Chapter 2
Basic Predictor Feedback

In this chapter we introduce the basic idea of a PDE backstepping design for systems with input delay. We treat the input delay as a transport PDE, an elementary first-order hyperbolic PDE. Our design yields a classical control formula obtained through various other approaches—modified Smith predictor (mSP), finite spectrum assignment (FSA), and the Artstein–Kwon–Pierson "reduction" approach. The backstepping approach is distinct because it provides a construction of an infinite-dimensional transformation of the actuator state, which yields a cascade system of transformed stable actuator dynamics and stabilized plant dynamics. Our design results in the construction of an explicit Lyapunov–Krasovskii functional and an explicit exponential stability estimate.

The basic ideas introduced in this chapter are the core for all the developments in the rest of the book. They are made parameter-adaptive, when the delay is unknown, in Part II, extended to nonlinear plants in Part III, and extended to PDE plants in Part IV. They are also converted to solve dual problems, such as observer design in the presence of sensor delay in Chapter 3.

We do not deal with the original "Smith predictor" (SP) [201] in detail in this chapter, as it is a rather different tool than the mSP/FSA/reduction approach. While these approaches are inspired mainly by full-state feedback considerations (though they do extend to output feedback problems), the original Smith predictor is a frequency-domain idea, inspired by different considerations than the ones we pursue here. However, in Sections 3.4 and 3.5 we present a side discussion that connects an observer-based predictor feedback design for systems with input delay with the classical Smith predictor.

We start with a basic idea of predictor feedback in Section 2.1. We then introduce a backstepping-based predictor design in Section 2.2 and explain in Section 2.3 that it results in the same controller as the mSP/FSA/reduction approaches, but with an additional benefit of providing a Lyapunov function. The predictor feedback design is illustrated with examples in Section 2.5. The heart of this chapter is stability analysis, which is presented in Section 2.4, with the aid of a backstepping-based Lyapunov function, and in Section 2.6 without using a Lyapunov function.

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2.1 Basic Idea of Predictor Feedback Design for ODE Systems with Actuator Delay

We consider the linear infinite-dimensional system

$$\dot{X}(t) = AX(t) + BU(t-D), \tag{2.1}$$

where $X \in \mathbb{R}^n$, (A,B) is a controllable pair, and the input signal $U(t)$ is delayed by D units of time.

Given a stabilizing gain vector K for the undelayed system, namely, given a vector K such that the matrix $A+BK$ is Hurwitz, our wish is to have a control that achieves

$$U(t-D) = KX(t). \tag{2.2}$$

This control can be alternatively written as

$$U(t) = KX(t+D), \tag{2.3}$$

and it appears to be nonimplementable since it requires future values of the state. However, with the variation-of-constants formula, treating the current state $X(t)$ as the initial condition, we have

$$X(t+D) = \mathrm{e}^{AD}X(t) + \int_{t-D}^{t} \mathrm{e}^{A(t-\theta)}BU(\theta)d\theta, \quad \forall t \geq 0. \tag{2.4}$$

This yields a feedback law

$$U(t) = K\left[\mathrm{e}^{AD}X(t) + \int_{t-D}^{t} \mathrm{e}^{A(t-\theta)}BU(\theta)d\theta\right], \quad \forall t \geq 0, \tag{2.5}$$

which is implementable, but it is infinite-dimensional, since it contains the distributed delay term involving past controls, $\int_{t-D}^{t} \mathrm{e}^{A(t-\theta)}BU(\theta)d\theta$. The closed-loop system is delay-compensated,

$$\dot{X}(t) = (A+BK)X(t), \quad t \geq D, \tag{2.6}$$

but this is true only after the control "kicks in" at $t = D$. During the interval $t \in [0,D]$, the system state is governed by

$$X(t) = \mathrm{e}^{At}X(0) + \int_{0}^{t} \mathrm{e}^{A(t-\tau)}BU(\tau - D)d\tau, \quad \forall t \in [0,D]. \tag{2.7}$$

The feedback law (2.5) was introduced within the framework of "finite spectrum assignment" [121, 135] and the "reduction approach" [8]. In the next section we derive the same control law, but in a considerably more complicated way, which will pay dividends later on by providing us with an explicit Lyapunov–Krasovskii function and the ability to conduct stability analysis in the time domain.

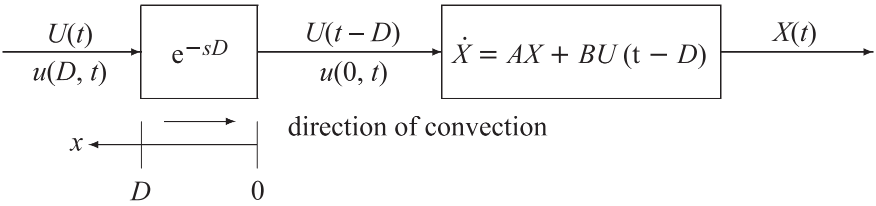

Fig. 2.1 Linear system $\dot{X}(t) = AX(t) + BU(t - D)$ with actuator delay D.

2.2 Backstepping Design Via the Transport PDE

The delay in the system (2.1) can be modeled by the following first-order hyperbolic PDE, also referred to as the "transport PDE":

$$u_t(x,t) = u_x(x,t), \tag{2.8}$$

$$u(D,t) = U(t). \tag{2.9}$$

The solution to this equation is

$$u(x,t) = U(t + x - D), \tag{2.10}$$

and therefore the output

$$u(0,t) = U(t - D) \tag{2.11}$$

gives the delayed input. The system (2.1) can now be written as

$$\dot{X}(t) = AX(t) + Bu(0,t). \tag{2.12}$$

Equations (2.8)–(2.12) form an ODE–PDE cascade that is driven by the input U from the boundary of the PDE (Fig. 2.1).

Suppose a static state feedback control has been designed for a system with no delay (i.e., with $D = 0$) such that

$$U(t) = KX(t) \tag{2.13}$$

is a stabilizing controller; i.e., the matrix $(A + BK)$ is Hurwitz. Consider the backstepping transformation

$$w(x,t) = u(x,t) - \int_0^x q(x,y)u(y,t)dy - \gamma(x)^T X(t) \tag{2.14}$$

with which we want to map the system (2.8)–(2.12) into the target system

$$\dot{X}(t) = (A + BK)X(t) + Bw(0,t), \tag{2.15}$$

$$w_t(x,t) = w_x(x,t), \tag{2.16}$$
$$w(D,t) = 0\,. \tag{2.17}$$

The reason for selecting the transformation (2.14) is the following. The ODE-PDE system

$$\dot{X}(t) = AX(t) + Bu(0,t), \tag{2.18}$$
$$u_t(x,t) = u_x(x,t), \tag{2.19}$$
$$u(D,t) = U(t) \tag{2.20}$$

has a block-lower-triangular structure, where the key "off-diagonal" component is the potentially unstable plant dynamics $AX(t)$. The transformation (2.14) is selected also to have a lower-triangular part. This transformation is to be understood as a part of the complete 2×2 transformation

$$(X,u) \mapsto (X,w), \tag{2.21}$$

which has a lower-triangular form

$$\begin{bmatrix} X \\ w \end{bmatrix} = \begin{bmatrix} I_{n\times n} & 0_{n\times[0,D]} \\ \Gamma & \mathscr{Q} + I_{[0,D]\times[0,D]} \end{bmatrix} \begin{bmatrix} X \\ u \end{bmatrix}, \tag{2.22}$$

where $I_{n\times n}$ denotes the identity matrix, $I_{[0,D]\times[0,D]}$ denotes the identity operator on the functions $u(x,t)$ of the argument $x \in [0,D]$, the symbol Γ denotes the operator

$$\Gamma:\ X(t) \mapsto \gamma(x)^T X(t), \tag{2.23}$$

and the symbol $\mathscr{Q}$ denotes the Volterra operator

$$\mathscr{Q}:\ u(x,t) \mapsto \int_0^x q(x,y)u(y,t)dy. \tag{2.24}$$

So, due to the lower-triangularity of $\mathscr{Q}$, the overall transformation $(X,u) \mapsto (X,w)$ is lower-triangular. Furthermore, the diagonal of this transformation is the identity operator,

$$\mathrm{Id} = \mathrm{diag}\left\{I_{n\times n},\ I_{[0,D]\times[0,D]}\right\}. \tag{2.25}$$

Due to the triangular structure, the transformation (2.22) is not only suitable for converting the system (2.18)–(2.20) into the target form (2.15)–(2.17)—with the help of an appropriate boundary feedback law—but is also invertible, as we shall see soon.

Let us now calculate the time and spatial derivatives of the transformation (2.14):

$$\begin{aligned} w_x(x,t) = {} & u_x(x,t) - q(x,x)u(x,t) - \int_0^x q_x(x,y)u(y,t)dy \\ & - \gamma'(x)^T X(t), \end{aligned} \tag{2.26}$$

$$
\begin{aligned}
w_t(x,t) &= u_t(x,t) - \int_0^x q(x,y)u_t(y,t)dy \\
&\quad - \gamma(x)^T \left[AX + Bu(0)\right] &(2.27)\\
&= u_x(x,t) - q(x,x)u(x,t) + q(x,0)u(0,t) \\
&\quad + \int_0^x q_y(x,y)u(y,t)dy - \gamma(x)^T \left[AX + Bu(0,t)\right]. &(2.28)
\end{aligned}
$$

Subtracting (2.26) from (2.28), we get

$$
\begin{aligned}
&\int_0^x (q_x(x,y) + q_y(x,y))u(y,t)dy \\
&\quad + \left[q(x,0) - \gamma(x)^T B\right] u(0,t) \\
&\quad + \left[\gamma'(x)^T - \gamma(x)^T A\right] X(t) = 0. &(2.29)
\end{aligned}
$$

This equation should be valid for all u and X, so we have three conditions:

$$q_x(x,y) + q_y(x,y) = 0, \tag{2.30}$$

$$q(x,0) = \gamma(x)^T B, \tag{2.31}$$

$$\gamma'(x) = A^T \gamma(x). \tag{2.32}$$

The first two conditions form a first-order hyperbolic PDE and the third one is a simple ODE. To find the initial condition for this ODE, let us set $x = 0$ in (2.14), which gives

$$w(0,t) = u(0,t) - \gamma(0)^T X(t). \tag{2.33}$$

Substituting this expression into (2.15), we get

$$\dot{X}(t) = AX(t) + Bu(0,t) + B\left(K - \gamma(0)^T\right) X(t). \tag{2.34}$$

Comparing this equation with (2.12), we have

$$\gamma(0) = K^T. \tag{2.35}$$

Therefore, the solution to the ODE (2.32) is $\gamma(x) = \mathrm{e}^{A^T x} K^T$, which gives

$$\gamma(x)^T = K\mathrm{e}^{Ax}. \tag{2.36}$$

A general solution to (2.30) is

$$q(x,y) = \phi(x-y), \tag{2.37}$$

where the function ϕ is determined from (2.31). We get

$$q(x,y) = K\mathrm{e}^{A(x-y)}B. \tag{2.38}$$

We can now plug the gains $\gamma(x)$ and $q(x,y)$ into the transformation (2.14) and set $x = D$ to get the control law:

$$u(D,t) = \int_0^D K\mathrm{e}^{A(D-y)}Bu(y,t)dy + K\mathrm{e}^{AD}X(t)\,. \tag{2.39}$$

2.3 On the Relation Among the Backstepping Design, the FSA/Reduction Design, and the Original Smith Controller

The controller (2.39) is given in terms of the transport delay state $u(y,t)$. Using (2.10), one can also derive the representation in terms of the input signal $U(t)$:

$$U(t) = K\left[\mathrm{e}^{AD}X(t) + \int_{t-D}^{t}\mathrm{e}^{A(t-\theta)}BU(\theta)d\theta\right], \tag{2.40}$$

which is identical to the controller (2.5) in Section 2.1.

The controller (2.40) was first derived in the years 1978–1982 in the framework of "finite spectrum assignment" [121, 135] and the "reduction approach" [8]. The idea of the reduction approach is to introduce the "predictor state"

$$P(t) = X(t+D)\,, \tag{2.41}$$

which is alternatively defined as

$$P(t) = \mathrm{e}^{AD}X(t) + \int_{t-D}^{t}\mathrm{e}^{A(t-\theta)}BU(\theta)d\theta\,, \tag{2.42}$$

and study the control of the reduced finite-dimensional system

$$\dot{P}(t) = AP(t) + BU(t)\,, \quad t \geq 0\,. \tag{2.43}$$

The resulting control law is simply

$$U(t) = KP(t)\,, \quad t \geq 0\,, \tag{2.44}$$

and it is also written as (2.40). The predictor transformation (2.42) and the simple, intuitive design based on the reduction approach do not equip the designer with a tool for Lyapunov–Krasovskii stability analysis. The reason for this is that the transformation $P(t)$ is only a transformation of the ODE state $X(t)$, rather than also providing a suitable change of variable for the infinite-dimensional actuator state $u(x,t)$. As a result, the analysis in [8, 121, 135] does not capture the entire system consisting of the ODE plant and the infinite-dimensional subsystem of the input delay. As we shall see in Section 2.4, the backstepping construction permits a stability analysis of the complete feedback system with the cascade PDE-ODE plant

(2.15)–(2.17) and the infinite-dimenasional control law (2.40), resulting in an exponential stability estimate in the appropriate norm for this system.

It is important also to appreciate the difference between the original Smith predictor [201] and the modified Smith predictors such as "finite spectrum assignment" [176, 121, 135], the "reduction" approach [8], and the latest incarnation of the same design, in which it appears in this book—"backstepping." The original Smith predictor [201] was a frequency-domain design, so it is not trivial to establish a parallel between it and the subsequent modifications. However, the main idea of the Smith predictor, if one were to develop it for a state-space problem, leads to feedback of the form

$$U(t) = K\left[X(t) + \int_0^t e^{A(t-\theta)}BU(\theta)d\theta - \int_{-D}^{t-D} e^{A(t-D-\theta)}BU(\theta)d\theta\right]. \qquad (2.45)$$

This feedback law is different than the (modified) predictor feedback pursued in this book, which is given in the form (2.40). The Smith predictor is suitable for compensating the effect of input delay on set-point regulation problems for stable plants. However, it is well known that the Smith predictor offers no stability guarantee for unstable plants.

We do not pursue the explanation for the potential feedback instability under the original Smith predictor feedback for unstable plants. This argument gets complex in the same manner as the positive stability argument for the feedback (2.40), which is given in the next section.

2.4 Stability of Predictor Feedback

Now we study closed-loop stability, both in the transformed variables where exponential stability is nearly obvious, and in the original variables where it is less easily evident.

From this point on, and throughout the book, we use the following notion of exponential stability.

Definition 2.1. Consider the (evolution equation) system

$$\dot{z}(t) = \mathscr{A}z(t), \qquad (2.46)$$

where $z(t)$ belongs to a (possibly infinite-dimensional) Banach space $\mathscr{B}$, and $\mathscr{A}$ is the system's infinitesimal generator. Let $\|\cdot\|_{\mathscr{B}}$ denote a norm associated with $\mathscr{B}$. The equilibrium $z = 0$ of system (2.46) is said to be *exponentially stable* if there exist positive constants ρ and α such that

$$\|z(t)\|_{\mathscr{B}} \le \rho e^{-\alpha t}\|z_0\|_{\mathscr{B}}, \quad \forall t \ge 0, \qquad (2.47)$$

where z_0 denotes the initial condition $z(0)$.

Now we state and prove an exponential stability result for the closed-loop system with the predictor feedback.

Theorem 2.1. *The closed-loop system consisting of the plant* (2.8)*,* (2.9)*,* (2.12) *with the controller* (2.39) *is exponentially stable at the origin in the sense of the norm*

$$\left(|X(t)|^2 + \int_0^D u(x,t)^2 dx\right)^{1/2}. \tag{2.48}$$

Proof. First we prove that the origin of the target system (2.15)–(2.17) is exponentially stable. Consider a Lyapunov–Krasovskii functional

$$V(t) = X(t)^T P X(t) + \frac{a}{2}\int_0^D (1+x) w(x,t)^2\, dx, \tag{2.49}$$

where $P = P^T > 0$ is the solution to the Lyapunov equation

$$P(A+BK) + (A+BK)^T P = -Q \tag{2.50}$$

for some $Q = Q^T > 0$, and the parameter $a > 0$ is to be chosen later. We have

$$\begin{aligned}\dot V(t) &= X(t)^T((A+BK)^T P + P(A+BK))X(t)\\ &\quad + 2X(t)^T PBw(0,t) - \frac{a}{2}w(0,t)^2 - \frac{a}{2}\int_0^D w(x,t)^2\, dx\\ &\le -X(t)^T Q X(t) + \frac{2}{a}|X(t)^T PB|^2 - \frac{a}{2}\int_0^D w(x,t)^2\, dx.\end{aligned} \tag{2.51}$$

Let us choose

$$a = \frac{4\lambda_{\max}(PBB^T P)}{\lambda_{\min}(Q)}, \tag{2.52}$$

where $\lambda_{\min}$ and $\lambda_{\max}$ are the minimum and maximum eigenvalues of the corresponding matrices. Then

$$\begin{aligned}\dot V(t) &\le -\frac{\lambda_{\min}(Q)}{2}|X(t)|^2 - \frac{2\lambda_{\max}(PBB^T P)}{\lambda_{\min}(Q)}\int_0^D w(x,t)^2\, dx\\ &\le -\frac{\lambda_{\min}(Q)}{2}|X(t)|^2 - \frac{2\lambda_{\max}(PBB^T P)}{(1+D)\lambda_{\min}(Q)}\int_0^D (1+x)w(x,t)^2\, dx\\ &= -\frac{\lambda_{\min}(Q)}{2}|X(t)|^2 - \frac{a}{2(1+D)}\int_0^D (1+x)w(x,t)^2\, dx\\ &\le -\min\left\{\frac{\lambda_{\min}(Q)}{2\lambda_{\max}(P)}, \frac{1}{1+D}\right\} V(t).\end{aligned} \tag{2.53}$$

So we obtain

$$\dot V(t) \le -\mu V(t), \tag{2.54}$$

where

$$\mu = \min\left\{\frac{\lambda_{\min}(Q)}{2\lambda_{\max}(P)}, \frac{1}{1+D}\right\}. \tag{2.55}$$

Thus, the closed-loop system is exponentially stable in the sense of the full state norm

$$\left(|X(t)|^2 + \int_0^D w(x,t)^2 dx\right)^{1/2}, \tag{2.56}$$

i.e., in the transformed variable (X, w). To show exponential stability in the sense of the norm $(|X(t)|^2 + \int_0^D u(x,t)^2 dx)^{1/2}$, we need the inverse of the transformation (2.14). One can show with calculations similar to (2.28)–(2.38) that such a transformation is

$$u(x,t) = w(x,t) + \int_0^x K e^{(A+BK)(x-y)} B w(y,t)\, dy + K e^{(A+BK)x} X(t). \tag{2.57}$$

Let us now denote the backstepping transformation and its inverse in compact form as

$$w(x,t) = u(x,t) - \int_0^x m(x-y) u(y,t) dy - KM(x)X(t), \tag{2.58}$$

$$u(x,t) = w(x,t) + \int_0^x n(x-y) w(y,t) dy + KN(x)X(t), \tag{2.59}$$

where

$$m(s) = KM(s)B, \tag{2.60}$$

$$n(s) = KN(s)B, \tag{2.61}$$

$$M(x) = e^{Ax}, \tag{2.62}$$

$$N(x) = e^{(A+BK)x}, \tag{2.63}$$

or even more compactly as

$$w(x,t) = u(x,t) - m(x) \star u(x,t) - KM(x)X(t), \tag{2.64}$$

$$u(x,t) = w(x,t) + n(x) \star w(x,t) + KN(x)X(t), \tag{2.65}$$

where $\star$ denotes the convolution operation in x. To derive a stability bound, we need to relate the norm

$$\left(|X(t)|^2 + \textstyle\int_0^D u(x,t)^2 dx\right)^{1/2}$$

to the norm

$$\left(|X(t)|^2 + \textstyle\int_0^D w(x,t)^2 dx\right)^{1/2},$$

and then the norm

$$\left(|X(t)|^2+\int_0^D w(x,t)^2dx\right)^{1/2}$$

to

$$\sqrt{V(t)}=\left(X(t)^TPX(t)+\tfrac{a}{2}\int_0^D(1+x)w(x,t)^2\,dx\right)^{1/2}.$$

We start from the latter, as it is easier, and obtain

$$\psi_1\left(|X(t)|^2+\int_0^D w(x,t)^2dx\right)\le V(t)\le\psi_2\left(|X(t)|^2+\int_0^D w(x,t)^2dx\right),\quad(2.66)$$

where

$$\psi_1=\min\left\{\lambda_{\min}(P),\frac{a}{2}\right\},\qquad(2.67)$$

$$\psi_2=\max\left\{\lambda_{\max}(P),\frac{a(1+D)}{2}\right\}.\qquad(2.68)$$

It is easy to show, using (2.64) and (2.65), that

$$\int_0^D w(x,t)^2dx\le\alpha_1\int_0^D u(x,t)^2dx+\alpha_2|X(t)|^2,\qquad(2.69)$$

$$\int_0^D u(x,t)^2dx\le\beta_1\int_0^D w(x,t)^2dx+\beta_2|X(t)|^2,\qquad(2.70)$$

where

$$\alpha_1=3\left(1+D\|m\|^2\right),\qquad(2.71)$$

$$\alpha_2=3\|KM\|^2,\qquad(2.72)$$

$$\beta_1=3\left(1+D\|n\|^2\right),\qquad(2.73)$$

$$\beta_2=3\|KN\|^2,\qquad(2.74)$$

and $\|\cdot\|$ denotes the $L_2[0,D]$ norm. Hence, we obtain

$$\phi_1\left(|X(t)|^2+\int_0^D u(x,t)^2dx\right)\le|X(t)|^2+\int_0^D w(x,t)^2dx,\qquad(2.75)$$

$$|X(t)|^2+\int_0^D w(x,t)^2dx\le\phi_2\left(|X(t)|^2+\int_0^D u(x,t)^2dx\right),\qquad(2.76)$$

where

$$\phi_1=\frac{1}{\max\{\beta_1,\beta_2+1\}},\qquad(2.77)$$

$$\phi_2=\max\{\alpha_1,\alpha_2+1\}.\qquad(2.78)$$

Combining the above inequalities, we get

$$\phi_1\psi_1\left(|X(t)|^2+\int_0^D u(x,t)^2dx\right)\le V(t)\le\phi_2\psi_2\left(|X(t)|^2+\int_0^D u(x,t)^2dx\right). \tag{2.79}$$

Hence, with (2.54), we get

$$|X(t)|^2+\int_0^D u(x,t)^2dx\le\frac{\phi_2\psi_2}{\phi_1\psi_1}\mathrm{e}^{-\mu t}\left(|X(0)|^2+\int_0^D u(x,0)^2dx\right), \tag{2.80}$$

which completes the proof of exponential stability. □

Remark 2.1. It is clear that the exponential stability estimate (2.80) is conservative. The decay rate $\mu/2$, where μ is defined in (2.55), seems like it could be much lower than $\min_i\{\mathrm{Re}\{-\lambda_i(A+BK)\}\}$. The overshoot coefficient $\phi_2\psi_2/\phi_1\psi_1$ looks equally conservative, though it is clear that its value must be large since the plant runs in an open loop until the control kicks in at $t=D$. Despite the conservatism in the Lyapunov analysis, it does quantitatively capture the dependence on time and on the initial conditions in the chosen norm (note that this choice is not unique). This cannot be said for [121, 135, 8], where stability is not even claimed in precise terms, but, instead, only a statement on eigenvalues is made.

2.5 Examples of Predictor Feedback Design

Example 2.1. Consider the second-order plant

$$Y(s)=\frac{\mathrm{e}^{-Ds}}{s^2+1}U(s), \tag{2.81}$$

i.e., the system

$$\ddot{\eta}(t)+\eta(t)=U(t-D). \tag{2.82}$$

This is a neutrally stable system with eigenvalues on the imaginary axis. Its state-space form is

$$\dot{\xi}_1(t)=\xi_2(t), \tag{2.83}$$

$$\dot{\xi}_2(t)=-\xi_1(t)+U(t-D), \tag{2.84}$$

where

$$\xi_1=\eta, \tag{2.85}$$

$$\xi_2=\dot{\eta}, \tag{2.86}$$

and

$$X=\begin{bmatrix}\xi_1\\ \xi_2\end{bmatrix}. \tag{2.87}$$

The objective with this system would be to add some damping, so the nominal feedback would be just a simple derivative control,

$$U(t) = -h\dot{\eta}(t), \qquad h > 0; \tag{2.88}$$

i.e., the nominal feedback gain vector is

$$K = [0 \quad -h]. \tag{2.89}$$

The predictor-based version of this feedback employs a matrix exponential

$$e^{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}(t-\theta)} = \begin{bmatrix} \cos(t-\theta) & \sin(t-\theta) \\ -\sin(t-\theta) & \cos(t-\theta) \end{bmatrix} \tag{2.90}$$

and can be obtained as

$$U(t) = h\eta(t)\sin(D) - h\dot{\eta}(t)\cos D - h\int_{t-D}^{t} \cos(t-\theta)U(\theta)d\theta. \tag{2.91}$$

Note that for $D = 0$, this feedback reduces to $U = -h\dot{\eta}$. The time response of the closed-loop system (2.83)–(2.91) is given by

$$\eta(t) = \cos(t)\eta(0) + \sin(t)\dot{\eta}(0) + \int_{t-D}^{t} \sin(t-\theta)U(\theta)d\theta, \tag{2.92}$$

$$\dot{\eta}(t) = -\sin(t)\eta(0) + \cos(t)\dot{\eta}(0) + \int_{t-D}^{t} \cos(t-\theta)U(\theta)d\theta \tag{2.93}$$

until $t = D$, and then an exponentially damped oscillatory response

$$\begin{bmatrix} \eta(t) \\ \dot{\eta}(t) \end{bmatrix} = e^{\begin{bmatrix} 0 & 1 \\ -1 & -h \end{bmatrix}(t-D)} \begin{bmatrix} \eta(D) \\ \dot{\eta}(D) \end{bmatrix} \tag{2.94}$$

for $t \geq D$.

Example 2.2. Figure 2.2 presents the simulation results for system (2.1) for

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 1 & -2 & -2 \\ 0 & 1 & -1 \end{bmatrix}, \tag{2.95}$$

$$B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \tag{2.96}$$

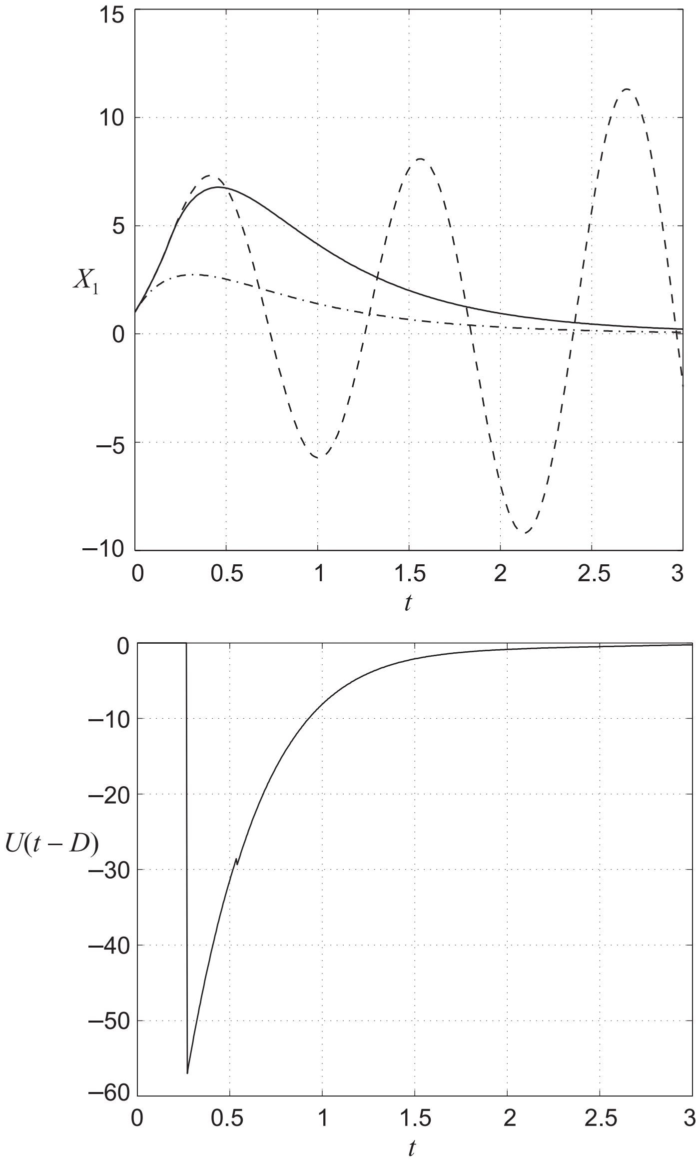

Fig. 2.2 The closed-loop response of the finite-dimensional system with actuator delay. Top: state evolution with nominal LQR controller in the absence of the delay (dash-dotted); with nominal LQR controller in the presence of the delay (dashed); with the backstepping controller in the presence of the delay (solid). Bottom: delayed control input.

This system is unstable at the origin with eigenvalues given by

$$\sigma_1 = 2, \tag{2.97}$$

$$\sigma_{2,3} = -1.5 \pm 1.4j\,. \tag{2.98}$$

One can see that the nominal LQR controller, with

$$Q = I_{3\times 3}, \tag{2.99}$$
$$R = 1, \tag{2.100}$$

does not stabilize the system when a small delay

$$D = 0.3 \tag{2.101}$$

is present. The predictor controller (2.39), which compensates the input delay, stabilizes the system. The larger transient is due to the fact that in the beginning the input to the system is zero because of the delay.

2.6 Stability Proof Without a Lyapunov Function

In this section we consider the problem of stability analysis without relying on the backstepping transformation and on a Lyapunov–Krasovskii function. This is a somewhat more compact proof of exponential stability; however, it comes with two caveats:

- This form of stability analysis does not extend from delay systems to systems involving more complex PDEs.
- A stability proof that avoids a construction of a Lyapunov function deprives the designer from the usual benefits of having a Lyapunov function—a study of robustness to modeling uncertainties, quantification of disturbance attenuation gains, inverse optimal redesign, and adaptive control design.

Nevertheless, we present this stability analysis so that the reader is aware of multiple options and alternatives for obtaining time-domain estimates for exponential stability.

We consider the closed-loop system

$$\dot{X}(t) = AX(t) + BU(t-D), \tag{2.102}$$
$$U(t) = K\left[\mathrm{e}^{AD}X(t) + \int_{t-D}^{t} \mathrm{e}^{A(t-\theta)}BU(\theta)d\theta\right] \tag{2.103}$$

and prove the following result.

Theorem 2.2. *Let $A+BK$ be a Hurwitz matrix such that*

$$\left|\mathrm{e}^{(A+BK)(t-t_0)}\right| \le G\mathrm{e}^{-g(t-t_0)}, \quad \forall t \ge t_0, \tag{2.104}$$

where $g > 0$, $G \ge 1$, and t_0 has any finite value. The solutions to system (2.102), (2.103) satisfy

$$\Xi(t) \le \Gamma \mathrm{e}^{-gt}\Xi(0), \quad \forall t \ge 0, \tag{2.105}$$

where $\Xi(t)$ denotes

$$\Xi(t) = |X(t)| + \sup_{\tau \in [t-D,t]} |U(\tau)| \tag{2.106}$$

and the overshoot coefficient Γ is given by

$$\Gamma = (1+|B|D)\left(1+(1+|K|)G\mathrm{e}^{|A|D}\right)\mathrm{e}^{gD}. \tag{2.107}$$

Proof. With the variation-of-constants formula, we write the solution to (2.102) as

$$X(t) = \mathrm{e}^{At}X_0 + \int_0^t \mathrm{e}^{A(t-\tau)}BU(\tau-D)d\tau. \tag{2.108}$$

Taking a 2-norm of both sides, we obtain

$$\begin{aligned} |X(t)| &\le \mathrm{e}^{|A|t}\left(|X_0| + \int_0^t |B||U(\tau-D)|d\tau\right) \\ &\le \mathrm{e}^{|A|t}\left(|X_0| + |B|t \sup_{\theta\in[-D,t-D]} |U(\theta)|\right), \quad \forall t \ge 0. \end{aligned} \tag{2.109}$$

This yields

$$\begin{aligned} |X(t)| &\le \mathrm{e}^{|A|D}\left(|X_0| + |B|D \sup_{\theta\in[-D,0]} |U(\theta)|\right) \\ &\le \mathrm{e}^{|A|D}\left(|X_0| + |B|D \sup_{\theta\in[-D,0]} |U(\theta)|\right)\mathrm{e}^{g(D-t)}, \quad \forall t \in [0,D], \end{aligned} \tag{2.110}$$

and, in particular,

$$|X(D)| \le \mathrm{e}^{|A|D}\left(|X_0| + |B|D \sup_{\theta\in[-D,0]} |U(\theta)|\right). \tag{2.111}$$

With the predictor feedback (2.103), we have

$$\dot{X}(t) = (A+BK)X(t), \quad \forall t \ge D. \tag{2.112}$$

Hence, with (2.104), we have

$$|X(t)| \le G|X(D)|\mathrm{e}^{-g(t-D)}, \quad \forall t \ge D. \tag{2.113}$$

Substituting (2.111), we get

$$|X(t)| \le G\mathrm{e}^{|A|D}\left(|X_0| + |B|D \sup_{\theta\in[-D,0]} |U(\theta)|\right)\mathrm{e}^{g(D-t)}, \quad \forall t \ge D. \tag{2.114}$$

Since $G \geq 0$, in view of (2.110), the same inequality holds for $t \in [0,D]$, so we obtain

$$|X(t)| \leq G\mathrm{e}^{|A|D}\left(|X_0| + |B|D \sup_{\theta\in[-D,0]} |U(\theta)|\right)\mathrm{e}^{g(D-t)}, \quad \forall t \geq 0. \tag{2.115}$$

We have arrived at an inequality from which one immediately gets an exponentially decaying (in time) bound on $|X(t)|$ in terms of the norm $\Xi(0)$. However, we need a bound on $\Xi(t)$ that incorporates the entire state of the closed-loop system. Toward this end, we observe that the control input (2.103) actually represents

$$U(t) = KX(t+D), \quad \forall t \geq 0. \tag{2.116}$$

With (2.115), we get

$$|U(t)| \leq G|K|\mathrm{e}^{|A|D}\left(|X_0| + |B|D \sup_{\theta\in[-D,0]} |U(\theta)|\right)\mathrm{e}^{-gt}, \quad \forall t \geq 0. \tag{2.117}$$

However, we need an estimate in terms of $\sup_{\theta\in[t-D,t]} |U(\theta)|$. For $t \geq D$, such an estimate immediately follows from (2.117), namely,

$$\sup_{\theta\in[t-D,t]} |U(\theta)| \leq G|K|\mathrm{e}^{|A|D}\left(|X_0| + |B|D \sup_{\theta\in[-D,0]} |U(\theta)|\right)\mathrm{e}^{g(D-t)}, \quad \forall t \geq D. \tag{2.118}$$

Now we turn our attention to estimating $\sup_{\theta\in[t-D,t]} |U(\theta)|$ over $t \in [0,D]$. We split the interval $[t-D,t]$ in the following manner:

$$\begin{aligned}
\sup_{\theta\in[t-D,t]} |U(\theta)| &\leq \sup_{\theta\in[t-D,0]} |U(\theta)| + \sup_{\theta\in[0,t]} |U(\theta)| \\
&\leq \sup_{\theta\in[-D,0]} |U(\theta)| + \sup_{\theta\in[0,t]} |U(\theta)| \\
&\leq \sup_{\theta\in[-D,0]} |U(\theta)| + G|K|\mathrm{e}^{|A|D}\left(|X_0| + |B|D \sup_{\theta\in[-D,0]} |U(\theta)|\right) \\
&\leq G|K|\mathrm{e}^{|A|D}|X_0| + \left(1 + G|K||B|D\mathrm{e}^{|A|D}\right) \sup_{\theta\in[-D,0]} |U(\theta)|, \\
&\qquad \forall t \in [0,D].
\end{aligned} \tag{2.119}$$

We upper-bound this expression to achieve uniformity with (2.118):

$$\sup_{\theta\in[t-D,t]} |U(\theta)| \leq \left[G|K|\mathrm{e}^{|A|D}|X_0| + \left(1 + G|K||B|D\mathrm{e}^{|A|D}\right) \sup_{\theta\in[-D,0]} |U(\theta)|\right]\mathrm{e}^{g(D-t)}, \quad \forall t \in [0,D]. \tag{2.120}$$

In fact, the same bound holds for both $t \in [0,D]$ (2.120) and $t \geq D$ (2.118):

$$\sup_{\theta\in[t-D,t]} |U(\theta)| \leq \left[G|K|e^{|A|D}|X_0| + \left(1 + G|K||B|De^{|A|D}\right) \sup_{\theta\in[-D,0]} |U(\theta)| \right] e^{g(D-t)}, \quad \forall t \geq 0. \tag{2.121}$$

Now adding the bound (2.115), we get

$$\begin{aligned} &|X(t)| + \sup_{\theta\in[t-D,t]} |U(\theta)| \\ &\quad \leq e^{g(D-t)} \Big[(1+|K|)Ge^{|A|D}|X_0| \\ &\qquad + \left(1 + (1+|K|)Ge^{|A|D}|B|D\right) \sup_{\theta\in[-D,0]} |U(\theta)| \Big], \qquad \forall t \geq 0. \end{aligned} \tag{2.122}$$

By majorizing this expression to extract a factor of $|X_0| + \sup_{\theta\in[-D,0]} |U(\theta)|$ on the right-hand side, we obtain

$$\begin{aligned} &|X(t)| + \sup_{\theta\in[t-D,t]} |U(\theta)| \\ &\quad \leq (1+|B|D)\left(1 + (1+|K|)Ge^{|A|D}\right) e^{g(D-t)} \left(|X_0| + \sup_{\theta\in[-D,0]} |U(\theta)| \right), \quad \forall t \geq 0, \end{aligned} \tag{2.123}$$

which completes the proof of the theorem. □

While Theorem 2.2 establishes exponential stability in terms of the $l_2 \times L_\infty[t-D,t]$ norm $|X(t)| + \sup_{\tau\in[t-D,t]} |U(\tau)|$, one might also be interested in how stability would be proved in the sense of the $l_2 \times L_2[t-D,t]$ norm $|X(t)| + \left(\int_{t-D}^{t} U^2(\theta)d\theta\right)^{1/2}$. This result is established next.

Theorem 2.3. *Let (2.104) hold. Then the solutions to system (2.102), (2.103) satisfy*

$$\Upsilon(t) \leq \Gamma e^{-gt} \Upsilon(0), \quad \forall t \geq 0, \tag{2.124}$$

where $\Upsilon(t)$ denotes

$$\Upsilon(t) = |X(t)| + \left(\int_{t-D}^{t} U^2(\theta)d\theta \right)^{1/2} \tag{2.125}$$

and the overshoot coefficient Γ is given by

$$\Gamma = \left(1 + |B|\sqrt{D}\right)\left(1 + \left(1 + \frac{|K|}{\sqrt{2g}}\right) Ge^{|A|D}\right) e^{gD}. \tag{2.126}$$

Proof. With the variation-of-constants formula and the Cauchy–Schwartz inequality, we obtain

$$|X(t)| \le \mathrm{e}^{|A|t}\left(|X_0| + \int_0^t |B||U(\tau - D)|d\tau\right)$$
$$\le \mathrm{e}^{|A|t}\left(|X_0| + |B|\sqrt{t}\left(\int_{-D}^{t-D} U^2(\theta)d\theta\right)^{1/2}\right), \quad \forall t \ge 0. \tag{2.127}$$

This yields

$$|X(t)| \le \mathrm{e}^{|A|D}\left(|X_0| + |B|\sqrt{D}\|U\|_{L_2[-D,0]}\right)$$
$$\le \mathrm{e}^{|A|D}\left(|X_0| + |B|\sqrt{D}\|U\|_{L_2[-D,0]}\right)\mathrm{e}^{g(D-t)}, \quad \forall t \in [0,D], \tag{2.128}$$

and, in particular,

$$|X(D)| \le \mathrm{e}^{|A|D}\left(|X_0| + |B|\sqrt{D}\|U\|_{L_2[-D,0]}\right). \tag{2.129}$$

With the predictor feedback (2.103), we have for $t \ge D$

$$|X(t)| \le G|X(D)|\mathrm{e}^{-g(t-D)}$$
$$\le G\mathrm{e}^{|A|D}\left(|X_0| + |B|\sqrt{D}\|U\|_{L_2[-D,0]}\right)\mathrm{e}^{g(D-t)}, \quad \forall t \ge D. \tag{2.130}$$

Since $G \ge 0$, in view of (2.128), the same inequality holds for $t \in [0,D]$, so we obtain

$$|X(t)| \le G\mathrm{e}^{|A|D}\left(|X_0| + |B|\sqrt{D}\|U\|_{L_2[-D,0]}\right)\mathrm{e}^{g(D-t)}, \quad \forall t \ge 0. \tag{2.131}$$

From (2.116) and with (2.131), we get

$$|U(t)| \le G|K|\mathrm{e}^{|A|D}\left(|X_0| + |B|\sqrt{D}\|U\|_{L_2[-D,0]}\right)\mathrm{e}^{-gt}, \quad \forall t \ge 0. \tag{2.132}$$

However, we ultimately need an estimate of $\|U\|_{L_2[t-D,t]}$. For $t \ge D$, such an estimate immediately follows from (2.132), namely,

$$\|U\|_{L_2[t-D,t]} \le G|K|\mathrm{e}^{|A|D}\left(|X_0| + |B|\sqrt{D}\|U\|_{L_2[-D,0]}\right)\left(\int_{t-D}^t \mathrm{e}^{-2g\tau}d\tau\right)^{1/2}$$
$$= G|K|\mathrm{e}^{|A|D}\left(|X_0| + |B|\sqrt{D}\|U\|_{L_2[-D,0]}\right)\left(\frac{1}{2g}\left(\mathrm{e}^{2g(D-t)} - \mathrm{e}^{-2gt}\right)\right)^{1/2}$$
$$\le G|K|\mathrm{e}^{|A|D}\left(|X_0| + |B|\sqrt{D}\|U\|_{L_2[-D,0]}\right)\frac{1}{\sqrt{2g}}\mathrm{e}^{g(D-t)}, \quad \forall t \ge D. \tag{2.133}$$

Now we turn our attention to estimating $\|U\|_{L_2[t-D,t]}$ over $t \in [0,D]$. We split the interval $[t-D,t]$ in the following manner:

$$\begin{aligned}
\|U\|_{L_2[t-D,t]} &\le \|U\|_{L_2[t-D,0]} + \|U\|_{L_2[0,t]} \\
&\le \|U\|_{L_2[-D,0]} + \|U\|_{L_2[0,t]} \\
&\le \|U\|_{L_2[-D,0]} + G|K|\mathrm{e}^{|A|D}\left(|X_0| + |B|\sqrt{D}\|U\|_{L_2[-D,0]}\right)\left(\int_0^t \mathrm{e}^{-2g\tau}d\tau\right)^{1/2} \\
&\le \|U\|_{L_2[-D,0]} + G|K|\mathrm{e}^{|A|D}\left(|X_0| + |B|\sqrt{D}\|U\|_{L_2[-D,0]}\right)\frac{1}{\sqrt{2g}} \\
&\le \frac{G}{\sqrt{2g}}|K|\mathrm{e}^{|A|D}|X_0| + \left(1 + \frac{G}{\sqrt{2g}}|K||B|\sqrt{D}\mathrm{e}^{|A|D}\right)\|U\|_{L_2[-D,0]}\,, \\
&\qquad \forall t \in [0,D]\,.
\end{aligned} \tag{2.134}$$

We upper-bound this expression to achieve uniformity with (2.133):

$$\begin{aligned}
&\|U\|_{L_2[t-D,t]} \\
&\quad \le \left[\frac{G}{\sqrt{2g}}|K|\mathrm{e}^{|A|D}|X_0| + \left(1 + \frac{G}{\sqrt{2g}}|K||B|\sqrt{D}\mathrm{e}^{|A|D}\right)\|U\|_{L_2[-D,0]}\right]\mathrm{e}^{g(D-t)}\,, \\
&\qquad \forall t \in [0,D]\,.
\end{aligned} \tag{2.135}$$

In fact, the same bound holds for both $t \in [0,D]$ (2.135) and $t \ge D$ (2.133):

$$\begin{aligned}
&\|U\|_{L_2[t-D,t]} \\
&\quad \le \left[\frac{G}{\sqrt{2g}}|K|\mathrm{e}^{|A|D}|X_0| + \left(1 + \frac{G}{\sqrt{2g}}|K||B|\sqrt{D}\mathrm{e}^{|A|D}\right)\|U\|_{L_2[-D,0]}\right]\mathrm{e}^{g(D-t)}\,, \\
&\qquad \forall t \ge 0\,.
\end{aligned} \tag{2.136}$$

Now adding the bound (2.131), we get

$$\begin{aligned}
&|X(t)| + \|U\|_{L_2[t-D,t]} \\
&\quad \le \mathrm{e}^{g(D-t)}\left[\left(1 + \frac{|K|}{\sqrt{2g}}\right)G\mathrm{e}^{|A|D}|X_0|\right. \\
&\qquad \left. + \left(1 + \left(1 + \frac{|K|}{\sqrt{2g}}\right)G|B|\sqrt{D}\mathrm{e}^{|A|D}\right)\|U\|_{L_2[-D,0]}\right]\,, \qquad \forall t \ge 0\,.
\end{aligned} \tag{2.137}$$

By majorizing this expression to extract a factor of $|X_0| + \|U\|_{L_2[-D,0]}$ on the right-hand side, we obtain

$$
\begin{aligned}
&|X(t)| + \|U\|_{L_2[t-D,t]} \\
&\quad \le \left(1+|B|\sqrt{D}\right)\left(1+\left(1+\frac{|K|}{\sqrt{2g}}\right)G\mathrm{e}^{|A|D}\right)\mathrm{e}^{g(D-t)}\left(|X_0| + \|U\|_{L_2[-D,0]}\right), \\
&\qquad \forall t \ge 0, \qquad (2.138)
\end{aligned}
$$

which completes the proof of the theorem. □

2.7 Backstepping Transformation in the Standard Delay Notation

In Section 2.2 we derived the backstepping transformation and its inverse, respectively, as

$$
w(x,t) = u(x,t) - \int_0^x K\mathrm{e}^{A(x-y)}Bu(y,t)dy - K\mathrm{e}^{Ax}X(t), \qquad (2.139)
$$

$$
u(x,t) = w(x,t) + \int_0^x K\mathrm{e}^{(A+BK)(x-y)}Bw(y,t)\,dy + K\mathrm{e}^{(A+BK)x}X(t). \qquad (2.140)
$$

This derivation was performed using the transport PDE representation (2.8), (2.9) of the delay dynamics, which provides both a physical intuition and a mathematically clear setting for formulating the backstepping transformation and the subsequent stability analysis in Section 2.4.

However, as most readers in the field of control of delay systems are not accustomed to the PDE notation, we present here an alternative view of the backstepping transformation, based purely on standard delay notation. For the reader's convenience, we repeat here the closed-loop system consisting of the plant

$$
\dot{X}(t) = AX(t) + BU(t-D) \qquad (2.141)
$$

and of the predictor feedback law

$$
U(t) = K\left[\mathrm{e}^{AD}X(t) + \int_{t-D}^{t}\mathrm{e}^{A(t-\theta)}BU(\theta)d\theta\right]. \qquad (2.142)
$$

We express the future state $X(t+D)$ using the current state $X(t)$ as the initial condition, and using the controls $U(\theta)$ from the past time window $[t-D,t]$, based on the variation of constants formula applied to (9.1) as

$$
\begin{aligned}
X(t+D) &= \mathrm{e}^{AD}X(t) + \int_t^{t+D}\mathrm{e}^{A(t+D-\eta)}BU(\eta-D)d\eta \\
&= \mathrm{e}^{AD}X(t) + \int_{t-D}^{t}\mathrm{e}^{A(t-\theta)}BU(\theta)d\theta. \qquad (2.143)
\end{aligned}
$$

Hence, the feedback law (2.142) is simply

$$U(t) = KX(t+D), \qquad \text{for all } t \geq 0. \tag{2.144}$$

We start by noting that, while $U(t)$ is the value of the control signal at time t, the function $U(\theta), \theta \in [t-D,t]$, namely, the function $U(\cdot)$ on the entire sliding window $[t-D,t]$, is the state of the actuator. The state $U(\theta), \theta \in [t-D,t]$, is infinite-dimensional, since it is a function, rather than a vector. To introduce the backstepping transformation in the standard delay notation, we consider the function

$$W(\theta) = U(\theta) - KX(\theta+D) \tag{2.145}$$

at time $\theta \in [t-D,t]$. From (2.144), with $\theta = t$, it follows that

$$W(t) = 0, \qquad \text{for all } t \geq 0. \tag{2.146}$$

However, in general we have

$$W(\theta) \neq 0, \qquad \text{for } \theta \in [-D,0]. \tag{2.147}$$

We develop next an alternative representation of $W(\theta)$, different than (2.145), which will serve as a definition of the actuator state for all $\theta \in [t-D,t]$ and all $t \geq 0$. Towards that end, similar to (2.143), using the variation-of-constants formula with initial time $t-D$, initial state $X(t)$, and current time θ, from (2.141) we obtain

$$X(\theta+D) = \mathrm{e}^{A(\theta+D-t)}X(t) + \int_{t-D}^{\theta} \mathrm{e}^{A(\theta-\sigma)}BU(\sigma)d\sigma, \tag{2.148}$$

for all $\theta \in [t-D,t]$ and all $t \geq 0$.

We are now ready to introduce the backstepping transformation $U \mapsto W$ of the actuator state. By substituting (2.148) into (2.145), we arrive at

$$W(\theta) = U(\theta) - K\left[\int_{t-D}^{\theta} \mathrm{e}^{A(\theta-\sigma)}BU(\sigma)d\sigma + \mathrm{e}^{A(\theta+D-t)}X(t)\right], \tag{2.149}$$

where $\theta \in [t-D,t]$ and $t \geq 0$. In the transformation (2.149), it is not helpful to view $W(\theta)$ as a value of a function but as a transformation of a function $W(\theta), [t-D,t]$, into another function $U(\theta), \theta \in [t-D,t]$.

Next we introduce a representation of the closed-loop system where U is replaced by W, which is an alternative representation of the target system (2.18)–(2.20). Setting $\theta = t-D$ in (2.149), solving the resulting equation as $U(t-D) = KX(t) + W(t-D)$, and substituting this expression into (2.141), we arrive at the target system

$$\dot{X}(t) = (A+BK)X(t) + BW(t-D), \tag{2.150}$$

$$W(t) = 0, \tag{2.151}$$

which is satisfied for all $t \geq 0$ and where (2.146) is repeated for the sake of clarity in further exposition. Thanks to the property (2.151), from (2.150) we obtain that

$$\dot{X}(t) = (A + BK)X(t), \qquad \text{for all } t \geq D, \tag{2.152}$$

which means that the delay is perfectly compensated in D seconds, namely, the system evolves as if the delay were absent after D seconds.

Since $W(\cdot)$ has a finite support $[-D, 0]$ and the X-system (2.150) is exponentially stable, the target system (2.150), (2.151) is exponentially stable. We proved this fact in Section 2.4. In order for exponential stability to also hold for the original system, namely, for the system whose state is $X(t), U(\theta), \theta \in [t-D, t]$, it is necessary that the transformation (2.149) be invertible. The inverse of (2.149) is given explicitly as

$$U(\theta) = W(\theta) + K\left[\int_{t-D}^{\theta} \mathrm{e}^{(A+BK)(\theta-\sigma)} BW(\sigma)d\sigma + \mathrm{e}^{(A+BK)(\theta+D-t)}X(t)\right], \tag{2.153}$$

where $\theta \in [t-D, t]$ and $t \geq 0$. Hence, since the target system (2.150), (2.151) is exponentially stable, the actuator state (2.153) also exponentially converges to zero.

To rigorously prove exponential stability, in the standard delay system notation, a Lyapunov–Krasovskii functional of the target system (2.150), (2.151) is given as

$$V(t) = X(t)^T PX(t) + \frac{a}{2}\int_{t-D}^{t} (1 + \theta + D - t)W(\theta)^2 d\theta, \tag{2.154}$$

where

$$a = \frac{4\lambda_{\max}(PBB^T P)}{\lambda_{\min}(Q)}. \tag{2.155}$$

The Lyapunov functional (2.154) depends on the state variables (X, W) in a simple diagonal-like manner with no cross-terms involving X and W and with a dependence on W, which is only a temporally scaled norm of this quantity. However, V is a functional of the state of the original system $X(t), U(\theta), \theta \in [t-D, t]$, and its expanded form is far from simple. The dependence of (2.154) on the variables (X, U), through the transformation (2.149), is

$$\begin{aligned}
V(t) = {} & X(t)^T PX(t) + \frac{a}{2}\int_{t-D}^{t} (1 + \theta + D - t)U(\theta)^2 d\theta \\
& + \frac{a}{2}X^T(t)\left(\int_{t-D}^{t} (1 + \theta + D - t)\mathrm{e}^{A^T(\theta+D-t)}K^T K\mathrm{e}^{A(\theta+D-t)}d\theta\right)X(t) \\
& + \frac{a}{2}\int_{t-D}^{t} (1 + \theta + D - t)\left(K\int_{t-D}^{\theta} \mathrm{e}^{A(\theta-\sigma)}BU(\sigma)d\sigma\right)^2 d\theta \\
& - 2\frac{a}{2}\int_{t-D}^{t} (1 + \theta + D - t)U(\theta)K\int_{t-D}^{\theta} \mathrm{e}^{A(\theta-\sigma)}BU(\sigma)d\sigma d\theta
\end{aligned}$$

$$
\begin{aligned}
&-2\frac{a}{2}\left(\int_{t-D}^{t}(1+\theta+D-t)U(\theta)K\mathrm{e}^{A(\theta+D-t)}d\theta\right)X(t)\\
&+2\frac{a}{2}\left(\int_{t-D}^{t}(1+\theta+D-t)K\int_{t-D}^{\theta}\mathrm{e}^{A(\theta-\sigma)}BU(\sigma)d\sigma K\mathrm{e}^{A(\theta+D-t)}d\theta\right)X(t),
\end{aligned}
\tag{2.156}
$$

containing cross terms involving X and several nested integrals of U. A Lyapunov functional of a form such as (2.156) would not be possible to obtain without first constructing the backstepping transformation (2.149).

2.8 Notes and References

This chapter is an expanded version of the backstepping desing results for LTI-ODE systems with input and output delays in [118]. We recover the classical predictor-based feedback law studied in numerous papers [176, 121, 135, 8, 165, 45, 234, 89] (see also [60, 193] for recent surveys). The backstepping construction enables a quantification of exponential stability in a norm suitable for the closed-loop infinite-dimensional system.

We want to explain our preference for the term “predictor feedback” over the rather common term “finite spectrum assignment,” which we consider to be somewhat misleading. It neglects the fact that the system

$$w_t(x,t) = w_x(x,t), \tag{2.157}$$

$$w(D,t) = 0 \tag{2.158}$$

has its own spectrum, with complex poles whose real parts are at negative infinity. Even if one accepted that having a spectrum at negative infinity is somehow equivalent to not having a spectrum at all, stability characterization based on spectrum alone is imprecise as it neglects the effect of eigenvectors and eigenfunctions.

Related to the issue of spectrum and stability, one should note that it can be proved that

$$\|w(t)\|_{L_p[0,D]} \le \mathrm{e}^{b(D-t)}\|w_0\|_{L_p[0,D]} \tag{2.159}$$

for any $b > 0$ and any $p \in [1,\infty]$ (Section 11.4). This is not a well-known fact, and its importance is that it reflects the trade-off between the decay rate and the overshoot coefficient. The decay rate b can be viewed as arbitrarily fast, at the expense of having a very large overshoot coefficient e^{bD}.

The bound (2.159) is tight for

$$t = D, \tag{2.160}$$

$$p = \infty, \tag{2.161}$$

$$w_0(x) \equiv \text{const}. \tag{2.162}$$

The bound is very conservative for both $t \gg D$ and $t \ll D$. The conservativeness for $t \gg D$ comes from the fact that $w(x,t)$ is identically zero for $t > D$. The conservativeness for $t \ll D$ comes from the fact that

$$||w(t)||_{L_p[0,D]} \leq ||w_0||_{L_p[0,D]} \tag{2.163}$$

actually holds for all $t \leq D$.

The stability of the entire infinite-dimensional state of a predictor-based feedback system, quantified in Section 2.4 in a $2 \times L_2$ norm, can be characterized in the Lyapunov sense using any of the $L_p[0,D]$ norms for the actuator state. For instance, in Section 2.6 we provided a stability characterization in a $2 \times L_\infty$ norm, though not with the aid of a Lyapunov function. A Lyapunov function can be constructed in that and other norms using the tools we introduce in Section 11.4.

Sections 2.4 and 2.6 contrast two options we have in performing a time-domain exponential stability analysis of the feedback system with predictor feedback. The approach pursued in Section 2.6, which avoids the use of a Lyapunov function with the help of the facts that

$$\begin{aligned} \dot{X}(t) &= (A+BK)X(t)\,, \qquad && t \geq D\,, \\ U(t) &= KX(t+D)\,, && t \geq 0\,, \end{aligned} \tag{2.164}$$

does not extend (in an obvious way) to the case where one encounters some modeling uncertainties (either parametric or additive disturbances) and does not endow the designer with a tool for inverse optimal redesign or for adaptive control design. Furthermore, the approach to proving stability without a Lyapunov function, given in Section 2.6, does not extend to the cases where the delay input dynamics are replaced by input dynamics modeled by a more complex PDE such as a heat equation (Chapter 15) or a wave equation (Chapter 16). In those cases we do not have a finite-time effect of the input dynamics, and therefore the stability analysis cannot be performed by calculating estimates over two distinct time intervals, $[0,D]$ and $[D,\infty)$. In those cases we employ Lyapunov functions constructed using the backstepping approach.

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