

Chapter 2

Examples of Optimization of Discrete Parameter Systems

The following chapter gives some examples of the general optimization problem (SO) introduced in the previous chapter. They all concern the problem of finding the cross-sectional areas of bars or beams, i.e. they are sizing problems. The list of such examples is the following:

1. Minimization of the weight of a two-bar truss subject to stress constraints.
2. Minimization of the weight of a two-bar truss subject to stress and instability constraints.
3. Minimization of the weight of a two-bar truss subject to stress and displacement constraints.
4. Minimization of the weight of a two-beam cantilever subject to a displacement constraint.
5. Minimization of the weight of a three-bar truss subject to stress constraints.
6. Minimization of the weight of a three-bar truss subject to a stiffness constraint.

A simple example of combined shape and sizing optimization of a two-bar truss is given in Exercise 2.5. Despite their simplicity, it turns out that these problems display several general features of structural optimization problems.

The solution methods we will use in this chapter are of a very simple nature, and are applicable only when solving optimization problems with one or two design variables. Later, in Chaps. 3–5, we will study solution methods that are suitable for larger problems, and resolve some of the problems presented in this chapter.

2.1 Weight Minimization of a Two-Bar Truss Subject to Stress Constraints

Consider the two-bar truss shown in Fig. 2.1. The bars have the same length L and Young's modulus E . The force $F > 0$, and for the angle α we assume $0 \leq \alpha \leq 90^\circ$. We are to minimize the weight under stress constraints. The design variables are the cross-sectional areas A_1 and A_2 . The objective function, i.e., the total weight of the truss, becomes

$$f(A_1, A_2) = (A_1 + A_2)\rho L, \quad (2.1)$$

where ρ is the density of the material. It may be noted that this particular objective function does not depend on any state variables. As design constraints we prescribe that the cross-sectional areas must, for obvious physical reasons, be non-negative, i.e.,

$$A_1 \geq 0, \quad A_2 \geq 0. \quad (2.2)$$

Fig. 2.1 Two-bar truss. Find the cross-sectional areas that minimize weight under stress constraints

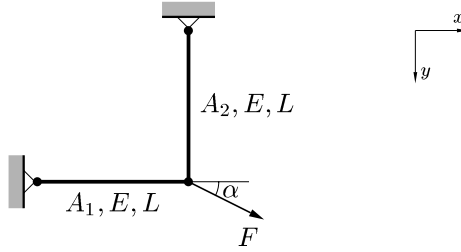
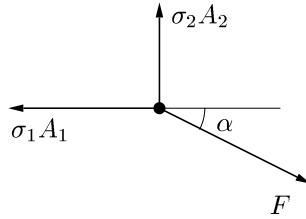


Fig. 2.2 Forces on the cut-out free node



In a truss problem of this type, the general approach would be to take the displacement vector \mathbf{u} of the free node as state variable and then establish a state constraint of the form $\mathbf{K}(A_1, A_2)\mathbf{u} = \mathbf{F}$ by making use of all three basic conditions of small displacement elasticity theory, i.e. *equilibrium* in terms of forces and stresses, *geometric conditions* relating the bars' elongations to the displacement vector, and a linear *constitutive law*. However, in this particular problem the number of bars equals the number of degrees-of-freedom, which implies that the bar forces, or stresses, may be obtained directly from the equilibrium equations. We say that the truss is *statically determinate*. Furthermore, the displacement is not present in the constraints nor in the objective function. Therefore, we do not need to formulate any constitutive or geometric relations in order to write down the optimization problem. The equilibrium equations are found from the free-body diagram of the free node as shown in Fig. 2.2. Equilibrium in the x - and y -directions gives

$$F \cos \alpha - \sigma_1 A_1 = 0, \quad F \sin \alpha - \sigma_2 A_2 = 0, \quad (2.3)$$

where we have opted to write the equations in terms of the bar stresses σ_1 and σ_2 directly, rather than first writing them in terms of the bar forces.

The state constraint involving stresses reads

$$|\sigma_i| \leq \sigma_0, \quad i = 1, 2, \quad (2.4)$$

where σ_0 is a maximum allowed stress level, the same in both tension and compression.

In summary, the particular version of the general (SO) problem that is at hand here is to find A_1 , A_2 , σ_1 and σ_2 such that (2.1) is minimized under the constraints (2.2), (2.3) and (2.4). In a nested version of this problem we eliminate σ_1 and σ_2 by using (2.3) in (2.4) to find

$$-\sigma_0 A_1 \leq F \cos \alpha \leq \sigma_0 A_1,$$

$$-\sigma_0 A_2 \leq F \sin \alpha \leq \sigma_0 A_2.$$

Since $F, \cos \alpha, \sin \alpha, A_1, A_2 \geq 0$ it is clear that the left-hand inequalities in these expressions are always satisfied, i.e., they are redundant and can be left out of the problem. Furthermore, the right-hand inequalities are

$$A_1 \geq \frac{F \cos \alpha}{\sigma_0}, \quad A_2 \geq \frac{F \sin \alpha}{\sigma_0},$$

which shows that the design constraints (2.2) are also redundant. We arrive at

$$(\text{SO})_{\text{nf}}^1 \quad \begin{cases} \min_{A_1, A_2} & A_1 + A_2 \\ \text{s.t.} & \begin{cases} A_1 \geq \frac{F \cos \alpha}{\sigma_0} \\ A_2 \geq \frac{F \sin \alpha}{\sigma_0} \end{cases} \end{cases}$$

where the constant factor ρL has been left out of the objective function since it does not affect the optimum values of A_1 and A_2 .

The problem $(\text{SO})_{\text{nf}}^1$ is a Linear Program (LP) in two variables and it is easily solved graphically as shown below. It should be noted that it is very unusual for a structural optimization problem to have a linear structure. In fact, it is even unusual for these problems to be convex. The fact that we find the LP structure in this case hinges on the simplicity of the constraints and objective function as well as on the statically determinate property.

In Fig. 2.3 a graphical solution of $(\text{SO})_{\text{nf}}^1$ is shown. In the A_1 - A_2 -plane we plot the lines defining the admissible domain. Next, we plot the line $A_1 + A_2 = \hat{f}(A_1, A_2) = \text{constant}$, representing the objective function. The solution is found when $\hat{f}(A_1, A_2)$ is given the smallest possible value that maintains part of the line in the admissible region. It is given by

$$A_1^* = \frac{F \cos \alpha}{\sigma_0}, \quad A_2^* = \frac{F \sin \alpha}{\sigma_0}.$$

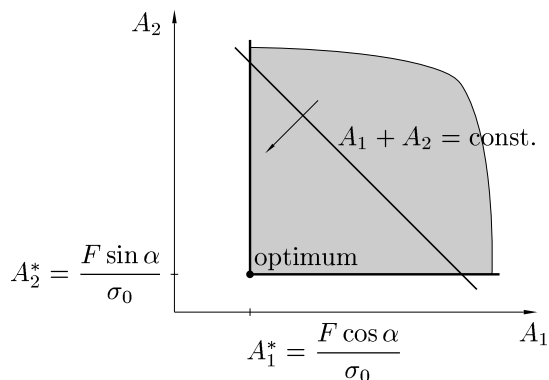


Fig. 2.3 Graphical solution of the problem

That is, both of the bars are fully used in tension: the stress is on the maximum level. It should be intuitively clear that this is a “good” structure from the point of view of using the least material.

Note that this problem, which is at the outset a sizing problem, is set so that topology may change: when $\alpha = 0$ or 90° one of the bars in the optimal solution “disappears.”

2.2 Weight Minimization of a Two-Bar Truss Subject to Stress and Instability Constraints

Consider a two-bar truss consisting of bars of length L and Young’s modulus E , placed at right angle according to Fig. 2.4. The force $F > 0$ is applied at an angle $\alpha = 45^\circ$. The problem is to find the circular cross-sectional areas A_1 and A_2 such that the weight of the truss is minimized under constraints on stresses and Euler buckling. The weight of the truss is

$$f(A_1, A_2) = \rho L(A_1 + A_2),$$

where ρ is the density of the material. The stress constraints are as usual

$$|\sigma_i| \leq \sigma_0, \quad i = 1, 2, \quad (2.5)$$

where $\sigma_0 > 0$ is the stress bound. Equilibrium for the free node gives the stresses in the bars as

$$\sigma_1 = \frac{F}{\sqrt{2}A_1}, \quad \sigma_2 = -\frac{F}{\sqrt{2}A_2},$$

so the stress constraints to be imposed in the optimization problem are

$$A_1 \geq \frac{F}{\sqrt{2}\sigma_0}, \quad A_2 \geq \frac{F}{\sqrt{2}\sigma_0}. \quad (2.6)$$

Clearly, these constraints imply that cross-sectional areas will be nonnegative so we do not need to impose such restrictions explicitly.

Concerning instability, we want to obtain a safety factor of 4 against Euler buckling. Such buckling can occur only in the second bar, since there is tensile stress in

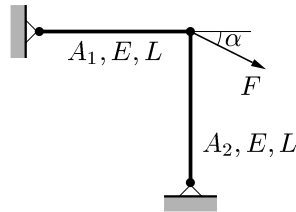


Fig. 2.4 A two-bar truss to be optimized under an instability constraint

the first bar. The buckling load for a hinged-hinged column is

$$P_c = \pi^2 \frac{EI}{L^2},$$

where for a circular cross section

$$I = \frac{A_2^2}{4\pi}.$$

Thus, the constraint

$$\frac{P_c}{4} \geq \sigma_2 A_2 = \frac{F}{\sqrt{2}}$$

becomes

$$A_2^2 \geq \frac{16FL^2}{\sqrt{2}\pi E}. \quad (2.7)$$

The optimization problem to be solved can, thus, be summarized as follows:

$$(\mathbb{SO})_{\text{nf}}^2 \quad \left\{ \begin{array}{l} \min_{A_1, A_2} A_1 + A_2 \\ \text{s.t.} \quad \left\{ \begin{array}{l} A_1 \geq \frac{F}{\sqrt{2}\sigma_0} \\ A_2 \geq \frac{F}{\sqrt{2}\sigma_0} \\ A_2^2 \geq \frac{16FL^2}{\sqrt{2}\pi E} \end{array} \right. \end{array} \right.$$

Depending on the values of the coefficients, the second or the third constraint will be active. Consider, for instance, the special case

$$\sigma_0 = \frac{E}{100}, \quad \sqrt{\frac{F}{\sigma_0}} = \frac{L}{4}.$$

Then, the constraints of $(\mathbb{SO})_{\text{nf}}^2$ become

$$A_1 \geq \frac{L^2}{16\sqrt{2}}, \quad A_2 \geq \frac{L^2}{16\sqrt{2}}, \quad A_2 \geq \frac{L^2}{10\sqrt{\sqrt{2}\pi}}$$

and since

$$1.6\sqrt{\sqrt{2}} > \sqrt{\pi} \quad \Leftrightarrow \quad \frac{L^2}{10\sqrt{\sqrt{2}\pi}} > \frac{L^2}{16\sqrt{2}},$$

it can be concluded that the optimum occurs when both the first and the third constraints are active, i.e., when

$$A_1^* = \frac{L^2}{16\sqrt{2}} \approx 0.044L^2, \quad A_2^* = \frac{L^2}{10\sqrt{\sqrt{2}\pi}} \approx 0.047L^2.$$

2.3 Weight Minimization of a Two-Bar Truss Subject to Stress and Displacement Constraints

Consider the truss in Fig. 2.5. The bars have lengths according to the figure, and consist of a material with Young's modulus E and density ρ . The force $F > 0$ and the angle $\alpha = 30^\circ$. We want to find the cross-sectional areas A_1 and A_2 such that the weight is minimized subject to stress constraints and a constraint on the tip displacement δ . The weight can be written

$$f(A_1, A_2) = \rho L \left(\frac{2}{\sqrt{3}} A_1 + A_2 \right). \quad (2.8)$$

The stress constraints are

$$|\sigma_i| \leq \sigma_0, \quad i = 1, 2, 3, \quad (2.9)$$

for a given stress bound $\sigma_0 > 0$. The displacement constraint is

$$\delta \leq \delta_0, \quad (2.10)$$

where

$$\delta_0 = \frac{\sigma_0 L}{E},$$

is a given bound on the tip displacement. The design constraints are

$$A_1 \geq 0, \quad A_2 \geq 0. \quad (2.11)$$

We are aiming at a nested formulation, and need to rewrite (2.9) and (2.10) in terms of cross-sectional areas. Equilibrium equations are obtained from Fig. 2.6.

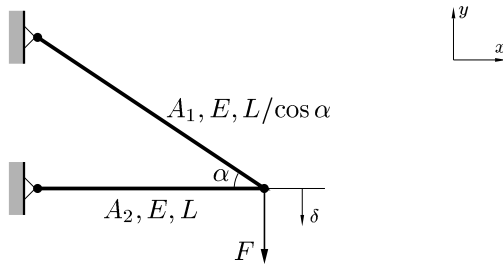
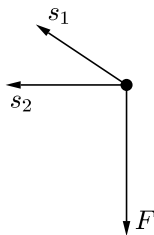


Fig. 2.5 Two bar truss subject to stress and displacement constraints

Fig. 2.6 Forces on the cut-out free node



The equations for the x - and y -directions become

$$-s_1 \cos \alpha - s_2 + F_x = 0, \quad s_1 \sin \alpha + F_y = 0,$$

where s_1 and s_2 are the bar forces, $F_x = 0$ and $F_y = -F$. These equations may be written in matrix form as

$$\begin{bmatrix} F_x \\ F_y \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} & 1 \\ -\frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \end{bmatrix}. \quad (2.12)$$

In symbolic matrix form this is written $\mathbf{F} = \mathbf{B}^T \mathbf{s}$. Here, superscript T denotes the transpose of a matrix; it will soon become apparent why we write (2.12) symbolically by use of the transpose of a matrix.

Since the number of bars equals the number of degrees-of-freedom, the truss is statically determinate, and we may obtain the bar forces \mathbf{s} by simply solving the equilibrium equations. From (2.12) we get

$$\mathbf{s} = \begin{bmatrix} s_1 \\ s_2 \end{bmatrix} = \mathbf{B}^{-T} \mathbf{F} = \begin{bmatrix} 2F \\ -\sqrt{3}F \end{bmatrix}. \quad (2.13)$$

In order to rewrite the displacement constraint (2.10) in terms of cross-sectional areas, we need to include geometric and constitutive conditions. In a small displacement theory, the elongations of the bars, δ_1 and δ_2 , are obtained by projecting the displacement vector $\mathbf{u} = [u_x \ u_y]^T$ of the free node on the unit vectors directed along the bars and pointing towards the free node:

$$\mathbf{e}_1 = \begin{bmatrix} \frac{\sqrt{3}}{2} \\ -\frac{1}{2} \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

The elongations thus become

$$\delta_1 = \mathbf{e}_1^T \mathbf{u} = \frac{\sqrt{3}}{2} u_x - \frac{1}{2} u_y, \quad \delta_2 = \mathbf{e}_2^T \mathbf{u} = u_x.$$

In matrix form this reads

$$\begin{bmatrix} \delta_1 \\ \delta_2 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ 1 & 0 \end{bmatrix} \begin{bmatrix} u_x \\ u_y \end{bmatrix}. \quad (2.14)$$

The—perhaps surprising—fact that occurs here is that the matrix of this equation is \mathbf{B} , i.e., the transpose of the matrix occurring in (2.12), so (2.14) can in symbolic matrix form be written as $\boldsymbol{\delta} = \mathbf{B}\mathbf{u}$. That \mathbf{B}^T and \mathbf{B} appear in this way in the equilibrium and geometric equations is not a coincidence: the same property holds in any truss problem and, in fact, given the right interpretation, in any small displacement structural problem. It is related to the validity of the work equation $\boldsymbol{\delta}^T \mathbf{s} = \mathbf{u}^T \mathbf{F}$, and it is a very economical fact since, given equilibrium, we can directly write down the geometric equations and vice versa.

Next, we need the constitutive equations. Hooke's law reads $\sigma_i = E\varepsilon_i$, where

$$\sigma_i = s_i / A_i, \quad \varepsilon_i = \delta_i / l_i,$$

are the stress and strain in bar i of length l_i . Combining these equations gives us the elongations in terms of the bar forces as

$$\delta_i = \frac{l_i s_i}{A_i E}. \quad (2.15)$$

From (2.13), and since $l_1 = 2L/\sqrt{3}$ and $l_2 = L$, we get

$$\boldsymbol{\delta} = \begin{bmatrix} \delta_1 \\ \delta_2 \end{bmatrix} = \begin{bmatrix} \frac{4FL}{\sqrt{3}A_1 E} \\ -\frac{\sqrt{3}FL}{A_2 E} \end{bmatrix}.$$

The displacements of the free node are thus given by

$$\mathbf{u} = \mathbf{B}^{-1} \boldsymbol{\delta} = \frac{FL}{E} \begin{bmatrix} -\frac{\sqrt{3}}{A_2} \\ -\frac{8}{\sqrt{3}A_1} - \frac{3}{A_2} \end{bmatrix}.$$

The tip displacement may now be written in terms of the cross-sectional areas as

$$\delta = -\mathbf{e}_y^T \mathbf{u} = \frac{FL}{E} \left(\frac{8}{\sqrt{3}A_1} + \frac{3}{A_2} \right),$$

where \mathbf{e}_y is the unit vector in the y -direction, so that (2.10) can be written

$$\frac{8}{\sqrt{3}A_1} + \frac{3}{A_2} \leq \frac{E\delta_0}{FL} = \frac{\sigma_0}{F}. \quad (2.16)$$

Regarding the stress constraints, we note from (2.13) and $F > 0$ that bar 1 is in tension and bar 2 in compression, so we need to consider only the stress constraints $s_1/A_1 \leq \sigma_0$ and $-s_2/A_2 \leq \sigma_0$, which with (2.13) lead to

$$A_1 \geq \frac{2F}{\sigma_0}, \quad A_2 \geq \frac{\sqrt{3}F}{\sigma_0}. \quad (2.17)$$

Since $F > 0$ and $\sigma_0 > 0$, we conclude that (2.11) are redundant: the optimal cross-sectional areas are strictly positive.

In summary, our problem is to minimize $f(A_1, A_2)$, according to (2.8), under constraints given by (2.16) and (2.17). Now, we will not treat this problem directly, but instead rewrite the problem by means of a change of variables. We do this to demonstrate the use of such ideas, since they will play an essential role in upcoming sections, and one may consider that the problem is also easier to solve in the new variables. These new, dimensionless variables are

$$x_1 = \frac{2F}{\sigma_0 A_1} > 0, \quad x_2 = \frac{\sqrt{3}F}{\sigma_0 A_2} > 0,$$

and the essential thing with these new variables is that they make the constraint (2.16) linear. Moreover, the new variables are scaled such that (2.17) becomes

$$1 \geq x_1, \quad 1 \geq x_2. \quad (2.18)$$

The displacement constraint (2.16) now becomes

$$\frac{4}{\sqrt{3}}x_1 + \sqrt{3}x_2 \leq 1, \quad (2.19)$$

and the objective function (2.8) is written as

$$f(A_1(x_1), A_2(x_2)) = \frac{\sqrt{3}\rho L F}{\sigma_0} \left(\frac{4}{3x_1} + \frac{1}{x_2} \right). \quad (2.20)$$

The constraint (2.19) gives the following estimates

$$1 \geq \frac{4}{\sqrt{3}}x_1 + \sqrt{3}x_2 \geq \sqrt{3}x_2, \quad 1 \geq \frac{4}{\sqrt{3}}x_1 + \sqrt{3}x_2 \geq \frac{4}{\sqrt{3}}x_1,$$

from which it is clear that (2.18) is redundant.

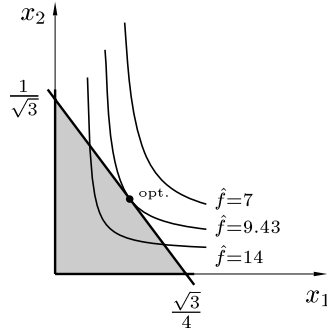
We have arrived at the following optimization problem

$$(\mathbb{SO})_{\text{nf}}^3 \quad \begin{cases} \min_{x_1, x_2} & \hat{f}(x_1, x_2) = \frac{4}{3x_1} + \frac{1}{x_2} \\ \text{s.t.} & \begin{cases} \frac{4}{\sqrt{3}}x_1 + \sqrt{3}x_2 \leq 1 \\ x_1 > 0, \quad x_2 > 0. \end{cases} \end{cases}$$

This problem is illustrated in Fig. 2.7, from which we conclude that constraint (2.19) is active. We write (2.19) as an equality and solve for x_2 to obtain

$$x_2 = \frac{1}{\sqrt{3}} - \frac{4}{3}x_1.$$

Fig. 2.7 Geometric illustration of $(\mathbb{SO})_{nf}^3$



This is then substituted into $\hat{f}(x_1, x_2)$, which becomes a function of x_1 for which we seek a stationary value. Such a stationary value is realized whenever

$$x_1 = \pm \left(\frac{1}{\sqrt{3}} - \frac{4}{3}x_1 \right).$$

The minus sign gives $x_1 = \sqrt{3}$ which is greater than 1 and, thus, not in the admissible domain. The plus sign gives the solution

$$x_1^* = \frac{\sqrt{3}}{7}, \quad x_2^* = \frac{\sqrt{3}}{7},$$

which, going back to the original variables, gives

$$A_1^* = \frac{14F}{\sqrt{3}\sigma_0} \approx \frac{8.1F}{\sigma_0}, \quad A_2^* = \frac{7F}{\sigma_0}.$$

2.4 Weight Minimization of a Two-Beam Cantilever Subject to a Displacement Constraint

Consider a cantilever beam, fixed at the left end and subject to a vertical force $F > 0$ at the right end. The beam consists of N segments, each of length L , so the total length of the cantilever is NL . Segment number N is to the left, at the built-in end, and segment 1 is at the free end. Each segment cross section has a hollow square form, see Fig. 2.8. The thickness of the material is t for all segments, and the length of the side of the square is x_A for segment $A = 1, \dots, N$. The bending moment of inertia, I_A can be calculated from classical formulas. If it is assumed that $t \ll x_A$, for all A , one finds:

$$I_A = \frac{x_A^4}{12} - \frac{(x_A - 2t)^4}{12} = \frac{2tx_A^3}{3}.$$

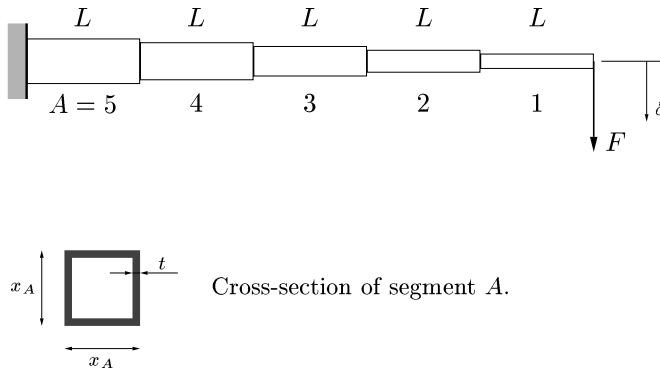


Fig. 2.8 The cantilever for 5 segments and the hollow square cross section

We want to minimize the weight of the beam under the constraint that the displacement at the tip, δ , is less than some prescribed value δ_0 . The design variables are the cross-sectional sizes x_A , $A = 1, \dots, N$. The weight when $t \ll x_A$ becomes

$$f(x_1, \dots, x_N) = L\rho \sum_{A=1}^N \left(x_A^2 - (x_A - 2t)^2 \right) = 4L\rho t \sum_{A=1}^N x_A,$$

where ρ is the density. The displacement at the tip of the beam can be seen as the sum of contributions from each segment, when other segments are considered as rigid, i.e.,

$$\delta = \sum_{A=1}^N \delta^{(A)}, \quad (2.21)$$

where $\delta^{(A)}$ is the displacement at the tip of the cantilever for a system where only segment A is elastic. Next, one concludes by simple geometry for small displacements, such that $\sin \theta_A \approx \theta_A$, that

$$\delta^{(A)} = \delta_A + (A - 1)L\theta_A, \quad (2.22)$$

where δ_A and θ_A are the displacement and the rotation at the right-hand side of segment A when only this segment is elastic, see Fig. 2.9. One calculates δ_A and θ_A by means of elementary beam theory as follows:

$$\delta_A = \frac{M_A L^2}{2EI_A} + \frac{F_A L^3}{3EI_A}, \quad (2.23)$$

$$\theta_A = \frac{M_A L}{EI_A} + \frac{F_A L^2}{2EI_A}, \quad (2.24)$$

where E is Young's modulus, and

$$M_A = (A - 1)LF, \quad F_A = F,$$

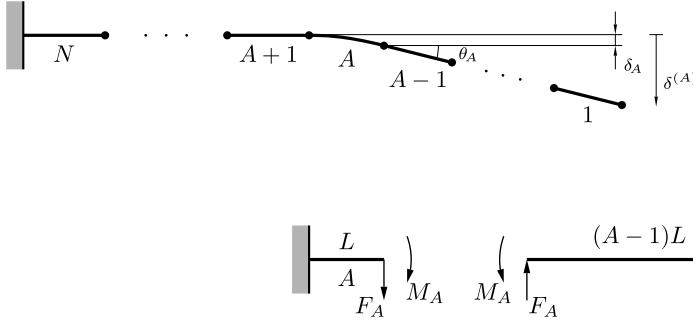


Fig. 2.9 The cantilever when only segment A is elastic

are the bending moment and the shear force at the right end of segment A . Putting (2.23) and (2.24) into (2.22), the result into (2.21) and using the above expression for I_A gives

$$\delta = \frac{3FL^3}{2Et} \sum_{A=1}^N \left(A^2 - A + \frac{1}{3} \right) \frac{1}{x_A^3}. \quad (2.25)$$

The present cantilever problem was originally formulated and solved analytically as well as numerically in Svanberg [34] for the case $N = 5$. Here we will be content with $N = 2$, which is easily solved analytically. For this case we have the following optimization problem:

$$(\mathbb{SO})_{\text{nf}}^4 \quad \begin{cases} \min_{x_1, x_2} f(x_1, x_2) = C_1(x_1 + x_2) \\ \text{s.t.} \quad \begin{cases} \frac{1}{x_1^3} + \frac{7}{x_2^3} \leq C_2 \\ x_1 > 0, \quad x_2 > 0, \end{cases} \end{cases}$$

where

$$C_1 = 4\rho Lt, \quad C_2 = \frac{2\delta_0 Et}{FL^3}.$$

Assuming equality in the nonstrict inequality constraint we solve this for x_2 . The result is put into $f(x_1, x_2)$, which becomes a function of x_1 only. Seeking a stationary value of this function gives the solution

$$x_1^* = \left(\frac{1 + 7^{1/4}}{C_2} \right)^{1/3}, \quad x_2^* = 7^{1/4} \left(\frac{1 + 7^{1/4}}{C_2} \right)^{1/3}.$$

Now, one may ask the question, what happens if we reverse the order of the structural measures in a problem of this kind, i.e., what if we minimize the tip dis-

placement under a constraint on the weight? We then have the following problem:

$$\begin{cases} \min_{x_1, x_2} & \frac{1}{x_1^3} + \frac{7}{x_2^3} \\ \text{s.t.} & \begin{cases} C_1(x_1 + x_2) \leq W \\ x_1 > 0, \quad x_2 > 0, \end{cases} \end{cases}$$

where W is some given allowable weight. This problem can be solved in the same way as $(\text{SO})_{\text{nf}}^4$. One finds the solution

$$x_1^{**} = \frac{W}{C_1} \left(\frac{1}{1 + 7^{1/4}} \right), \quad x_2^{**} = \frac{W}{C_1} \left(\frac{7^{1/4}}{1 + 7^{1/4}} \right),$$

and it can be concluded that the reversed problem gives a solution different from $(\text{SO})_{\text{nf}}^4$. However, it holds that

$$\frac{x_2^*}{x_1^*} = \frac{x_2^{**}}{x_1^{**}} = 7^{1/4} \approx 1.63.$$

Thus, the solution of $(\text{SO})_{\text{nf}}^4$ can be obtained by a scaling of the solution of the reversed problem and vice versa. This is a general property which will be discussed more thoroughly in Sect. 5.2.3.

2.5 Weight Minimization of a Three-Bar Truss Subject to Stress Constraints

Consider the three-bar truss shown in Fig. 2.10. The bars have Young's modulus E and the lengths are $l_1 = L$, $l_2 = L$, $l_3 = L/\beta$, where $\beta > 0$. In this example $\beta = 1$, but in the next section the same truss will be studied with $\beta = 1/10$. We will therefore perform all derivations for a general $\beta > 0$. The force $F > 0$. As for the two-bar truss in Sect. 2.1 we are to minimize the weight under stress constraints. The design variables are the cross-sectional areas A_1 , A_2 and A_3 , but for simplicity

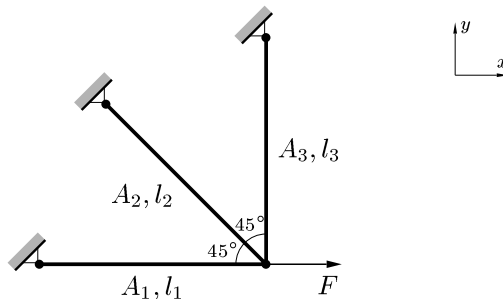


Fig. 2.10 Three-bar truss. Find the cross-sectional areas that minimize weight under stress constraints

we assume that

$$A_1 = A_3.$$

The objective function, which is the total weight of the truss, becomes

$$f(A_1, A_2) = \rho_1 L A_1 + \rho_2 L A_2 + \rho_3 \frac{L}{\beta} A_3 = L \left(\rho_1 + \frac{\rho_3}{\beta} \right) A_1 + L \rho_2 A_2, \quad (2.26)$$

where ρ_1 , ρ_2 and ρ_3 are the densities of the bars. The design constraints are

$$A_1 \geq 0, \quad A_2 \geq 0. \quad (2.27)$$

Concerning designs with A_1 or A_2 equal to zero, it is clearly impossible to have $A_1 = A_3 = 0$ since then there is no equilibrium possible as it would imply collapse of the structure under the given external load. On the other hand, $A_2 = 0$ is a valid design.

The state constraints are that the maximum absolute value of the stress in bar i must not exceed the values σ_i^{\max} , i.e.

$$|\sigma_i| \leq \sigma_i^{\max}, \quad i = 1, 2, 3. \quad (2.28)$$

The equilibrium equation is found by cutting out the free node as shown in Fig. 2.11. The equilibrium equations in the x - and y -directions become

$$-s_1 - \frac{s_2}{\sqrt{2}} + F = 0, \quad s_3 + \frac{s_2}{\sqrt{2}} = 0.$$

In matrix form these equations read

$$\begin{bmatrix} F \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{\sqrt{2}} & 0 \\ 0 & -\frac{1}{\sqrt{2}} & -1 \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix} \iff \mathbf{F} = \mathbf{B}^T \mathbf{s}. \quad (2.29)$$

Note that in contrast to the two-bar truss in Sect. 2.3, we cannot obtain the bar forces from the equilibrium equations alone since the number of bars exceeds the number of degrees-of-freedom. We say that the truss is *statically indeterminate*. In order to find the bar forces, or, rather, the stresses, that appear in the constraints, we need to make use of Hooke's law and the geometry conditions.

From (2.15) we have

$$s_i = \frac{E A_i \delta_i}{l_i}.$$

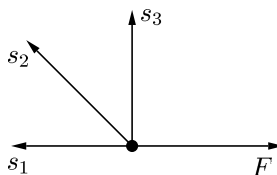


Fig. 2.11 Forces on the cut-out free node

We write these equations for all three bars in matrix form as

$$\mathbf{s} = \mathbf{D}\boldsymbol{\delta},$$

where

$$\mathbf{D} = \frac{E}{l} \begin{bmatrix} A_1 & 0 & 0 \\ 0 & A_2 & 0 \\ 0 & 0 & \beta A_1 \end{bmatrix}.$$

Since $\boldsymbol{\delta} = \mathbf{B}\mathbf{u}$, cf. the discussion following (2.14), the bar forces are obtained as

$$\mathbf{s} = \mathbf{D}\mathbf{B}\mathbf{u}. \quad (2.30)$$

The equilibrium equations (2.29) thus become

$$\mathbf{F} = \mathbf{B}^T \mathbf{s} = \mathbf{B}^T \mathbf{D}\mathbf{B}\mathbf{u} = \mathbf{K}\mathbf{u}, \quad (2.31)$$

where $\mathbf{K} = \mathbf{B}^T \mathbf{D}\mathbf{B}$ is the global stiffness matrix of the truss, which is easily calculated as

$$\mathbf{K} = \frac{E}{l} \begin{bmatrix} A_1 + \frac{A_2}{2} & -\frac{A_2}{2} \\ -\frac{A_2}{2} & \frac{A_2}{2} + \beta A_1 \end{bmatrix}.$$

From (2.31) we obtain the displacements of the free node as $\mathbf{u} = \mathbf{K}^{-1}\mathbf{F}$:

$$u_x = \frac{FL}{EA_1} \left(\frac{2\beta A_1 + A_2}{2\beta A_1 + (1 + \beta)A_2} \right), \quad (2.32)$$

$$u_y = \frac{FL}{EA_1} \left(\frac{A_2}{2\beta A_1 + (1 + \beta)A_2} \right). \quad (2.33)$$

Using (2.30), the stresses may be written as

$$\boldsymbol{\sigma} = \mathbf{A}\mathbf{s} = \mathbf{A}\mathbf{D}\mathbf{B}\mathbf{u},$$

where

$$\mathbf{A} = \begin{bmatrix} \frac{1}{A_1} & 0 & 0 \\ 0 & \frac{1}{A_2} & 0 \\ 0 & 0 & \frac{1}{A_1} \end{bmatrix}.$$

Some straightforward calculations give us the bar stresses as

$$\sigma_1 = \frac{F}{2\beta A_1 + (1 + \beta)A_2} \left(2\beta + \frac{A_2}{A_1} \right), \quad (2.34)$$

$$\sigma_2 = \frac{\sqrt{2}F\beta}{2\beta A_1 + (1 + \beta)A_2}, \quad (2.35)$$

$$\sigma_3 = -\frac{F\beta\frac{A_2}{A_1}}{2\beta A_1 + (1+\beta)A_2}. \quad (2.36)$$

Since $F, A_1, A_2 > 0$, we conclude that bars 1 and 2 are in tension and bar 3 is in compression, so only the stress constraints $\sigma_1 \leq \sigma_1^{\max}$, $\sigma_2 \leq \sigma_2^{\max}$ and $-\sigma_3^{\max} \leq \sigma_3$ need to be considered. In what follows we will put $\beta = 1$, i.e. $l_3 = L$. The stress constraint $\sigma_1 \leq \sigma_1^{\max}$ then takes the form

$$\frac{F(2A_1 + A_2)}{2A_1(A_1 + A_2)} \leq \sigma_1^{\max}. \quad (2.37)$$

The constraint $\sigma_2 \leq \sigma_2^{\max}$ reads

$$\frac{F}{\sqrt{2}(A_1 + A_2)} \leq \sigma_2^{\max}. \quad (2.38)$$

Naturally, this constraint should only be included if bar 2 is present, i.e. if $A_2 > 0$. Finally, the stress constraint $-\sigma_3^{\max} \leq \sigma_3$ is written as

$$\frac{FA_2}{2A_1(A_1 + A_2)} \leq \sigma_3^{\max}. \quad (2.39)$$

We have arrived at the following optimization problem

$$(\text{SO})_{\text{nf}}^5 \left\{ \begin{array}{l} \min_{A_1, A_2} (\rho_1 A_1 + \rho_2 A_2 + \rho_3 A_1) L \\ \text{s.t.} \left\{ \begin{array}{l} \frac{F(2A_1 + A_2)}{2A_1(A_1 + A_2)} - \sigma_1^{\max} \leq 0 \\ \frac{F}{\sqrt{2}(A_1 + A_2)} - \sigma_2^{\max} \leq 0 \quad \text{if } A_2 > 0 \\ \frac{FA_2}{2A_1(A_1 + A_2)} - \sigma_3^{\max} \leq 0 \\ A_1 > 0, \quad A_2 \geq 0. \end{array} \right. \end{array} \right.$$

In order to illustrate that all bars may not be present in the optimal truss, and that structural optimization problems may have more than one, and even an infinite number of solutions, we will solve this problem for five different cases by altering the density and the yield stress of the bars.

CASE A)

$\rho_1 = 2\rho_0$, $\rho_2 = \rho_3 = \rho_0$, $\sigma_1^{\max} = \sigma_2^{\max} = \sigma_3^{\max} = \sigma_0$.

By introducing the new dimensionless variables x_1 and x_2 as

$$x_1 = \frac{A_1 \sigma_0}{F}, \quad x_2 = \frac{A_2 \sigma_0}{F}, \quad (2.40)$$

we may write the optimization problem as

$$(\text{SO})_{\text{nf}}^{5a} \left\{ \begin{array}{l} \min_{x_1, x_2} 3x_1 + x_2 \\ \text{s.t.} \left\{ \begin{array}{ll} \frac{2x_1 + x_2}{2x_1(x_1 + x_2)} - 1 \leq 0 & (\sigma_1) \\ \frac{1}{\sqrt{2}(x_1 + x_2)} - 1 \leq 0 & \text{if } x_2 > 0 \quad (\sigma_2) \\ \frac{x_2}{2x_1(x_1 + x_2)} - 1 \leq 0 & (\sigma_3) \end{array} \right. \\ x_1 > 0, \quad x_2 \geq 0, \end{array} \right.$$

where, for simplicity, the objective function is the weight divided by the positive scalar $FL\rho_0/\sigma_0$. The problem is illustrated in Fig. 2.12. Note that the σ_2 -constraint is linear. It is clear that the σ_1 -constraint (2.37) is active at the solution, and that all other constraints are inactive. Thus

$$2x_1 + x_2 - 2x_1(x_1 + x_2) = 0,$$

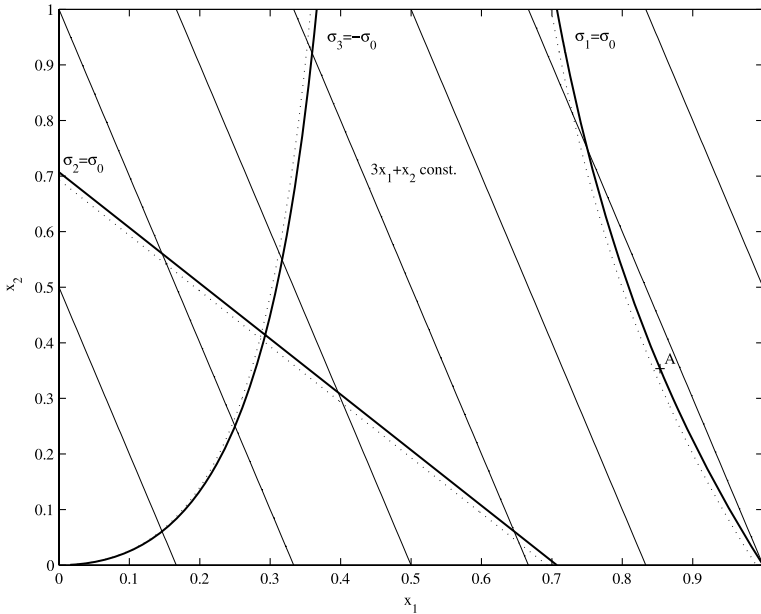


Fig. 2.12 Case a). A solid thick line with a dotted line alongside indicates a constraint; the region on the same side of the thick line as the corresponding dotted line is not part of the design space. The thin solid lines are iso-merit lines, i.e. all points on a thin line yield the same value of the objective function. Point A is the solution

which gives

$$x_2 = \frac{2x_1(x_1 - 1)}{1 - 2x_1}. \quad (2.41)$$

Substituting this into the objective function we find that the problem is reduced to minimizing

$$3x_1 + \frac{2x_1(x_1 - 1)}{1 - 2x_1},$$

for $x_1 > 0$. We find that this function has a stationary value for x_1 satisfying the second order equation

$$8x_1^2 - 8x_1 + 1 = 0.$$

The solution of this equation is

$$x_1^* = \frac{1}{2} \pm \frac{\sqrt{2}}{4},$$

where the minus sign is not valid since upon substitution into (2.41) it gives a negative x_2^* . Using the plus sign instead, gives

$$x_2^* = \frac{\sqrt{2}}{4}.$$

Reverting to the original area variables A_1 and A_2 , cf. (2.40), the optimal solution is

$$A_1^* = \frac{F}{2\sigma_0} \left(1 + \frac{1}{\sqrt{2}} \right), \quad A_2^* = \frac{F}{2\sqrt{2}\sigma_0},$$

and the corresponding optimum weight is

$$(3A_1^* + A_2^*)\rho_0 L = \frac{FL\rho_0}{\sigma_0} \left(\frac{3}{2} + \sqrt{2} \right).$$

CASE B)

$\rho_1 = \rho_2 = \rho_3 = \rho_0$, $\sigma_1^{\max} = \sigma_3^{\max} = 2\sigma_0$, $\sigma_2^{\max} = \sigma_0$.

Using the same dimensionless variables as for the previous case, we may write the problem as

$$(\text{SO})_{\text{nf}}^{5b} \left\{ \begin{array}{l} \min_{x_1, x_2} 2x_1 + x_2 \\ \text{s.t.} \left\{ \begin{array}{ll} \frac{2x_1 + x_2}{4x_1(x_1 + x_2)} - 1 \leq 0 & (\sigma_1) \\ \frac{1}{\sqrt{2}(x_1 + x_2)} - 1 \leq 0 & \text{if } x_2 > 0 \quad (\sigma_2) \\ \frac{x_2}{4x_1(x_1 + x_2)} - 1 \leq 0 & (\sigma_3) \\ x_1 > 0, \quad x_2 \geq 0, \end{array} \right. \end{array} \right.$$

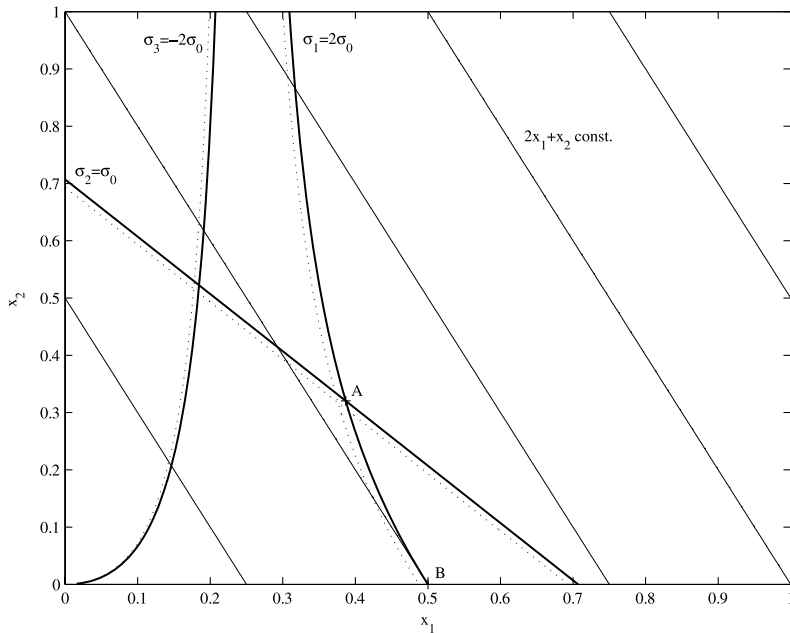


Fig. 2.13 Case b). Point B is the solution

see Fig. 2.13. It would appear that the solution is at the intersection A of the σ_1 - and σ_2 -constraints. However, we must keep in mind that the σ_2 -constraint is valid only for $x_2 > 0$. By deleting this constraint, it is evident from the figure, that the point B on the σ_1 -constraint curve, where x_2 is zero, gives the lowest weight that can be attained. This point is obtained by letting $x_2 = 0$ in the active σ_1 -constraint:

$$2x_1 - 4x_1^2 = 0,$$

which gives $x_1^* = 1/2$ as $x_1^* = 0$ is not a valid design. In the original variables, the optimum solution becomes

$$A_1^* = \frac{F}{2\sigma_0}, \quad A_2^* = 0,$$

with the optimal weight

$$\frac{FL\rho_0}{\sigma_0}.$$

CASE C)

$$\rho_1 = (2\sqrt{2} - 1)\rho_0, \quad \rho_2 = \rho_3 = \rho_0, \quad \sigma_1^{\max} = \sigma_3^{\max} = 2\sigma_0, \quad \sigma_2^{\max} = \sigma_0.$$

The density of bar 1 is now increased somewhat as compared to case b). This will alter the objective function but not the constraints:

$$(\text{SO})_{\text{nf}}^{5c} \quad \begin{cases} \min_{x_1, x_2} & 2\sqrt{2}x_1 + x_2 \\ \text{s.t.} & \text{the constraints in } (\text{SO})_{\text{nf}}^{5b}, \end{cases}$$

which is illustrated in Fig. 2.14. It is not evident from the figure whether the solution is at the intersection A between the σ_1 - and σ_2 -constraints, or the point B corresponding to a design without bar 2. Point A may be calculated by solving the two equations obtained when equality is satisfied in the σ_1 - and σ_2 -constraints, which leads to

$$x_1^* = \frac{4 + \sqrt{2}}{14}, \quad x_2^* = \frac{6\sqrt{2} - 4}{14}.$$

Point B is $x_1^{**} = 1/2$, $x_2^{**} = 0$. It turns out that these two points yield the same value of the objective function, and thus, there are two solutions to this problem! In the original variables, the solutions are written

$$A_1^* = \frac{F}{\sigma_0} \left(\frac{4 + \sqrt{2}}{14} \right), \quad A_2^* = \frac{F}{\sigma_0} \left(\frac{6\sqrt{2} - 4}{14} \right),$$

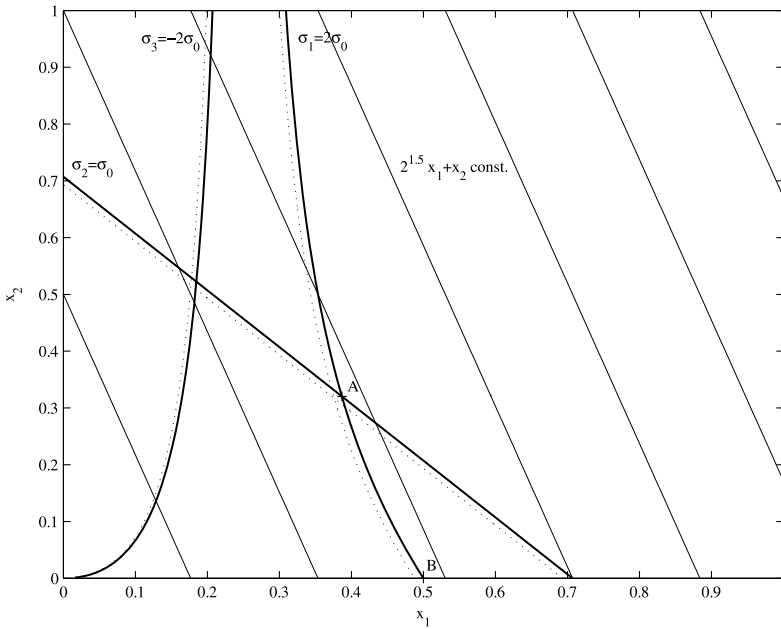


Fig. 2.14 Case c). Points A and B are the solutions

$$A_1^{**} = \frac{F}{2\sigma_0}, \quad A_2^{**} = 0,$$

with the optimum weight

$$\frac{\sqrt{2}FL\rho_0}{\sigma_0}.$$

CASE D)

$$\rho_1 = 3\rho_0, \quad \rho_2 = \rho_3 = \rho_0, \quad \sigma_1^{\max} = \sigma_3^{\max} = 2\sigma_0, \quad \sigma_2^{\max} = \sigma_0.$$

Again, the density of bar 1 is increased. The optimization problem becomes

$$(\text{SO})_{\text{nf}}^{5d} \quad \begin{cases} \min_{x_1, x_2} & 4x_1 + x_2 \\ \text{s.t.} & \text{the constraints in } (\text{SO})_{\text{nf}}^{5b}. \end{cases}$$

In Fig. 2.15, we see that the σ_1 - and σ_2 -constraints are active at the solution. This point has already been calculated for case c) as

$$A_1^* = \frac{F}{\sigma_0} \left(\frac{4 + \sqrt{2}}{14} \right), \quad A_2^* = \frac{F}{\sigma_0} \left(\frac{6\sqrt{2} - 4}{14} \right),$$

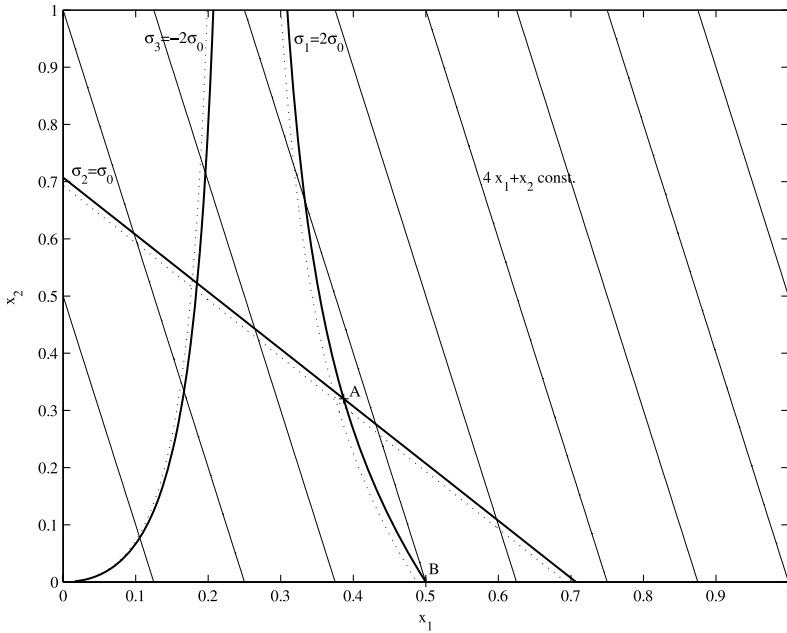


Fig. 2.15 Case d). Point A is the solution

which gives the optimal weight

$$\frac{FL\rho_0}{\sigma_0} \left(\frac{6 + 5\sqrt{2}}{7} \right).$$

CASE E)

$\rho_1 = \rho_3 = \rho_0$, $\rho_2 = 2\rho_0$, $\sigma_1^{\max} = \sigma_3^{\max} = 2\sigma_0$, $\sigma_2^{\max} = \sigma_0$.

Finally, we modify case b) by doubling the density of bar 2, which leads to the problem

$$(\mathbb{SO})_{\text{nf}}^{5e} \quad \begin{cases} \min_{x_1, x_2} & x_1 + x_2 \\ \text{s.t.} & \text{the constraints in } (\mathbb{SO})_{\text{nf}}^{5b}, \end{cases}$$

see Fig. 2.16. The solution point is point *B*, with the optimal truss lacking bar 2:

$$A_1^* = \frac{F}{2\sigma_0}, \quad A_2^* = 0,$$

with the optimal weight

$$\frac{FL\rho_0}{\sigma_0}.$$

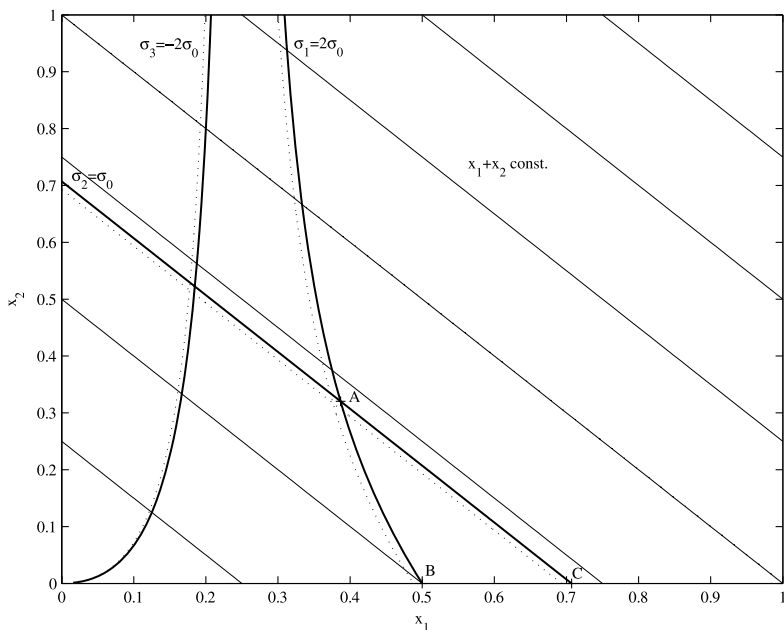


Fig. 2.16 Case e). Point *B* is the solution

This is the same solution as for case b). The reason that we get the same solution although we have doubled the density of bar 2 is of course that bar 2 is not present in the optimal trusses.

Assume now that A_2 is not allowed to become too small: $A_2 \geq 0.1F/\sigma_0$, i.e. $x_2 \geq 0.1$. Since the σ_2 -constraint curve is parallel to the iso-merit lines, we conclude that in this case there will be an infinite number of solutions, namely all points on the line between A and C in Fig. 2.16 for which $x_2 \geq 0.1$! Here, C is the point with $x_1 = 1/\sqrt{2}$ and $x_2 = 0$.

2.6 Weight Minimization of a Three-Bar Truss Subject to a Stiffness Constraint

In this section, the weight of the three-bar truss in the previous section will be minimized under a stiffness constraint; the two-norm of the displacement vector has to be lower than a prescribed value $\delta_0 > 0$, i.e. $\mathbf{u}^T \mathbf{u} \leq \delta_0^2$. The scalar $\beta = 1/10$, i.e. bar 3 is 10 times longer than bars 1 and 2. The displacements of the free node are given in (2.32)–(2.33). Inserting $\beta = 1/10$ into these expressions we get

$$\mathbf{u} = \frac{FL}{EA_1(2A_1 + 11A_2)} \begin{pmatrix} 2A_1 + 10A_2 \\ 10A_2 \end{pmatrix},$$

so that the stiffness constraint may be written as

$$\mathbf{u}^T \mathbf{u} = \frac{F^2 L^2 (4A_1^2 + 200A_2^2 + 40A_1 A_2)}{E^2 A_1^2 (2A_1 + 11A_2)^2} \leq \delta_0^2.$$

The density of all bars is ρ_0 , which gives the objective function

$$W = \rho_0 L (11A_1 + A_2).$$

Dimensionless variables are introduced according to

$$x_i = \frac{E\delta_0}{FL} A_i, \quad i = 1, 2. \quad (2.42)$$

Writing the optimization problem in terms of these variables leads to

$$(\mathbb{SO})_{\text{nf}}^6 \quad \begin{cases} \min_{x_1, x_2} & 11x_1 + x_2 \\ \text{s.t.} & \begin{cases} \frac{4x_1^2 + 200x_2^2 + 40x_1 x_2}{x_1^2 (2x_1 + 11x_2)^2} - 1 \leq 0 \\ x_1 > 0, \quad x_2 \geq 0, \end{cases} \end{cases}$$

where we have scaled the objective function by a factor $E\delta_0/(\rho_0 FL^2)$. This problem is illustrated in Fig. 2.17. At first glance it would appear that the solution is

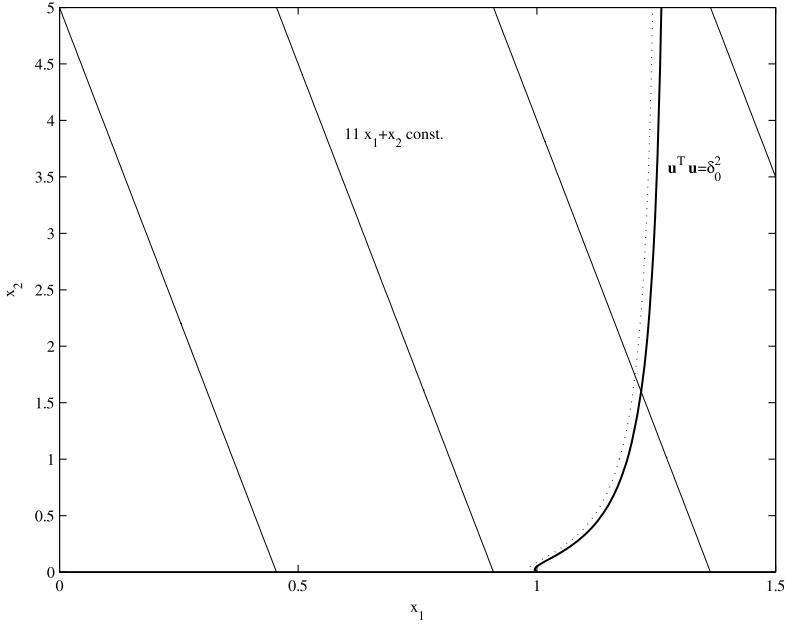


Fig. 2.17 Illustration of problem $(\text{SO})_{\text{nf}}^6$

$x_1 = 1, x_2 = 0$. The zoom plots in Fig. 2.18 reveal, however, that this is not the case. The solution may be obtained by first solving the active stiffness constraint equation for x_2 in terms of x_1 , and then solving the highly nonlinear one-dimensional optimization problem in the variable x_1 obtained by insertion of the expression for x_2 into the objective function. The solution of the problem is $x_1^* = 0.995, x_2^* = 0.0169$. By using (2.42), the corresponding optimal cross-sectional areas are obtained. As a much simpler alternative solution procedure for the two-dimensional optimization problem $(\text{SO})_{\text{nf}}^6$ at hand, we can simply produce finer and finer zoom plots similar to those in Fig. 2.18 and read off the solution.

Since the optimum thickness of bar 2 is very small, it is interesting to investigate how much heavier the optimum structure would be if bar 2 were removed. With no bar 2, the stiffness constraint reads

$$\frac{1}{x_1^2} - 1 \leq 0,$$

whereas the objective function becomes $11x_1$. Thus, with bar 2 removed, the optimal solution is $x_1^* = 1$ and the corresponding (scaled) weight is 11. With bar 2 present, the optimum weight is 10.965, i.e. only 0.3% less than with no bar 2. Since the production cost of the truss would most certainly be significantly less with no bar 2 present, it would make little sense to manufacture the truss with bar 2 included. This serves to illustrate that one should never uncritically accept a solution obtained by performing structural optimization. Finally, we remark that it would have been

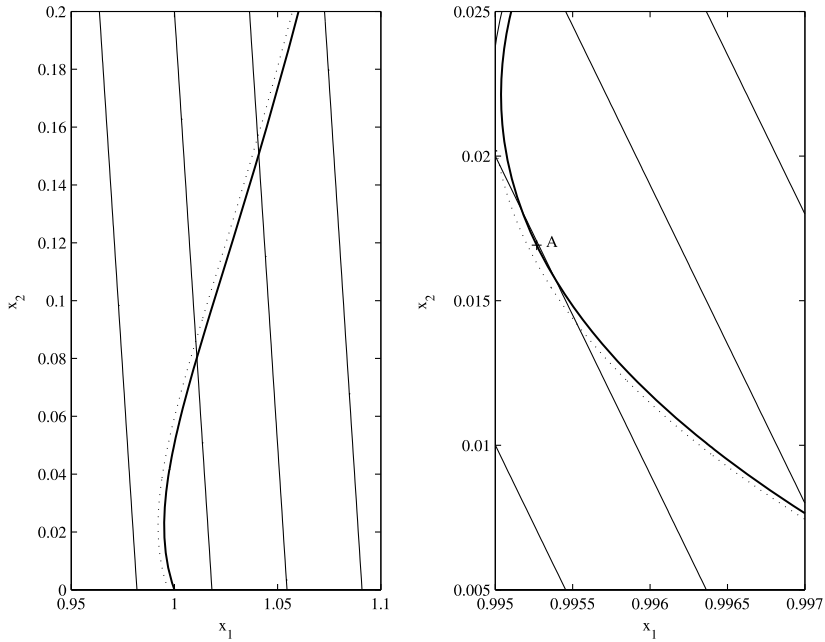


Fig. 2.18 Point A is the solution of problem $(\mathbb{SO})_{\text{nf}}^6$

possible to avoid an optimal solution with a very thin bar 2 if the minimization of the manufacturing cost had, somehow, been included in the optimization problem.

2.7 Exercises

Exercise 2.1 What happens if $F < 0$ in the example of Sect. 2.1?

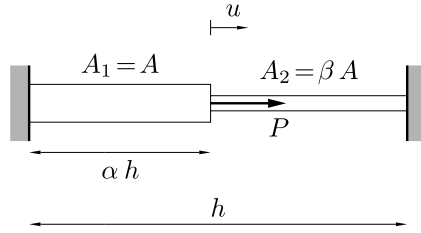
Exercise 2.2 If the length of the second bar in the example of Sect. 2.5 is changed, the optimum topology of the truss changes: the optimum area of the second bar is zero for $l_2 \geq L$ given $\beta = 1$, $\rho_i = \rho_0$, and $\sigma_i^{\max} = \sigma_0$, $i = 1, 2, 3$. Verify this for a special case, e.g., $l_2 = \sqrt{2}L$.

Exercise 2.3 How does the solution of the example of Sect. 2.5 change if the maximum allowable stress in compression is lower than that in tension?

Exercise 2.4 Verify the details leading to the solutions (x_1^*, x_2^*) and (x_1^{**}, x_2^{**}) in Sect. 2.4.

Exercise 2.5 The stiffness of the two-bar truss subjected to the force $P > 0$ in Fig. 2.19 should be maximized by minimizing the displacement u of the free node.

Fig. 2.19 The one-dimensional two-bar truss of Exercise 2.5



Young's modulus is E for both bars. The volume of the truss is not allowed to exceed the value V_0 . The total length of the bars is h , and bar 1 has length αh , where α is a scalar between α_{\min} and α_{\max} . The cross-sectional areas of the bars are $A_1 = A$ and $A_2 = \beta A$, where β is a scalar. The design variables are α and β . Since α determines the “shape” of the truss, and β the cross-sectional area of bar 2, the problem to be solved is a combined shape and sizing optimization problem.

a) Show that the problem may be formulated as the following mathematical programming problem:

$$\begin{cases} \min_{\alpha, \beta} & \frac{\alpha(1 - \alpha)}{1 - \alpha + \alpha\beta} \\ \text{s.t.} & \begin{cases} g_1 = \alpha + (1 - \alpha)\beta - \frac{V_0}{Ah} \leq 0 \\ \alpha_{\min} \leq \alpha \leq \alpha_{\max}, \quad \beta \geq 0. \end{cases} \end{cases}$$

Let $V_0/(Ah) = 1$ and $\alpha_{\max} = 1$. Show that the set $\{(\alpha, \beta) : g_1 \leq 0, \alpha_{\min} \leq \alpha \leq 1, \beta \geq 0\} = \{(\alpha, \beta) : \alpha_{\min} \leq \alpha \leq 1, 0 \leq \beta \leq 1\} \cup \{(\alpha, \beta) : \alpha = 1, \beta > 1\}$. Solve the problem for arbitrary α_{\min} .

b) Let $V_0/(Ah) = 1.2$ and $\alpha_{\min} = 0.2$. Solve the problem for $\alpha_{\max} = 0.6$ and $\alpha_{\max} = 0.8$.

An Introduction to Structural Optimization

Christensen, P.; Klarbring, A.

2009, X, 214 p., Hardcover

ISBN: 978-1-4020-8665-6