I.2 TYPICAL BEHAVIOR FOR ONE MAP

Before we study parametrized families of maps, we want to analyze individual maps. We are interested in the possible behavior of the successive images of an initial point x_0 on the interval $[-1,1]$ for a fixed map f. For this we first outline a graphical method for determining the iterates $x_n = f^n(x_0)$. Here, we define $f^n(x_0) = f(f^{n-1}(x_0))$. The following Figure I.5 shows how this is done through the rule: Go from x_0 to the graph of the function, from the graph to the diagaonal, from the diagonal to the graph,... .

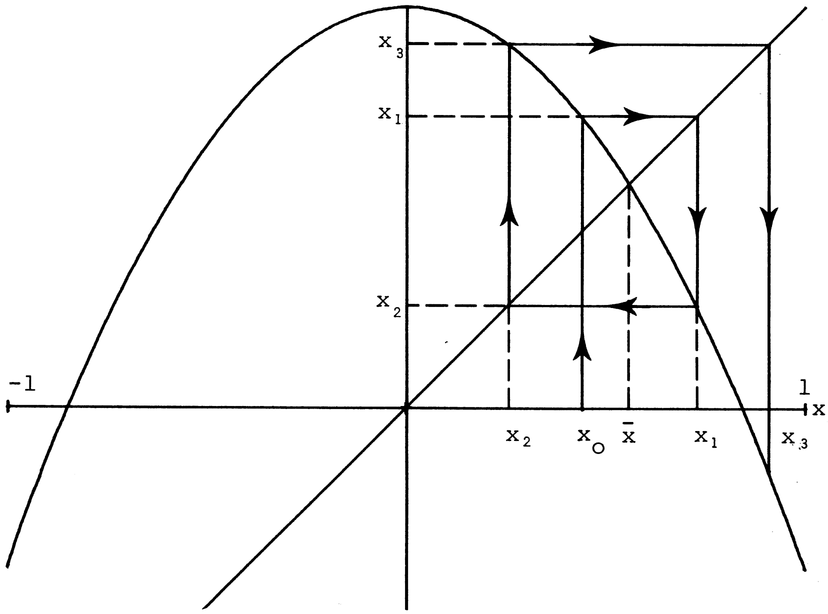

Figure I.5. $f(x) = 1 - 1.4\,x^2$

Note that the point marked $\bar{x}$ does not move at all; $f^n(\bar{x}) = f(\bar{x}) = \bar{x}$. We shall call $\bar{x}$ a fixed point of f. Physically speaking, this means that if the system is at $\bar{x}$ at some

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time, it will remain there forever. Going back to our continuous systems the Poincaré map will have a fixed point, if the system has a <u>closed</u> (and hence periodic) <u>orbit</u>. Such orbits are sometimes called <u>limit cycles</u>.

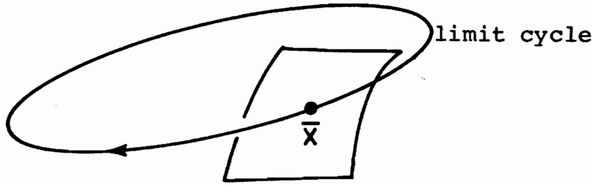


Figure I.6.

Let us now consider another map.

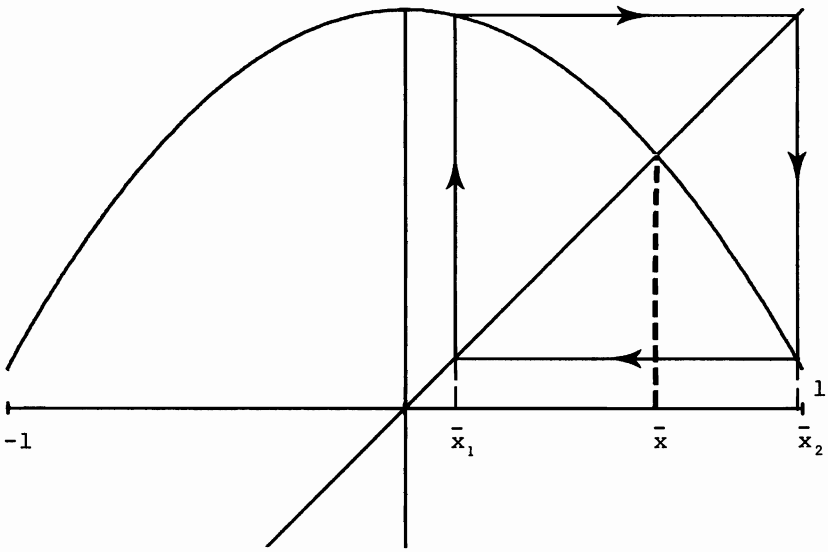


Figure I.7. $f(x) = 1 - .9\,x^2$.

This map has the property that the point $\bar{x}_1$ satisfies $f(\bar{x}_1) = \bar{x}_2$ and $f(\bar{x}_2) = \bar{x}_1$ or in other terms $f^2(\bar{x}_1) = \bar{x}_1$, and $f^2(\bar{x}_2) = \bar{x}_2$. One says f has a <u>periodic orbit</u> of period 2 (which is implied by the fact that f^2 has two fixed points namely $\bar{x}_1$, $\bar{x}_2$ which are not fixed points for f).

Again, we have an analogous picture for flows.

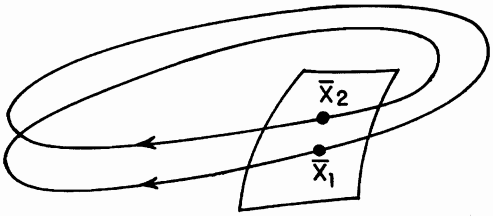


Figure I.8.

We come now to a very important point of the discussion. If a fixed point (or a periodic point) is to be relevant for observations in the dynamical system described by f, we must see whether it persists under small perturbations. There are different kinds of perturbations envisageable, namely

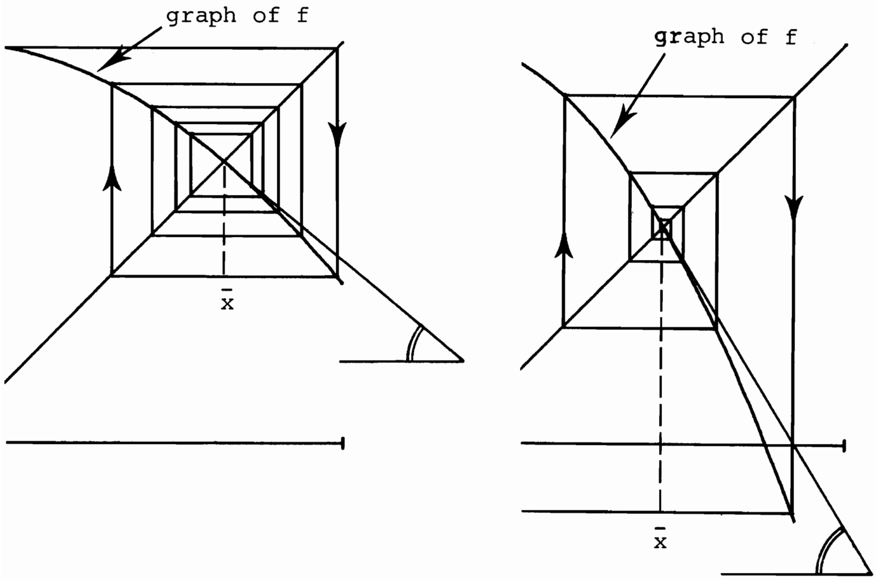
(1) perturbation of the intial point: "Do systems in nearby initial states evolve similarly?"

(2) perturbations of the function f: "f is only approximately known."

(3) stochastic perturbations: "The true equation is not $x_n = f(x_{n-1})$ but there will be noise terms which can be modelled by saying that $x_n = f(x_{n-1}) + r(x_{n-1})$ where $r(x)$ is a small random step, i.e., a variation of $f(x_{n-1})$ with some a priori probability distribution."

We shall analyze below in great detail the Case (1). The motivations for this are two-fold: (A) There are some situations where, if (1) is under control, then (2) and (3) do not have an interesting effect: i.e., small perturbations of f or small random steps do not affect the qualitative behavior of the system. (B) There are situations where (2) and (3) are ill-understood and no results are available, (cf. Kifer [1974] for a positive result in this direction).

However, we insist that considering large random forces or large perturbations of f is a totally different enterprise, because one is in fact changing the whole problem, and this has nothing to do with small perturbations of the system, which are of main interest to us. We now concentrate, as announced, on perturbations of the initial point. Let us thus analyze the neighborhood of a fixed point. There are two basic situations, with regards to the long-time (i.e., f^n for n large) evolution of a point near a fixed point.



a) Figure I.9. b)

We see that if the slope $f'(x)$ of f at $\bar{x}$ satisfies $|f'(\bar{x})| < 1$ (angle <45° with horizontal, Fig. I.9A) then a point x_0 near $\bar{x}$ will have the property that $\lim_{n\to\infty} f^n(x_0) = \bar{x}$ and in fact this is true for all choices of x_0 in a sufficiently small neighborhood of $\bar{x}$. If, on the other hand, as in Fig. I.9B, $|f'(\bar{x})| > 1$, then there is a neighborhood $\mathcal{U}$

of $\bar{x}$ such that $f^{n_0}(x_0) \notin \mathcal{U}$ after some number n_0 of steps, when $x_0 \neq \bar{x}$. The number n_0 of steps depends on x_0 and on the neighborhood. Note that the global form of f can be such that $f^{n_0+k}(x_0)$ is again in the neighborhood $\mathcal{U}$ for some later k, depending on x_0 but this is not of concern to us now. In the case $|f'(\bar{x})| < 1$ we call $\bar{x}$ a <u>stable fixed point</u>, in the opposite case $\bar{x}$ is called an <u>unstable fixed point</u>. The case of $|f'(\bar{x})| = 1$ will be dealt with in Section II.4. The same kind of analysis can be made for periodic points. E.g., in the case of a periodic point $\bar{x}_1$ of period 2, we saw that $f^2(\bar{x}_1) = \bar{x}_1$, so we get a fixed point if we consider f^2 instead of f. Thus the condition for stability becomes

$$|f^{2'}(\bar{x}_1)| \leq 1 \quad .$$

By the chain rule of differentiation,

$$f^{2'}(\bar{x}_1) = f'(f(\bar{x}_1))f'(\bar{x}_1) = f'(\bar{x}_1)f'(\bar{x}_2)\,(=f^{2'}(\bar{x}_2)),$$

i.e., the derivative is the <u>product</u> of derivatives along the periodic orbit. We give an example of a stable periodic orbit of period 2.

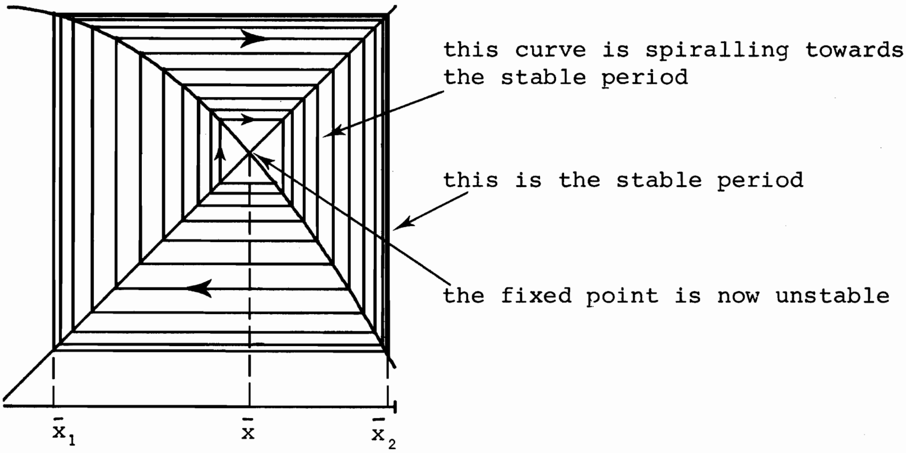


Figure I.10.

Of course, all this may be easily translated to the case of periodic orbits of arbitrary finite length. Also, we can pass again to flows, and speak of <u>stable limit cycles</u> and <u>unstable limit cycles</u>.

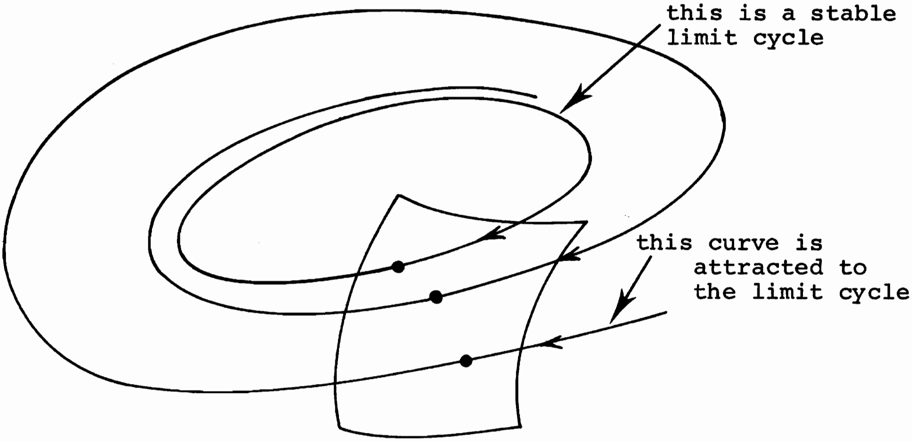


Figure I.11.

In accordance with the point of view (1), we see that <u>stable periodic orbits are relevant for physical systems</u>, because many initial points will eventually show the same behavior for large n. Namely, if x_0 and x_0' are two initial points near to a stable fixed point $\bar{x}$, they will satisfy

$$\lim_{n\to\infty} f^n(x_0) = \lim_{n\to\infty} f^n(x_0') = \bar{x}$$

i.e., irrespective of the initial point, the system will reach the final state $\bar{x}$.

We can now ask and answer two important questions.

<u>Q1</u>: Do all points converge to some stable periodic orbit?

<u>Q2</u>: Can there be several distinct stable periodic orbits for one map?

The answer to Q2 is, under the hypotheses of negative Schwarzian derivative:

TH1 <u>There can be at most one stable periodic orbit</u> (II.4.2).

Note the "at most"! We shall see later that there are maps (in fact many) which have <u>no</u> stable periodic orbit. We shall call them <u>aperiodic maps</u>. (But all continuous maps from [-1,1] to itself have at least one unstable or stable fixed point). For the moment, we shall concentrate on those maps with a stable periodic orbit. Then it is reasonable to ask question Q1. The answer is, of course, "no". Not all points converge to a stable periodic orbit. It suffices to look at Fig. I.10. There we have a stable periodic orbit $\bar{x}_1, \bar{x}_2$ but we also have the point $\bar{x}$, defined by $f(\bar{x}) = \bar{x}$: obviously it is an unstable fixed point (slope >45°, and TH1 above) but the point $x_0 = \bar{x}$ will satisfy $f^n(x_0) = \bar{x}$ and hence it does <u>not</u> tend to $\bar{x}_1$, $\bar{x}_2$. So we see that the "no" to question Q1 is unavoidable for many maps. But there is a more reasonable alternative question.

Q1': How many points converge to the stable periodic orbit?

with the answer:

TH2 <u>The measure of those points which do not converge to the stable periodic orbit is zero</u>. (II.5.7)

Measure is Lebesgue measure, and if you are not familiar with this concept, you can replace the statement by the weaker one that those points which do <u>not</u> converge to the stable periodic orbit do not form subintervals of the line [-1,1].

How can we determine whether a given map f has a stable period or not? First of all, this question should not suggest that by looking at the analytic form of f (even if $f(x) = 1-\mu x^2$) one can decide on the occurrence or absence of a stable periodic orbit. This is in fact a hopeless problem.

We should rather ask how to find all the stable periodic orbits whose periods are not too long. One has the following criterion.

TH3 If f has a stable periodic orbit, then the initial point 0 will be attracted to it. (II.4.2)

(Recall that 0 is the point for which $f' = 0$.) In other words, 0 never belongs to the exceptional set described in the preceding discussion. It is now legitimate to ask whether this criterion is ever useful. In fact it is--at least for short periods--and furthermore, it provides us with a tool to construct a map without a stable periodic orbit. This map is the map $x \to 1 - 2x^2$ (Fig. I.12).

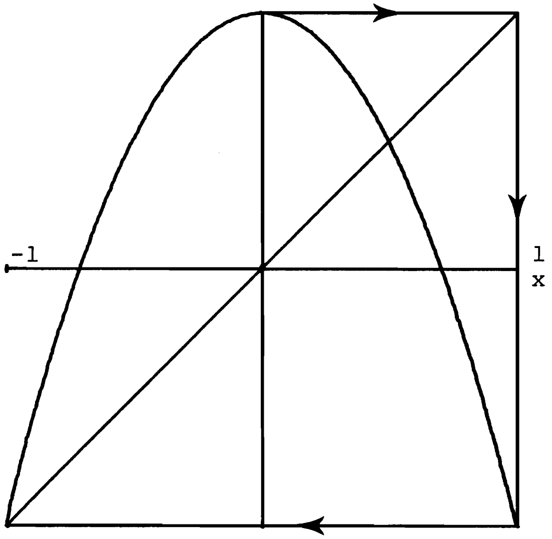


Figure I.12. $f(x) = 1 - 2x^2$

If we take $x_0 = 0$, then $x_1 = f(x_0) = 1$ and $x_2 = f^2(x_0) = -1$ and then $x_n = -1$ for all $n \geq 2$. Therefore the image of $x_0 = 0$ "settles down" after 2 iterations. But -1 (which is obviously a fixed point of the map f) is not a stable fixed point. Assume that f has a stable periodic orbit somewhere.

Then the images of 0 will not be attracted to it, because they stay at -1, and -1 is not part of the periodic orbit, since it is a fixed point. We arrive thus at a contradiction and it follows that f does not have a stable periodic orbit.

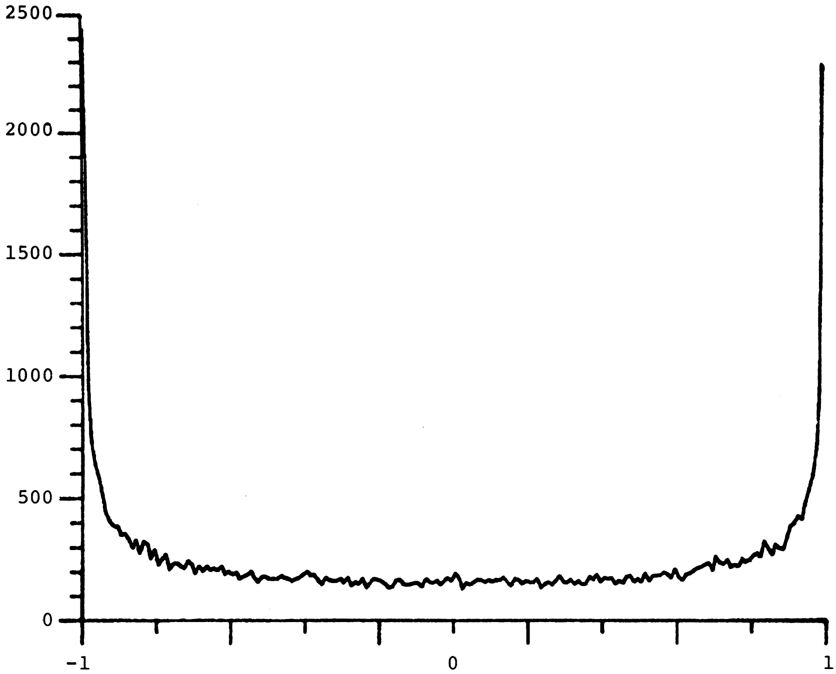
It is now natural to ask a new question. What happens to a typical initial point when there is no stable periodic orbit? Two essential cases have been studied, but it is not known whether there are other typical cases.

The first case occurs for $f(x) = 1 - 2x^2$, the case we have just examined. Let us perform the following experiment. We take as initial point x_0 "any" point (not $x_0 = 0$), and iterate it 50000 times. Then we plot the histogram for the number of points which have fallen in each of the 200 intervals $[n/100,(n+1)/100])$, $n = -100,-99,\ldots,99$. The result is shown in Fig. I.13. This curve is very near to $(\pi(1-y^2)^{1/2})^{-1}$, see Figure I.14. [The map $f(x) = 1 - 2x^2$ was studied by Ulam and von Neumann [1947]. If one takes as new coordinates

$$y = \frac{4}{\pi}\left(\arcsin\sqrt{\frac{x+1}{2}}\right) - 1 \quad ,$$

then, in these new coordinates, f takes the form $\hat{f}(y) = 1 - 2|y|$. This function has obviously no stable periodic orbits (the derivative is everywhere ± 2).]

The kind of behavior exhibited by the function f is called ergodic behavior. It says, in essence, that most points visit every region of phase space with about equal probability. But in fact, our example is much more chaotic in the following sense. Consider two nearby points x_0 and x_0' and their respective evolutions $x_n = f^n(x_0)$, $x_n' = f^n(x_0')$. For our function f, in general, no matter how near x_0 is to x_0', for some n, the points x_n and x_n' will eventually be noticeably separated. Ruelle [1978(1)] has coined the term sensitive dependence on initial conditions for this kind of behavior (II.7). It explains in a very appealing way the apparent incompatibility



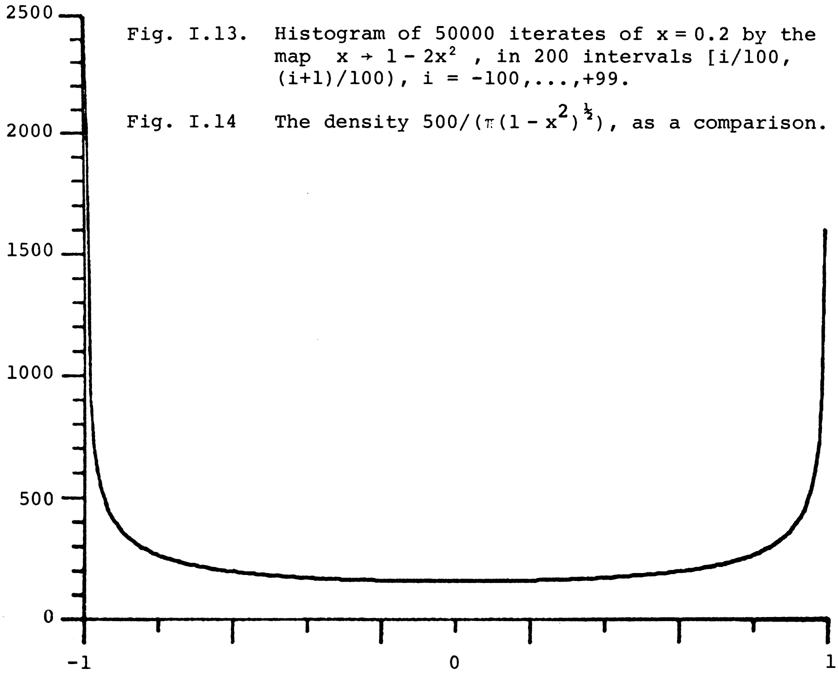


Fig. I.13. Histogram of 50000 iterates of $x = 0.2$ by the map $x \to 1 - 2x^2$, in 200 intervals $[i/100, (i+1)/100)$, $i = -100,\ldots,+99$.

Fig. I.14 The density $500/(\pi(1-x^2)^{\frac{1}{2}})$, as a comparison.

between the determinacy of a system, and the unpredictability of its time evolution. In fact, any imprecision of our knowledge of x_0, no matter how small, will eventually show up on the scale of the interval. Furthermore, this amplification of error can be quite violent and rapid, and for our previously discussed example, each iteration amplifies the error by two, since the derivative of $1-2|x|$ equals ± 2, and

$$\hat{f}(x_0) - \hat{f}(x_0') \sim \hat{f}'(x_0)(x_0-x_0') = \pm 2(x_0-x_0') \quad .$$

Another nice way to say this has been illustrated by Shaw [1978]: One can view the sensitivity to initial conditions as forgetting where a point comes from. Let us perform the following game to see this. For the map $x \to 1-2x^2$ we choose 10000 initial points $x_0^{(1)},\ldots,x_0^{(10000)}$ near $-1/4$, (more precisely $x_0^{(m)} = -1/4 + (m-1)10^{-15}$) and we ask how many of the points $x_n^{(m)} = f^n(x_0^{(m)})$ have <u>not</u> hit the "target" where we define the target to be the interval $(-0.22, -.2)$. If they hit it, they are "dead" and we pursue only the fate of the "survivors ".

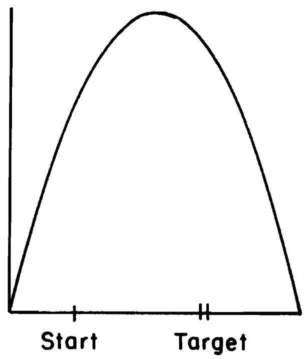


Figure 1.15

We obtain an exponential

$$\text{survivors} \sim 10000 \exp(-n/\tau) ,$$

where the lifetime τ is simply given by the probability of hitting the target when the points are distributed according to the histogram, Fig. I.13. Since the theoretical distribution is $P(y) = 1/(\pi\sqrt{1-y^2})$ we find

$$\tau \sim (0.02P(x))^{-1} = 50/\pi \sqrt{1-0.2^2} \sim 154$$

iterations. This fits the observed slope very well. On the other hand, since the precision of our information decreases by a factor of 2 per iteration, we shall have totally forgotten the initial data after about

$$-\log(10^{-15} \cdot 10^4)/\log 2 \sim 36$$

iterations provided the initial interval does not hit the target right away. So during the first 36 iterations, the information is well retained and then an exponential falloff can be observed. See Figure I.16.

We come now to a <u>second case</u> (which will turn out to be quite rare but very crucial). This case occurs for the function $f(x) = 1 - 1.401155\ldots x^2$. This function can be shown to have no stable periodic orbits. The histogram looks like in Figure I.17.

Note that this is not the histogram of a long stable periodic orbit. But it can be shown that for almost all initial choices of x_0 we will obtain the same histogram. That is, almost all initial x_0's are attracted to the same stable, but <u>nonperiodic</u> orbit (III.3). The volume occupied by this orbit is of (Lebesgue-) measure zero, i.e., the orbit occupies no length (volume). This is in contrast to the first case (of the function $f(x) = 1 - 2x^2$). Furthermore, there is another marked difference: The map in question does <u>not</u> have sensitive dependence with respect to initial conditions. In

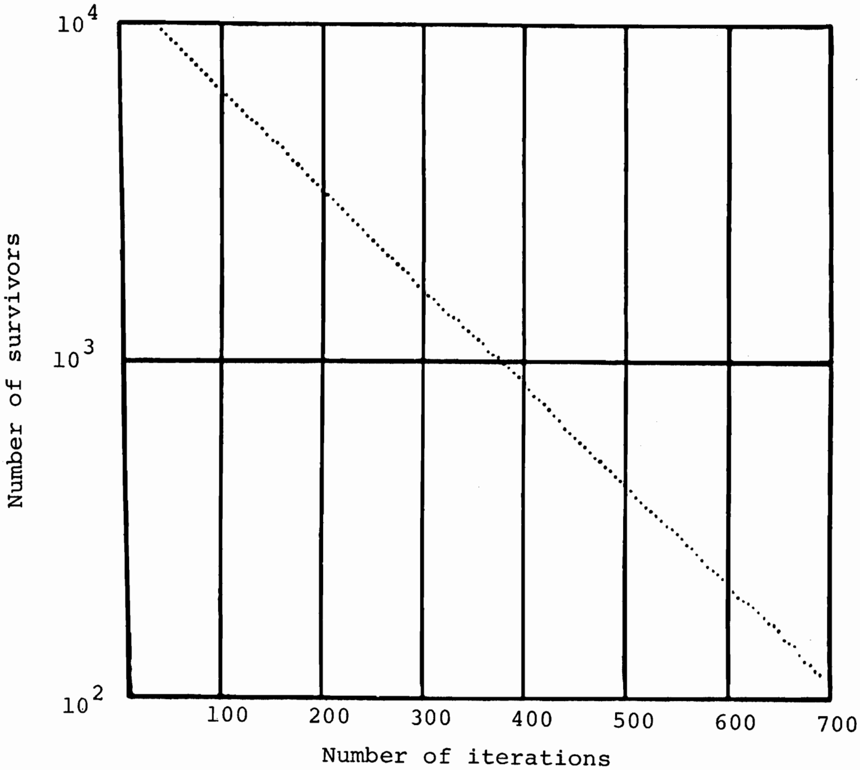
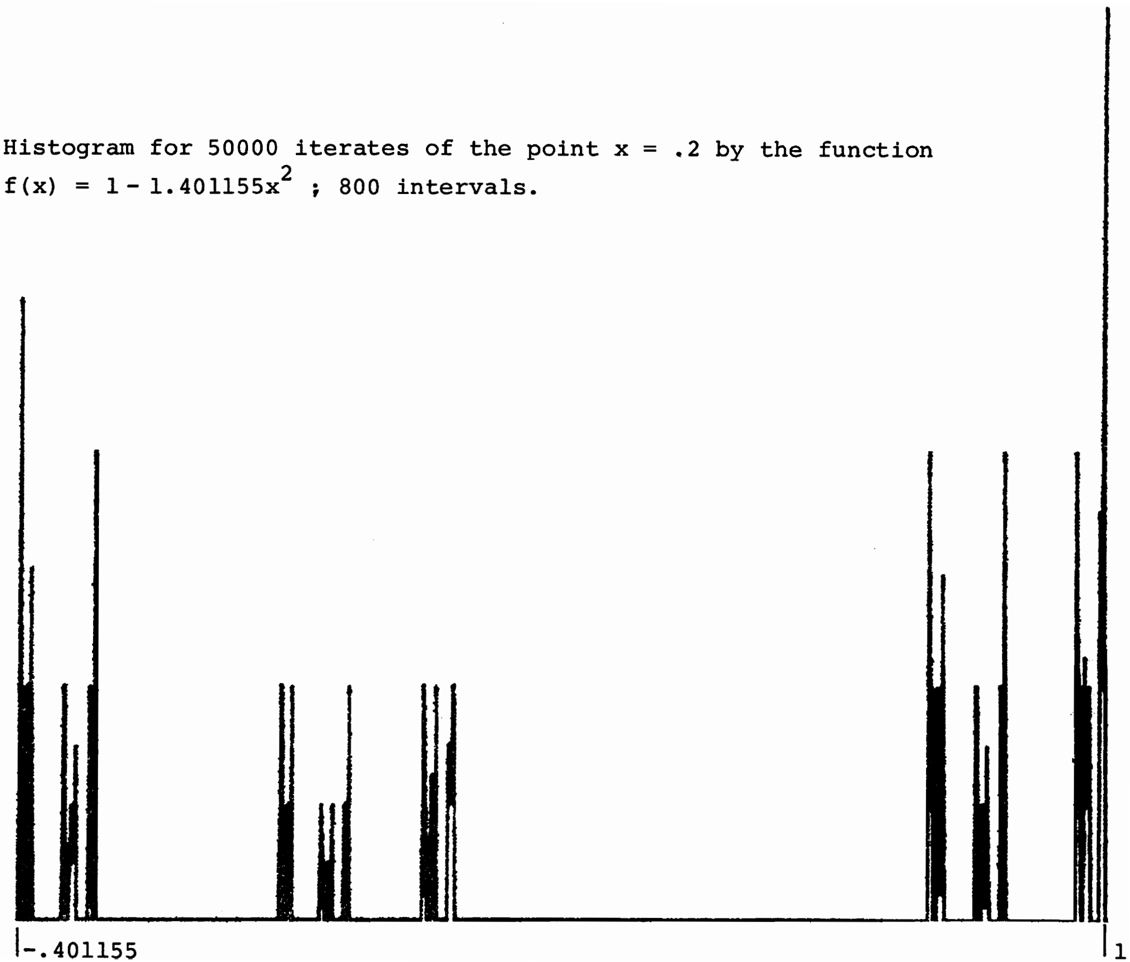


Figure I.16.

fact the orbits of nearby points stay close for almost all choices of the initial points. We have thus an ergodic but not a mixing behavior of the map in this case.

We wish to reformulate the "sensitive dependence" in a more measurable fashion. If we want to know how much we should expect two very close points to separate, we are naturally led to examine the derivative $Df^n = d/dx\ f^n$ of f^n. In the map $f(x) = 1 - 2|x|$ which we have analyzed before, it is obvious from the chain rule of differentiation that $|D\ f^n(x_0)| = 2^n$ unless one of the points $f^k(x_0)$

Fig. I.17. Histogram for 50000 iterates of the point $x = .2$ by the function $f(x) = 1 - 1.401155x^2$; 800 intervals.



equals zero, and then the quantity $Df^n(x_0)$ is undefined for $n > k$. We see thus that for <u>most</u> initial points the derivative along the orbit diverges like an exponential, namely like $\exp(n \log 2)$. The number $\log 2$ will then be called the <u>characteristic exponent</u> or <u>Liapunov exponent</u> of the map f. Now it so happens that many of the maps which show behavior similar to $x \to 1 - 2x^2$ have positive characteristic exponents (while the maps like $x \to 1 - 1.401155...x^2$ do have zero characteristic exponents). If a map has an invariant ergodic probability density (II.8) like $[\pi(1-y^2)^{1/2}]^{-1}$ in the case of the map $1 - 2x^2$, then for almost all initial points the quantity $\lim_{n\to\infty} 1/n \log|Df^n(x_0)|$ may exist and be positive, say equal to $\alpha > 0$. This will then have the interpretation that in the mean, two initial points which are very close will start to separate at the rate α^n during n iterations. The exact relations between invariant measures, characteristic exponents and sensitive dependence on initial conditions as well as topological and Kolmogorov-Sinai entropy are very subtle and not yet totally clarified, (cf. II.8).

IMPORTANT REMARK. When we have discussed above the "typical" behavior of a map, we have always insisted on analyzing what happens to <u>most</u> initial points. This is motivated by the fact that we want to make general statements about the behavior of dynamical systems. It happens very often that while <u>most</u> of the initial points show a very <u>regular</u> behavior (i.e., they approach a stable periodic orbit), some other initial points--<u>very few</u> in the sense of Lebesgue measure--behave rather in the ergodic way described above. Such a situation has been described in the paper by Li and Yorke [1975] "Period three implies chaos" (cf. II.3). They show, among other things, that if a map has a (stable or unstable) orbit of period three, then the map in question shows sensitive dependence with respect to initial conditions for an uncountable set of pairs of initial points. But--and maybe the paper did not make this sufficiently clear--for <u>most</u> other points this need not be the case, and hence from a physical point of view the chaotic behavior may be essentially unobservable.

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