that, under assumptions of the Closed Range Theorem, every sequence $x_n \in K$ such that $A(x_n)$ is weakly converging, has a weak cluster point □

For the convenience of the reader, we list below some useful calculus of support functions and barrier cones[4].

[4]See [35, Chapter 3] for instance.

Table 2.1: **Properties of Support Functions.**

(1)	▷	If $K \subset L \subset X$, then $b(L) \subset b(K)$ and $\sigma_K \leq \sigma_L$
(2)	▷	If $K_i \subset X, \quad i \in I$, then $b(\overline{co}(\bigcup_{i \in I} K_i)) \subset \bigcap_{i \in I} b(K_i)$ $\sigma(\overline{co}(\bigcup_{i \in I} K_i), p) = \sup_{i \in I} \sigma_{K_i}(p)$
(3)	▷	If $K_i \subset X_i, \quad (i = 1, \ldots, n)$, then $b(\prod_{i=1}^n K_i) = \prod_{i=1}^n b(K_i)$ $\sigma(\prod_{i=1}^n K_i, (p_1, \ldots, p_n)) = \sum_{i=1}^n \sigma_{K_i}(p_i)$
(4)a)	▷	If $A \in \mathcal{L}(X, Y)$, then $b(\overline{A(K)}) = A^{\star^{-1}} b(K)$ $\sigma_{\overline{A(K)}}(p) = \sigma_K(A^\star p)$
(4)b)	▷	If K_1 and K_2 are contained in X, then $b(K_1 + K_2) = b(K_1) \cap b(K_2)$ $\sigma_{K_1 + K_2}(p) = \sigma_{K_1}(p) + \sigma_{K_2}(p)$ In particular, if $K \subset X$ and P is a cone, then $b(K + P) = b(K) \cap P^-$ and $\sigma_{K+P}(p) = \sigma_K(p)$ if $p \in P^-$ and $+\infty$ if not
(5)	▷	If $L \subset X$ and $M \subset Y$ are *closed convex* subsets and $A \in \mathcal{L}(X, Y)$ is a continuous linear operator such that the following *constraint qualification condition* $0 \in \mathrm{Int}(M - A(L))$ holds true, then $b(L \cap A^{-1}(M)) = b(L) + A^\star b(M)$ and $\forall\, p \in b(L \cap A^{-1}(M))$, $\exists\, \overline{q} \in Y^\star$ such that $\sigma_{L \cap A^{-1}(M)}(p) = \sigma_L(p - A^\star \overline{q}) + \sigma_M(\overline{q})$ $= \inf_{q \in Y^\star}(\sigma_L(p - A^\star q) + \sigma_M(q))$
(5)a)	▷	If $M \subset Y$ is a *closed convex* subset and if $A \in \mathcal{L}(X, Y)$ is a continuous linear operator such that $0 \in \mathrm{Int}(\mathrm{Im}(A) - M)$, then $b(A^{-1}(M)) = A^\star b(M)$ and, for every $p \in b(A^{-1}(M))$, there exists $\overline{q} \in b(M)$ such that $\sigma_{A^{-1}(M)}(p) = \sigma_M(\overline{q}) = \inf_{A^\star q = p}(\sigma_M(q))$
(5)b)	▷	If K_1 and K_2 are closed convex subsets of X such that $0 \in \mathrm{Int}(K_1 - K_2)$, then $b(K_1 \cap K_2) = b(K_1) + b(K_2)$ and $\forall\, p \in b(K_1 \cap K_2)$, $\exists\, \overline{q}_i \in X^\star, (i = 1, 2)$ such that $\sigma_{K_1 \cap K_2}(p) = \sigma_{K_1}(\overline{q}_1) + \sigma_{K_2}(\overline{q}_2)$ $= \inf_{p = p_1 + p_2}(\sigma_{K_1}(p_1) + \sigma_{K_2}(p_2))$

2.5 Transposition of Closed Convex Process

Definition 2.5.1 (Transpose of a Process) *Let X, Y be Banach spaces, $F : X \rightsquigarrow Y$ be a process. Its* left-transpose *(in short, its* transpose*) $F^\star$ is the closed convex process from $Y^\star$ to $X^\star$ defined by*

$$p \in F^\star(q) \iff \forall\, x \in X,\ \forall\, y \in F(x),\ \ < p, x > \ \leq\ < q, y > \tag{2.7}$$

In particular, the transpose $F^\star$ of a linear process F is defined by

$$p \in F^\star(q) \iff \forall\, x \in X, \forall\, y \in F(x),\ \ < p, x > = < q, y >$$

The graph of the transpose $F^\star$ of F is related to the polar cone of the graph of F in the following way:

Lemma 2.5.2 (Graph of the Transpose) *Let us consider Banach spaces X, Y and let $F : X \rightsquigarrow Y$ be a process. Then*

$$(q, p) \in \mathrm{Graph}(F^\star) \iff (p, -q) \in (\mathrm{Graph}(F))^-$$

In the case of linear processes, we observe that $p \in F^\star(q)$ if and only if $(p, -q)$ belongs to $\mathrm{Graph}(F)^\perp$ and we see at once that the *bitranspose* of a closed linear process F coincides with F.

The definition of a bitranspose of a convex process is not symmetric: If $G : Y^\star \rightsquigarrow X^\star$ is a convex process, we define its transpose $G^\star : X \rightsquigarrow Y$ by the formula

$$(-y, x) \in (\mathrm{Graph}(G))^-$$

(instead of the formula $(y, -x) \in (\mathrm{Graph}(G))^-$ obtained by exchanging the roles of X and $Y^\star$, Y and $X^\star$ respectively.)

With this definition, *the bitranspose of a closed convex process F coincides with F.*

We provide now a formula for transposing the product of closed convex processes.

Theorem 2.5.3 (Transpose of a Product) *Let W, X, Y, Z be Banach spaces, F be a closed convex process from X to Y, $A \in \mathcal{L}(W, X)$ and $B \in \mathcal{L}(Y, Z)$ be continuous linear operators. Assume that*

$$\mathrm{Im}(A) - \mathrm{Dom}(F) = X \tag{2.8}$$

Then the transpose of BFA is equal to:

$$(BFA)^\star = A^\star F^\star B^\star$$

Proof — First, we prove that the formula

$$(BF)^\star = F^\star B^\star$$

always holds true, since the graph of BF is equal to $(\mathbf{1}\times B)\mathrm{Graph}(F)$. Consequently, thanks to the Bipolar Theorem 2.4.3

$$((\mathbf{1}\times B)\mathrm{Graph}(F))^- = (\mathbf{1}\times B)^{\star^{-1}}(\mathrm{Graph}(F)^-)^-$$

so that $(p, -q)$ belongs to $\mathrm{Graph}(BF)^-$ if and only if $(p, -B^\star q)$ belongs to $\mathrm{Graph}(F)^-$, i.e., if and only if p belongs to $F^\star(B^\star q)$.

The graph of FA being equal to $(A\times\mathbf{1})^{-1}\mathrm{Graph}(F)$, we need to assume the constraint qualification property

$$\mathrm{Im}(A\times\mathbf{1}) - \mathrm{Graph}(F) = X\times Y \tag{2.9}$$

for deducing from the properties of polar cones that

$$(\mathrm{Graph}(FA))^- = (A^\star\times\mathbf{1})(\mathrm{Graph}(F)^-)$$

If this is the case, we infer that $r \in (FA)^\star(q)$ if and only if there exists $p \in F^\star(q)$ such that $r = A^\star p$.

It remains now to check that assumption (2.8) implies the constraint qualification property (2.9.)

Indeed, let (x, y) belong to $X\times Y$. Since x can be expressed as $x = Ax_0 - x_1$ where x_0 belongs to W and x_1 to the domain of F, we can write

$$(x, y) = (Ax_0, y_0) - (x_1, y_1)$$

where $y_1 \in F(x_1)$ and $y_0 = y + y_1$. Hence (x, y) belongs to $\mathrm{Im}(A\times\mathbf{1}) - \mathrm{Graph}(F)$.

We then deduce from the above proof the conclusion of the theorem. □

Corollary 2.5.4 *Let X, Y and Z be Banach spaces, F a closed convex process from X to Z, G a closed convex process from Y to Z and $A \in \mathcal{L}(X,Y)$. If*

$$A(\mathrm{Dom}(F)) - \mathrm{Dom}(G) \;=\; Y \tag{2.10}$$

then $(F + GA)^\star = F^\star + A^\star G^\star$.

Proof — We set

$$H(x) \;:=\; F(x) + G(Ax), \quad B(y,z) \;:=\; y + z$$

so that the set-valued map H can be written $H = B(F \times G)(\mathbf{1} \times A)$. Since $\mathrm{Dom}(F \times G)$ is equal to $\mathrm{Dom}(F)\times$ $\mathrm{Dom}(G)$, assumption (2.10) implies that

$$\mathrm{Im}(\mathbf{1} \times A) - \mathrm{Dom}(F \times G) \;=\; X \times Y$$

Therefore from Theorem 2.5.3 follows that $H^\star = (\mathbf{1}\times A)^\star (F \times G)^\star B^\star$. Since

$$(F \times G)^\star \;=\; F^\star \times G^\star \;\;\&\;\; (\mathbf{1} \times A)^\star \;=\; \mathbf{1} + A^\star$$

we infer that $H^\star = F^\star + A^\star G^\star$. □

Corollary 2.5.5 (Transpose of the Restriction) *Let X, Y be Banach spaces, $F : X \rightsquigarrow Y$ be a closed convex process and $K \subset X$ be a closed convex cone. Assume that*

$$K - \mathrm{Dom}(F) \;=\; X$$

Then the transpose of the restriction $F|_K$ of F to K is given by

$$(F|_K)^\star(q) \;=\; \begin{cases} F^\star(q) \;+\; K^- & \text{if } \; q \in \mathrm{Dom}(F^\star) \\ \emptyset & \text{otherwise} \end{cases}$$

Proof — We apply Corollary 2.5.4 with $A = \mathbf{1}$ and G defined by

$$G(x) \;=\; \begin{cases} \{0\} & \text{if } x \in K \\ \emptyset & \text{otherwise} \end{cases}$$

whose domain is K and whose transpose is the constant set-valued map defined by $G^\star(q) = K^-$. □

We shall adapt to the case of closed convex processes the Bipolar Theorem 2.4.3. To this end, we begin by stating the following simple result.

Proposition 2.5.6 *Let X, Y be Banach spaces and $F : X \leadsto Y$ be a process. Then*

$$(\mathrm{Im}(F))^- \;=\; -F^{\star^{-1}}(0) \quad \& \quad F(0) \;=\; (\mathrm{Dom}(F^\star))^+$$

Therefore, if F is a convex process, the image of F is dense if and only if the kernel $F^{\star^{-1}}(0)$ of its transpose is equal to 0.

Furthermore a convex process F is surjective if $F^{\star^{-1}}(0) = \{0\}$ and either the dimension of Y is finite or the image of F is closed.

Proof — To say that q belongs to the polar cone of the image of F amounts to saying that the pair $(0, q)$ belongs to the polar cone of the graph of F, i.e., that $(-q, 0)$ belongs to the graph of the transpose $F^\star$, in other words, that 0 belongs to $F^\star(-q)$. The proof of the second statement is naturally analogous. □

The extension of the Bipolar Theorem 2.4.3 to closed convex processes is then a consequence of Corollary 2.5.5 and Proposition 2.5.6.

Theorem 2.5.7 (Bipolar Theorem) *Consider Banach spaces X, Y and let $F : X \leadsto Y$ be a closed convex process, and $K \subset X$ be a cone satisfying* $\mathrm{Dom}(F) - K = X$. *Then*

$$(F(K))^- \;=\; -F^{\star^{-1}}(K^+)$$

Proof — We apply Proposition 2.5.6 to the restriction $F|_K$ whose image is $F(K)$ and whose transpose is $F^\star(\cdot) + K^-$ thanks to Corollary 2.5.5. □

The above condition is obviously satisfied when the domain of F is the whole space. In this case, we obtain

Corollary 2.5.8 *Let $F : X \leadsto Y$ be a strict closed convex process. Then*

$$\mathrm{Dom}(F^\star) \;=\; F(0)^+$$

and $F^\star$ is upper hemicontinuous (see Definition 2.6.2 below) with bounded closed convex images, mapping the unit ball to the ball of radius $\|F\|$. In particular, $F^\star(0) = \{0\}$.

The restriction of $F^\star$ to the vector space

$$\mathrm{Dom}(F^\star) \cap (\mathrm{Dom}(F^\star))$$

is single-valued and linear.

Proof — We observe that

$$\forall\, q \in \mathrm{Dom}(F^\star), \quad \sup_{p \in F^\star(q)} \|p\| \ \leq\ \|F\| \|q\|$$

because for all $x \in \mathrm{Dom}(F) = X$ and for all $p \in F^\star(q)$, we have, by definitions of the transpose and the norm of a closed convex process

$$\begin{cases} \|p\| & := \ \sup_{x \in B} < p, x > \leq \ \sup_{x \in B} \inf_{y \in F(x)} < q, y > \\ & \leq \ \sup_{x \in B} \inf_{y \in F(x)} \|q\| \|y\| \ = \ \|F\| \|q\| \end{cases}$$

Then $F^\star$ maps bounded sets to bounded sets. Therefore, the cone $F^\star(0)$ being bounded, is equal to $\{0\}$. Since the domain of F is the whole space, the assumptions of Proposition 2.6.4 below are met, so that for all $x \in X$, the function $q \mapsto \sigma(F^\star(q), x)$ is upper semicontinuous.

The domain of $F^\star$ is closed, thanks to the Closed Range Theorem 2.4.4 applied to the projection $\pi_{Y^\star}$ from $X^\star \times Y^\star$ and the cone $\mathrm{Graph}(F^\star)$. Then

$$\mathrm{Dom}(F^\star) \ = \ \pi_{Y^\star}\,(\mathrm{Graph}(F^\star))$$

is closed because, the domain of F being equal to X,

$$\mathrm{Im}\,((\pi_{Y^\star})^\star) \ - \ (\mathrm{Graph}(F^\star))^- \ = X \times Y$$

This and Proposition 2.5.6 imply that $\mathrm{Dom}(F^\star) = F(0)^+$.

If q belongs to both $\mathrm{Dom}(F^\star)$ and $-\mathrm{Dom}(F^\star)$, then

$$F^\star(q) + F^\star(-q) \ \subset \ F^\star(0) \ = \ \{0\}$$

Hence $F^\star(q)$ contains only one element and $F^\star(q) = -F^\star(-q)$. □

Example: Case of linear processes

When $F : X \leadsto Y$ is a linear process from a Banach space X to another Y, then, from the very definition of the adjoint, it follows that $F^\star$ is also a linear process from $Y^\star$ into $X^\star$. Hence its domain is a subspace of $Y^\star$ and, by Proposition 2.5.6, $\overline{\mathrm{Dom}(F^\star)} = F(0)^\perp$.

In general the space $\mathrm{Dom}(F^\star)$ is not closed. However this is always the case when Y has a finite dimension.

Furthermore Corollary 2.5.8 implies that when F is strict, then

$$\mathrm{Dom}(F^\star) \; = \; F(0)^\perp$$

and $F^\star$ is a linear operator from the subspace $F(0)^\perp$ into $X^\star$.

When $X = Y$, a closed subspace $P \subset \mathrm{Dom}(F)$ is called *invariant* under F if $F(P) \subset P$. We have the following

Proposition 2.5.9 *Assume that* $\mathrm{Dom}(F^\star)$ *is closed. If a closed subspace P is invariant under F, then its orthogonal space $P^\perp$ is invariant under $F^\star$.*

Consequently, if both $\mathrm{Dom}(F)$ *and* $\mathrm{Dom}(F^\star)$ *are closed, then a closed subspace P is invariant under F, if and only if its orthogonal space $P^\perp$ is invariant under $F^\star$.*

Proof — Let $P \subset \mathrm{Dom}(F)$ be an invariant closed subspace. Then $F(0) \subset P$ and therefore

$$P^\perp \; \subset \; F(0)^\perp \; = \; \mathrm{Dom}(F^\star)^{\perp\perp} \; = \; \mathrm{Dom}(F^\star)$$

Fixing $q \in P^\perp$ and $p \in F^\star(q)$, for every $x \in P$, $y \in F(x) \subset P$ we have

$$< p, x > = < q, y > = \; 0$$

Thus $p \in P^\perp$ and we have proved that if a subspace P is invariant under F, then its orthogonal $P^\perp$ is invariant under the adjoint process $F^\star$. Since $F = F^{\star\star}$ the last statement follows. □

For every $x \in X$ define recursively

$$F^1(x) \; = \; F(x) \text{ and for every integer } n \geq 1, \; F^{n+1}(x) \; = \; F(F^n(x))$$

Then for every n, $F^n(0)$ is a subspace of X and

$$F^n(0) \; \subset \; F^{n+1}(0)$$

Consider the subspace

$$Q \; = \; \bigcup_{n \geq 1} F^n(0)$$

Clearly $F(Q) \subset Q$. If F is strict, then it is Lipschitz on X. Thus $F(\overline{Q}) \subset \overline{Q}$ and therefore in this case $\overline{Q}$ is the smallest closed subspace of X invariant under F. We also have the following implication

$$F^k(0) = F^{k+1}(0) \implies \forall\, n \in \mathbf{N},\ F^k(0) = F^{k+n}(0) \qquad (2.11)$$

In particular this yields that when $X = \mathbf{R}^n$, then $Q = F^n(0)$. Indeed in this case $\{F^k(0)\}_{k\geq 1}$ is a nondecreasing sequence of subspaces of $\mathbf{R}^n$ satisfying (2.11.) The dimension of $\mathbf{R}^n$ being equal to n, the last claim follows.

Thus $F^n(0)$ is the smallest subspace of $\mathbf{R}^n$ invariant under F. □

The sum of two closed convex processes or the product BF is not necessarily closed. We have to provide sufficient conditions implying that they are still closed.

Proposition 2.5.10 (Closed Graph Criterion) *Let X, Y and Z be reflexive Banach spaces, $F : X \rightsquigarrow Y$ and $G : X \rightsquigarrow Z$ be closed convex processes and $B \in \mathcal{L}(Y, Z)$ be a continuous linear operator. If*

$$B^\star(\mathrm{Dom}(G^\star)) - \mathrm{Dom}(F^\star) = Y^\star$$

then the convex process $BF + G$ is closed.

Proof — This is a consequence of the Closed Range Theorem 2.4.4 with $K := \mathrm{Graph}(F \times G)$ and the continuous linear operator A defined by $A(x, y, z) := (x, By + z)$. □

Proposition 2.5.11 *Let X, Y be Banach spaces, $G : X \rightsquigarrow Y$ be a closed convex process and $P \subset X$ and $Q \subset Y$ be closed convex cones. Let us consider the convex process F defined by*

$$F(x) := \begin{cases} G(x) + Q & \text{if } x \in P \\ \emptyset & \text{if } x \notin P \end{cases}$$

It is closed when we suppose that

$$\mathrm{Dom}(G^\star) + Q^- = Y^\star$$

If we assume that $\mathrm{Dom}(G) - P = X$, *then its transpose is given by*

$$F^\star(q) := \begin{cases} G^\star(q) + P^- & \text{if } q \in Q^+ \\ \emptyset & \text{if } q \notin Q^+ \end{cases}$$

Proof — We observe that F is the sum of the closed convex process G and the closed convex process H defined by $H(x) := Q$ if $x \in P$ and $\emptyset$ if $x \notin P$, whose domain is P and whose transpose is defined by $H^\star(q) := P^-$ if $q \in Q^+$ and $\emptyset$ if not. Corollary 2.5.4 ends the proof. □

2.6 Upper Hemicontinuous Maps

We associate with a set-valued map F from a metric space X to a normed space Y the family of functions

$$x \mapsto \sigma(F(x), p) := \sup_{y \in F(x)} < p, y >$$

indexed by the continuous linear functionals $p \in Y^\star$.

We observe that

Corollary 2.6.1 *If a set-valued map F from a metric space X to a normed space Y is weakly upper semicontinuous and has compact values* (resp. *lower semicontinuous*), *then the function*

$$(x, q) \in X \times Y^\star \mapsto \sigma(F(x), q)$$

is upper semicontinuous (resp. *lower semicontinuous*).

Therefore, it is quite convenient to introduce the following definition.

Definition 2.6.2 (Upper Hemicontinuous Map) *We shall say that a set-valued map $F : X \rightsquigarrow Y$ is* upper hemicontinuous *at $x_0 \in$* Dom(F) *if and only if for any $p \in Y^\star$, the function $x \mapsto \sigma(F(x), p)$ is upper semicontinuous at x_0.*

It is said to be upper hemicontinuous if and only if it is upper hemicontinuous at every point of Dom(F).

Proposition 2.6.3 *The graph of an upper hemicontinuous set-valued map with* closed convex *values is* closed.

Proof — Let us consider elements $(x_n, y_n) \in \text{Graph}(F)$ converging to a pair (x, y). Then, for every $p \in Y^\star$,

$$< p, y > = \lim_{n \to \infty} < p, y_n > \leq \limsup_{n \to \infty} \sigma(F(x_n), p) \leq \sigma(F(x), p)$$

by the upper semicontinuity of $x \mapsto \sigma(F(x), p)$. This inequality implies that $y \in F(x)$ since these subsets are closed and convex, thanks to the Separation Theorem 2.4.2. We have shown that (x, y) belongs to Graph(F), which ends the proof. □

It is useful to compare the support functions of $F(x)$ and $F^\star(q)$.

Proposition 2.6.4 (Support Function of the Transpose) *Let X, Y be Banach spaces and $F : X \rightsquigarrow Y$ be a closed convex process. Then for every $x \in X$ and $q \in Y^\star$*

$$\sigma(F^\star(q), x) \; + \; \sigma(F(x), -q) \; \leq \; 0$$

Furthermore for every x_0 in the interior of the domain of F there exists $p_0 \in F^\star(q_0)$ such that $< p_0, x_0 >$ is equal to the common value

$$\sigma(F^\star(q_0), x_0) \; = \; -\sigma(F(x_0), -q_0) \tag{2.12}$$

In the same way, if q_0 belongs to the interior of the domain of $F^\star$, there exists $y_0 \in F(x_0)$ such that $< q_0, y_0 >$ is equal to the common value (2.12.)

Therefore for every $x \in Dom(F)$, the function $q \mapsto \sigma(F^\star(q), x)$ is upper semicontinuous on the interior of the domain of $F^\star$ and for every $q \in Dom(F^\star)$ the function $x \mapsto \sigma(F(x), -q)$ is upper semicontinuous on the interior of the domain of F.

Proof — The first claim follows from the very definition of the support function.

Denoting by π_X the projector from $X \times Y$ to X, we can write

$$\sigma(F(x_0), q_0) \; = \; \sigma(\mathrm{Graph}(F) \cap \pi_X^{-1}(x_0), (0, q_0))$$

We apply the formula (5) of Table 2.1 on the support functions of an intersection and inverse image with $L := \mathrm{Graph}(F)$, $M := \{x_0\}$ and $A := \pi_X$. The constraint qualification assumption is satisfied because 0 belongs to the interior of

$$\pi_X(\mathrm{Graph}(F)) \; - \; x_0 \; = \; \mathrm{Dom}(F) \; - \; x_0$$

by assumption. Then

$$\sigma(F(x_0), q_0) \; = \; \inf_{p \in X^\star} (< p, x_0 > + \sigma(\mathrm{Graph}(F), (0, q_0) - \pi_X^\star(p)))$$

We observe that $\pi_X^\star(p) = (p, 0)$ and that

$$\sigma(\text{Graph}(F), (-p, q_0)) \;=\; 0$$

if and only if $p \in F^\star(q_0)$, so that (2.12) ensues. Since the function $-\sigma(F(x_0), \cdot)$ is upper semicontinuous, the proof of the last claim ensues. $\square$

Springer
http://www.springer.com/978-0-8176-4847-3

Set-Valued Analysis
Aubin, J.-P.; Frankowska, H.
2009, XXI, 461 p., Softcover
ISBN: 978-0-8176-4847-3
A product of Birkhäuser Basel