2

Linear Algebra

The FT is a linear operator defined, for our purposes, on finite-dimensional inner product spaces. Given a finite Abelian group G, we will define the FT (in Chapter 4) to be a linear operator on a finite-dimensional inner product space associated with G. More generally, in this chapter, we define an association of sets with inner product spaces. We also define dual bases and a special type of linear operator, i.e., a type of operator that carries orthonormal bases to orthonormal bases. These operators are then formulated in terms of orthonormal bases and the dual of these bases.

The following definition will be used throughout this book: For any nonempty set S and any complex-valued function f defined on S, the *complex conjugate* of f, denoted by $\bar{f}$, is defined, for $s \in S$, by $\bar{f}(s) = \overline{f(s)}$.

2.1 Inner Product Spaces

Let V be a complex vector space, i.e., a vector space over the field of complex numbers $\mathbb{C}$. An *inner product* in V is a function $\langle \cdot, \cdot \rangle \colon V \times V \to \mathbb{C}$ which is required to satisfy the following properties: for x, y, $z \in V$ and $c \in \mathbb{C}$,

$$
\begin{aligned}
\langle x, y \rangle &= \overline{\langle y, x \rangle} && \text{(conjugate symmetric)},\\
\langle x, x \rangle &> 0 \quad \text{if } x \neq 0 && \text{(positive)},\\
\langle x, x \rangle = 0 &\Rightarrow x = 0 && \text{(definite)},\\
\langle cx + y, z \rangle &= c\langle x, z \rangle + \langle y, z \rangle && \text{(linear in the first variable)}.
\end{aligned}
$$

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A vector space in which an inner product is defined is called an *inner product space.*

Example 2.1.1. The complex Euclidean vector space $\mathbb{C}^n$ is an inner product space with the inner product defined by

$$\langle x, y\rangle = \sum_{j=1}^{n} x_j \bar{y}_j,$$

where $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ are vectors in $\mathbb{C}^n$.

Suppose that V is a complex inner product space. The *norm* (or *length*) of a vector $x \in V$, denoted by $\|x\|$, is defined to be the (nonnegative) number $\sqrt{\langle x, x\rangle}$. Two vectors x and y in V are said to be *orthogonal* or *perpendicular* (in symbols, $x \perp y$) if $\langle x, y\rangle = 0$. The linear, positive and definite properties of the inner product imply that the zero vector is the only vector that is orthogonal to every vector in V. Consequently, the norm of the zero vector is equal to zero. A nonzero vector x is called a *unit vector* if $\|x\| = 1$. A subset E of V is called an *orthonormal set* if every vector in E is a unit vector and if every vector in E is orthogonal to every other vector in E. If, in addition to being an orthonormal set, E is a basis of V, then E is called an *orthonormal basis*.

There is a very useful inequality which guarantees that the absolute value of the inner product of two vectors is never greater than the product of the norms of the vectors involved. The mentioned inequality is known as Schwarz's inequality. Although we will use only Schwarz's inequality (in the remark at the end of Section 2.2 below, and in Sections 5.2 and 5.3), we also list other well-known inequalities and identities involving norm of vectors in the following theorem.

Theorem 2.1.1. *Suppose that V is a complex inner product space. The following inequalities and identities hold: for any x, $y \in V$,*

(i) (*Bessel's inequality*) *if $\{e_j \mid j = 1, \ldots, k\}$ is an orthonormal subset of V, then*

$$\sum_{j=1}^{k} |\langle x, e_j\rangle|^2 \leq \|x\|^2,$$

equality holds if and only if $x = \sum_{j=1}^{k} \langle x, e_j\rangle e_j$;

(ii) (*Schwarz's inequality*) $|\langle x, y\rangle| \le \|x\|\|y\|$, *furthermore, if* $y \neq 0$, *then equality holds if and only if* $x = cy$, *where* $c = \langle x, y\rangle / \|y\|^2$;

(iii) (*Triangle inequality*) $\|x + y\| \le \|x\| + \|y\|$, *furthermore, if* $y \neq 0$, *then equality holds if and only if* $x = cy$ *for some nonnegative constant* c;

(iv) (*Pythagorean theorem*) $\|x + y\|^2 = \|x\|^2 + \|y\|^2$ *if* $x \perp y$;

(v) (*Parallelogram law*) $\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2$.

Proof. (i) For the Bessel inequality, we note that

$$\begin{aligned} 0 &\le \left\| x - \sum_{s=1}^{k} \langle x, e_s\rangle e_s \right\|^2 \\ &= \|x\|^2 - \sum_{s=1}^{k} \langle x, e_s\rangle\langle e_s, x\rangle - \sum_{s=1}^{k} \overline{\langle x, e_s\rangle}\langle x, e_s\rangle \\ &\quad + \sum_{s,\,t=1}^{k} \langle x, e_s\rangle \overline{\langle x, e_t\rangle}\langle e_s, e_t\rangle \\ &= \|x\|^2 - \sum_{s=1}^{k} |\langle x, e_s\rangle|^2. \end{aligned}$$

(ii) The Schwarz inequality holds trivially if $y = 0$. For $y \neq 0$ it is a special case of the Bessel inequality, in which the orthonormal set is taken to be the set $\{y/\|y\|\}$ consisting of only one vector.

(iii) We use the Schwarz inequality to prove the triangle inequality. Denote the real part of a complex number z by $\operatorname{Re} z$. Since

$$\|x + y\|^2 = \|x\|^2 + \|y\|^2 + 2\mathrm{Re}\langle x, y\rangle \tag{2.1}$$

$$\le \|x\|^2 + \|y\|^2 + 2|\langle x, y\rangle| \quad \text{(by the fact that } \operatorname{Re} z \le |z|) \tag{2.2}$$

$$\le \|x\|^2 + \|y\|^2 + 2\|x\|\|y\| \quad \text{(by the Schwarz inequality)} \tag{2.3}$$

$$= (\|x\| + \|y\|)^2,$$

the triangle inequality follows.

If $y \neq 0$, then $\|x+y\| = \|x\| + \|y\|$ if and only if we have equality in (2.2) and (2.3) or, equivalently, $\mathrm{Re}\langle x, y\rangle = \langle x, y\rangle = \|x\|\|y\|$. By the Schwarz inequality, the latter equality is equivalent to $x = cy$, where $c = \langle x, y\rangle/\|y\|^2 \geq 0$.

The remaining statements (iv) and (v), that is, the Pythagorean theorem and the parallelogram law, follow from (2.1). ∎

There is a simple geometric interpretation of the Bessel inequality. Since the sum $\sum_{j=1}^{k}\langle x, e_j\rangle e_j$ is the *orthogonal projection* of x in the subspace spanned by the orthonormal vectors e_j, $j = 1, \ldots, k$, the Bessel inequality states that the norm of any vector x is always greater than the norm of its orthogonal projection in any finite-dimensional subspace, unless the subspace in consideration contains x, in which case x and its orthogonal projection are identical.

Remark. We shall use the same notation for inner products in all inner product spaces; consequently, we shall use the same notation to denote norms in all inner product spaces.

Let $\Lambda\colon V \to W$ be a linear operator, where W is also a complex inner product space. The operator Λ is said to be an operator on V if $W = V$, a *linear functional* if $W = \mathbb{C}$, and an *isometry* if it is one-to-one, onto, and preserves the inner product, i.e.,

$$\langle \Lambda(x), \Lambda(y)\rangle = \langle x, y\rangle$$

for all x, $y \in V$. It is easy to verify that the inverse of an isometry is also an isometry. Hence, we can speak of an isometry between two inner product spaces. Two complex inner product spaces V and W are said to be *isometric* (in symbols, $V \simeq W$) if there is an isometry between them.

2.2 Linear Functionals and Dual Spaces

Suppose that V is a complex inner product space (not necessarily finite-dimensional). The set V^* of linear functionals on V is a complex vector space with respect to the pointwise definition of addition and scalar multiplication defined as follows: for f, $g \in V^*$

and $c \in \mathbb{C}$, the sum of f and g, denoted by $f+g$, and the scalar multiplication of f by c, denoted by cf, are defined by

$$(f+g)(x) = f(x) + g(x)$$
$$cf(x) = c(f(x))$$

for all $x \in V$. The vector space V^* is called the *dual space* of V.

To exhibit some elements of V^*, for each $y \in V$, we define the function $\ell_y \colon V \to \mathbb{C}$ by setting $\ell_y(x) = \langle x, y \rangle$. Since the inner product is linear in the first variable, ℓ_y is a linear functional on V, that is, $\ell_y \in V^*$. In fact, every linear functional on V can be obtained in this way if V is finite-dimensional. This is the main content of the next theorem, which is a special case of a famous theorem known as the Riesz representation theorem.

Theorem 2.2.1. *Let V be a finite-dimensional complex inner product space. The function $\ell \colon V \to \mathbb{C}$ is a linear functional if and only if there is a unique vector y in V such that $\ell(x) = \langle x, y \rangle$ for all x in V.*

Proof. It remains to show only that if ℓ is a linear functional on V, then there is a unique $y \in V$ such that $\ell(x) = \langle x, y \rangle$ for every $x \in V$. Let $n = \dim V$ and let $\{b_j\}_{j=1}^n$ be an orthonormal basis for V. If $x \in V$, then x can be written uniquely as

$$x = \sum_{j=1}^{n} \langle x, b_j \rangle b_j.$$

Since ℓ is linear, we have

$$\begin{aligned}
\ell(x) &= \sum_{j=1}^{n} \langle x, b_j \rangle \ell(b_j) \\
&= \sum_{j=1}^{n} \langle x, \bar{\ell}(b_j) b_j \rangle \\
&= \left\langle x, \sum_{j=1}^{n} \bar{\ell}(b_j) b_j \right\rangle \\
&= \langle x, y \rangle,
\end{aligned}$$

where $y = \sum_{j=1}^n \bar{\ell}(b_j) b_j$. To prove the uniqueness of y, assume that there is another $y' \in V$ such that $\ell(x) = \langle x, y' \rangle$ for all x in

V. It follows that $\langle x, y - y' \rangle = 0$ for every vector x in V, whence $y - y' = 0$ or $y = y'$. ■

By Theorem 2.2.1, there is a one-to-one correspondence between V and V^*, which is given by $v \leftrightarrow \ell_v$, where $\ell_v(x) = \langle x, v \rangle$ for all $x \in V$. Since

$$\ell_{cv} = \bar{c}\ell_v \quad \text{and} \quad \ell_{v+v'} = \ell_v + \ell_{v'}, \tag{2.4}$$

for all v, $v' \in V$ and $c \in \mathbb{C}$, the correspondence $v \leftrightarrow \ell_v$, which is conjugate linear, induces an inner product in V^* defined in terms of the inner product in V by the equation

$$\langle \ell_v, \ell_{v'} \rangle = \overline{\langle v, v' \rangle}. \tag{2.5}$$

Consequently, the relation $\|\ell_v\| = \|v\|$ holds for every $v \in V$; i.e., every linear functional on V has finite norm or, equivalently, bounded.

For each $v \in V$, the linear functional ℓ_v, called the *dual* of v, is often denoted by v^*. With this notation, we have

$$v^*(x) = \langle x, v \rangle. \tag{2.6}$$

In general, bases of V induce bases of V^*. Furthermore, orthonormal bases induce orthonormal bases. A special case is illustrated next. Suppose that $n = \dim V$ and $E = \{e_j \mid j = 1, \ldots, n\}$ is an orthonormal basis for V. Since every element of V^* is of the form v^* for some

$$v = \sum_{j=1}^{n} \langle v, e_j \rangle e_j \in V,$$

by (2.4) we have

$$v^* = \sum_{j=1}^{n} \overline{\langle v, e_j \rangle}\, e_j^*.$$

It follows that the set $E^* = \{e_j^* \mid j = 1, \ldots, n\}$ spans the space V^*. Moreover, the relation (2.5) implies that E^* is an orthonormal set, hence it is an orthonormal basis of V^*. Consequently, we have $\dim V = \dim V^*$. The basis E^* is called the *dual basis* of E.

Remark. As mentioned, Theorem 2.2.1 is the finite-dimensional case of the Riesz representation theorem. The main conclusion of Theorem 2.2.1 is that every linear functional ℓ is given in terms of the inner product. Consequently, ℓ is bounded. Observe that any linear functional ℓ defined in terms of the inner product as $\ell(x) = \langle x, y\rangle$ for some fixed y is bounded regardless of the dimension of V. That is, $|\ell(x)| \leq \|x\|\|y\|$ for all x. This fact follows from the Schwarz inequality. Thus, to modify the statement of Theorem 2.2.1 to get a general version of the Riesz theorem for infinite-dimensional Hilbert spaces we must add the hypothesis that ℓ is bounded. For a beautiful introduction to the topic of Hilbert spaces and a nice proof of the Riesz representation theorem see [4].

2.3 A Special Class of Linear Operators

It is simpler to define a general family of operators of which the FT is a member than to define the FT itself. This is what we do in this section.

Let S be any nonempty finite set and let V_S be the set of all complex-valued functions defined on S. Then V_S is a complex vector space with respect to the pointwise definition of addition and scalar multiplication. Furthermore, V_S becomes an inner product space with an inner product defined by setting

$$\langle f, g\rangle = \sum_{s \in S} f(s)\bar{g}(s).$$

With this definition, it is simple to construct an orthonormal basis for V_S. For each $s \in S$, let $\delta_s \colon S \to \mathbb{C}$ be the function defined by

$$\delta_s(t) = \begin{cases} 1 & \text{if } s = t, \\ 0 & \text{if } s \neq t. \end{cases}$$

Then it is obvious that the set $\Delta_S = \{\delta_s \mid s \in S\}$ is an orthonormal basis for V_S, called the *standard basis*. Since S is a finite set, V_S is a finite-dimensional complex inner product space. In fact, $V_S \simeq \mathbb{C}^n$, where $n = |S|$. Hence, S can serve as an index set for any basis of V_S.

Suppose, in addition to Δ_S, that $B_S = \{B_s \mid s \in S\}$ is another orthonormal basis of V_S. Since every $x \in V_S$ can be written uniquely as

$$x = \sum_{s \in S} \langle x, B_s \rangle B_s = \sum_{s \in S} B_s B_s^*(x),$$

the identity operator on V_S can be expressed uniquely in terms of the basis B_S and its dual B_S^* as

$$I = \sum_{s \in S} B_s B_s^*. \tag{2.7}$$

In terms of the dual basis Δ_S^*, we have

$$B_s^* = \sum_{t \in S} \langle B_s^*, \delta_t^* \rangle \delta_t^* = \sum_{t \in S} \langle \delta_t, B_s \rangle \delta_t^*,$$

whence

$$I = \sum_{s,\, t \in S} \langle \delta_t, B_s \rangle B_s \delta_t^*.$$

It follows that the image of any $x \in V_S$ under any linear operator Λ on V_S is given by

$$\Lambda(x) = \sum_{s,\, t \in S} \langle \delta_t, B_s \rangle \Lambda(B_s) \delta_t^*(x).$$

Hence,

$$\Lambda = \sum_{s,\, t \in S} \langle \delta_t, B_s \rangle \Lambda(B_s) \delta_t^*. \tag{2.8}$$

In equation (2.8), for each $s \in S$, $\Lambda(B_s)$ can be any vector in V_S. Now we single out an operator that maps B_s to the unique element of the basis Δ_S that is associated with B_s in a very natural way: for a fixed $s \in S$, by (2.7),

$$B_s = \sum_{t \in S} B_t B_t^*(B_s) = \sum_{t \in S} \delta_s(t) B_t. \tag{2.9}$$

The uniqueness of this expression (of B_s in the basis B_S) induces a one-to-one correspondence $B_s \leftrightarrow \delta_s$, which is independent of any enumeration (or indexing of elements) of the basis B_S. Through this correspondence, we define a linear operator $\mathcal{F}$ on V_S by setting $\mathcal{F}(B_s) = \delta_s$ for every $s \in S$.

The next theorem follows from the definition of $\mathcal{F}$ and equation (2.8).

Theorem 2.3.1. *Assume the following*:

(a1) *S is a nonempty finite set and V_S is the associated inner product space of complex-valued functions on S;*
(a2) *$\Delta_S = \{\delta_s \mid s \in S\}$ and $B_S = \{B_s \mid s \in S\}$ are two orthonormal bases of V_S, where Δ_S is the standard basis;*
(a3) *$\mathcal{F}$ is the linear operator on V_S such that $\mathcal{F}(B_s) = \delta_s$ for every $s \in S$, where δ_s is the unique vector in Δ_S associated with B_s by equation* (2.9).

Then

(c1) $\mathcal{F} = \sum_{s,\,t \in S} \langle \delta_t, B_s \rangle \delta_s \delta_t^*$,
(c2) *$\mathcal{F}$ is an isometry, and*
(c3) *$\mathcal{F}f(s) = \langle f, B_s \rangle$, for any $f \in V_S$. (Here we write $\mathcal{F}f$ for $\mathcal{F}(f)$.)*

The complex number $\langle f, B_s \rangle$ is called the *s-coefficient* of f in the orthonormal basis B_S.

If G is a finite Abelian group, the FT on G is the linear operator $\mathcal{F}$ described in Theorem 2.3.1 with respect to a particular orthonormal basis B_G, which we will define in the next chapter.

Exercises.

5. Let V be a complex vector space, not necessarily finite-dimensional. Is every non-identically zero linear functional on V surjective?

6. Let V be a finite-dimensional complex vector space, not necessarily an inner product space.

(i) Assume that f and g are linear functionals on V and that $f(x) = 0$ whenever $g(x) = 0$. Show that $f = cg$ for some constant c.

(ii) Let $\{b_1, \ldots, b_n\}$ be a basis of V and let $\{c_1, \ldots, c_n\}$ be any set of constants. Show that there is a unique linear functional f on V such that $f(b_j) = c_j$ for $j = 1, \ldots, n$.
(iii) Let x be a nonzero vector in V. Prove that there is a linear functional f on V such that $f(x) = 1$.
(iv) Let f be a nonzero linear functional on V. Prove that there is at least one vector $x \in V$ such that $f(x) = 1$.
(v) Let $f_1, \ldots, f_n$ be linear functionals on V, where $n < \dim V$. Prove that there is a nonzero vector $x \in V$ such that $f_j(x) = 0$ for $j = 1, \ldots, n$.

7. Let V be an inner product space, not necessarily finite-dimensional, with the underlying field of scalars $\mathbb{F}$, where either $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$. Let x and y be two vectors in V, prove the following statements:

(i) If $\mathbb{F} = \mathbb{R}$ and $\|x\| = \|y\|$, then $(x + y) \perp (x - y)$.
(ii) If $\mathbb{F} = \mathbb{R}$, then $x \perp y$ if and only if $\|x+y\|^2 = \|x\|^2+\|y\|^2$.
(iii) If $\mathbb{F} = \mathbb{C}$, then $x \perp y$ if and only if $\|x + cy\|^2 = \|x\|^2 + \|cy\|^2$ for every complex number c.

8. Let $\mathcal{F}^{-1}$ denote the inverse of $\mathcal{F}$. For $f \in V_S$, prove that

$$f = \sum_{s \in S} \langle \mathcal{F} f, \delta_s \rangle B_s \quad \text{and} \quad \mathcal{F}^{-1} f = \sum_{s \in S} \langle f, \delta_s \rangle B_s.$$

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