2

QUANTUM ELECTRODYNAMICS

2.1. BASIC LAGRANGIAN AND HAMILTONIAN FORMALISM FOR THE ELECTROMAGNETIC FIELD

The author studied the dynamics of the electromagnetic field in a lagrangian framework; the Lagrangian density L was deduced from a least action principle and, following a canonical formalism, the Hamiltonian density H was then obtained.

$$\delta \int L \, \mathrm{d}s \, \mathrm{d}t = 0,$$

$$\frac{1}{c}\dot{\varphi} + \boldsymbol{\nabla} \cdot \boldsymbol{A} = 0,$$

$$\begin{aligned} L \quad = \quad & \frac{1}{8\pi}\left\{-\frac{1}{c^2}\dot{\varphi}^2 + |\boldsymbol{\nabla}\,\varphi|^2 + \frac{1}{c^2}(\dot{A}_x^2 + \dot{A}_y^2 + \dot{A}_z^2)\right. \\ & \left. - |\boldsymbol{\nabla}\,A_x|^2 - |\boldsymbol{\nabla}\,A_y|^2 - |\boldsymbol{\nabla}\,A_z|^2\right\}. \end{aligned}$$

$$\begin{array}{ll} \varphi, & P_0 = -\dfrac{1}{4\pi c^2}\dot{\varphi}, \\ A_x, & P_x = \dfrac{1}{4\pi c^2}\dot{A}_x, \\ A_y, & P_y = \dfrac{1}{4\pi c^2}\dot{A}_y, \\ A_z, & P_z = \dfrac{1}{4\pi c^2}\dot{A}_z, \end{array}$$

$$\begin{aligned} \Box\,\varphi &= 0, \\ \Box\,\boldsymbol{A} &= 0. \end{aligned}$$

$$
\begin{aligned}
\mathcal{E} &= -\nabla \varphi - \frac{1}{c}\dot{\boldsymbol{A}}. \\
\mathcal{H} &= \nabla \times \boldsymbol{A}.
\end{aligned}
$$

$$
\begin{aligned}
H &= P_0\dot{\varphi} + P_x\dot{A}_x + P_y\dot{A}_y + P_z\dot{A}_z - L \\
&= \frac{1}{8\pi}\left\{-\frac{1}{c^2}\dot{\varphi}^2 - |\nabla \varphi|^2 + \frac{1}{c^2}(\dot{A}_x^2 + \dot{A}_y^2 + \dot{A}_z^2) + |\nabla A_x|^2 \right. \\
&\quad \left. + |\nabla A_y|^2 + |\nabla A_z|^2 \right\} \\
&= 2\pi c^2(-P_0^2 + P_x^2 + P_y^2 + P_z^2) \\
&\quad + \frac{1}{8\pi}\left(-|\nabla \varphi|^2 + |\nabla A_x|^2 + |\nabla A_y|^2 + |\nabla A_z|^2\right),
\end{aligned}
$$

$$
4\pi c P_0 = \nabla \cdot \boldsymbol{A},
$$

$$
\frac{1}{c}\dot{\varphi} + \nabla \cdot \boldsymbol{A} = 0,
$$

$$
\nabla^2 \varphi + \frac{1}{c}\nabla \cdot \dot{\boldsymbol{A}} = 0.
$$

$$
\begin{aligned}
\int H \,\mathrm{d}s &= \frac{1}{8\pi}\int\left\{-(\nabla \cdot \boldsymbol{A})^2 - |\nabla \varphi|^2 + \frac{1}{c^2}(\dot{A}_x^2 + \dot{A}_y^2 + \dot{A}_z^2) \right. \\
&\qquad \left. + |\nabla A_x^2| + |\nabla A_y|^2 + |\nabla A_z|^2 \right\}\mathrm{d}s \\
&= \frac{1}{8\pi}\int\left\{-(\nabla \cdot \boldsymbol{A})^2 + \varphi\,\nabla^2 \varphi + \frac{1}{c^2}(\dot{A}_x^2 + \dot{A}_y^2 + \dot{A}_z^2) \right. \\
&\qquad \left. - \boldsymbol{A} \cdot \nabla^2 \boldsymbol{A} \right\}\mathrm{d}s.
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{E} &= -\nabla \varphi - \frac{1}{c}\dot{\boldsymbol{A}}, \\
\int \mathcal{E}^2 \mathrm{d}s &= \int\left\{|\nabla \varphi|^2 + \frac{2}{c}(\nabla \varphi)\cdot\dot{\boldsymbol{A}} + \frac{1}{c^2}(\dot{A}_x^2 + \dot{A}_y^2 + \dot{A}_z^2)\right\}\mathrm{d}s \\
&= \int\left\{-\varphi\,\nabla^2 \varphi - \frac{2}{c}\varphi\,\nabla \cdot \dot{\boldsymbol{A}} + \frac{1}{c^2}(\dot{A}_x^2 + \dot{A}_y^2 + \dot{A}_z^2)\right\}\mathrm{d}s \\
&= \int\left\{\varphi\,\nabla^2 \varphi + \frac{1}{c^2}(\dot{A}_x^2 + \dot{A}_y^2 + \dot{A}_z^2)\right\}\mathrm{d}s,
\end{aligned}
$$

$$\begin{aligned}
\mathcal{H} &= \boldsymbol{\nabla} \times \boldsymbol{A}, \\
\int \mathcal{H}^2 \mathrm{d}s &= \int |\boldsymbol{\nabla} \times \boldsymbol{A}|^2 \mathrm{d}s = \int \boldsymbol{A} \cdot \boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \boldsymbol{A} \; \mathrm{d}s \\
&= \int \left\{ \boldsymbol{A} \cdot \boldsymbol{\nabla} \left(\boldsymbol{\nabla} \cdot \boldsymbol{A} \right) - \boldsymbol{A} \cdot \boldsymbol{\nabla}^2 \boldsymbol{A} \right\} \mathrm{d}s \\
&= \int \left\{ -(\boldsymbol{\nabla} \cdot A)^2 - \boldsymbol{A} \cdot \boldsymbol{\nabla}^2 \boldsymbol{A} \right\} \mathrm{d}s,
\end{aligned}$$

[1]

$$\int H \, \mathrm{d}s = \frac{1}{8\pi} \int (\mathcal{E}^2 + \mathcal{H}^2) \, \mathrm{d}s.$$

2.2. ANALOGY BETWEEN THE ELECTROMAGNETIC FIELD AND THE DIRAC FIELD

In the following pages, the author explored the possibility of describing the electromagnetic field in full analogy with what usually done for a Dirac field. In a three-dimensional formalism, he then introduced a wavefunction $\boldsymbol{\psi}$ *in terms of the electric and magnetic fields* $\boldsymbol{E}, \boldsymbol{H}$ *(and, more specifically, in terms of quantities* $\boldsymbol{E} \pm i\boldsymbol{H}$*), and its dynamics (for free fields) was developed in close analogy with the Dirac procedure for spin-1/2 fields. Commutation (rather than anticommutation) rules for Dirac-like matrices were adopted, and energy eigenvalues and eigenvectors were calculated.*
For further details, see R. Mignani, M. Baldo and E. Recami, Lett. Nuovo Cim. **11** (1974) 568; E. Giannetto, Atti del IX Congresso Nazionale di Storia della Fisica, edited by F. Bevilacqua (Milan, 1988) 173; S. Esposito, Found. Phys. **28** (1998) 231.

[1]@ *In the original manuscript, the author pointed out that, from:*

$$\frac{1}{c}\dot{\varphi} + \boldsymbol{\nabla} \cdot A = 0, \quad \Box \varphi = 0,$$

it follows that:

$$\boldsymbol{\nabla}^2 \varphi + \frac{1}{c} \boldsymbol{\nabla} \cdot \dot{A} = 0.$$

$$4\pi\rho - \boldsymbol{\nabla}\cdot\boldsymbol{E} = 0, \qquad \boldsymbol{\nabla}\cdot\boldsymbol{H} = 0,$$

$$4\pi\boldsymbol{I} + \frac{1}{c}\frac{\partial\boldsymbol{E}}{\partial t} = \boldsymbol{\nabla}\times\boldsymbol{H}, \qquad -\frac{1}{c}\frac{\partial\boldsymbol{H}}{\partial t} = \boldsymbol{\nabla}\times\boldsymbol{E}.$$

$$\begin{aligned}
\psi_1 &= E_1 - iH_1 = E_x - iH_x,\\
\psi_2 &= E_2 - iH_2 = E_y - iH_y,\\
\psi_3 &= E_3 - iH_3 = E_z - iH_z.
\end{aligned}$$

$$\boxed{\boldsymbol{\nabla}\cdot\boldsymbol{\psi} = \boldsymbol{\nabla}\cdot\boldsymbol{E} - i\boldsymbol{\nabla}\cdot\boldsymbol{H} = 4\pi\rho.} \tag{1}$$

$$\begin{aligned}
\boldsymbol{\nabla}\times\boldsymbol{\psi} = \boldsymbol{\nabla}\times\boldsymbol{E} - i\boldsymbol{\nabla}\times\boldsymbol{H} \quad &= \quad -\frac{1}{c}\frac{\partial\boldsymbol{H}}{\partial t} - \frac{1}{c}\frac{\partial\boldsymbol{E}}{\partial t} - 4\pi i\boldsymbol{I}\\
&= \quad -\frac{i}{c}\left(\frac{\partial\boldsymbol{E}}{\partial t} - i\frac{\partial\boldsymbol{H}}{\partial t}\right) - 4\pi i\boldsymbol{I},
\end{aligned}$$

$$\boxed{4\pi\boldsymbol{I} + \frac{1}{c}\frac{\partial\boldsymbol{\psi}}{\partial t} = +i\boldsymbol{\nabla}\times\boldsymbol{\psi}.} \tag{2}$$

The Maxwell equations are given by:

$$\boxed{\begin{aligned}
\frac{1}{c}\frac{\partial\boldsymbol{\psi}}{\partial t} - i\boldsymbol{\nabla}\times\boldsymbol{\psi} + 4\pi\boldsymbol{I} \quad &= \quad 0,\\
\boldsymbol{\nabla}\cdot\boldsymbol{\psi} - 4\pi\rho = 0.&
\end{aligned}}$$

$$\begin{aligned}
\frac{1}{c}\frac{\partial\psi_1}{\partial t} - i\frac{\partial\psi_3}{\partial y} + i\frac{\partial\psi_2}{\partial z} + 4\pi I_x \quad &= \quad 0,\\
\frac{1}{c}\frac{\partial\psi_2}{\partial t} - i\frac{\partial\psi_1}{\partial z} + i\frac{\partial\psi_3}{\partial x} + 4\pi I_y \quad &= \quad 0,\\
\frac{1}{c}\frac{\partial\psi_3}{\partial t} - i\frac{\partial\psi_2}{\partial x} + i\frac{\partial\psi_1}{\partial y} + 4\pi I_z \quad &= \quad 0,\\
\frac{\partial\psi_1}{\partial x} + \frac{\partial\psi_2}{\partial y} + \frac{\partial\psi_3}{\partial z} - 4\pi\rho \quad &= \quad 0.
\end{aligned}$$

Without charge:

$$\left\{\begin{array}{rcl}
\dfrac{W}{c}\psi_1 + ip_y\psi_3 - ip_z\psi_2 &=& 0,\\[2ex]
\dfrac{W}{c}\psi_2 + ip_z\psi_1 - ip_x\psi_3 &=& 0,\\[2ex]
\dfrac{W}{c}\psi_3 + ip_x\psi_2 - ip_y\psi_1 &=& 0,\\[1ex]
\hline
p_x\psi_1 + p_y\psi_2 + p_z\psi_3 &=& 0.
\end{array}\right.$$

[2]

$$\left(\frac{W}{c} + \alpha_x p_x + \alpha_y p_y + \alpha_z p_z\right)\psi = 0. \tag{3}$$

$$\alpha_x = \begin{vmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & +i & 0 \end{vmatrix}, \quad \alpha_y = \begin{vmatrix} 0 & 0 & +i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{vmatrix},$$

$$\alpha_z = \begin{vmatrix} 0 & -i & 0 \\ +i & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix}, \quad \mathbf{1} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}.$$

[3]

$$\begin{array}{rcl}
\alpha_x\alpha_y - \alpha_y\alpha_x &=& -i\alpha_z,\\
[\alpha_x, \alpha_z]_- &=& +i\alpha_y,\\
[\alpha_y, \alpha_z]_- &=& i\alpha_x.
\end{array}$$

$$\beta_x = |1 \;\; 0 \;\; 0|, \quad \beta_y = |0 \;\; 1 \;\; 0|, \quad \beta_z = |0 \;\; 0 \;\; 1|.$$

$$(\beta_x p_x + \beta_y p_y + \beta_z p_z)\,\psi = 0. \tag{4}$$

Following the Dirac method, the eigenvalues of the Maxwell equation are obtained from:

[2]@ *The line before the fourth equation means that it is deduced from the previous three equations.*

[3]@ *Note that the signs on the RHS of the following two equations were wrong: correctly, we have* $\alpha_x\alpha_y - \alpha_y\alpha_x = i\alpha_z$ *and* $[\alpha_x, \alpha_z]_- = -i\alpha_y$.

$$\begin{vmatrix} W/c & -ip_z & ip_y \\ ip_z & W/c & -ip_x \\ -ip_y & ip_x & W/c \end{vmatrix} = 0,$$

$$\left(\frac{W}{c}\right)^3 - p^2\,\frac{W}{c} = 0,$$

$$\frac{W}{c} = \begin{cases} p, \\ -p, \\ 0, \end{cases}$$

$p = \sqrt{p_x^2 + p_y^2 + p_z^2}.$

W/c	ψ_1	ψ_2	ψ_3
p	$p_y^2 + p_z^2$	$-p_x p_y - ipp_z$	$-p_x p_z + ipp_y$
$-p$	$p_y^2 + p_z^2$	$-p_x p_y + ipp_z$	$-p_x p_z - ipp_y$
0	p_x	p_y	p_z

For $t = 0$:

$$\begin{aligned} \psi_1 &= a\,\delta(x - x_0)\delta'(y - y_0)\delta'(r - r_0), \\ \psi_2 &= b\,\delta'(x - x_0)\delta(y - y_0)\delta'(z - z_0), \\ \psi_3 &= -(a + b)\,\delta'(x - x_0)\delta'(y - y_0)\delta(z - z_0). \end{aligned}$$

$$\frac{\partial \psi_1}{\partial x} + \frac{\partial \psi_2}{\partial y} + \frac{\partial \psi_3}{\partial z} = 0.$$

$$\psi_1(x, y, z) = \int A(x_0, y_0, z_0)\,\delta(x - x_0)\delta'(y - y_0)\delta'(z - z_0)\,\mathrm{d}x_0\mathrm{d}y_0\mathrm{d}z_0,$$

$$\psi_2(x, y, z) = \int B(x_0, y_0, z_0)\,\delta'(x - x_0)\delta(y - y_0)\delta'(z - z_0)\,\mathrm{d}x_0\mathrm{d}y_0\mathrm{d}z_0,$$

$$\psi_3(x, y, z) = \int -(A + B)\,\delta'(x - x_0)\delta'(y - y_0)\delta(z - z_0)\,\mathrm{d}x_0\mathrm{d}y_0\mathrm{d}z_0.$$

$$\psi_1 = \frac{\partial^2 A}{\partial y \partial z}, \quad \psi_2 = \frac{\partial^2 B}{\partial z \partial x}, \quad \psi_3 = -\frac{\partial^2 (A + B)}{\partial x \partial y};$$

$$\frac{\partial\psi_1}{\partial x}=\frac{\partial^3 A}{\partial x\partial y\partial z},\quad \frac{\partial\psi_2}{\partial y}=\frac{\partial^2 B}{\partial x\partial y\partial z},\quad \frac{\partial\psi_3}{\partial z}=-\frac{\partial^2(A+B)}{\partial x\partial y\partial z}.$$

$$\begin{aligned}
&\frac{\partial' A}{\partial y\partial z}=\psi_1,\\
&\frac{\partial A}{\partial y}=\int\psi_1\mathrm{d}z+f_y,\\
&A=A_0+F_1(x,y)+F_2(x,z);\\
&\frac{\partial^2 B}{\partial z\partial x}=\psi_2,\\
&B=B_0+F_3(x,y)+F_4(y,z).
\end{aligned}$$

$$\psi_3=-\frac{\partial^2(A+B)}{\partial x\partial y}=-\frac{\partial^2(A_0+B_0)}{\partial x\partial y}+F(x,y).$$

By substituting the expressions:

$$\psi_1=\frac{\partial^2 A}{\partial y\partial z},\quad \psi_2=\frac{\partial^2 B}{\partial z\partial x},\quad \psi_3=\frac{\partial^2 C}{\partial x\partial y},$$

into the Maxwell equations, we get:

$$\begin{aligned}
\frac{1}{c}\frac{\partial^3 A}{\partial y\partial z\partial t}-i\frac{\partial^3 C}{\partial x\partial^2 y}+i\frac{\partial^2 B}{\partial x\partial^2 z}&=0,\\
\frac{1}{c}\frac{\partial^3 B}{\partial z\partial x\partial t}-i\frac{\partial^3 A}{\partial y\partial^2 z}+i\frac{\partial^3 C}{\partial y\partial^2 x}&=0,\\
\frac{1}{c}\frac{\partial^3 C}{\partial x\partial y\partial t}-i\frac{\partial^3 B}{\partial z\partial^2 x}+i\frac{\partial^3 A}{\partial z\partial^2 y}&=0;\\
\frac{\partial^3(A+B+C)}{\partial x\partial y\partial z}&=0.
\end{aligned}$$

$$\boxed{A+B+C=0.}$$

$$\begin{aligned}
\frac{\partial}{\partial y}\left(\frac{1}{c}\frac{\partial^2}{\partial z\partial t}+i\frac{\partial^2}{\partial x\partial y}\right)A+i\frac{\partial}{\partial x}\left(\frac{\partial^2}{\partial^2 y}+\frac{\partial^2}{\partial^2 z}\right)B=0,\\
\frac{\partial}{\partial x}\left(\frac{1}{c}\frac{\partial^2}{\partial z\partial t}-i\frac{\partial^2}{\partial x\partial y}\right)B-i\frac{\partial}{\partial y}\left(\frac{\partial^2}{\partial^2 x}+\frac{\partial^2}{\partial^2 z}\right)A=0,
\end{aligned}$$

$$\left[\frac{\partial}{\partial y}\left(\frac{1}{c}\frac{\partial^2}{\partial x\partial t}-i\frac{\partial^2}{\partial y\partial z}\right)A-\frac{\partial}{\partial x}\left(\frac{1}{c}\frac{\partial^2}{\partial y\partial t}+i\frac{\partial^2}{\partial_x\partial_z}\right)B=0.\right]$$

$$\begin{aligned} A &= -a\,\mathrm{e}^{i(\gamma_1 x+\gamma_2 y+\gamma_3 z)}, \\ B &= -b\,\mathrm{e}^{i(\gamma_1 x+\gamma_2 y+\gamma_3 z)}, \\ C &= -c\,\mathrm{e}^{i(\gamma_1 x+\gamma_2 y+\gamma_3 z)}; \end{aligned}$$

$$\begin{aligned} \psi_1 &= a\,\gamma_2\gamma_3\,\mathrm{e}^{i(\gamma_1 x+\gamma_2 y+\gamma_3 z)}, \\ \psi_2 &= b\,\gamma_3\gamma_1\,\mathrm{e}^{i(\gamma_1 x+\gamma_2 y+\gamma_3 z)}, \\ \psi_3 &= c\,\gamma_1\gamma_2\,\mathrm{e}^{i(\gamma_1 x+\gamma_2 y+\gamma_3 z)}. \end{aligned}$$

2.3. ELECTROMAGNETIC FIELD: PLANE WAVE OPERATORS

Plane wave expansion of the electromagnetic field was considered in a way similar to what is usually done for a Dirac or a Klein-Gordon field. In the second part, the author again introduced a sort of photon wave field Ψ, in close analogy to the Dirac field for a spin-1/2 particle and in a full Lorentz-invariant formalism. The properties of this field are deduced from general group-theoretic arguments.

$$\begin{array}{lll} \varphi, & P_0 = -\dfrac{1}{4\pi c^2}\dot{\varphi}, & \dot{\varphi} = 4\pi c^2 P_0; \\ A_x, & P_x = \dfrac{1}{4\pi c^2}\dot{A}_x, & \dot{A}_x = 4\pi c^2 P_x; \\ A_y, & P_y = \dfrac{1}{4\pi c^2}\dot{A}_y, & \dot{A}_y = 4\pi c^2 P_y; \\ A_z, & P_z = -\dfrac{1}{4\pi c^2}\dot{A}_z, & \dot{A}_z = 4\pi c^2 P_z; \\ P_0, & -\varphi, & \dot{P}_0 = -\dfrac{1}{4\pi}\boldsymbol{\nabla}^2\,\varphi; \\ P_x, & -A_x, & \dot{P}_x = \dfrac{1}{4\pi}\boldsymbol{\nabla}^2\,A_x; \end{array}$$

$$\begin{array}{lll} P_y, & -A_y, & \dot{P}_y = \dfrac{1}{4\pi}\boldsymbol{\nabla}^2\,A_y; \\ P_z, & -A_z & \dot{P}_z = \dfrac{1}{4\pi}\boldsymbol{\nabla}^2\,A_z. \end{array}$$

[4]

$$
\begin{aligned}
U_0(\gamma) &= \int \mathrm{e}^{-2\pi i(\gamma_1 x+\gamma_2 y+\gamma_3 z)}\varphi(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z,\\
U_x(\gamma) &= \int \mathrm{e}^{-2\pi i\boldsymbol{\gamma}\cdot\boldsymbol{q}} A_x(q)\,\mathrm{d}q.\\
U_y(\gamma) &= \int \mathrm{e}^{-2\pi i\boldsymbol{\gamma}\cdot\boldsymbol{q}} A_y(q)\,\mathrm{d}q,\\
U_z(\gamma) &= \int \mathrm{e}^{-2\pi i\boldsymbol{\gamma}\cdot\boldsymbol{q}} A_z(q)\,\mathrm{d}q.
\end{aligned}
$$

$$
\int L(q)\,\mathrm{d}q = \int M(\gamma)\,d\gamma,
$$

[5]

$$
\begin{aligned}
M \quad = \quad & \frac{1}{8\pi}\left\{-\frac{1}{c^2}\overline{\dot{U}}_0\dot{U}_0 + 4\pi^2\gamma^2\overline{U}_0U_0 + \frac{1}{c^2}(\overline{\dot{U}}_x\dot{U}_x + \overline{\dot{U}}_y\dot{U}_y + \overline{\dot{U}}_z\dot{U}_z)\right.\\
& \left. - 4\pi^2\gamma^2(\overline{U}_xU_x + \overline{U}_yU_y + \overline{U}_zU_z)\right\}.
\end{aligned}
$$

$$
\begin{aligned}
&U_0, \qquad V_0 = -\frac{1}{4\pi c^2}\overline{\dot{U}}_0,\\
&U_x, \qquad V_x = \frac{1}{4\pi c^2}\overline{\dot{U}}_x,\\
&U_y, \qquad V_y = \frac{1}{4\pi c^2}\overline{\dot{U}}_y,\\
&U_z, \qquad V_z = \frac{1}{4\pi c^2}\overline{\dot{U}}_z.
\end{aligned}
$$

$$
\begin{aligned}
U &= (U_x, U_y, U_z), \quad V = (V_x, V_y, V_z),\\
\dot{U} &= (\dot{U}_x, \dot{U}_y, \dot{U}_z), \quad \dot{V} = (\dot{V}_x, \dot{V}_y, \dot{V}_z),
\end{aligned}
$$

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[4]@ *In the original manuscript, the author considered in what follows the role of the operators* $\boldsymbol{\nabla}^2 = L^2$ *and* $L = \sqrt{\nabla^2}$. *He denoted with* $\boldsymbol{q}$ *the vector* (x, y, z).

[5]@ *A bar over a quantity denotes complex conjugation.*

$$
\begin{aligned}
\overline{U}_0(\gamma) &= U_0(-\gamma),\\
\overline{\dot{U}}_0(\gamma) &= \dot{U}_0(-\gamma),\\
\overline{U}(\gamma) &= U(-\gamma),\\
\overline{\dot{U}}(\gamma) &= \dot{U}(-\gamma),\\
\overline{V}(\gamma) &= V(-\gamma),\\
\overline{\dot{V}}(\gamma) &= \dot{V}(-\gamma),\\
\overline{V}_0(\gamma) &= V_0(-\gamma),\\
\overline{\dot{V}}_0(\gamma) &= \dot{V}_0(-\gamma).
\end{aligned}
$$

$$
\begin{aligned}
&\frac{1}{c^2}\ddot{U}_0 + 4\pi^2\gamma^2 U_0 = 0,\\
&\frac{1}{c^2}\ddot{U} + 4\pi^2\gamma^2 U = 0,\\
&\frac{1}{c}\dot{U}_0 + 2\pi i(\gamma_1 U_x + \gamma_2 U_y + \gamma_3 U_z) = 0,\\
&2\pi i\gamma^2 U_0 + \frac{1}{c}(\gamma_1 \dot{U}_x + \gamma_2 \dot{U}_y + \gamma_3 \dot{U}_z) = 0.
\end{aligned}
$$

[6]

$$
\begin{aligned}
\psi_0(\gamma) &= \int \mathrm{e}^{-2\pi i\boldsymbol{\gamma}\cdot\boldsymbol{q}} \cdot \frac{1}{2c\sqrt{h}}\left(\sqrt{2\pi\gamma c}\,\varphi(q) + \frac{i}{\sqrt{2\pi\gamma c}}\,\dot{\varphi}(q)\right)\mathrm{d}q,\\
\psi_x(\gamma) &= \int \mathrm{e}^{-2\pi i(\boldsymbol{\gamma}\cdot\boldsymbol{q}} \cdot \frac{1}{2c\sqrt{h}}\left(\sqrt{2\pi\gamma c}\,A_x(q) + \frac{i}{\sqrt{2\pi\gamma c}}\,\dot{A}_x(q)\right)\mathrm{d}q,\\
\psi_y(\gamma) &= \int \mathrm{e}^{-2\pi i\boldsymbol{\gamma}\cdot\boldsymbol{q}} \cdot \frac{1}{2c\sqrt{h}}\left(\sqrt{2\pi\gamma c}\,A_y(q) + \frac{i}{\sqrt{2\pi\gamma c}}\,\dot{A}_y(q)\right)\mathrm{d}q,\\
\psi_z(\gamma) &= \int \mathrm{e}^{-2\pi i\boldsymbol{\gamma}\cdot\boldsymbol{q}} \cdot \frac{1}{2c\sqrt{h}}\left(\sqrt{2\pi\gamma c}\,A_z(q) + \frac{i}{\sqrt{2\pi\gamma c}}\,\dot{A}_z(q)\right)\mathrm{d}q.
\end{aligned}
$$

$$
\begin{aligned}
\varphi(q) &= c\sqrt{h}\int \frac{1}{\sqrt{2\pi\gamma c}}\left[\psi_0(\gamma) + \overline{\psi}_0(-\gamma)\right] e^{2\pi i\boldsymbol{\gamma}\cdot\boldsymbol{q}}\mathrm{d}\gamma,\\
\dot{\varphi}(q) &= \frac{c\sqrt{h}}{i}\int \sqrt{2\pi\gamma c}\left[\psi_0(\gamma) - \overline{\psi}_0(-\gamma)\right] \mathrm{e}^{2\pi i\boldsymbol{\gamma}\cdot\boldsymbol{q}}\mathrm{d}\gamma,
\end{aligned}
$$

[6]@ *Probably, the author proceeded in analogy with the Dirac field .*

$$A_x(q) = c\sqrt{h}\int \frac{1}{\sqrt{2\pi\gamma c}}\,[\psi_x(\gamma) + \overline{\psi}_x(-\gamma)]\,\mathrm{e}^{2\pi i\boldsymbol{\gamma}\cdot\boldsymbol{q}}\mathrm{d}\gamma,$$
$$\ldots,$$
$$\dot{A}_x(q) = \frac{c\sqrt{h}}{i}\int \sqrt{2\pi\gamma c}\,[\psi_x(\gamma) - \overline{\psi}_x(-\gamma)]\,\mathrm{e}^{2\pi i\boldsymbol{\gamma}\cdot\boldsymbol{q}}\mathrm{d}\gamma,$$
$$\ldots,$$

[7]

$$\begin{aligned}\Box\,\varphi &= \frac{1}{c^2}\ddot{\varphi} - \boldsymbol{\nabla}^2\,\varphi \\ &= \frac{\sqrt{h}}{c\,i}\int\sqrt{2\pi\gamma c}\Big\{\dot{\psi}_0(\gamma) - \dot{\overline{\psi}}_0(-\gamma) \\ &\qquad + 2\pi\gamma c\,i\,\psi_0(\gamma) + 2\pi\gamma c\,i\,\overline{\psi}_0(-\gamma)\Big\}\,e^{2\pi i\boldsymbol{\gamma}\cdot\boldsymbol{q}}\mathrm{d}\gamma.\end{aligned}$$

$$\begin{aligned}&\psi_0(\gamma) = -2\pi\gamma c\,i\,\psi_0(\gamma), \quad \dot{\overline{\psi}}_0(\gamma) = 2\pi\gamma c\,i\,\overline{\psi}_0(\gamma),\\ &\dot{\psi}(\gamma) = -2\pi\gamma c\,i\,\psi_x(\gamma), \quad \dot{\overline{\psi}}_x(\gamma) = 2\pi\gamma c\,i\,\overline{\psi}_x(\gamma),\\ &\ldots.\end{aligned}$$

$$\begin{aligned}&\frac{1}{8\pi}\int\Big\{-\frac{1}{c^2}\dot{\varphi}^2 - |\boldsymbol{\nabla}\,\varphi|^2 + \frac{1}{c^2}(\dot{A}_x^2 + \dot{A}_y^2 + \dot{A}_z^2) \\ &\qquad + |\boldsymbol{\nabla}\,A_x|^2 + |\boldsymbol{\nabla}\,A_y|^2 + |\boldsymbol{\nabla}\,A_z|^2\Big\}\,\mathrm{d}q \\ &= \int h\gamma c\Big\{-\frac{\psi_0(\gamma)\overline{\psi}_0(\gamma) + \overline{\psi}_0(\gamma)\psi_0(\gamma)}{2} + \frac{\psi_x(\gamma)\overline{\psi}_x(\gamma) + \overline{\psi}_x(\gamma)\psi_x(\gamma)}{2} \\ &\qquad + \frac{\psi_y(\gamma)\overline{\psi}_y(\gamma) + \overline{\psi}_y(\gamma)\psi_y(\gamma)}{2} + \frac{\psi_z(\gamma)\overline{\psi}_z(\gamma) + \overline{\psi}_z(\gamma)\psi_z(\gamma)}{2}\Big\}\,\mathrm{d}\gamma,\end{aligned}$$

$$\boxed{\begin{aligned}W = \int h\gamma c\,\{&-\psi_0(\gamma)\overline{\psi}_0(\gamma) + \overline{\psi}_x(\gamma)\psi_x(\gamma) \\ &+ \overline{\psi}_y(\gamma)\psi_y(\gamma) + \overline{\psi}_z(\gamma)\psi_z(\gamma)\}\,\mathrm{d}\gamma.\end{aligned}}$$

[7]@ *In the original manuscript, the author also cited the following (seeming) identity, whose meaning in this general framework is not clear:*

$$\begin{aligned}0 &= \dot{\varphi}(q) - \dot{\varphi}(q) \\ &= c\sqrt{h}\int\frac{1}{\sqrt{2\pi\gamma c}}\Big\{\dot{\psi}_0(\gamma) + \dot{\overline{\psi}}_0(-\gamma) + 2\pi\gamma c\,i\,\psi_0(q) - 2\pi\gamma c\,i\,\overline{\psi}_0(-\gamma)\Big\}\,e^{2\pi i\boldsymbol{\gamma}\cdot\boldsymbol{q}}\mathrm{d}\gamma.\end{aligned}$$

$$\boxed{\begin{array}{l}\psi_0(\gamma)\overline{\psi}_0(\gamma') - \overline{\psi}_0(\gamma')\psi_0(\gamma) = -\delta(\gamma-\gamma'),\\ \psi_x(\gamma)\overline{\psi}_x(\gamma') - \overline{\psi}_x(\gamma')\psi_x(\gamma) = +\delta(\gamma-\gamma'),\\ \ldots.\end{array}}$$

$$\begin{aligned}&\boldsymbol{\nabla}^2\,\varphi + \frac{1}{c}\boldsymbol{\nabla}\cdot\dot{\boldsymbol{A}} =\\ &= c\sqrt{h}\int 2\pi\sqrt{\frac{2\pi\gamma}{c}}\left\{-\gamma[\psi_0(\gamma)+\overline{\psi}_0(-\gamma)] + \gamma_x[\psi_x(\gamma)-\overline{\psi}_x(-\gamma)]\right.\\ &\qquad\left. + \gamma_y[\psi_y(\gamma)-\overline{\psi}_y(-\gamma)] + \gamma_z[\psi_z(\gamma)-\overline{\psi}_z(-\gamma)]\right\} \mathrm{e}^{2\pi i\boldsymbol{\gamma}\cdot\boldsymbol{q}}\mathrm{d}\gamma,\\ &\frac{1}{c}\dot{\varphi} + \boldsymbol{\nabla}\cdot\boldsymbol{A}\\ &= \frac{ch}{\sqrt{i}}\int\sqrt{\frac{2\pi}{\gamma c}}\left\{\gamma[\psi_0(\gamma)-\overline{\psi}_0(-\gamma)] - \gamma_x[\psi_x(\gamma)-\overline{\psi}_x(-\gamma)]\right.\\ &\qquad\left. - \gamma_y[\psi_y(\gamma)-\overline{\psi}_y(-\gamma)] - \gamma_z[\psi_z(\gamma)-\overline{\psi}_z(-\gamma)]\right\} \mathrm{e}^{2\pi i\boldsymbol{\gamma}\cdot\boldsymbol{q}}\mathrm{d}\gamma,\end{aligned}$$

$$\gamma\psi_0 - \gamma_x\psi_x - \gamma_y\psi_y - \gamma_z\psi_z = 0,$$

$$\gamma\overline{\psi}_0 - \gamma_x\overline{\psi}_x - \gamma_y\overline{\psi}_y - \gamma_z\overline{\psi}_z = 0,$$

$$\psi_0 = \psi_0(\gamma),\ \psi_x = \psi_x(\gamma),\ \ldots,\ \overline{\psi}_0 = \overline{\psi}_0(\gamma),\ \overline{\psi}_x = \overline{\psi}_x(\gamma),\ \ldots.$$

2.3.1 Dirac Formalism

$$\Psi = (\psi_0,\ \psi_x,\ \psi_y,\ \psi_z),$$

$$H = -\frac{h}{2\pi i}\,\frac{\partial}{\partial t},\quad p_x = \frac{h}{2\pi i}\,\frac{\partial}{\partial x},\quad p_z = \frac{h}{2\pi i}\,\frac{\partial}{\partial y},\quad p_z = \frac{h}{2\pi i}\,\frac{\partial}{\partial z};$$

$$S_x = \begin{vmatrix} 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & -1\\ 0 & 0 & 1 & 0\end{vmatrix},\quad S_y = \begin{vmatrix} 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\end{vmatrix},$$

$$S_z = \begin{vmatrix} 0 & 0 & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\end{vmatrix};$$

$$T_x = \begin{vmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix}, \quad T_y = \begin{vmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix},$$

$$T_z = \begin{vmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{vmatrix}.$$

1) $\Psi' = H\Psi = h\gamma_c\Psi$

2) $\Psi' = p_x\Psi = h\gamma_x\Psi$

3) $\Psi' = p_y\Psi = h\gamma_y\Psi$

4) $\Psi' = p_z\Psi = h\gamma_z\Psi$

5) $$\Psi' = S_x\Psi = \left\{ -\gamma_y \frac{\partial}{\partial\gamma_z} + \gamma_z \frac{\partial}{\partial\gamma_y} + \begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{vmatrix} \right\} \Psi$$

6) $$\Psi' = S_y\Psi = \left\{ -\gamma_z \frac{\partial}{\partial\gamma_z} + \gamma_x \frac{\partial}{\partial\gamma_z} + \begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{vmatrix} \right\} \Psi$$

7) $$\Psi' = S_z\Psi = \left\{ -\gamma_x \frac{\partial}{\partial\gamma_y} + \gamma_y \frac{\partial}{\partial\gamma_x} + \begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix} \right\} \Psi$$

8) $$\Psi' = T_x\Psi = \left\{ -\gamma \frac{\partial}{\partial\gamma_x} - \frac{\gamma_x}{2\gamma} + \begin{vmatrix} 0 & 0 & 0 & 0 \\ 1 & -\gamma_x/\gamma & 0 & 0 \\ 0 & -\gamma_y/\gamma & 0 & 0 \\ 0 & -\gamma_z/\gamma & 0 & 0 \end{vmatrix} - 2\pi i\, ct\, \gamma_x \right\} \Psi$$

$$9)\qquad \Psi' = T_y \Psi = \left\{ -\gamma \frac{\partial}{\partial \gamma_y} - \frac{\gamma_y}{2\gamma} + \begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\gamma_x/\gamma & 0 \\ 1 & 0 & -\gamma_y/\gamma & 0 \\ 0 & 0 & -\gamma_z/\gamma & 0 \end{vmatrix} - 2\pi i\, ct\, \gamma_y \right\} \Psi$$

$$10)\qquad \Psi' = T_z \Psi = \left\{ -\gamma \frac{\partial}{\partial \gamma_z} - \frac{\gamma_z}{2\gamma} + \begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\gamma_x/\gamma \\ 0 & 0 & 0 & -\gamma_y/\gamma \\ 1 & 0 & 0 & -\gamma_z/\gamma \end{vmatrix} - 2\pi i\, ct\, \gamma_z \right\} \Psi$$

$\psi_0 = 0,$
$\Psi = (\psi_x,\, \psi_y,\, \psi_z).$
$\underline{\gamma} = (\gamma_x,\, \gamma_y,\, \gamma_z),\ \gamma = \sqrt{\gamma_x^2 + \gamma_y^2 + \gamma_z^2}.$

$$(\gamma',\, \gamma'_x,\, \gamma'_y,\, \gamma'_z) = C(\gamma,\, \gamma_x,\, \gamma_y,\, \gamma_z),$$

$C = \|c_{ik}\|\ (i,\, k = 0,\, 1,\, 2,\, 3)$

$$c_{00}^2 - \sum_{i=1}^{3} c_{0i}^2 = 1,$$

$$c_{00}\, c_{i0} - \sum_{k=1}^{3} c_{0k}\, c_{ik} = 0, \quad (i = 1,\, 2,\, 3),$$

$$c_{i0}\, c_{k0} - \sum_{k=10}^{3} c_{ik}\, c_{ki} = -\partial_{ik}, \quad (i,\, k = 1,\, 2,\, 3).$$

$$\Psi'(\gamma') = \mathrm{e}^{-2\pi i c(\gamma' - \gamma)t} \sqrt{\frac{\gamma}{\gamma'}}\, D\, \Psi(\gamma),$$

$D = \|d_{ik}\|\ (i, k = 1,\, 2,\, 3)$

$$d_{11} = c_{11} - \frac{\gamma'_x}{\gamma'}\, c_{01}, \quad d_{21} = c_{21} - \frac{\gamma'_y}{\gamma'}\, c_{01}, \quad d_{31} = c_{31} - \frac{\gamma'_z}{\gamma'}\, c_{01},$$

$$d_{12} = c_{12} - \frac{\gamma'_x}{\gamma'}\, c_{02}, \quad d_{22} = c_{22} - \frac{\gamma'_y}{\gamma'}\, c_{02}, \quad d_{32} = c_{32} - \frac{\gamma'_z}{\gamma'}\, c_{02},$$

$$d_{13} = c_{13} - \frac{\gamma'_x}{\gamma'}\, c_{03}, \quad d_{23} = c_{23} - \frac{\gamma'_y}{\gamma'}\, c_{03}, \quad d_{33} = c_{33} - \frac{\gamma'_z}{\gamma'}\, c_{03}.$$

$$\gamma'_x\Psi'_x + \gamma'_y\Psi'_y + \gamma'_z\Psi'_z = \sqrt{\frac{\gamma}{\gamma'}}\,\mathrm{e}^{-2\pi c(\gamma'-\gamma)t}\,(\gamma_x\Psi_x + \gamma_y\Psi_y + \gamma_z\Psi_z).$$

$$S_x = -\gamma_y\,\frac{\partial}{\partial\gamma_z} + \gamma_z\,\frac{\partial}{\partial\gamma_y} + \begin{vmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{vmatrix},$$

$$S_y = -\gamma_z\,\frac{\partial}{\partial\gamma_x} + \gamma_x\,\frac{\partial}{\partial\gamma_z} + \begin{vmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{vmatrix},$$

$$S_z = -\gamma_x\,\frac{\partial}{\partial\gamma_y} + \gamma_y\,\frac{\partial}{\partial\gamma_x} + \begin{vmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix},$$

$$T_x = -\gamma\,\frac{\partial}{\partial\gamma_x} - \frac{\gamma_x}{2\gamma} - 2\pi i\,c\,\gamma_x t - \begin{vmatrix} \gamma_x/\gamma & 0 & 0 \\ \gamma_y/\gamma & 0 & 0 \\ \gamma_z/\gamma & 0 & 0 \end{vmatrix},$$

$$T_y = -\gamma\,\frac{\partial}{\partial\gamma_y} - \frac{\gamma_y}{2\gamma} - 2\pi i\,c\,\gamma_y t - \begin{vmatrix} 0 & \gamma_x/\gamma & 0 \\ 0 & \gamma_y/\gamma & 0 \\ 0 & \gamma_y/\gamma & 0 \end{vmatrix},$$

$$T_z = -\gamma\,\frac{\partial}{\partial\gamma_z} - \frac{\gamma_z}{2\gamma} - 2\pi i\,c\gamma_z t - \begin{vmatrix} 0 & 0 & \gamma_x/\gamma \\ 0 & 0 & \gamma_y/\gamma \\ 0 & 0 & \gamma_z/\gamma \end{vmatrix},$$

$$\boxed{\gamma_x\psi_x + \gamma_y\psi_y + \gamma_z\psi_z = 0.}$$

2.4. QUANTIZATION OF THE ELECTROMAGNETIC FIELD

In what follows,[8] *the author considered the quantization of the electromagnetic field inside a box, obtaining the usual equations in terms of*

[8]@ *In the original manuscript, the title of this section is "Dispersion".*

oscillators. Particular care was devoted to distinguish the role of the right-handed polarized states from that of the left-handed ones.

$$\boxed{\nabla \cdot \boldsymbol{E} = \nabla \cdot \boldsymbol{C} = 0.}$$

$\mathrm{d}S = \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$:

$$\frac{1}{8\pi} \int \left(E^2 - H^2\right) \mathrm{d}S\,\mathrm{d}t = \text{minimum},$$

$\varphi = 0$.

$$\boldsymbol{E} = -\frac{1}{c}\frac{\partial \boldsymbol{C}}{\partial t}, \quad \boldsymbol{H} = \nabla \times \boldsymbol{C};$$

$$\delta\boldsymbol{E} = -\frac{1}{c}\frac{\partial}{\partial t}\delta\boldsymbol{C}, \quad \delta\boldsymbol{H} = \nabla \times \delta\boldsymbol{C}.$$

$$\begin{aligned} \frac{1}{c}\frac{\partial \boldsymbol{H}}{\partial t} + \nabla \times \boldsymbol{E} &= 0, \\ \frac{1}{c}\frac{\partial \boldsymbol{E}}{\partial t} &= \nabla \times \boldsymbol{H} = \nabla \times \nabla \times \boldsymbol{C} = \nabla\left(\nabla \cdot \boldsymbol{C}\right) - \nabla^2 \boldsymbol{C} \\ &= -\nabla^2 \boldsymbol{C}. \end{aligned}$$

Conjugate variables:

$$C_x, \quad C_y, \quad C_z;$$

$$-\frac{1}{4\pi c}E_x, \quad -\frac{1}{4\pi c}E_y, \quad -\frac{1}{4\pi c}E_z.$$

$$H = \frac{1}{8\pi}\int (E^2 + H^2)\,\mathrm{d}S.$$

Let us consider the electromagnetic field confined inside a cube with side k, *its volume being* $S = k^3$:

$$\gamma_1 = \frac{n_1}{k}, \quad \gamma_2 = \frac{n_2}{k}, \quad \gamma_3 = \frac{n_3}{k}.$$

$$\mathrm{d}N = 2k^3\,\mathrm{d}\gamma_1\,\mathrm{d}\gamma_2\,\mathrm{d}\gamma_3.$$

$$v = c\gamma.$$

$$\gamma = \sqrt{\gamma_1^2 + \gamma_2^2 + \gamma_3^2} = \frac{v}{c}.$$

$$\begin{aligned}
\boldsymbol{A}_s^1 &= \boldsymbol{k}_1 \cos 2\pi(\gamma_1 x + \gamma_2 y + \gamma_3 z) + \boldsymbol{k}_2 \sin 2\pi(\gamma_1 x + \gamma_2 y + \gamma_3 z),\\
\boldsymbol{A}_s^2 &= -\boldsymbol{k}_1 \sin 2\pi(\gamma_1 x + \gamma_2 y + \gamma_3 z) + \boldsymbol{k}_2 \cos 2\pi(\gamma_1 x + \gamma_2 y + \gamma_3 z),\\
\boldsymbol{A}_s^3 &= \boldsymbol{k}_1 \cos 2\pi(\gamma_1 x + \gamma_2 y + \gamma_3 z) - \boldsymbol{k}_2 \sin 2\pi(\gamma_1 x + \gamma_2 y + \gamma_3 z),\\
\boldsymbol{A}_s^4 &= \boldsymbol{k}_1 \sin 2\pi(\gamma_1 x + \gamma_2 y + \gamma_3 z) + \boldsymbol{k}_2 \cos 2\pi(\gamma_1 x + \gamma_2 y + \gamma_3 z);
\end{aligned}$$

$\boldsymbol{A}_s^1$ and $\boldsymbol{A}_s^2$ correspond to right-handed, circularly polarized waves, while $\boldsymbol{A}_s^3$ and $\boldsymbol{A}_s^4$ correspond to the left-handed ones.
The direction of $s = (v_1, v_2, v_3)$ is defined by the right-handed direction of $\boldsymbol{k}_1, \boldsymbol{k}_2$. Note that $\gamma_1, \gamma_2, \gamma_3$ are given apart from a simultaneous change of sign!

$$\boxed{\begin{array}{rcl} s & \longrightarrow & -s,\\ \boldsymbol{k}_1,\ \boldsymbol{k}_2 & \longrightarrow & \boldsymbol{k}_2,\ \boldsymbol{k}_1. \end{array}}$$

$$\boxed{\begin{array}{c} \boldsymbol{A}_{-s}^1 = \boldsymbol{A}_s^2,\\[1em] \boldsymbol{A}_{-s}^2 = \boldsymbol{A}_s^1,\\[1em] \boldsymbol{A}_{-s}^3 = \boldsymbol{A}_s^4,\\[1em] \boldsymbol{A}_{-s}^4 = \boldsymbol{A}_s^3. \end{array}}$$

$|\boldsymbol{k}_1| = 1, |\boldsymbol{k}_2| = 1;\ S = k^3.$

$$\boxed{\begin{array}{c} \boldsymbol{C} = \sum a_s^i \boldsymbol{A}_s^i,\\[1em] \boldsymbol{E} = \sum b_s^i \boldsymbol{A}_s^i. \end{array}}$$

Notice that, in these sums, the terms corresponding to s and those corresponding to $-s$ give the same contribution: $s \equiv -s$. The terms with s and $-s$ are counted only once; the sign of s is defined by the right-handed rotation of $\boldsymbol{k}_1, \boldsymbol{k}_2$!

$$\boxed{a_s^i b_s^i - b_s^i a_s^i = \frac{2hc}{iS}.}$$

$$b_s^i = -\frac{1}{c}\dot{a}_s^i, \qquad a_s^i = \frac{1}{4\pi\gamma^2 c}\dot{b}_s^i.$$

$$\ddot{a}_s^i + 4\pi^2\gamma^2c^2a_s^i = 0, \qquad \ddot{b}_s^i + 4\pi^2\gamma^2c^2b_s^i = 0,$$

$\gamma^2c^2 = \nu^2.$

$$\boxed{\begin{array}{rcl} \dot{a}_s^i & = & -cb_s^i, \\ \dot{b}_s^i & = & 4\pi^2\gamma^2 c\, a_s^i. \end{array}}$$

$$H = \sum_{s,i} \frac{4\pi^2\gamma^2 a_s^{i2} + b_s^{i2}}{8\pi} S.$$

$$\dot{a}_s^i = -\frac{4\pi c}{S}\frac{\partial H}{\partial b_s^i}, \qquad \dot{b}_s^i = \frac{4\pi c}{S}\frac{\partial H}{\partial a_s^i}.$$

$$\boxed{\begin{array}{ll} p_s^i = \sqrt{\dfrac{\nu S\pi}{hc}}a_s^i, & q_s^i = \sqrt{\dfrac{S}{4\pi\nu hc}}b_s^i, \\[2ex] a_s^i = \sqrt{\dfrac{hc}{\nu S\pi}}p_s^i, & b_s^i = \sqrt{\dfrac{4\pi\nu hc}{S}}q_s^i. \end{array}}$$

$$\boxed{H = \sum_{\nu,i}\frac{1}{2}(p_s^{i2} + q_s^{i2})h\nu.}$$

$$\boxed{p_s^i q_s^i - q_s^i p_s^i = \frac{1}{i}, \quad a_s^i b_s^i - b_s^i a_s^i = \frac{2hc}{iS}.}$$

$$\dot{p}_s^i = -2\pi\nu q_s^i = -\frac{2\pi}{h}\frac{\partial H}{\partial q_s^i}, \quad \dot{q}_s^i = 2\pi\nu p_s^i = \frac{2\pi}{h}\frac{\partial H}{\partial p_s^i}.$$

$$\begin{array}{lllll}
s & \to & p_s^R = \dfrac{p_s' - q_s^2}{\sqrt{2}}, & q_s^R = \dfrac{q_s' + p_s'}{\sqrt{2}}, & p_s^R q_s^R - q_s^R p_s^R = \dfrac{1}{i}; \\
-s & \to & P_{-s}^R = \dfrac{p_s^2 - q_s'}{\sqrt{2}}, & q_{-s}^R = \dfrac{q_s^2 + p_s'}{\sqrt{2}}, & p_{-s}^R q_{-s}^R - q_{-s}^R p_{-s}^R = \dfrac{1}{i}; \\
s & \to & p_s^L = \dfrac{p_s^4 - q_s^3}{\sqrt{2}}, & q_s^L = \dfrac{q_s^4 + p_s^3}{\sqrt{2}}, & p_s^L q_s^L - q_s^L p_s^L = \dfrac{1}{i}; \\
-s & \to & p_{-s}^L = \dfrac{p_s^3 - q_s^4}{\sqrt{2}}, & q_{-s}^L = \dfrac{q_s^3 + p_s^4}{\sqrt{2}}, & p_{-s}^L q_{-s}^L - q_{-s}^L p_{-s}^L = \dfrac{1}{i}.
\end{array}$$

From now on, *the terms with* s are distinct from *those with* $-s$!

$$p_s^R q_s^R - q_s^R p_s^R = \frac{1}{i}, \quad p_s^L q_s^L - q_s^L p_s^L = \frac{1}{i}.$$

$$\begin{array}{r|l}
a_s = \dfrac{p_s^R - i q_s^R}{\sqrt{2}} & b_s = \dfrac{p_s^L - i q_s^L}{\sqrt{2}} \\
a_s^\star = \dfrac{p_s^R + i q_s^R}{\sqrt{2}} & b_s^\star = \dfrac{p_s^L + i q_s^L}{\sqrt{2}} \\
a_s a_s^\star - a_s^\star a_s = 1 & b_s b_s^\star - b_s^\star b_s = 1 \\
a_s^\star a_s = \dfrac{1}{2}(p_s^{D\,2} + q_s^{D\,2}) - \dfrac{1}{2} & b_s^\star b_s = \dfrac{1}{2}(p_s^{S\,2} + q_s^{S\,2}) - \dfrac{1}{2} \\
a_s^\star a_s = n_s, \quad (n_s = 0, 1, 2, \ldots) & b_s^\star b_s = n_s' \\
a_s(n_s, n_s + 1) = \sqrt{n_s + 1} & b_s(n_s', n_s' + 1) = \sqrt{n_s' + 1} \\
a_s^\star(n_s, n_s - 1) = \sqrt{n_s} & b_s(n_s', n_{s-1}') = \sqrt{n_s'}
\end{array}$$

$$p_s^R = \frac{a_s + a_s^\star}{\sqrt{2}}, \quad p_s^L = \frac{b_s + b_s^\star}{\sqrt{2}};$$

$$q_s^R = i\frac{a_s - a_s^\star}{\sqrt{2}}, \quad q_s^L = i\frac{b_s + b_s^\star}{\sqrt{2}}.$$

$$\boxed{\begin{aligned} W &= \frac{1}{2}\sum_{s,i}\frac{1}{2}h\nu_s(p_s^{i\,2}+q_s^{i\,2}) \\ &= \sum_s \frac{1}{2}h\nu_s(p_s^{D\,2}+q_s^{D\,2})+\sum_s \frac{1}{2}h\nu_s(p_s^{S\,2}+q_s^{S\,2}) \\ &= \sum_s h\nu_s(n_s+n_s')\,(+\text{ an infinite constant}). \end{aligned}}$$

$$p_s^1=\frac{p_s^R+q_{-s}^R}{\sqrt{2}}, \qquad q_s^1=\frac{q_s^R-p_{-s}^R}{\sqrt{2}},$$

$$p_s^2=\frac{p_{-s}^R+q_s^R}{\sqrt{2}}, \qquad q_s^2=\frac{q_{-s}^R-p_s^R}{\sqrt{2}},$$

$$p_s^3=\frac{p_{-s}^L+q_s^L}{\sqrt{2}}, \qquad q_s^3=\frac{q_{-s}^L-p_s^L}{\sqrt{2}},$$

$$p_s^4=\frac{p_{-s}^L+q_{-s}^L}{\sqrt{2}}, \qquad q_s^4=\frac{q_s^L-p_{-s}^L}{\sqrt{2}}$$

(in the LHS s and $-s$ are gathered together, while on the RHS they are kept distinct).

$$p_s^1=\frac{1}{2}[a_s+ia_{-s}+a_s^\star-ia_{-s}^\star], \qquad q_s^1=\frac{1}{2}[ia_s-a_{-s}-ia_s^\star-a_{-s}^\star],$$

$$p_s^2=\frac{1}{2}[a_{-s}+ia_s+a_{-s}^\star-ia_s^\star], \qquad q_s^2=\frac{1}{2}[ia_s-a_s-ia_s^\star-a_s^\star],$$

$$p_s^3=\frac{1}{2}[b_{-s}+ib_s+b_{-s}^\star-ib_s^\star], \qquad q_s^3=\frac{1}{2}[ib_{-s}-b_s-ib_{-s}^\star-b_s^\star],$$

$$p_s^4=\frac{1}{2}[b_s+ib_{-s}+b_s^\star-ib_s^\star], \qquad q_s^4=\frac{1}{2}[ib_s-b_{-s}-ib_s^\star-b_{-s}^\star].$$

$$\dot{a}_s=\ldots, \quad \dot{b}_s=\ldots, \quad \dot{a}_s^*=\ldots, \quad \dot{b}_s^*=\ldots.$$

In what follows, the orthogonal functions A_s^i are defined for all the values of s (see page 73); the indices of $\boldsymbol{k}_1$, $\boldsymbol{k}_2$ are given in such a way that the vectors $\boldsymbol{k}_1$, $\boldsymbol{k}_2$, s form a right-handed trihedron. The vectors $\boldsymbol{k}_1$ and $\boldsymbol{k}_2$ transform one into the other by changing s into $-s$. Each function $\boldsymbol{A}_s^i$ is counted twice, due to the relations:

$$\boldsymbol{A}_s^1=\boldsymbol{A}_{-s}^2, \quad \boldsymbol{A}_s^2=\boldsymbol{A}_{-s}^1, \quad \boldsymbol{A}_s^3=\boldsymbol{A}_{-s}^4, \quad \boldsymbol{A}_s^4=\boldsymbol{A}_{-s}^3.$$

$$\boxed{\begin{aligned} \boldsymbol{C} &= \frac{c}{2}\sqrt{\frac{h}{\pi S}}\sum_s \frac{1}{\sqrt{\nu_s}}[(a_s + a_s^\star)\boldsymbol{A}_s^1 + i(a_s - a_s^\star)\boldsymbol{A}_s^2 \\ &\qquad + i(b_s - b_s^\star)\boldsymbol{A}_s^3 + (b_s + b_s^\star)\boldsymbol{A}_s^4], \\ \boldsymbol{E} &= \sqrt{\frac{\pi h}{S}}\sum_s \sqrt{\nu_s}\,[i(a_s - a_s^\star)\boldsymbol{A}_s^1 - (a_s + a_s^\star)\boldsymbol{A}_s^2 \\ &\qquad - (b_s + b_s^\star)\boldsymbol{A}_s^3 + i(b_s - b_s^\star)\boldsymbol{A}_s^4]. \end{aligned}}$$

$$\begin{array}{ll} a_s(n_s, n_{s+1}) = \sqrt{n_s + 1}, & b_s(n_s', n_{s+1}') = \sqrt{n_s' + 1}, \\ a_s^\star(n_s, n_{s-1}) = \sqrt{n_s}, & b_s(n_s', n_{s-1}') = \sqrt{n_s'}, \\ a_s a_s^\star - a_s^\star a_s = 1, & b_s b_s^\star - b_s^\star b_s = 1, \\ a_s^\star a_s = n_s, & b_s^\star b_s = n_s'. \end{array}$$

$$\begin{aligned} W &= \frac{1}{4}\sum_s h\nu_s[\cancel{a}_s^2 + \cancel{a}_s^{*2} + a_s a_s^* + a_s^* a_s - \cancel{a}_s^2 - \cancel{a}_s^{*2} + a_s a_s^* + a_s^* a_s \\ &\qquad - \cancel{b}_s^2 - \cancel{b}_s^{*2} + b_s b_s^* + b_s^* b_s + \cancel{b}_s^2 + \cancel{b}_s^{*2} + b_s b_s^* + b_s^* b_s] \\ &= \sum_s h\nu_s(n_s + n_{s+1}') = \boxed{\sum_s h\nu_s(n_s + \mathcal{N}_s)} + \text{an infinite constant,} \end{aligned}$$

with:

$$\boxed{\begin{aligned} n_s &= a_s^* a_s, \\ \mathcal{N}_s &= b_s^* b_s. \end{aligned}}$$

By absorbing the *infinite* constant *into W, we have:*

$$\boxed{W_R = \sum_s h\nu_s(n_s + \mathcal{N}_s).}$$

We have used $\mathcal{N}_s$ instead of n_s': n_s corresponds to right-handed polarized waves, while $\mathcal{N}_s$ to the left-handed ones.

2.5. CONTINUATION I: ANGULAR MOMENTUM

The author continued[9] *to study the quantization of the electromagnetic field, obtaining explicit expressions for the matrix elements of the creation and the annihilation operators (in the number operator representation) and for the angular momentum of the field. Transformation properties of the n-photon states* ψ *were quickly outlined at the end of this Section.*

$$\boxed{\begin{aligned} \boldsymbol{C} &= \sum_k \sqrt{\frac{2hc}{k}}\, p_k \boldsymbol{f}_k, \\ \boldsymbol{E} &= \sum_k \sqrt{2hck}\, q_k \boldsymbol{f}_k. \end{aligned}}$$

$$\dot{q}_k = kc\, p_k, \quad \dot{p}_k = -kc\, q_k.$$

$$\begin{aligned} \frac{1}{c}\frac{\partial \boldsymbol{C}}{\partial t} &= \sum_k \sqrt{\frac{2h}{ck}}\, \dot{p}_k \boldsymbol{f}_k = -\boldsymbol{E} = -\sum_k \sqrt{2hck}\, q_k \boldsymbol{f}_k, \\ \frac{1}{c}\frac{\partial \boldsymbol{E}}{\partial t} &= \sum_k \sqrt{\frac{2hk}{c}}\, \dot{q}_k \boldsymbol{f}_k = -\boldsymbol{\nabla}^2 \boldsymbol{C} = \sum_k k\sqrt{2hck}\, p_k \boldsymbol{f}_k. \end{aligned}$$

$$\boxed{\begin{aligned} \dot{q}_k &= \frac{2\pi}{h}\frac{\partial W}{\partial p_k}, \\ \dot{p}_k &= -\frac{2\pi}{h}\frac{\partial W}{\partial q_k}. \end{aligned}}$$

$$W = \sum h\nu_k\, \frac{1}{2}(p_k^2 + q_k^2) = \sum_k \frac{h}{2\pi} ck\, \frac{1}{2}(p_k^2 + q_k^2).$$

[9]@ *In the original manuscript, the title of this section is "Irradiation".*

$$\begin{aligned} \dot{q}_k &= -\frac{2\pi i}{h}(q_k W - W q_k), \\ \dot{p}_k &= -\frac{2\pi i}{h}(p_k W - W q_k); \end{aligned}$$

$$\begin{aligned} i(q_k W - W q_k) &= \frac{\partial W}{\partial p_k}, \\ -i(p_k W - W p_k) &= \frac{\partial W}{\partial q_k}; \end{aligned}$$

$$\begin{aligned} -i(q_k p_k - p_k q_k) &= 1, \\ +i(p_k q_k - q_k p_k) &= 1. \end{aligned}$$

$$\boxed{p_k q_k - q_k p_k = \frac{1}{i}.}$$

$$\boxed{W = \sum_k h\nu_k \left(n_k + \frac{1}{2}\right) = \sum h\nu_k \frac{p_k^2 + q_k^2}{2}.}$$

$$\frac{1}{2}(p_k^2 + q_k^2) = \frac{p_k + iq_k}{\sqrt{2}} \frac{p_k - iq_k}{\sqrt{2}} + \frac{1}{2},$$

$$a_k = \frac{p_k - iq_k}{\sqrt{2}}, \quad a_k^* = \frac{p_k + iq_k}{\sqrt{2}}.$$

$$a_k a_k^* - a_k^* a_k = \frac{i}{2}(p_k q_k - q_k p_k + p_k q_k - q_k p_k) = 1.$$

$$\boxed{\begin{aligned} a_k^* a_k &= n_k, \\ a_k a_k^* &= n_k + 1. \end{aligned}}$$

$$\boxed{\begin{aligned} a_k &= \frac{p_k - iq_k}{\sqrt{i}}, & p_k &= \frac{a_k + a_k^*}{\sqrt{2}}, \\ a_k^* &= \frac{p_k + iq_k}{\sqrt{i}}, & q_k &= \frac{a_k^* - a_k}{i\sqrt{2}}. \end{aligned}}$$

$$a_k = \begin{vmatrix} 0 & 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & \sqrt{2} & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & \sqrt{3} & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & \sqrt{4} & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & \sqrt{5} & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{vmatrix},$$

$$a_k^* = \begin{vmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & \sqrt{2} & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & \sqrt{3} & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & \sqrt{4} & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & \sqrt{5} & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{vmatrix};$$

$$a_k^* a_k = \begin{vmatrix} 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 2 & 0 & \dots \\ 0 & 0 & 0 & 3 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{vmatrix},$$

$$a_k a_k^* = \begin{vmatrix} 1 & 0 & 0 & 0 & \dots \\ 0 & 2 & 0 & 0 & \dots \\ 0 & 0 & 3 & 0 & \dots \\ 0 & 0 & 0 & 4 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{vmatrix};$$

$$p_k = \begin{vmatrix} 0 & 1/\sqrt{2} & 0 & \dots \\ 1/\sqrt{2} & 0 & 1 & \dots \\ 0 & 1 & 0 & \dots \\ \dots & \dots & \dots & \dots \end{vmatrix},$$

$$q_k = \begin{vmatrix} 0 & i/\sqrt{2} & 0 & \dots \\ -i/\sqrt{2} & 0 & -i & \dots \\ 0 & i & 0 & \dots \\ \dots & \dots & \dots & \dots \end{vmatrix}.$$

$$\boldsymbol{C}' = \boldsymbol{C} + \epsilon\, S\, \boldsymbol{C}, \quad \boldsymbol{E}' = \boldsymbol{E} + \epsilon\, S\, \boldsymbol{E}.$$

$$p_r' = p_r + \varepsilon \sum_s S_{rs} p_s, \quad q_r' = q_r + \varepsilon \sum_s S_{rs} q_s.$$

$$\boxed{S_{rs} = -S_{sr}.}$$

$$\begin{aligned}
&\psi = \psi(n_1, n_2, \ldots),\\
&\psi' = \psi + \frac{T}{i}\varepsilon\psi;\\
&q' = q + \frac{\varepsilon}{i}(qT - Tq),\\
&p' = p + \frac{\varepsilon}{i}(pT - Tp).
\end{aligned}$$

$$\begin{aligned}
p_r T - T p_r &= i \sum S_{rs} p_s,\\
q_r T - T q_r &= i \sum S_{rs} q_s.
\end{aligned}$$

$$\boxed{T = \sum_{rs} S_{rs}\, p_r q_s.}$$

T is the angular momentum in units $h/2\pi$.

$$T = \sum S_{rs} p_r q_s = \sum_{r<s} S_{rs}(p_r q_s - p_s q_r).$$

$$\begin{aligned}
p_r q_s - p_s q_r &= \frac{1}{2i}(a_r^* a_s^* - a_r a_s - a_r^* a_s + a_r a_s^*\\
&\qquad - a_s^* a_r^* - a_s a_r + a_s^* a_r - a_s a_r^*)\\
&= \frac{1}{i}(a_r a_s^* - a_s a_r^*).
\end{aligned}$$

$$T = \sum_{r<s} \frac{1}{i}(a_r a_s^* - a_s a_r^*) S_{rs}.$$

For n photons:

$$\psi = \psi(n_1, n_2, \ldots)\, \delta\left(\sum n_i - n\right).$$

For $n = 1$, $\psi = \psi(n_1, n_2, \ldots)$ and all n_i but one vanish, and the non-zero number is equal to 1:

$$\begin{aligned}
&\psi(1, 0, 0, 0, 0, \ldots) = c_1,\\
&\psi(0, 1, 0, 0, 0, \ldots) = c_2,\\
&\psi(0, 0, 1, 0, 0, \ldots) = c_3,\\
&\ldots .
\end{aligned}$$

$$\boxed{\psi = (c_1, c_2, c_3, \ldots).}$$

$$\psi' = T\psi.$$

$$\begin{aligned} \sum_{r<s} S_{rs}(a_r a_s^* - a_s a_r^*) &= \sum_{r,s} S_{rs} a_r a_s^*, \\ \frac{1}{i} \sum_{rs} S_{rs} a_r a_s^* \psi &= (c_1', c_2', \ldots). \end{aligned}$$

$$c_s' = \frac{1}{i} \sum S_{rs} c_r = i \sum S_{sr} c_r.$$

$$\boxed{c_r' = i \sum S_{rs} c_s.}$$

2.6. CONTINUATION II: INCLUDING THE MATTER FIELDS

What had been studied in the Sect. 2.4 was tentatively generalized here to the case of an electromagnetic field interacting with a charged Dirac field ψ. As above, the scalar potential is assumed to be zero, $\varphi = 0$, and again the box volume is $S = k^3$.

Dirac equations:

$$\left[\frac{W}{c} + \rho_3\, \boldsymbol{\sigma} \cdot \left(\boldsymbol{p} + \frac{e}{c} \boldsymbol{C}\right) + \rho_1 mc\right] \psi = 0.$$

$p = (p_x, p_y, p_z)$. For plane waves, p_x, p_y, p_z are constant.

$$\psi_p^r = (\psi_1, \psi_2, \psi_3, \psi_4) = \mathrm{e}^{(2\pi i/h)(p_x x + p_y y + p_z z)} (\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4),$$

$$\frac{W}{c} = \begin{cases} +\sqrt{m^2c^2 + p^2}, & \text{for } r = 1, 2, \\ -\sqrt{m^2c^2 + p^2}, & \text{for } r = 3, 4. \end{cases}$$

The spinor factors are given in the following table:

$\epsilon_1\sqrt{2S\left(1+\frac{W}{c}\frac{p_z}{m^2c^2}+\frac{p^2}{m^2c^2}\right)}$	$\epsilon_2\sqrt{2S\,(\dots)}$	$\epsilon_3\sqrt{2S\,(\dots)}$	$\epsilon_4\sqrt{2S\,(\dots)}$
1	0	$-\frac{W/c+p_z}{mc}$	$-\frac{p_x+ip_y}{mc}$
$\frac{p_x-ip_y}{mc}$	$-\frac{W/c+p_z}{mc}$	0	1
1	0	$-\frac{W/c+p_z}{mc}$	$-\frac{p_x+ip_y}{mc}$
$\frac{p_x-ip_y}{mc}$	$-\frac{W/c+p_z}{mc}$	0	1

$$\boxed{\begin{gathered} p_x = g_1\frac{h}{k}, \quad p_y = g_2\frac{h}{k}, \quad p_z = g_3\frac{h}{k}; \\ g_1,\ g_2,\ g_3 = 0, \pm 1, \pm 2, \pm 3, \dots. \end{gathered}}$$

$$\boxed{\begin{aligned} H &= -c\rho_3\,\boldsymbol{\sigma}\cdot\boldsymbol{p} - \rho_1\,mc^2 + \sum_s h\nu_s(n_s+\mathcal{N}_s) - e\rho_3\,\boldsymbol{\sigma}\cdot\boldsymbol{C} \\ &= H_0 - e\rho_3\,\boldsymbol{\sigma}\cdot\boldsymbol{C} = H_0 + H_1. \end{aligned}}$$

$H_1 = -e\rho_3\,\boldsymbol{\sigma}\cdot\boldsymbol{C}$. *Quantities* $n_s, \mathcal{N}_s$ are the numbers of the right-handed and left-handed polarized waves, respectively.

$$\begin{aligned} \left\langle p, r, n_i, \mathcal{N}_i | H_0 | p', r', n_i', \mathcal{N}_i' \right\rangle &= \delta(p-p')\,\delta(r-r_i)\,\delta(n-n')\,\delta(\mathcal{N}-\mathcal{N}') \\ &\quad \times W^{p,r}_{\text{electr.}} + \sum_s h\nu_s(n_s+\mathcal{N}_s). \end{aligned}$$

Expression for $\rho_3\boldsymbol{\sigma}$ on *the states* $\psi_p^1, \psi_p^2, \psi_p^3, \psi_p^4$:

$$\rho_3\sigma_x = \begin{vmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{vmatrix}, \quad \rho_3\sigma_y = \begin{vmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{vmatrix},$$

$$\rho_3\sigma_z = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}.$$

$$\begin{aligned}
\psi_p^1 &= (1,0,0,0)\,\mathrm{e}^{(2\pi i/h)(p_x x + p_y y + p_z z)}, \\
\psi_p^2 &= (0,1,0,0)\,\mathrm{e}^{(2\pi i/h)(p_x x + p_y y + p_z z)}, \\
\psi_p^3 &= (0,0,1,0)\,\mathrm{e}^{(2\pi i/h)(p_x x + p_y y + p_z z)}, \\
\psi_p^4 &= (0,0,0,1)\,\mathrm{e}^{(2\pi i/h)(p_x x + p_y y + p_z r)}.
\end{aligned}$$

2.7. QUANTUM DYNAMICS OF ELECTRONS INTERACTING WITH AN ELECTROMAGNETIC FIELD

The dynamics of a system composed of interacting electrons and photons is considered in the realm of Quantum Field Theory (Klein-Gordon theory). The electrons are described by a field ψ (or P, deduced from ψ), while the electromagnetic field is described in terms of the potential $(\varphi, \boldsymbol{C})$. An expression for the quantized Hamiltonian is given, along with the commutation rules for creation/annihilation operators.

For a charge $-e$ we have:

$$\left[\left(-\frac{h}{2\pi i c}\frac{\partial}{\partial t} + \frac{e}{c}\varphi\right)^2 - \sum_x \left(\frac{h}{2\pi i}\frac{\partial}{\partial x} + \frac{e}{c}C_x\right)^2 - m^2c^2\right]\psi = 0.$$

$$\begin{aligned}
P &= \frac{h^2}{8\pi^2 c^2 m}\left(\frac{\partial}{\partial t} + \frac{2\pi i}{h}e\,\varphi\right)\overline{\psi}, \\
\overline{P} &= \frac{h^2}{8\pi^2 c^2 m}\left(\frac{\partial}{\partial t} - \frac{2\pi i}{h}e\,\varphi\right)\psi.
\end{aligned}$$

$$\begin{aligned}
&\left[\frac{1}{c^2}\left(\frac{\partial}{\partial t} - \frac{2\pi i}{h}e\,\varphi\right)^2 - \sum_x \left(\frac{\partial}{\partial x} + \frac{2\pi i}{hc}e\,C_x\right)^2 + \frac{4\pi^2}{h^2}m^2c^2\right]\psi = 0, \\
&\left[\frac{1}{c^2}\left(\frac{\partial}{\partial t} + \frac{2\pi i}{h}e\,\varphi\right)^2 - \sum_x \left(\frac{\partial}{\partial x} - \frac{2\pi i}{hc}e\,C_x\right)^2 + \frac{4\pi^2}{h^2}m^2c^2\right]\overline{\psi} = 0.
\end{aligned}$$

$$\boldsymbol{\nabla}^2 C_x - \frac{\partial}{\partial x}\boldsymbol{\nabla}\cdot\boldsymbol{C} = \frac{\partial C_x}{\partial y^2} + \frac{\partial^2 C_x}{\partial r^2} - \frac{\partial^2 C_y}{\partial x \partial y} - \frac{\partial^2 C_z}{\partial x \partial z}.$$

$$\begin{aligned}&\left[\frac{h^2}{8\pi^2mc^2}\left(\frac{\partial}{\partial t}-\frac{2\pi i}{h}e\,\varphi\right)^2\right.\\&\left.\quad-\frac{h^2}{8\pi^2m}\sum_x\left(\frac{\partial}{\partial x}+\frac{2\pi i}{hc}e\,C_x\right)^2+\frac{1}{2}mc^2\right]\psi=0,\end{aligned}\tag{1}$$

$$\begin{aligned}&\left[\frac{h^2}{8\pi^2mc^2}\left(\frac{\partial}{\partial t}+\frac{2\pi i}{h}e\,\varphi\right)^2\right.\\&\left.\quad-\frac{h^2}{8\pi^2m}\sum_x\left(\frac{\partial}{\partial x}-\frac{2\pi i}{hc}e\,C_x\right)^2+\frac{1}{2}mc^2\right]\overline{\psi}=0.\end{aligned}\tag{2}$$

$$\left(\frac{\partial}{\partial t}-\frac{2\pi i}{h}e\,\varphi\right)\overline{P}=-\frac{1}{2}mc^2\psi+\frac{h^2}{8\pi^2m}\sum_x\left(\frac{\partial}{\partial x}+\frac{2\pi i}{hc}e\,C_x\right)^2\psi,\tag{3}$$

$$\left(\frac{\partial}{\partial t}+\frac{2\pi i}{h}e\,\varphi\right)P=-\frac{1}{2}mc^2\overline{\psi}+\frac{h^2}{8\pi^2m}\sum_x\left(\frac{\partial}{\partial x}-\frac{2\pi i}{hc}e\,C_x\right)\overline{\psi},\tag{4}$$

$$\left(\frac{\partial}{\partial t}+\frac{2\pi i}{h}e\,\varphi\right)\overline{\psi}=\frac{8\pi^2mc^2}{h^2}P,\tag{5}$$

$$\left(\frac{\partial}{\partial t}-\frac{2\pi i}{h}e\,\varphi\right)\psi=\frac{8\pi^2mc^2}{h^2}\overline{P}.\tag{6}$$

$$\begin{aligned}&\rho=\frac{he}{4\pi imc^2}\left[\overline{\psi}\left(\frac{\partial}{\partial t}-\frac{2\pi i}{hc}e\,\varphi\right)\psi-\psi\left(\frac{\partial}{\partial t}+\frac{2\pi i}{hc}e\,\varphi\right)\overline{\psi}\right],\\&i_x=-\frac{he}{4\pi imc}\left[\overline{\psi}\left(\frac{\partial}{\partial x}+\frac{2\pi i}{hc}e\,\varphi\right)\psi-\psi\left(\frac{\partial}{\partial x}-\frac{2\pi i}{hc}\mathrm{d}x\right)\overline{\psi}\right],\\&\cdots\ .\end{aligned}$$

$$\mathrm{d}\tau=\mathrm{d}V\mathrm{d}t.$$

[10]

[10]@ *Notice that, more appropriately, one should write* $\mathrm{d}^4\tau=\mathrm{d}^3V\mathrm{d}t$, *since* $\mathrm{d}\tau$ *denotes the 4-dimensional volume element, while drm*V *is the 3-dimensional space volume element.*

$$
\begin{aligned}
\delta \int \Bigg\{ & \frac{h^2}{8\pi^2 m} \left[\frac{1}{c^2} \left(\frac{\partial}{\partial t} + \frac{2\pi i}{h} e\,\varphi \right) \overline{\psi} \, \left(\frac{\partial}{\partial t} - \frac{2\pi i}{h} e\,\varphi \right) \psi \right. \\
& \left. - \sum_x \left(\frac{\partial}{\partial x} - \frac{2\pi i x}{hc} e\,C_x \right) \overline{\psi} \cdot \left(\frac{\partial}{\partial x} + \frac{2\pi i}{hc} e\,C_x \right) \psi \right] \\
& - \frac{1}{2} m c^2 \overline{\psi} \psi + \frac{1}{8\pi} \left(\left| \frac{1}{c} \frac{\partial \boldsymbol{C}}{\partial t} + \boldsymbol{\nabla}\,\varphi \right|^2 - |\boldsymbol{\nabla} \times \boldsymbol{C}|^2 \right) \Bigg\} \mathrm{d}\tau = 0.
\end{aligned}
\tag{7}
$$

From this, the variation with respect to $\overline{\psi}$ or ψ gives Eq. (1) or (2), respectively. The variation with respect to φ yields:

$$
\begin{aligned}
& -\frac{1}{4\pi} \sum_x \frac{\partial}{\partial x} \left(\frac{\partial \varphi}{\partial x} + \frac{1}{c} \frac{\partial C}{\partial t} \right) \\
& - \frac{he}{4\pi i m c^2} \left[\overline{\psi} \left(\frac{\partial}{\partial t} - \frac{2\pi i}{h} e\,\varphi \right) \psi - \psi \left(\frac{\partial}{\partial t} + \frac{2\pi i}{h} e\,\varphi \right) \overline{\psi} \right] = 0,
\end{aligned}
$$

$$
\frac{1}{4\pi} \boldsymbol{\nabla} \cdot \boldsymbol{E} - \rho = 0. \tag{8}
$$

The variation with respect to C_x instead gives:

$$
\begin{aligned}
& -\frac{1}{4\pi c} \frac{\partial}{\partial t} \left(\frac{\partial \varphi}{\partial x} + \frac{1}{c} \frac{\partial C_x}{\partial t} \right) - \frac{1}{4\pi} \left[\frac{\partial}{\partial y} \left(\frac{\partial C_y}{\partial x} - \frac{\partial C_x}{\partial y} \right) \right. \\
& \left. - \frac{\partial}{\partial z} \left(\frac{\partial C_x}{\partial z} - \frac{\partial C_z}{\partial x} \right) \right] - \frac{he}{4\pi i m c} \left[\overline{\psi} \left(\frac{\partial}{\partial x} + \frac{2\pi i}{hc} e\,C_x \right) \psi \right. \\
& \left. - \psi \left(\frac{\partial}{\partial x} - \frac{2\pi i}{hc} e\,C_x \right) \overline{\psi} \right] = 0,
\end{aligned}
$$

$$
\frac{1}{4\pi c} \frac{\partial E_x}{\partial t} - \frac{1}{4\pi} \left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial r} \right) + i_x = 0, \tag{9}
$$

and similarly for the other components.

$$
\begin{aligned}
A &= \frac{1}{8\pi} \sum_x \left(\frac{1}{c} \frac{\partial C_x}{\partial t} + \frac{\partial \varphi}{\partial x} \right)^2 \\
&= \frac{1}{8\pi} \frac{1}{c^2} \sum_x \left(\frac{\partial C_x}{\partial t} \right)^2 + \frac{1}{8\pi} \left(\frac{\partial \varphi}{\partial x} \right)^2 + \frac{1}{4\pi c} \sum_x \frac{\partial C_x}{\partial t} \frac{\partial \varphi}{\partial x}, \\
B &= \frac{1}{4\pi c^2} \sum_x \left(\frac{\partial C_x}{\partial t} \right)^2 + \frac{1}{4\pi c} \sum_x \frac{\partial C_x}{\partial t} \frac{\partial \varphi}{\partial x},
\end{aligned}
$$

$$B - A \;=\; \frac{1}{8\pi c^2}\sum_x \left(\frac{\partial C_x}{\partial t}\right)^2 - \frac{1}{8\pi}\sum_x \left(\frac{\partial \varphi}{\partial x}\right)^2 .$$

Without matter fields, *the conjugate Hamiltonian variables are:*

$$\begin{array}{ll} C_x, & -\dfrac{1}{4\pi c}E_x; \\ C_y, & -\dfrac{1}{4\pi c}E_y; \\ C_z, & -\dfrac{1}{4\pi c}E_z; \\ \varphi, & 0 \end{array}$$

[11]

$$\mathcal{H} = \frac{1}{8\pi}|\boldsymbol{\nabla}\times\boldsymbol{C}|^2 + \frac{1}{8\pi}E^2 + \frac{1}{4\pi}\sum_x \frac{\partial\varphi}{\partial x}E_x,$$

$$\begin{aligned} &\dot{E}_x = c\left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right), \\ &\dot{C}_x = -cE_x - c\frac{\partial\varphi}{\partial x}, \quad E_x = -\frac{\partial\varphi}{\partial x} - \frac{1}{c}\frac{\partial C_x}{\partial t}, \\ &\dot{\varphi} = \ldots \\ &\dot{0} = 0 = -\frac{1}{4\pi}\boldsymbol{\nabla}\cdot\boldsymbol{E}. \end{aligned}$$

In the following we consider a particle with charge $-e$ *and assume* $\varphi = 0$.

$$\delta\int L\mathrm{d}\tau = 0, \quad \text{with } \mathrm{d}\tau = \mathrm{d}V\mathrm{d}t.$$

$$\begin{aligned} &\delta\int\left\{\frac{h^2}{8\pi^2 m}\left[\frac{1}{c^2}\frac{\partial}{\partial t}\overline{\psi}\;\frac{\partial}{\partial t}\psi\right.\right. \\ &\left.-\sum_x\left(\frac{\partial}{\partial x} - \frac{2\pi i}{hc}e\,C_x\right)\overline{\psi}\;\left(\frac{\partial}{\partial x} + \frac{2\pi i}{hc}e\,C_x\right)\psi\right] \\ &\left.-\frac{1}{2}mc^2\,\overline{\psi}\psi + \frac{1}{8\pi}\left[\frac{1}{c}\left|\frac{\partial C}{\partial t}\right|^2 - |\boldsymbol{\nabla}\times C|^2\right]\right\}\mathrm{d}\tau = 0. \end{aligned} \tag{7'}$$

[11]@ *In the following, the author looked for the variable conjugate to* φ.

$$\begin{aligned}
&\psi, \qquad P = \frac{h^2}{8\pi^2 mc^2}\frac{\partial}{\partial t}\overline{\psi};\\
&\overline{\psi}, \qquad \overline{P} = \frac{h^2}{8\pi^2 mc}\frac{\partial}{\partial t}\psi;
\end{aligned}$$

$$\begin{aligned}
&C_x, \qquad -\frac{E_x}{4\pi c} = \frac{1}{4\pi c^2}\frac{\partial C_x}{\partial t}:\\
&C_y, \qquad -\frac{E_y}{4\pi c} = \frac{1}{4\pi c^2}\frac{\partial C_y}{\partial t};\\
&C_z, \qquad -\frac{E_z}{4\pi c} = \frac{1}{4\pi c^2}\frac{\partial C_z}{\partial t}.
\end{aligned}$$

$$\begin{aligned}
H &= \int\left[\frac{8\pi^2 mc^2}{h^2}\overline{P}P + \frac{1}{2}mc^2\,\overline{\psi}\psi + \frac{h^2}{8\pi^2 m}\sum_x\left(\frac{\partial}{\partial x} - \frac{2\pi i}{hc}e\,C_x\right)\overline{\psi}\right.\\
&\quad\left.\times\left(\frac{\partial}{\partial x} + \frac{2\pi i}{hc}e\,C_x\right)\psi + \frac{1}{8\pi}(E^2+H^2)\right]\mathrm{d}V,\\
H &= \int\left[\frac{8\pi^2 mc^2}{h^2}\overline{P}P + \frac{1}{2}mc^2\,\overline{\psi}\psi + \frac{h^2}{8\pi^2 m}\boldsymbol{\nabla}\,\overline{\psi}\cdot\boldsymbol{\nabla}\,\psi + \right.\\
&\quad + \frac{hc}{4\pi imc}\boldsymbol{C}\cdot(\overline{\psi}\boldsymbol{\nabla}\,\psi - \psi\boldsymbol{\nabla}\,\overline{\psi})\\
&\quad\left.+\frac{c^2}{2mc^2}|\boldsymbol{C}|^2\overline{\psi}\psi + \frac{1}{8\pi}(E^2+H^2)\right]\mathrm{d}V.
\end{aligned}$$

$$\begin{aligned}
\rho &= \frac{2\pi i}{h}e(\psi P - \overline{\psi}\overline{P}),\\
\boldsymbol{i} &= -\frac{he}{4\pi imc}\left[\overline{\psi}\left(\boldsymbol{\nabla} + \frac{2\pi i}{hc}e\,\boldsymbol{C}\right)\psi - \psi\left(\boldsymbol{\nabla} - \frac{2\pi C}{hc}e\,\boldsymbol{C}\right)\overline{\psi}\right]\\
&= -\frac{he}{4\pi imc}(\overline{\psi}\boldsymbol{\nabla}\,\psi - \psi\boldsymbol{\nabla}\,\overline{\psi}) - \frac{c^2}{mc^2}\overline{\psi}\psi\,\boldsymbol{C}.
\end{aligned}$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{f}'_k = 0, \quad \boldsymbol{f}_\lambda = \boldsymbol{\nabla}\,\varphi_\lambda; \quad \boldsymbol{\nabla}^2\,\varphi_\lambda + \lambda^2\varphi_\lambda = 0.$$

$$\left\{\begin{array}{l}
\boldsymbol{\nabla}^2\,\boldsymbol{f}_\lambda + \lambda^2\boldsymbol{f}_\lambda = 0,\\
\boldsymbol{\nabla}^2\,\boldsymbol{f}'_k + k^2\boldsymbol{f}'_k = 0.
\end{array}\right.$$

$$\int \boldsymbol{f}_\lambda \cdot \boldsymbol{f}_{\lambda'}\,\mathrm{d}V = \delta_{\lambda\lambda'},$$
$$\int \boldsymbol{f}'_k \cdot \boldsymbol{f}'_{k'}\,\mathrm{d}V = \delta_{kk'},$$
$$\int \boldsymbol{f}_\lambda \cdot \boldsymbol{f}_k\,\mathrm{d}V = 0;$$

$$\int \varphi_\lambda \varphi_{\lambda'}\,\mathrm{d}V = \frac{1}{\lambda'^2}\int \boldsymbol{f}_\lambda \cdot \boldsymbol{f}_{\lambda'}\,\mathrm{d}V = \frac{1}{\lambda^2}\delta_{\lambda\lambda'},$$
$$\lambda\varphi_\lambda = u_\lambda; \qquad \int u_\lambda u_{\lambda'}\,\mathrm{d}V = \delta_{\lambda\lambda'}.$$

$$\begin{aligned} \psi &= \sum \left[A_\lambda (q_\lambda + Q_\lambda) + iB_\lambda (p_\lambda - P_\lambda)\right]\lambda\varphi_\lambda, \quad (A_\lambda = B_\lambda) \\ P &= \sum \left[C_\lambda (p_\lambda + P_\lambda) + iD_\lambda (q_\lambda - Q_\lambda)\right]\lambda\varphi_\lambda; \quad (C_\lambda = D_\lambda) \end{aligned}$$

$$\begin{aligned} \int \overline{P} P\,\mathrm{d}V &= \sum \left[C_\lambda^2 (p_\lambda + P_\lambda)^2 + D_\lambda^2 (q_\lambda - Q_\lambda)^2\right], \\ \int \overline{\psi}\psi\,\mathrm{d}V &= \sum \left[A_\lambda^2 (q_\lambda + Q_\lambda)^2 + B_\lambda^2 (p_\lambda - P_\lambda)^2\right]. \end{aligned}$$

$$\begin{aligned} &\frac{8\pi^2 mc^2}{h^2}\int \overline{P} P\,\mathrm{d}V + \frac{1}{2m}\left(m^2c^2 + \lambda^2\frac{h^2}{4\pi^2}\right)\int \overline{\psi}\psi\,\mathrm{d}V \\ &= \frac{8\pi^2 mc^2}{h^2}\sum_\lambda \left[C_\lambda^2 (p_\lambda + P_\lambda)^2 + D_\lambda^2 (q_\lambda - Q_\lambda)^2\right] \\ &+ \frac{1}{2m}\left(m^2c^2 + \lambda^2\frac{h^2}{4\pi^2}\right)\sum_\lambda \left[A_\lambda^2 (q_\lambda + Q_\lambda)^2 + B_\lambda^2 (p_\lambda - P_\lambda)^2\right] \\ &= \sum_\lambda \left[\frac{1}{2}p_\lambda^2 + \frac{1}{2}q_\lambda^2 + \frac{1}{2}P_\lambda^2 + \frac{1}{2}Q_\lambda^2\right] c\sqrt{m^2c^2 + \lambda^2\frac{h^2}{4\pi^2}}, \end{aligned}$$

$$\frac{8\pi^2 mc^2}{h^2}C_\lambda^2 + \frac{1}{2m}\left(m^2c^2 + \lambda^2\frac{h^2}{4\pi^2}\right)B_\lambda^2 = \frac{1}{2}c\sqrt{m^2c^2 + \lambda^2\frac{h^2}{4\pi^2}},$$
$$\frac{8\pi^2 mc^2}{h^2}D_\lambda^2 + \frac{1}{2m}\left(m^2c^2 + \lambda^2\frac{h^2}{4\pi^2}\right)A_\lambda^2 = \frac{1}{2}c\sqrt{m^2c^2 + \lambda^2\frac{h^2}{4\pi^2}},$$

$$\frac{8\pi^2 mc^2}{h^2} C_\lambda^2 = \frac{1}{2m}\left(m^2c^2 + \lambda^2 \frac{h^2}{4\pi^2}\right) B_\lambda^2,$$
$$\frac{8\pi^2 mc^2}{h^2} D_\lambda^2 = \frac{1}{2m}\left(m^2c^2 + \lambda^2 \frac{h^2}{4\pi^2}\right) A_\lambda^2,$$

$$A_\lambda^2 = B_\lambda^2 = \frac{mc}{2\sqrt{m^2c^2 + \lambda^2 \frac{h^2}{4\pi^2}}}\,,$$

$$C_\lambda^2 = D_\lambda^2 = \frac{h^2}{32\pi^2 mc}\sqrt{m^2c^2 + \lambda^2 \frac{h^2}{4\pi^2}}\,.$$

$$\boxed{\begin{aligned} \psi &= \frac{1}{\sqrt{2}}\sum_\lambda \sqrt{\frac{mc}{\sqrt{m^2c^2+\lambda^2h^2/4\pi^2}}}\left[q_\lambda + q'_\lambda + i\left(p_\lambda - p'_\lambda\right)\right] u_\lambda, \\ P &= \frac{h}{4\pi\sqrt{2}}\sum_\lambda \sqrt{\frac{\sqrt{m^2c^2+\lambda^2h^2/4\pi^2}}{mc}}\left[p_\lambda + p'_\lambda + i\left(q_\lambda - q'_\lambda\right)\right] u_\lambda. \end{aligned}}$$

$$\begin{aligned} 4/i &= 2(p_\lambda q_\lambda - q_\lambda p_\lambda) + 2(p'_\lambda q'_\lambda - q'_\lambda p'_\lambda) \pm 2i(q_\lambda q'_\lambda - q'_\lambda q_\lambda) \\ &\quad \mp 2i(p_\lambda p'_\lambda - p'_\lambda p_\lambda), \\ 0 &= (p_\lambda q_\lambda - q_\lambda p_\lambda) - (p'_\lambda q'_\lambda - q'_\lambda p'_\lambda) + (p_\lambda q'_\lambda - q'_\lambda p_\lambda) - (p'_\lambda q_\lambda - q_\lambda p'_\lambda), \\ 0 &= (p_\lambda q_\lambda - q_\lambda p_\lambda) - (p'_\lambda q'_\lambda - q'_\lambda p'_\lambda) - (p_\lambda q'_\lambda - q'_\lambda p_\lambda) - (p'_\lambda q_\lambda - q_\lambda p'_\lambda), \\ 0 &= (p_\lambda q'_\lambda - q'_\lambda p_\lambda) + (p'_\lambda q_\lambda - q_\lambda p'_\lambda) \pm (p_\lambda p'_\lambda - p'_\lambda p_\lambda) \pm (q_\lambda q'_\lambda - q'_\lambda q_\lambda). \end{aligned}$$

$$\boxed{\begin{array}{rclrcl} p_\lambda q_\lambda - q_\lambda p_\lambda &=& 1/i, & p'_\lambda q'_\lambda - q'_\lambda p'_\lambda &=& 1/i, \\ p_\lambda q'_\lambda - q'_\lambda p_\lambda &=& 0, & p'_\lambda q_\lambda - q_\lambda p'_\lambda &=& 0, \\ p_\lambda p'_\lambda - p'_\lambda p_\lambda &=& 0, & q_\lambda q'_\lambda - q'_\lambda q_\lambda &=& 0. \end{array}}$$

$$-Ze \;=\; \int \rho\, \mathrm{d}V = \frac{2\pi i}{h} e \int (\psi P - \overline{\psi}\,\overline{P})\, \mathrm{d}V,$$

$$\begin{aligned}
Z &= -\frac{2\pi i}{h}\int(\psi P - \overline{\psi P})\,\mathrm{d}V \\
&= \sum_\lambda\left(\frac{1}{2}p_\lambda^2 + \frac{1}{2}q_\lambda^2 - \frac{1}{2}p_\lambda^{\prime 2} - \frac{1}{2}q_\lambda^{\prime 2}\right) \\
&= \sum_\lambda\left[\left(\frac{1}{2}p_\lambda^2 + \frac{1}{2}q_\lambda^2 - \frac{1}{2}\right) - \left(\frac{1}{2}p_\lambda^{\prime 2} + \frac{1}{2}q_\lambda^{\prime 2} - \frac{1}{2}\right)\right] \\
&= \sum_\lambda(N_\lambda - N'_\lambda) = \sum_\lambda Z_\lambda.
\end{aligned}$$

$$\boxed{\boxed{\begin{aligned}
H &= H_M + H_R, \\
H_M &= H_M^0 + H_M^1,
\end{aligned}}}$$

where H_M and H_R account for the matter and radiation field contribution to the Hamiltonian, respectively. H_M^0 is the free particle Hamiltonian, while H_M^1 describes the particle interaction and that between particles and light quanta.

$$\begin{aligned}
&N_\lambda = \frac{1}{2}p_{\lambda^2} + \frac{1}{2}q_{\lambda^2} - \frac{1}{2}, \\
&N'_\lambda = \frac{1}{2}p'_{\lambda^2} + \frac{1}{2}q'_{\lambda^2} - \frac{1}{2}, \\
&Z_\lambda = N_\lambda - N'_\lambda.
\end{aligned}$$

$$\boxed{\begin{aligned}
H_M^0 &= \sum_\lambda\left(\frac{1}{2}p_\lambda^2 + +\frac{1}{2}q_{\lambda^2}\frac{1}{2}p'_{\lambda^2} + \frac{1}{2}q'_{\lambda^2}\right)c\sqrt{m^2c^2 + \lambda^2\frac{h^2}{4\pi^2}} \\
&= \sum_\lambda(N_\lambda + N'_\lambda)c\sqrt{m^2c^2 + \lambda^2\frac{h^2}{4\pi^2}} + \textit{zero point energy.}
\end{aligned}}$$

[12]

[12]@ *In the original manuscript, some expressions were written in terms of ν instead of k, but the warning "use k instead of ν" appears. We have therefore chosen to use the symbol k throughout.*

$$
\begin{aligned}
\boldsymbol{C} &= \sum_k A_k Q_k \boldsymbol{f}_k + \sum_\lambda B_\lambda P_\lambda \boldsymbol{f}_\lambda, \\
-\boldsymbol{E} &= \sum_k C_k P_k \boldsymbol{f}_k - \sum_\lambda D_\lambda Q_\lambda \boldsymbol{f}_\lambda
\end{aligned}
$$

$(\boldsymbol{\nabla} \times \boldsymbol{f}_\lambda = 0)$.

$$
\int E^2 \mathrm{d}V = \sum_k C_k^2 P_k^2 + \sum_\lambda D_\lambda^2 Q_\lambda^2 .
$$

$$
\begin{aligned}
H_x &= \frac{\partial C_z}{\partial y} - \frac{\partial C_y}{\partial z}, \\
H_x^2 &= \left(\frac{\partial C_z}{\partial y}\right)^2 + \left(\frac{\partial C_H}{\partial r}\right)^2 - 2\frac{\partial C_x}{\partial y}\frac{\partial C_y}{\partial x}, \\
H^2 &= \sum_x |\boldsymbol{\nabla}\, C_x|^2 - \sum_{xy} \frac{\partial C_x}{\partial y}\frac{\partial C_y}{\partial x} = \sum_k A_k^2 N^2 Q_k^2 .
\end{aligned}
$$

$$
\int H^2 \mathrm{d}V = \ldots .
$$

$$
\boxed{
\begin{aligned}
P_k Q_k - Q_k P_k &= 1/i, \\
P_\lambda Q_\lambda - Q_\lambda P_\lambda &= 1/i.
\end{aligned}
}
$$

$$
\begin{aligned}
\frac{C_k^2}{8\pi} &= \frac{1}{2}\frac{hck}{2\pi}, \\
\frac{A_k^2}{8\pi} &= \frac{1}{2}\frac{hck}{2\pi}, \\
\frac{D_{\lambda^2}}{8\pi} &= \frac{1}{2};
\end{aligned}
$$

$$
\begin{aligned}
C_k &= \sqrt{2hck}, \\
A_k &= \sqrt{\frac{2hc}{k}}, \\
D_\lambda &= \sqrt{4\pi} = 2\sqrt{\pi}, \\
B_\lambda &= \frac{hc}{\sqrt{\pi}}.
\end{aligned}
$$

$$\mathcal{N}_k = \frac{1}{2}(P_k^2 + Q_k^2) - \frac{1}{2}.$$

$$\boxed{\begin{aligned} \boldsymbol{C} &= \sum_k \sqrt{\frac{2hc}{k}} Q_k \boldsymbol{f}'_k + \sum_\lambda \frac{hc}{\sqrt{\pi}} P_\lambda \boldsymbol{f}_\lambda, \\ -\boldsymbol{E} &= \sum_k \sqrt{2hck} P_k \boldsymbol{f}'_k - \sum_\lambda \sqrt{4\pi} Q_\lambda \boldsymbol{f}_\lambda. \end{aligned}}$$

$$\boxed{\nu_k = c\,\frac{k}{2\pi}.}$$

$$\boxed{\begin{aligned} H_R &= \frac{1}{2}\sum_k h\frac{ck}{2\pi}(Q_k^2 + P_k^2) + \frac{1}{2}\sum_\lambda Q_\lambda^2 \\ &= \sum_k \frac{1}{2}(P_k^2 + Q_k^2)\, h\nu_k + \frac{1}{2}\sum_\lambda Q_\lambda^2 \\ &= \sum_k \mathcal{N} h\nu_k + \sum_\lambda \frac{1}{2} Q_\lambda^2 + \text{ rest energy.} \end{aligned}}$$

---

[13]

$$\boldsymbol{\nabla}\, u_\lambda = \boldsymbol{\nabla}\, \lambda\varphi_\lambda = \lambda \boldsymbol{f}_\lambda,$$

$$\boldsymbol{\nabla}\, \psi = \frac{1}{\sqrt{2}}\sum_\lambda \sqrt{\frac{mc}{\sqrt{m^2c^2 + \lambda^2h^2/4\pi^2}}}\,\left[q_\lambda + q'_\lambda + i(p_\lambda - p'_\lambda)\right]\lambda \boldsymbol{f}_\lambda,$$

$$\begin{aligned} \overline{\psi}\boldsymbol{\nabla}\, \psi - \psi\boldsymbol{\nabla}\, \overline{\psi} = -imc\sum_{\lambda\lambda'} & \frac{1}{\sqrt[4]{(m^2c^2 + \lambda^2h^2/4\pi^2)\,(m^2c^2 + \lambda' h^2/4\pi^2)}} \\ & \times \left[(p_\lambda - p'_\lambda)(q_{\lambda'} + q'_{\lambda'}) - (p_{\lambda'} - p'_{\lambda'})(q_\lambda + q'_\lambda)\right]\lambda' u_\lambda \boldsymbol{f}_{\lambda'}. \end{aligned}$$

$$\boldsymbol{\nabla}\cdot \varphi_\lambda \boldsymbol{f}_{\lambda'} = \boldsymbol{f}_\lambda \cdot \boldsymbol{f}_{\lambda'} - \lambda' 2\varphi_\lambda\varphi_{\lambda'}.$$

[13]@ *In the original manuscript, the expression* $\boldsymbol{\nabla}\, u_\lambda = \boldsymbol{\nabla}\, \lambda u_\lambda = \lambda f_\lambda$ *was written down, which is evidently incorrect.*

$$\begin{aligned}
\psi &= \frac{1}{\sqrt{2}} \sum_\lambda \sqrt{\frac{mc}{\sqrt{m^2c^2 + h^2\lambda^2/4\pi^2}}} \left[q_\lambda + q'_\lambda + i(p_\lambda - p'_\lambda)\right] u_\lambda, \\
P &= \frac{h}{4\pi\sqrt{2}} \sum_\lambda \sqrt{\frac{\sqrt{m^2c^2 + h^2\lambda^2/4\pi^2}}{mc}} \left[p_\lambda + p'_\lambda + i(q_\lambda - q'_\lambda)\right] u_\lambda.
\end{aligned}$$

[14]

$$a_\lambda = \frac{1}{\sqrt{2}}(q_\lambda + ip_\lambda), \qquad b_\lambda = \frac{1}{\sqrt{2}}(q'_\lambda + ip'_\lambda),$$

$$\overline{a}_\lambda = \frac{1}{\sqrt{2}}(q_\lambda - ip_\lambda), \qquad \overline{b}_\lambda = \frac{1}{\sqrt{2}}(q'_\lambda - ip'_\lambda).$$

$$\begin{aligned}
[a_\lambda, \overline{a}_\mu] - [\overline{b}_\lambda, b_\mu] - [a_\lambda, b_\mu] - [\overline{b}_\lambda, \overline{a}_\mu] &= 2\delta_{\lambda\mu}, \\
-[\overline{a}_\lambda, a_\mu] + [b_\lambda, \overline{b}_\mu] + [\overline{a}_\lambda, \overline{b}_\mu] - [b_\lambda, a_\mu] &= 2\delta_{\lambda\mu}.
\end{aligned}$$

$$[x, y] = xy \mp yx,$$

where the upper/lower sign refers to Einstein/Fermi particles.

$$\begin{aligned}
&[a_\lambda, a_\mu] + [\overline{b}_\lambda, \overline{b}_\mu] + [a_\lambda, \overline{b}_\mu] + [\overline{b}_\lambda, a_\mu] = 0, \\
&[\overline{a}_\lambda, \overline{a}_\mu] + [b_\lambda, b_\mu] + [\overline{a}_\lambda, b_\mu] + [b_\lambda, \overline{a}_\mu] = 0, \\
&[a_\lambda, \overline{a}_\mu] + [\overline{b}_\lambda, b_\mu] + [a_\lambda, b_\mu] + [\overline{b}_\lambda, \overline{a}_\mu] = 0, \\
&[\overline{a}_\lambda, \overline{a}_\mu] + [b_\lambda, b_\mu] - [\overline{a}_\lambda, b_\mu] - [b_\lambda, \overline{a}_\mu] = 0, \\
&[a_\lambda, \overline{a}_\mu] + [\overline{b}_\lambda, b_\mu] - [a_\lambda, b_\mu] - [\overline{b}_\lambda, \overline{a}_\mu] = 0, \\
&[a_\lambda, a_\mu] + [\overline{b}_\lambda, \overline{b}_\mu] - [a_\lambda, \overline{b}_\mu] - [\overline{b}_\lambda, a_\mu] = 0, \\
&[a_\lambda, a_\mu] - [\overline{b}_\lambda, \overline{b}_\mu] + [\overline{b}_\lambda, a_\mu] - [a_\lambda, \overline{b}_\mu] = 0, \\
&[\overline{a}_\lambda, \overline{a}_\mu] - [b_\lambda, b_\mu] + [b_\lambda, \overline{a}_\mu] - [\overline{a}_\lambda, b_\mu] = 0.
\end{aligned}$$

2.8. CONTINUATION

$$\begin{aligned}
\psi &= \sum_\lambda \sqrt{\frac{mc}{\sqrt{m^2c^2 + h^2\lambda^2/4\pi^2}} mc}\, (a_\lambda + \overline{b}_\lambda)\, u_\lambda, \\
P &= \frac{hi}{4\pi} \sum_\lambda \sqrt{\frac{\sqrt{m^2c^2 + h^2\lambda^2/4\pi^2}}{mc}}\, (\overline{a}_\lambda - b_\lambda)\, u_\lambda,
\end{aligned}$$

[14]@ *In the original manuscript the simple formulas* $(a - ib)(a + ib) = a^2 + b^2 + i(ab - ba)$ *and* $(a + ib)(a - ib) = a^2 + b^2 - i(ab - ba)$ *are noted on the side.*

$$\begin{aligned}\overline{\psi} &= \sum_{\lambda} \sqrt{\frac{mc}{\sqrt{m^2c^2 + h^2\lambda^2/4\pi^2}}} \, (\overline{a}_\lambda + b_\lambda) \, u_\lambda, \\ \overline{P} &= -\frac{hi}{4\pi} \sum_{\lambda} \sqrt{\frac{\sqrt{m^2c^2 + h^2\lambda^2/4\pi^2}}{mc}} \, (a_\lambda - \overline{b}_\lambda) \, u_\lambda.\end{aligned}$$

From the commutation relations reported at the end of the previous Section, we deduce that:

$$\begin{aligned}&[a_\lambda, a_\mu] + \left[\overline{b}_\lambda, \overline{b}_\mu\right] = 0, \\ &\left[a_\lambda, \overline{b}_\mu\right] + \left[\overline{b}_\lambda, a_\mu\right] = 0, \\ &[a_\lambda, \overline{a}_\mu] + \left[\overline{b}_\lambda, b_\mu\right] = 0, \\ &[a_\lambda, b_\mu] + \left[\overline{b}_\lambda, \overline{a}_\mu\right] = 0, \\ &[\overline{a}_\lambda, \overline{a}_\mu] + [b_\lambda, b_\mu] = 0, \\ &[\overline{a}_\lambda, b_\mu] + [b_\lambda, \overline{a}_\mu] = 0, \\ &[a_\lambda, a_\mu] + \left[\overline{b}_\lambda, b_\mu\right] = 0, \\ &[b_\lambda, b_\mu] + [\overline{a}_\lambda, b_\mu] = 0;\end{aligned}$$

$$\begin{aligned}[a_\lambda, \overline{a}_\mu] - \left[\overline{b}_\lambda, b_\mu\right] &= 2\delta_{\lambda\mu}, \\ [\overline{a}_\lambda, a_\mu] - \left[b_\lambda, \overline{b}_\mu\right] &= -2\delta_{\lambda\mu}.\end{aligned}$$

$$\begin{aligned}0 = [a + ib, a + ib] &= [a, a] - [b, b] + i[a, b] + i[b, a], \\ 0 = [a - ib, a - ib] &= [a, a] - [b, b] - i[a, b] - i[b, a], \\ 0 = [a + ib, a - ib] &= [a, a] + [b, b] - i[a, b] + i[b, a], \\ 0 = [a - ib, a + ib] &= [a, a] + [b, b] + i[a, b] - i[b, a];\end{aligned}$$

$$[a, a] = [b, b] = [a, b] = [b, a] = 0.$$

2.9. QUANTIZED RADIATION FIELD

The author again considered the quantization of the electromagnetic field, but using now another expansion in a basis different from that adopted in Sects. 2.4, 2.5. In the original manuscript, the present Section and the following four Sections are placed in the Quaderno 17 just after what has been here reported in Sect. 7.1.

$$\boldsymbol{E} = -\frac{1}{c}\frac{\partial \boldsymbol{C}}{\partial t}, \quad \frac{1}{c^2}\frac{\partial^2 \boldsymbol{C}^2}{\partial t^2} = \boldsymbol{\nabla}^2\,\boldsymbol{C} = -\frac{1}{c}\frac{\partial \boldsymbol{E}}{\partial t}.$$

$$C_x, \qquad C_y, \qquad C_z;$$

$$-\frac{E_x}{4\pi c}, \quad -\frac{E_y}{4\pi c}, \quad -\frac{E_z}{4\pi c}.$$

$$\begin{aligned}
&\gamma_1,\, \gamma_2,\, \gamma_3 = 0, \pm 1, \pm 2, \ldots; \\
&\gamma = \frac{c}{k}\sqrt{\gamma_1^2 + \gamma_2^2 + \gamma_3^2}; \\
&p_x = \frac{h}{k}\gamma_1, \quad p_y = \frac{h}{k}\gamma_2, \quad p_z = \frac{h}{k}\gamma_3.
\end{aligned}$$

$$|k_s| = 1, \qquad \boxed{\boldsymbol{k}_s = \boldsymbol{k}_{-s}.}$$

$$\boxed{\boldsymbol{f}_s = \boldsymbol{k}_s\ \mathrm{e}^{2\pi i\left(\gamma_i^s x/k + \gamma_2^s y/k + \gamma_3^s z/k\right)}\ \frac{1}{\sqrt{k^3}}.}$$

[15]

$$\boxed{\begin{aligned} \boldsymbol{C} &= \sum a_s \boldsymbol{f}_s, \\ \boldsymbol{E} &= \sum b_s \boldsymbol{f}_s. \end{aligned}}$$

$$\boxed{\begin{aligned} a_s &= \tilde{a}_{-s}, \\ b_s &= \tilde{b}_{-s}. \end{aligned}}$$

$$a_s a_{s'} - a_{s'} a_s = 0,$$

$$b_s b_{s'} - b_{s'} b_s = 0,$$

$$\boxed{a_s \tilde{b}_{s'} - \tilde{b}_{s'} a_s = \frac{2hc}{i}\delta_{s,s'}.}$$

[15]@ *In the original manuscript, the normalization factor* $1/\sqrt{k^3}$ *is incorrectly treated as a denominator instead of a numerator.*

$$\dot{\boldsymbol{C}} = -c\,\boldsymbol{E} = \sum -c\,b_s \boldsymbol{f}_s; \quad \dot{\boldsymbol{E}} = -c\,\boldsymbol{\nabla}^2\,\boldsymbol{C} = \sum \frac{4\pi^2\nu_s^2}{c} a_s \boldsymbol{f}_s.$$

$$\boxed{\begin{aligned} \dot{a}_s &= -c\,b_s, \\ \dot{b}_s &= \frac{4\pi^2\nu_s^2}{c} a_s. \end{aligned}}$$

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\left(a_s + \frac{c}{2\pi\nu_s i} b_s\right) &= -c\,b_s - 2\pi\nu_s i\,a_s = -2\pi\nu_s i\left(a_s + \frac{c}{2\pi\nu_s i} b_s\right), \\ \frac{\mathrm{d}}{\mathrm{d}t}\left(a_s - \frac{c}{2\pi\nu_s i} b_s\right) &= -c\,b_s + 2\pi\nu_s i\,a_s = 2\pi\nu_s i\left(a_s - \frac{c}{2\pi\nu_s i} b_s\right). \end{aligned}$$

$$\boxed{\begin{aligned} A_s &= a_s + \frac{c}{2\pi\nu_s i} b_s, \\ B_s &= a_s - \frac{c}{2\pi\nu_s i} b_s; \end{aligned}}$$

$$\begin{aligned} \dot{A}_s &= -2\pi\nu_s i\ A_s, \\ \dot{B}_s &= 2\pi\nu_s i\ B_s; \end{aligned}$$

$$\boxed{\begin{aligned} \tilde{A}_s &= B_{-s}, \\ \tilde{B}_s &= A_{-s}. \end{aligned}}$$

$$\begin{aligned} A_s B_s - B_s A_s &= 0, \\ \tilde{A}_s \tilde{B}_s - \tilde{B}_s \tilde{A}_s &= 0, \\ A_s \tilde{B}_s - \tilde{B}_s A_s &= 0, \end{aligned}$$

$$\boxed{A_s \tilde{A}_s - \tilde{A}_s A_s = \frac{2hc^2}{\pi\nu_s},}$$

$$\boxed{B_s \tilde{B}_s - \tilde{B}_s B_s = -\frac{2hc^2}{\pi\nu_s}.}$$

$$
\begin{aligned}
A_s A_t - A_t A_s &= 0, \\
B_s B_t - B_t B_s &= 0, \\
\tilde{A}_s \tilde{A}_t - \tilde{A}_t \tilde{A}_s &= 0, \\
\tilde{B}_s \tilde{B}_t - \tilde{B}_t \tilde{B}_s &= 0, \\
A_s \tilde{B}_t - \tilde{B}_t A_s &= 0, \\
\tilde{A}_s B_t - B_t \tilde{A}_s &= 0,
\end{aligned}
$$

$$\boxed{A_s \tilde{A}_t - \tilde{A}_t A_s = \frac{2hc^2}{\pi \nu_s} \delta_{st},}$$

$$\boxed{B_s \tilde{B}_t - \tilde{B}_t B_s = -\frac{2hc^2}{\pi \nu_s} \delta_{st}.}$$

$$\boxed{Z_s = \frac{1}{c} \sqrt{\frac{\pi \nu_s}{2h}} A_s.}$$

$$\boxed{\begin{aligned}
Z_s Z_t - Z_t Z_s &= 0, \\
\tilde{Z}_s \tilde{Z}_t - \tilde{Z}_t \tilde{Z}_s &= 0, \\
Z_s \tilde{Z}_t - \tilde{Z}_t Z_s &= \delta_{st}.
\end{aligned}}$$

$$\boxed{\tilde{Z}_s Z_s = n_s.}$$

$$\boxed{\begin{aligned}
&< n_s | Z_s | n_{s+1} >= \sqrt{n_s + 1}, \\
&< n_s | \tilde{Z}_s | n_{s-1} >= \sqrt{n_s}.
\end{aligned}}$$

[[16]]

$$A_s = c \sqrt{\frac{2h}{\pi \nu_s}} \; Z_s,$$

[16]@ *In the original manuscript, the unidentified Ref. 5.45 is here alluded to.*

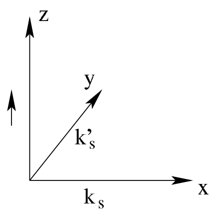
$$\boxed{\begin{aligned} a_s &= \frac{A_s + \tilde{A}_{-s}}{2} = c\sqrt{\frac{2h}{\pi\nu_s}}\ \frac{Z_s + \tilde{Z}_{-s}}{2}, \\ b_s &= \frac{2\pi\nu_s i}{c}\frac{A_s - \tilde{A}_s}{2} = i\sqrt{2h\pi\nu_s}\ (Z_s - \tilde{Z}_{-s}). \end{aligned}}$$

$$\begin{aligned} W_s &= \frac{1}{8\pi}\sum\left(\tilde{b}_s b_s + \frac{4\pi^2\nu_s^2}{c^2}\tilde{a}_s a_s\right) \\ &= \frac{1}{8\pi}\sum 2h\pi\nu_s\left[(\tilde{Z}_s - Z_{-s})(Z_s - \tilde{Z}_{-s}) + (\tilde{Z}_s + Z_{-s})(Z_s + \tilde{Z}_{-s})\right] \\ &= \frac{1}{4}\sum h\nu_s\{2\tilde{Z}_s Z_s + 2Z_{-s}\tilde{Z}_{-s}\} \\ &= \boxed{\sum h\nu_s \frac{\tilde{Z}_s Z_s + Z_{-s}\tilde{Z}_{-s}}{2} = \sum\left(n_s + \frac{1}{2}\right)h\nu_s.} \end{aligned}$$

$$\begin{aligned} \boldsymbol{f}_s &= \frac{1}{k^{3/2}}\,\mathrm{e}^{2\pi i\left(\gamma_1^s x/k + \gamma_2^s y/k + \gamma_3^s z/k\right)}\boldsymbol{k}_s, \\ \boldsymbol{f}_{-s} &= \frac{1}{k^{3/2}}\,\mathrm{e}^{-2\pi i\left(\gamma_1^s x/k + \gamma_2^s y/k + \gamma_3^s z/k\right)}\boldsymbol{k}_s = \overline{\boldsymbol{f}}_s. \end{aligned}$$

$$\boldsymbol{f}_s = \frac{1}{k^{3/2}}\,\mathrm{e}^{2\pi i\boldsymbol{\gamma}_s\cdot\boldsymbol{r}/k}\boldsymbol{k}_s,$$

with $\boldsymbol{r} = (x, y, z)$.



$$\boxed{\begin{aligned} \boldsymbol{C} &= \sum_s \frac{c}{2}\sqrt{\frac{2h}{\pi\nu_s}}\ (Z_s\boldsymbol{f}_s + \tilde{Z}_s\overline{\boldsymbol{f}}_s), \\ \boldsymbol{E} &= \sum_s i\sqrt{2h\pi\nu_s}\ (Z_s\boldsymbol{f}_s - \tilde{Z}_s\overline{\boldsymbol{f}}_s). \end{aligned}}$$

$$\boxed{\begin{aligned}E^2(r) = -\frac{2h\pi}{k^3}\sum_{s,t}\sqrt{\nu_s\nu_t}\;\boldsymbol{k}_s\cdot\boldsymbol{k}_t\Big\{&Z_sZ_t\;\mathrm{e}^{2\pi i(\boldsymbol{\gamma}_s+\boldsymbol{\gamma}_t)\cdot\boldsymbol{r}/k}\\ &+\tilde{Z}_s\tilde{Z}_t\;\mathrm{e}^{-2\pi i(\boldsymbol{\gamma}_s+\boldsymbol{\gamma}_t)\cdot\boldsymbol{r}/k} - Z_s\tilde{Z}_t\;\mathrm{e}^{2\pi i(\boldsymbol{\gamma}_s-\boldsymbol{\gamma}_t)\cdot\boldsymbol{r}/k}\\ &-\tilde{Z}_sZ_t\;\mathrm{e}^{2\pi i(-\boldsymbol{\gamma}_s+\boldsymbol{\gamma}_t)\cdot\boldsymbol{r}/k}\Big\}.\end{aligned}}$$

[17]

$$\boxed{\begin{aligned}H^2(r) = -\frac{2h\pi}{k^3}\sum_{s,t}\sqrt{\nu_s\nu_t}\;\boldsymbol{k}'_s\cdot\boldsymbol{k}'_t\Big\{&Z_sZ_t\;\mathrm{e}^{2\pi i(\boldsymbol{\gamma}_s+\boldsymbol{\gamma}_t)\cdot\boldsymbol{r}/k}\\ &+\tilde{Z}_s\tilde{Z}_t\;\mathrm{e}^{-2\pi i(\boldsymbol{\gamma}_s+\boldsymbol{\gamma}_t)\cdot\boldsymbol{r}/k} - Z_s\tilde{Z}_t\;\mathrm{e}^{2\pi i(\boldsymbol{\gamma}_s-\boldsymbol{\gamma}_t)\cdot\boldsymbol{r}/k}\\ &-\tilde{Z}_sZ_t\;\mathrm{e}^{2\pi i(-\boldsymbol{\gamma}_s+\boldsymbol{\gamma}_t)\cdot\boldsymbol{r}/k}\Big\}.\end{aligned}}$$

2.10. WAVE EQUATION OF LIGHT QUANTA

Quantized fields of the electromagnetic interaction were again considered in these pages, with an emphasis (the name of this Section is the original one) on the definition of a wavefunction ψ for the photon. Matrix elements of the annihilation and creation operators $Z, \tilde{Z}$ were reported in the subsequent Section, along with quantum expressions for the photon energy and angular momentum.
[18]

$$\boxed{\begin{gathered}\boldsymbol{C} = \sum a_s\boldsymbol{f}_s, \quad \boldsymbol{E} = \sum b_s\boldsymbol{f}_s;\\ a_s = c\sqrt{\frac{2h}{\pi\nu_s}}\;\frac{Z_s+\overline{Z}_{-s}}{2}, \quad b_s = i\sqrt{2h\pi\nu_s}\;(Z_s-\overline{Z}_{-s}).\end{gathered}}$$

$$\boldsymbol{f}_s = \frac{1}{k^{3/2}}\;\mathrm{e}^{2\pi i\gamma^s\cdot r/h}\boldsymbol{k}_s,$$

$$\boxed{\overline{\boldsymbol{f}}_s = \boldsymbol{f}_{-s}.}$$

[17] $C \sim (\mathrm{e}^{2\pi i\gamma r/k}, 0, 0), \quad H \sim (0, 2\pi i(\gamma/k)\,\mathrm{e}^{2\pi\gamma r/k}, 0)$.
[18] @ *The original manuscript alludes here to the unidentified Ref. 11.20.*

$$\begin{aligned} \gamma^s &= (\gamma_1^s, \gamma_2^s, \gamma_3^s), \\ \gamma_1, \gamma_2, \gamma_3 &= 0, \pm 1, \pm 2, \pm 3, \ldots; \end{aligned}$$

$$\boxed{\nu_s = \frac{c}{k}\gamma^s, \quad h\nu_s = \frac{hc}{k}\ \gamma^s.}$$

$$\boxed{\boldsymbol{\psi} = \sum Z_s \boldsymbol{f}_s.}$$

$$\boxed{\begin{aligned} &\boldsymbol{C} = \sum_s c\sqrt{\frac{2h}{\pi\nu_s}}\ \frac{Z_s + \overline{Z}_{-s}}{2}\boldsymbol{f}_s = \sum_s c\sqrt{\frac{2h}{\pi\nu_s}}\ \frac{Z_s\boldsymbol{f}_s + \overline{Z}_s\overline{\boldsymbol{f}}_s}{2}, \\ &\boldsymbol{E} = \sum_s i\sqrt{2h\pi\nu_s}\ (Z_s - \overline{Z}_{-s})\boldsymbol{f}_s = \sum_s i\sqrt{2h\pi\nu_s}\ (Z_s\boldsymbol{f}_s - \overline{Z}_s\overline{\boldsymbol{f}}_s). \end{aligned}}$$

2.11. CONTINUATION

$\boldsymbol{\nabla}\cdot\boldsymbol{C} = 0.$

$$\begin{aligned} \frac{1}{c}\frac{\partial \boldsymbol{E}}{\partial t} &= \boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\boldsymbol{C} = -\boldsymbol{\nabla}^2\,\boldsymbol{C}, \\ \frac{1}{c}\frac{\partial \boldsymbol{H}}{\partial t} - \boldsymbol{\nabla}\times\boldsymbol{E} &= \frac{1}{c}\frac{\partial}{\partial t}\boldsymbol{\nabla}\times\boldsymbol{C}. \end{aligned}$$

$$\begin{aligned} \boldsymbol{C} &= \sum c\ \sqrt{\frac{h}{2\pi\nu_s}}\ (Z_s\boldsymbol{f}_s + \tilde{Z}_s\overline{\boldsymbol{f}}_s), \\ \frac{\partial \boldsymbol{C}}{\partial t} &= \sum c\ \sqrt{\frac{h}{2\pi\nu_s}}\ (\dot{Z}_s\boldsymbol{f}_s + \dot{\tilde{Z}}_s\overline{\boldsymbol{f}}_s), \\ \boldsymbol{\nabla}^2\,\boldsymbol{C} &= \sum \frac{2\pi\nu_s}{c}\ \sqrt{2h\pi\nu_s}\ (Z_s\boldsymbol{f}_s + \tilde{Z}_s\overline{\boldsymbol{f}}_s); \\ \boldsymbol{E} &= \sum i\ \sqrt{2h\pi\nu_s}\ (Z_s\boldsymbol{f}_s - \tilde{Z}_s\overline{\boldsymbol{f}}_s), \\ \frac{\partial \boldsymbol{E}}{\partial t} &= \sum i\ \sqrt{2h\pi\nu_s}\ (\dot{Z}_s\boldsymbol{f}_s - \dot{\tilde{Z}}_s\overline{\boldsymbol{f}}_s). \end{aligned}$$

$$i\sqrt{2h\pi\nu_s}\ (\dot{Z}_s - \dot{\tilde{Z}}_{-s}) - 2\pi\nu_s\sqrt{2h\pi\nu_s}\ (Z_s + \tilde{Z}_{-s}) = 0,$$
$$\sqrt{\frac{h}{2\pi\nu_s}}\ (\dot{Z}_s + \dot{\tilde{Z}}_{-s}) + i\sqrt{2h\pi\nu_s}(Z_s - \tilde{Z}_{-s}) = 0.$$

$$\begin{aligned}
\dot{Z}_s - \dot{\tilde{Z}}_{-s} &= -2\pi i\nu_s\ (Z_s + \tilde{Z}_{-s}),\\
\dot{Z}_s + \dot{\tilde{Z}}_{-s} &= -2\pi i\nu_s\ (Z_s - \tilde{Z}_{-s}).
\end{aligned}$$

$$\boxed{\dot{Z}_s = -2\pi i\nu_s Z_s,}\quad \boxed{\dot{\tilde{Z}}_s = 2\pi i\nu_s \tilde{Z}_s,}\quad \boxed{\dot{\tilde{Z}}_{-s} = 2\pi i\nu_s \tilde{Z}_{-s}.}$$

$$\begin{aligned}
\int \frac{E^2}{8\pi}\ \mathrm{d}\tau &= \sum \frac{h\nu_s}{4}(Z_s - \tilde{Z}_{-s})(\tilde{Z}_s - Z_{-s})\\
&= \sum \frac{h\nu_s}{4}(Z_s\tilde{Z}_s + \tilde{Z}_{-s}Z_{-s} - Z_sZ_{-s} - \tilde{Z}_{-s}\tilde{Z}_s)\\
&= \sum \frac{h\nu_s}{2}\left(\frac{Z_s\tilde{Z}_s + \tilde{Z}_s\tilde{Z}_s}{2} - \frac{Z_sZ_{-s} + \tilde{Z}_s\tilde{Z}_{-s}}{2}\right).
\end{aligned}$$

$$\begin{aligned}
\int \frac{H^2}{8\pi}\ \mathrm{d}\tau &= \sum \frac{h\nu_s}{4}(Z_s + \tilde{Z}_{-s})(\tilde{Z}_s + Z_{-s})\\
&= \sum \frac{h\nu_s}{4}(Z_s\tilde{Z}_s + \tilde{Z}_{-s}\tilde{Z}_{-s} + Z_sZ_{-s} + \tilde{Z}_{-s}\tilde{Z}_s)\\
&= \sum \frac{h\nu_s}{2}\left(\frac{Z_s\tilde{Z}_s + \tilde{Z}_sZ_s}{2} + \frac{Z_sZ_{-s} + \tilde{Z}_s\tilde{Z}_{-s}}{2}\right).
\end{aligned}$$

$$\boxed{\int \frac{E^2 + H^2}{8\pi}\ \mathrm{d}\tau = \sum h\nu_s \frac{Z_s\tilde{Z}_s + \tilde{Z}_sZ_s}{2}.}$$

$$\begin{aligned}
\mathrm{e}^{iLx}\ (0,0,1) &= \boldsymbol{f}_s,\\
iL\mathrm{e}^{iLx}(0,-1,0) &= \boldsymbol{\nabla}\times\boldsymbol{f}_s
\end{aligned}$$

$$\boldsymbol{f}_{-s}\times\boldsymbol{\nabla}\times\boldsymbol{f}_s = iL(1,0,0).$$

Let us denote with $\boldsymbol{r}_s$ a unitary vector along the propagation direction:

$$\begin{aligned}
\int \frac{\boldsymbol{E}\times\boldsymbol{H}}{4\pi c}\,\mathrm{d}\tau &= \sum -\frac{h\nu_s}{2c}\,(Z_s-\tilde{Z}_{-s})(Z_s+\tilde{Z}_{-s})\boldsymbol{r}_s \\
&= \sum \frac{h\nu_s}{2c}\,\boldsymbol{r}_s\,(\tilde{Z}_sZ_s - Z_{-s}\tilde{Z}_{-s} - Z_{-s}Z_s - \tilde{Z}_s\tilde{Z}_{-s}) \\
&= \boxed{\sum \frac{h\nu_s}{c}\,\boldsymbol{r}_s\,\frac{\tilde{Z}_sZ_s + Z_s\tilde{Z}_{-s}}{2}.}
\end{aligned}$$

$$Z_s\tilde{Z}_s - \tilde{Z}_sZ_s = 1.$$

$\tilde{Z}_sZ_s = X.$

$$\begin{array}{ll}
Z_sX - XZ_s = (Z_s,X) = Z_s, & Z_{ik}(X_k - X_i) = 1, \\
\tilde{Z}_sX - X\tilde{Z}_s = (\tilde{Z}_s,X) = -\tilde{Z}_s, & \tilde{Z}_{ik}(X_k - X_i) = -1.
\end{array}$$

$$\begin{aligned}
< X|Z|X+1 > &= f(X), \\
< X+1|\tilde{Z}|X > &= \tilde{f}(X).
\end{aligned}$$

$$\begin{aligned}
< X|\tilde{Z}Z|X > &= < X|\tilde{Z}|X-1 >< X-1|Z|X >= |f(X-1)|^2, \\
< X|Z\tilde{Z}|X > &= < X|Z|X+1 >< X+1|\tilde{Z}|X >= |f(X)|^2;
\end{aligned}$$

$$\begin{aligned}
|f(X)|^2 &= X+1, \\
|f(X_0)|^2 &= 1, \quad X_0 = 0.
\end{aligned}$$

$$\begin{aligned}
|f(X)|^2 &= |f(X-1)|^2 + 1, \\
|f(X_0)|^2 &= 1.
\end{aligned}$$

$$\begin{aligned}
< X_0|\tilde{Z}Z|X_0 > &= 0, \\
< X_0|Z\tilde{Z}|X_0 > &= |f(0)|^2.
\end{aligned}$$

$$\boxed{\begin{aligned}
\tilde{Z}_sZ_s &= n_s, \quad (n_s = 0,1,2\ldots) \\
< n_s|Z_s|n_s+1 > &= \sqrt{n_s+1}, \\
< n_s+1|\tilde{Z}_s|n_s > &= \sqrt{n_s}.
\end{aligned}}$$

$\overline{\boldsymbol{f}}_s = \boldsymbol{f}_{-s}.$

$$\boxed{\begin{aligned} \int \frac{E^2+H^2}{8\pi}\,\mathrm{d}\tau &= \sum h\nu_s\left(n_s+\frac{1}{2}\right), \\ \int \frac{\boldsymbol{E}\times\boldsymbol{H}}{4\pi c}\,\mathrm{d}\tau &= \sum \frac{h\nu_s}{c}\,\boldsymbol{r}_s\left(n_s+\frac{1}{2}\right). \end{aligned}}$$

2.12. FREE ELECTRON SCATTERING

The interaction between electrons and electromagnetic radiation was here studied in detail, and expressions for the matrix elements of the interaction energy (as well as for the transition probability) were explicitly obtained. Some care was also devoted to the kinematics of the process here considered. The material reported in this Section starts with that present in Quaderno 17 on the page following 151bis, but the complete study of the subject starts at page 133 of the same Quaderno.

$$\left[\frac{W}{c}+\rho_1\,\boldsymbol{\sigma}\cdot\left(\boldsymbol{p}+\frac{e}{c}\boldsymbol{C}\right)+\rho_3\,mc\right]\psi=0.$$

Using Dirac coordinates:

$$\psi_r = u_r\,\frac{1}{\sqrt{k^3}}\,\mathrm{e}^{2\pi i\left(\mathit{\Gamma}^r_1 x/k+\mathit{\Gamma}^r_2 y/k+\mathit{\Gamma}^r_3 z/k\right)}.$$

$u_r = (u^r_1,\, u^r_2,\, u^r_3,\, u^r_r),\;\; \tilde{u}_u u_r = 1,\;\; \mathit{\Gamma} = \sqrt{\mathit{\Gamma}^r_1+\mathit{\Gamma}^r_2+\mathit{\Gamma}^r_3}.$

$$\boxed{E_r = \pm c\sqrt{m^2c^2+\frac{h^2}{k^2}\,\mathit{\Gamma}^2}\,.}$$

$$H = H_0+\mathcal{I},$$

$$H_0 = -c\;\rho_1\;\boldsymbol{\sigma}\cdot\boldsymbol{p}-\rho_3\;mc^2+\sum_s n_s h\nu_s,$$

$$\mathcal{I} = -c\;\rho_1\;\boldsymbol{\sigma}\cdot\frac{e}{c}\boldsymbol{C} = -e\,\rho_1\;\boldsymbol{\sigma}\cdot\boldsymbol{C}.$$

$$\boxed{\begin{aligned} <\ldots|H_0|\ldots> &= E_r + \sum n_s h\nu_s, \\ < r; n_s \ldots |\mathcal{I}| r'; n_s+1 \ldots > &= -\sqrt{n_s+1}\ \frac{ec}{2}\sqrt{\frac{2h}{\pi\nu_s}} \\ &\quad \times \int \widetilde{\psi}_r\ \rho_1 \boldsymbol{\sigma}\cdot\boldsymbol{f}_s\ \psi_{r'}\ \mathrm{d}\tau, \\ < r; n_s \ldots |\mathcal{I}| r'; n_s-1 \ldots > &= -\sqrt{n_s}\ \frac{ec}{2}\sqrt{\frac{2h}{\pi\nu_s}} \\ &\quad \times \int \widetilde{\psi}_r\ \rho_1\ \boldsymbol{\sigma}\cdot\boldsymbol{f}_{-s}\ \psi_{r'}\ \mathrm{d}\tau. \end{aligned}}$$

$$\begin{aligned} \psi_r &= u_r\ \frac{1}{k^{3/2}}\ \mathrm{e}^{2\pi i \mathit{\Gamma}^r\cdot r/k}, \\ \psi_{r'} &= u_{r'}\ \frac{1}{k^{3/2}}\ \mathrm{e}^{2\pi i \mathit{\Gamma}^{r'}\cdot r/k}, \\ \boldsymbol{f}_s &= \boldsymbol{k}_s\ \frac{1}{k^{3/2}}\ \mathrm{e}^{2\pi i \gamma^s\cdot r/k}, \\ \boldsymbol{f}_{-s} &= \boldsymbol{f}_s\ \frac{1}{k^{3/2}}\ \mathrm{e}^{-2\pi i \gamma^s\cdot r/k}. \end{aligned}$$

$\boldsymbol{k}_s = \boldsymbol{k}_{-s}$.

$$\boxed{\begin{aligned} \int \widetilde{\psi}_r\ \rho_1 \boldsymbol{\sigma}\cdot\boldsymbol{f}_s\ \psi_{r'}\ \mathrm{d}\tau &= k^{-7/2}\ \tilde{u}_r\ \rho_1\ \boldsymbol{\sigma}\cdot\boldsymbol{k}_s\ u_{r'} \\ &\quad \times \int \mathrm{e}^{2\pi i(\mathit{\Gamma}^{r'}+\gamma_s-\mathit{\Gamma}^r)\cdot r/k}\mathrm{d}\tau \\ &= \frac{\tilde{u}_r\ \rho_1\ \boldsymbol{\sigma}\cdot\boldsymbol{k}_s\ u_{r'}}{k^{7/2}}\ \delta_{\mathit{\Gamma}^r,\ \mathit{\Gamma}^{r'}+\gamma_s}, \end{aligned}}$$

$$\boxed{\int \widetilde{\psi}_r\ \rho_1 \boldsymbol{\sigma}\cdot\boldsymbol{f}_{-s}\ \psi_{r'}\ \mathrm{d}\tau = \frac{\tilde{u}_r\ \rho_1\ \boldsymbol{\sigma}\cdot\boldsymbol{k}_s\ u_{r'}}{k^{7/2}}\ \delta_{\mathit{\Gamma}^r,\ \mathit{\Gamma}^{r'}-\gamma_s}.}$$

$$\boxed{\begin{aligned} < r; n_s \ldots |\mathcal{I}| r'; n_s+1 \ldots > &= -\frac{ec}{2k^{3/2}}\sqrt{n_s+1}\ \sqrt{\frac{2h}{\pi\nu_s}} \\ &\quad \times\ \tilde{u}_r\ \rho_1\ \boldsymbol{\sigma}\cdot\boldsymbol{k}_s\ u_{r'}\ \delta_{\mathit{\Gamma}^r,\mathit{\Gamma}^{r'}+\gamma_s}, \\ < r; n_s \ldots |\mathcal{I}| r'; n_s-1 \ldots > &= -\frac{ec}{2k^{3/2}}\sqrt{n_s}\ \sqrt{\frac{2h}{\pi\nu_s}} \\ &\quad \times\ \tilde{u}_r\ \rho_1\ \boldsymbol{\sigma}\cdot\boldsymbol{k}_s\ u_{r'}\ \delta_{\mathit{\Gamma}^r,\mathit{\Gamma}^{r'}-\gamma_s}. \end{aligned}}$$

For $t = 0$: $a_1 = 1,\ a_2, \ldots = 0$.
For $t \to 0$:

$$\dot{a}_i = -\frac{2\pi i}{h}\ \mathrm{e}^{2\pi i (E_i - E_1)t/h}\ H_{i1};$$

$$\boxed{a_i = -\frac{1}{E_i - E_1}\left(\mathrm{e}^{2\pi i (E_i - E_1)t/h} - 1\right) H_{i1}.}$$

$H_{12} = 0.$

$$\begin{aligned}
\dot{a}_2 &= -\frac{2\pi i}{h} \sum_i \frac{-1}{E_i - E_1}\ \mathrm{e}^{2\pi i (E_2 - E_i)t/h}\ \left(\mathrm{e}^{2\pi i (E_i - E_1)t/h} - 1\right) H_{2i} H_{i1} \\
&= \frac{2\pi i}{h} \sum_i \frac{1}{E_i - E_1}\left(\mathrm{e}^{2\pi i (E_2 - E_1)t/h} - \mathrm{e}^{2\pi i (E_2 - E_i)t/h}\right) H_{2i} H_{i1};
\end{aligned}$$

$$\boxed{\begin{aligned}
a_2 \ = \ \sum_i \Bigg[& \frac{1}{(E_i - E_1)(E_2 - E_1)}\left(\mathrm{e}^{2\pi i (E_2 - E_1)t/h} - 1\right) \\
& - \frac{1}{(E_2 - E_i)(E_i - E_1)} \mathrm{e}^{2\pi i (E_2 - E_i)t} \Bigg] H_{2i} H_{i1}.
\end{aligned}}$$

$$\begin{array}{llcccl}
 & \text{electron} & & \text{radiation} & & \\
2 & b & & n_t = 1 & & \\
 & & \nearrow & & \searrow & \\
i, i' & r, r' & & & & n_t = 1,\ n_s = 1 \\
 & & \searrow & & \nearrow & \\
1 & a & & n_s = 1 & &
\end{array}$$

$$\boxed{\varGamma^a + \gamma^s = \varGamma^b + \gamma^t}\boxed{= \varGamma^r = \varGamma^{r'} + \gamma^s + \gamma^t}$$

s, t label the incident and the scattered quanta, respectively.

$$\boxed{\begin{aligned} < b;0,1\ldots|\mathcal{I}|r;0,0> &= -\frac{ec}{2k^{3/2}}\sqrt{\frac{2h}{\pi\nu_t}}\tilde{u}_b\ \rho_1\boldsymbol{\sigma}\cdot\boldsymbol{k}_t u_r, \\ < r';0,0\ldots|\mathcal{I}|a;1,0> &= -\frac{ec}{2k^{3/2}}\sqrt{\frac{2h}{\pi\nu_s}}\tilde{u}_{r'}\rho_1\boldsymbol{\sigma}\cdot\boldsymbol{k}_s u_a, \\ < b;0,1\ldots|\mathcal{I}|r;1,1> &= -\frac{ec}{2k^{3/2}}\sqrt{\frac{2h}{\pi\nu_s}}\tilde{u}_b\ \rho_1\boldsymbol{\sigma}\cdot\boldsymbol{k}_s u_r, \\ < r';1,1\ldots|\mathcal{I}|a;1,0> &= -\frac{ec}{2k^{3/2}}\sqrt{\frac{2h}{\pi\nu_t}}\tilde{u}_{r'}\rho_1\boldsymbol{\sigma}\cdot\boldsymbol{k}_t u_a. \end{aligned}}$$

The probability for a transition at a time t to occur is (taking into account only the term with the resonance denominator equal to $E_1 - E_2$ in the expression for a_2):

$$\boxed{P_{12} = \frac{\sin^2[\pi(E_2-E_1)t/h]}{(E_2-E_1)^2}\cdot 4\left|\sum_i \frac{H_{2i}H_{i1}}{E_i-E_1}\right|^2.}$$

$$\boxed{\begin{aligned} p_a = \frac{h}{k}\mathit{\Gamma}^a, &\quad p_r = \frac{h}{k}(\mathit{\Gamma}^a+\gamma^s), \\ p_b = \frac{h}{k}\mathit{\Gamma}^b, &\quad p_{r'} = \frac{h}{k}(\mathit{\Gamma}^b-\gamma^t). \end{aligned}}$$

$$\boxed{\begin{gathered} \mathit{\Gamma} = \mathit{\Gamma}^a+\gamma^s = \mathit{\Gamma}^b+\gamma^t, \\ \mathit{\Gamma}^b = \mathit{\Gamma}^a+\gamma^s-\gamma^t. \end{gathered}}$$

$$\begin{aligned} E_a &= c\sqrt{m^2c^2+\frac{h^2}{k^2}\mathit{\Gamma}^{a2}}, \\ E_b &= c\sqrt{m^2c^2+\frac{h^2}{k^2}\mathit{\Gamma}^{b2}}, \\ E_r &= \pm c\sqrt{m^2c^2+\frac{h^2}{k^2}\left(\mathit{\Gamma}^a+\gamma^s\right)^2}, \\ E_{r'} &= \pm c\sqrt{m^2c^2+\frac{h^2}{k^2}\left(\mathit{\Gamma}^b-\gamma^t\right)^2}. \end{aligned}$$

$$\boxed{\begin{aligned} E_1 &= c\sqrt{m^2c^2 + \frac{h^2}{k^2}\Gamma^{a2}} + h\nu_s, \\ E_2 &= c\sqrt{m^2c^2 + \frac{h^2}{k^2}(\Gamma^a + \gamma_s - \gamma_t)^2} + h\nu_t, \\ E_i &= \pm c\sqrt{m^2c^2 + \frac{h^2}{k^2}(\Gamma^a + \gamma_s)^2}, \\ E_{i'} &= \pm c\sqrt{m^2c^2 + \frac{h^2}{k^2}(\Gamma^a - \gamma^t)^2} + h\nu_s + h\nu_t. \end{aligned}}$$

Let us denote by u the spin function for a plane wave with momentum p_x, p_y, p_z and by u^0 that for a wave of zero momentum.

$$\boxed{u^p = \left[f_1 \mp f_2 \frac{\boldsymbol{\alpha}\cdot\boldsymbol{p}}{p}\right] u^0,}$$

where the upper/lower sign refers to positive/negative energy waves.

$$\boxed{\begin{gathered} f_1 = \sqrt{\frac{1+\sqrt{1+p^2/m^2c^2}}{2\sqrt{1+p^2/m^2c^2}}}, \quad f_2 = \sqrt{\frac{-1+\sqrt{1+p^2/m^2c^2}}{2\sqrt{1+p^2/m^2c^2}}}; \\ |f_1^2| + |f_2^2| = 1. \end{gathered}}$$

$$\boxed{\boldsymbol{\alpha} = \rho_1\,\boldsymbol{\sigma}.}$$

$$\boxed{\begin{aligned} u_b &= \left[f_1^b - f_2^b \frac{\boldsymbol{\alpha}\cdot\boldsymbol{p}_b}{p_b}\right] u_b^0, \qquad u_r \left[f_1^r \mp f_2^r \frac{\boldsymbol{\alpha}\cdot\boldsymbol{p}_r}{p_r}\right] u_r^0, \\ u_a &= \left[f_1^a - f_2^b \frac{\boldsymbol{\alpha}\cdot\boldsymbol{p}_a}{p_a}\right] u_a^0, \qquad u_{r'} \left[f_1^{r'} \mp f_2^{r'} \frac{\boldsymbol{\alpha}\cdot\boldsymbol{p}_{r'}}{p_{r'}}\right] u_{r'}^0. \end{aligned}}$$

We consider positive waves u_a, u_b.

[19]

1) Positive u_r:

$$\begin{aligned}
&\tilde{u}_b\ \boldsymbol{\alpha}\cdot\boldsymbol{k}_t\ u_r\ \tilde{u}_r\ \boldsymbol{\alpha}\cdot\boldsymbol{k}_s\ u_a \\
&\quad= \tilde{u}_b^0\left(f_1^b - f_2^b\frac{\boldsymbol{\alpha}\cdot\boldsymbol{p}_b}{p_b}\right)\boldsymbol{\alpha}\cdot\boldsymbol{k}_t\left(f_1^r - f_2^r\frac{\boldsymbol{\alpha}\cdot\boldsymbol{p}_r}{p_r}\right)u_r^0 \\
&\qquad\times\tilde{u}_r^0\left(f_1^r - f_2^r\frac{\boldsymbol{\alpha}\cdot\boldsymbol{p}_r}{p_r}\right)\boldsymbol{\alpha}\cdot\boldsymbol{k}_s\left(f_1^a - f_2^a\frac{\boldsymbol{\alpha}\cdot\boldsymbol{p}_a}{p_a}\right)u_a^0 \\
&\quad= \tilde{u}_b^0\left[f_1^b\boldsymbol{\alpha}\cdot\boldsymbol{k}_t f_2^r\frac{\boldsymbol{\alpha}\cdot\boldsymbol{p}_r}{p_r} + f_2^b\frac{\boldsymbol{\alpha}\cdot\boldsymbol{p}_b}{p_b}\boldsymbol{\alpha}\cdot\boldsymbol{k}_t f_1^r\right]u_r^0 \\
&\qquad\times\tilde{u}_r^0\left[f_1^r\boldsymbol{\alpha}\cdot\boldsymbol{k}_s f_2^a\frac{\boldsymbol{\alpha}\cdot\boldsymbol{p}_a}{p_a} + f_2^r\frac{\boldsymbol{\alpha}\cdot\boldsymbol{p}_r}{p_r}\boldsymbol{\alpha}\cdot\boldsymbol{k}_s f_1^a\right]u_a^0.
\end{aligned}$$

2) Negative u_r:

$$\begin{aligned}
&\tilde{u}_b\ \boldsymbol{\alpha}\cdot\boldsymbol{k}_t\ u_r\ \tilde{u}_r\ \boldsymbol{\alpha}\cdot\boldsymbol{k}_s\ u_a \\
&\quad= \tilde{u}_b^0\left[f_1^b f_1^r\boldsymbol{\alpha}\cdot\boldsymbol{k}_t - f_2^b f_2^r\frac{\boldsymbol{\alpha}\cdot\boldsymbol{p}_b}{p_b}\boldsymbol{\alpha}\cdot\boldsymbol{k}_t\frac{\boldsymbol{\alpha}\cdot\boldsymbol{p}_r}{p_r}\right]u_r^0 \\
&\qquad\times\tilde{u}_r^0\left[f_1^r f_1^a\boldsymbol{\alpha}\cdot\boldsymbol{k}_s - f_2^r f_2^a\frac{\boldsymbol{\alpha}\cdot\boldsymbol{p}_r}{p_r}\boldsymbol{\alpha}\cdot\boldsymbol{k}_s\frac{\boldsymbol{\alpha}\cdot\boldsymbol{p}_a}{p_a}\right]u_r^0.
\end{aligned}$$

3) Positive $u_{r'}$:

$$\tilde{u}_b\ \boldsymbol{\alpha}\cdot\boldsymbol{k}_s\ u_{r'}\ \tilde{u}_{r'}\boldsymbol{\alpha}\cdot\boldsymbol{k}_t\ u_a = \ldots$$

[which is obtained from 1) with the replacements $r\to r'$, $k_s\to k_t$, $k_t\to k_s$].

4) Negative $u_{r'}$:

$$\tilde{u}_b\ \boldsymbol{\alpha}\cdot\boldsymbol{k}_s\ u_{r'}\ \tilde{u}_{r'}\ \boldsymbol{\alpha}\cdot\boldsymbol{k}_t\ u_a = \ldots$$

[which is obtained from 1) with the replacements $r\to r'$, $k_s\to k_t$, $k_t\to k_s$].

[19]@ *The original manuscript alludes here to the unidentified Ref. 10.40.*

1)

$$\sum_{\text{positive } u_r} \tilde{u}_b \; \boldsymbol{\alpha} \cdot \boldsymbol{k}_t \; u_r \; \tilde{u}_r \boldsymbol{\alpha} \cdot \boldsymbol{k}_s \; u_a$$

$$\begin{aligned}
&= \tilde{u}_b^0 \left[f_1^b f_2^r \boldsymbol{\alpha} \cdot \boldsymbol{k}_t \frac{\boldsymbol{\alpha} \cdot \boldsymbol{p}_r}{p_r} + f_1^r f_2^b \frac{\boldsymbol{\alpha} \cdot \boldsymbol{p}_b}{p_b} \boldsymbol{\alpha} \cdot \boldsymbol{k}_t \right] \\
&\qquad \times \left[f_1^r f_2^a \boldsymbol{\alpha} \cdot \boldsymbol{k}_s \frac{\boldsymbol{\alpha} \cdot \boldsymbol{p}_a}{p_a} + f_2^r f_1^a \frac{\boldsymbol{\alpha} \cdot \boldsymbol{p}_r}{p_r} \boldsymbol{\alpha} \cdot \boldsymbol{k}_s \right] u_a^0 \\
&= \tilde{u}_b^0 \left[f_1^b f_2^r \boldsymbol{\sigma} \cdot \boldsymbol{k}_t \frac{\boldsymbol{\sigma} \cdot \boldsymbol{p}_r}{p_r} + f_2^b f_1^r \frac{\boldsymbol{\sigma} \cdot \boldsymbol{p}_b}{p_b} \boldsymbol{\sigma} \cdot \boldsymbol{k}_t \right] \\
&\qquad \times \left[f_1^r f_2^a \boldsymbol{\sigma} \cdot \boldsymbol{k}_s \frac{\boldsymbol{\sigma} \cdot \boldsymbol{p}_a}{p_a} + f_2^r f_1^a \frac{\boldsymbol{\sigma} \cdot \boldsymbol{p}_r}{p_r} \boldsymbol{\sigma} \cdot \boldsymbol{k}_s \right] u_a^0 .
\end{aligned}$$

$$(\boldsymbol{\sigma} \cdot \boldsymbol{k}_t)(\boldsymbol{\sigma} \cdot \boldsymbol{p}_r) = \boldsymbol{k}_t \cdot \boldsymbol{p}_r + i \boldsymbol{\sigma} \cdot \boldsymbol{k}_t \times \boldsymbol{p}_r ,$$

$$\begin{aligned}
&(\boldsymbol{\sigma} \cdot \boldsymbol{k}_t)(\boldsymbol{\sigma} \cdot \boldsymbol{p}_r)(\boldsymbol{\sigma} \cdot \boldsymbol{k}_s)(\boldsymbol{\sigma} \cdot \boldsymbol{p}_r) \\
&\quad = (\boldsymbol{k}_t \cdot \boldsymbol{p}_r)(\boldsymbol{k}_s \cdot \boldsymbol{p}_a) + i(\boldsymbol{k}_t \cdot \boldsymbol{p}_r)(\boldsymbol{\sigma} \cdot \boldsymbol{k}_s \times \boldsymbol{p}_a) \\
&\qquad + i(\boldsymbol{k}_s \cdot \boldsymbol{p}_a)(\boldsymbol{\sigma} \cdot \boldsymbol{k}_p \times p_r) - (\boldsymbol{\sigma} \cdot \boldsymbol{k}_t \times \boldsymbol{p}_r)(\boldsymbol{\sigma} \cdot \boldsymbol{k}_s \times \boldsymbol{p}_a) \\
&\quad = \boxed{\begin{array}{l} (\boldsymbol{k}_t \cdot \boldsymbol{p}_r)(\boldsymbol{k}_s \cdot \boldsymbol{p}_a) + i(\boldsymbol{k}_t \cdot \boldsymbol{p}_r)(\boldsymbol{\sigma} \cdot \boldsymbol{k}_s \times \boldsymbol{p}_a) \\ +i(\boldsymbol{k}_s \cdot \boldsymbol{p}_a)(\boldsymbol{\sigma} \cdot \boldsymbol{k}_t \times \boldsymbol{p}_r) - (\boldsymbol{k}_t \times \boldsymbol{p}_r)(\boldsymbol{k}_s \times \boldsymbol{p}_a) \\ -i[\boldsymbol{\sigma}, (\boldsymbol{k}_t \times \boldsymbol{p}_r) \times (\boldsymbol{k}_s \times \boldsymbol{p}_a)]. \end{array}}
\end{aligned}$$

For $u_a = u_a^0$, $p_a = 0$: $f_1^a = 1$, $f_2^a = 0$.

---

1)

$$\sum_{\text{positive } u_r} \tilde{u}_b \; \boldsymbol{\alpha} \cdot \boldsymbol{k}_t \; u_r \; \tilde{u}_r \boldsymbol{\alpha} \cdot \boldsymbol{k}_s \; u_a$$

$$= \tilde{u}_b^0 \left[f_1^b f_2^r \boldsymbol{\sigma} \cdot \boldsymbol{k}_t \frac{\boldsymbol{\sigma} \cdot \boldsymbol{p}_r}{p_r} + f_2^b f_1^r \frac{\boldsymbol{\sigma} \cdot \boldsymbol{p}_b}{p_b} \boldsymbol{\sigma} \cdot \boldsymbol{k}_t \right] f_2^r \frac{\boldsymbol{\sigma} \cdot \boldsymbol{p}_r}{p_r} \boldsymbol{\sigma} \cdot \boldsymbol{k}_s \; u_a^0 .$$

For $\boldsymbol{k}_s \cdot \boldsymbol{p}_r = 0$:

$$\begin{aligned}
(\boldsymbol{\sigma} \cdot \boldsymbol{k}_t)(\boldsymbol{\sigma} \cdot \boldsymbol{p}_r)(\boldsymbol{\sigma} \cdot \boldsymbol{p}_r)(\boldsymbol{\sigma} \cdot \boldsymbol{k}_s) &= p_r^2 (\boldsymbol{\sigma} \cdot \boldsymbol{k}_t)(\boldsymbol{\sigma} \cdot \boldsymbol{k}_s) \\
&= p_r^2 (\boldsymbol{k}_t \cdot \boldsymbol{k}_s) + i p_r^2 (\boldsymbol{\sigma} \cdot \boldsymbol{k}_t \times \boldsymbol{k}_s),
\end{aligned}$$

$$\begin{aligned}
(\boldsymbol{\sigma} \cdot \boldsymbol{p}_b)(\boldsymbol{\sigma} \cdot \boldsymbol{k}_t)(\boldsymbol{\sigma} \cdot \boldsymbol{p}_r)(\boldsymbol{\sigma} \cdot \boldsymbol{k}_s) &= (\boldsymbol{p}_b \cdot \boldsymbol{k}_t + i \boldsymbol{\sigma} \cdot \boldsymbol{p}_b \times \boldsymbol{k}_t) \; i \boldsymbol{\sigma} \cdot \boldsymbol{p}_r \times \boldsymbol{k}_s \\
&= -(\boldsymbol{p}_b \times \boldsymbol{k}_t) \cdot (\boldsymbol{p}_r \times \boldsymbol{k}_s) + i(\boldsymbol{p}_b \cdot \boldsymbol{k}_t)(\boldsymbol{\sigma} \cdot \boldsymbol{p}_r \times \boldsymbol{k}_s) \\
&\quad - i \boldsymbol{\sigma} \cdot (\boldsymbol{p}_b \times \boldsymbol{k}_t) \times (\boldsymbol{p}_r \times \boldsymbol{k}_s).
\end{aligned}$$

2)

$$\sum_{\text{negative } u_r} \tilde{u}_b\ \boldsymbol{\alpha}\cdot\boldsymbol{k}_t\ u_r\ \tilde{u}_r\ \boldsymbol{\alpha}\cdot\boldsymbol{k}_s\ u_a$$

$$= \tilde{u}_b^0 \left[f_1^b f_1^b \boldsymbol{\sigma}\cdot\boldsymbol{k}_t - f_2^b f_2^b \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}_b}{p_b}\boldsymbol{\sigma}\cdot\boldsymbol{k}_t \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}_r}{p_r}\right] f_1^r \boldsymbol{\sigma}\cdot\boldsymbol{k}_s u_a^0.$$

3)

$$\sum_{\text{positive } u_{r'}} \tilde{u}_b\ \boldsymbol{\alpha}\cdot\boldsymbol{k}_s\ u_{r'}\ \tilde{u}_{r'}\ \boldsymbol{\alpha}\cdot\boldsymbol{k}_t\ u_a$$

$$= \tilde{u}_b^0 \left[f_1^b f_2^{r'} \boldsymbol{\sigma}\cdot\boldsymbol{k}_s \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}_{r'}}{p_{r'}} - f_2^b f_1^{r'} \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}_b}{p_b}\boldsymbol{\sigma}\cdot\boldsymbol{k}_s\right] f_2^{r'} \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}_{r'}}{p_{r'}}\boldsymbol{\sigma}\cdot\boldsymbol{k}_t\ u_a^0.$$

4)

$$\sum_{\text{negative } u_{r'}} \tilde{u}_b\ \boldsymbol{\alpha}\cdot\boldsymbol{k}_s\ u_{r'}\ \tilde{u}_{r'}\ \boldsymbol{\alpha}\cdot\boldsymbol{k}_t\ u_a$$

$$= \tilde{u}_b^0 \left[f_1^b f_1^{r'} \boldsymbol{\sigma}\cdot\boldsymbol{k}_s - f_2^b f_2^{r'} \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}_b}{p_b}\boldsymbol{\sigma}\cdot\boldsymbol{k}_s \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}_r}{p_r}\right] f_1^{r'} \boldsymbol{\sigma}\cdot\boldsymbol{k}_t\ u_a^0.$$

Let us denote with $\overline{\eta}$ the average value with respect to u_b^0 and u_a^0:

$$\overline{|\tilde{u}_b^0 A u_a^0|^2} = \overline{\tilde{u}_b^0 A u_a^0 \tilde{A} u_b^0} = \frac{1}{2}\tilde{u}_b^0 A\tilde{A} u_a^0 = \frac{1}{4}[(A\tilde{A})_{11} + (A\tilde{A})_{22}].$$

$$\begin{aligned} A &= A_0 + i\boldsymbol{\sigma}\cdot\boldsymbol{B}, \\ A\tilde{A} &= [A_0 + i\boldsymbol{\sigma}\cdot\boldsymbol{B}][\overline{A}_0 - i\boldsymbol{\sigma}\cdot\overline{\boldsymbol{B}}] \\ &= A_0\overline{A}_0 + i\overline{A}_0\boldsymbol{\sigma}\cdot\boldsymbol{B} - iA_0\boldsymbol{\sigma}\cdot\overline{\boldsymbol{B}} + \boldsymbol{B}\cdot\tilde{\boldsymbol{B}} + i\boldsymbol{\sigma}\cdot\boldsymbol{B}\times\overline{\boldsymbol{B}}. \end{aligned}$$

$$\boxed{\overline{A\tilde{A}} = A_0\overline{A}_0 + \boldsymbol{B}\cdot\overline{\boldsymbol{B}}, \quad \overline{|\tilde{u}_0^b A u_0^a|^2} = \frac{1}{2}A_0\overline{A}_0 + \frac{1}{2}\boldsymbol{B}\cdot\overline{\boldsymbol{B}}.}$$

$\boldsymbol{\gamma}_s = (\gamma_s, 0, 0)$, $\boldsymbol{k}_s = (0,0,1)$, $\boldsymbol{\gamma}_t = (\gamma_t \sin\vartheta\cos\varphi, \gamma_t\sin\vartheta\sin\varphi, \gamma_t\cos\vartheta)$. Near the resonance we have:

$$\nu_t = \frac{\nu_s}{1 + \dfrac{h\nu_s}{mc^2}(1-\sin\vartheta\cos\varphi)}.$$

$$\boldsymbol{p}_r = \frac{h\nu_s}{c}\ (1,\ 0,\ 0), \quad \boldsymbol{p}_{r'} = -\frac{h\nu_t}{c}\ (\sin\vartheta\cos\varphi,\ \sin\vartheta\varphi,\ \cos\vartheta),$$

$$\boldsymbol{p}_b = \frac{h\nu_t}{c}\left(\left(1+\frac{h\nu_s}{mc^2}\right)(1-\sin\vartheta\cos\varphi),\ -\sin\vartheta\sin\varphi,\ -\cos\vartheta\right).$$

$$\begin{aligned}
E_1 &= mc^2 + h\nu_s, \\
E_1' &= \pm\sqrt{m^2c^4 + h^2\nu_t^2} + h\nu_s + h\nu_t, \\
E_i &= \pm\sqrt{m^2c^4 + h^2\nu_s^2}, \\
E_r &\sim E_1.
\end{aligned}$$

2.13. BOUND ELECTRON SCATTERING

Let us consider f bound electrons; the unperturbed energy of the system *interacting with an electromagnetic field* is $E_n + \sum_s n_s h\nu_s$. Denoting with $\psi_a(q_1, \ldots, q_f)$ the electron wavefunction corresponding to energy E_a, the interaction with the electromagnetic field *is described by*:

$$\boxed{\begin{aligned}
< a; n_s \ldots |\mathcal{I}| b; n_s + 1 \ldots > &= -e\ c\sqrt{\frac{h(n_s+1)}{2\pi\nu_s}} \\
&\quad \times \int \tilde{\psi}_a \sum_{i=1}^{f} \boldsymbol{\alpha}^i \cdot \boldsymbol{f}_s(q_1)\ \psi_f\ \mathrm{d}\tau, \\
< a; n_s \ldots |\mathcal{I}| b; n_s - 1 \ldots > &= -e\ c\sqrt{\frac{hn_s}{2\pi\nu_s}} \\
&\quad \times \int \tilde{\psi}_a \sum_{i=1}^{f} \boldsymbol{\alpha}^i, \boldsymbol{f}_s(q_1)\ \psi_f\ \mathrm{d}\tau.
\end{aligned}}$$

$\alpha^i = \rho_1^i\ \sigma^i$.

In first approximation, $\lambda \gg |q_i|$;

$$f_s(q_i) \sim f_s(0) = \frac{k_s}{k^{3/2}}.$$

For coherent scattering , by labelling with S, t the incident and scattered quantum, respectively, *with wave-vectors* k_s, k_t, *we have*:

$$\boxed{\begin{aligned} < a;0,1,\ldots|\mathcal{I}|b;0,0\ldots> &= -\frac{e\ c}{k^{3/2}}\sqrt{\frac{h}{2\pi\nu_t}}\int\tilde{\psi}_a\sum_{i=1}^{f}\boldsymbol{\alpha}^i\cdot\boldsymbol{k}_t\ \psi_b\ \mathrm{d}\tau, \\ < b;0,1,\ldots|\mathcal{I}|a;1,0\ldots> &= -\frac{e\ c}{k^{3/2}}\sqrt{\frac{h}{2\pi\nu_s}}\int\tilde{\psi}_b\sum_{i=1}^{f}\boldsymbol{\alpha}^i\cdot\boldsymbol{k}_s\ \psi_a\ \mathrm{d}\tau, \end{aligned}}$$

for resonant scattering, or otherwise

$$\boxed{\begin{aligned} < a;0,1,\ldots|\mathcal{I}|b;1,1\ldots> &= -\frac{e\ c}{k^{3/2}}\sqrt{\frac{h}{2\pi\nu_s}}\int\tilde{\psi}_a\sum_{i=1}^{f}\boldsymbol{\alpha}^i\cdot\boldsymbol{k}_s\ \psi_b\ \mathrm{d}\tau, \\ < b;1,1,\ldots|\mathcal{I}|a;1,0\ldots> &= -\frac{e\ c}{k^{3/2}}\sqrt{\frac{h}{2\pi\nu_t}}\int\tilde{\psi}_b\sum_{i=1}^{f}\boldsymbol{\alpha}^i\cdot\boldsymbol{k}_t\ \psi_a\ \mathrm{d}\tau. \end{aligned}}$$

For $t=0$: $a_1=1,\ a_2=0,\ n_i=0;\ \ H_{12}=0,\ H_{1i},H_{2i}\neq 0.$
For $t\sim 0$:

$$\dot{a}_i=-\frac{2\pi i}{h}H_{i1}\ \mathrm{e}^{2\pi i(E_i-E_1)t/h}-\frac{1}{2T}a_i.$$

$$\boxed{a_i=-\frac{\mathrm{e}^{-t/2T}}{E_i-E_1+(h/4\pi iT)}\left(\mathrm{e}^{2\pi i(E_i-E_1)t/h+t/2T}-1\right)H_{i1}.}$$

$$\boxed{t\gg T:\quad a_i=\frac{-H_{i1}}{E_i-E_1+(h/4\pi iT)}\ \mathrm{e}^{2\pi i(E_i-E_1)t/h}.}$$

$$\dot{a}_2=\frac{2\pi i}{h}\sum_i\frac{H_{2i}H_{i1}}{E_i-E_1+(h/4\pi iT)}\ \mathrm{e}^{2\pi i(E_i-E_1)t/h}.$$

$$\boxed{a_2=\left(\sum_i\frac{H_{2i}H_{i1}}{E_i-E_1+(h/4\pi iT)}\right)\frac{\mathrm{e}^{2\pi i(E_2-E_1)t/h}-1}{E_2-E_1}.}$$

When a variable magnetic field $H = H(t)$ is included *in the interaction, we have to consider also the* diagonal magnetic moments μ_i. For $H_x = H_y = 0, H_z = H(t)$:

$$\dot{a}_1 = \frac{2\pi i}{h} H(t)\ \mu_1\ a_1,$$

$$a_1 = \mathrm{e}^{(2\pi i)/h\ \mu_1 \int H \mathrm{d}t}.$$

$$\dot{a}_i = -\frac{2\pi i}{h} H_{i1}\ \mathrm{e}^{2\pi i (E_i - E_1)t/h}\ \mathrm{e}^{(2\pi i)/h\ \mu_1 \int H \mathrm{d}t} - \frac{1}{2T}\ a_i + \frac{2\pi i}{h} H\ \mu_i\ a_i.$$

$$\boxed{\begin{aligned} a_i &= \mathrm{e}^{-t/2T} \mathrm{e}^{(2\pi i)/h\ \mu_i \int H \mathrm{d}t} \left(-\frac{2\pi i}{h} H_{i1} \right) \\ &\times \left[\int \mathrm{e}^{2\pi i (E_i - E_1)t/h + t/2T + (2\pi i)/h\ (\mu_1 - \mu_i) \int H \mathrm{d}t} \mathrm{d}t + C \right]. \end{aligned}}$$

$$\boxed{\dot{a}_2 = -\frac{2\pi i}{h} \sum_i H_{2i}\ \mathrm{e}^{2\pi i (E_2 - E_1)t/h} a_i + \frac{2\pi i}{h} H \mu_2 a_2.}$$

$$\boxed{\begin{aligned} a_2 &= \left(-\tfrac{2\pi i}{h}\right) \mathrm{e}^{(2\pi i/h)\ \mu_2 \int H \mathrm{d}t} \\ &\times \left[\sum H_{2i} \int_0^t \mathrm{e}^{2\pi i (E_2 - E_i)t/h - (2\pi i/h)\ \mu_2 \int H \mathrm{d}t} a_i\ \mathrm{d}t \right]. \end{aligned}}$$

$$H = H_0 \cos 2\pi \nu t,$$

$$\int H \mathrm{d}t = \frac{H_0}{2\pi\nu} \sin 2\pi\nu t,$$

$$\frac{2\pi}{h} (\mu_1 - \mu_i) \int H \mathrm{d}t = \frac{H_0 (\mu_1 - \mu - i)}{h\nu} \sin 2\pi\nu t,$$

$$\boxed{\begin{array}{rcl} e^{(2\pi i/h)(\mu_1-\mu_i)\int H\mathrm{d}t} & = & e^{i[H_0(\mu_1-\mu_i)/h\nu]\sin 2\pi\nu t} \\ & = & e^{iA_i\sin 2\pi\nu t}, \end{array}}$$

$$\boxed{A_i = \frac{H_0(\mu_1-\mu_i)}{h\nu}.}$$

[20]

$$e^{iA_i\sin 2\pi\nu t} = c_0^i + c_i^i\ e^{2\pi\nu it} + c_{-1}^i\ e^{-2\pi\nu it} + c_2^i e^{4\pi\nu it} + c_{-2}^i\ e^{-4\pi\nu it} + \ldots.$$

$\omega = 2\pi\nu t$:

$$e^{iA_i\sin\omega} = c_0^i + c_1^i\ e^{i\omega} + c_{-1}^i\ e^{-i\omega} + c_2^i\ e^{2i\omega} + c_{-2}^i\ e^{-2i\omega} + \ldots.$$

$$c_0^i = \frac{1}{2\pi}\int_0^{2\pi} e^{iA_i\sin\omega}\ \mathrm{d}\omega.$$

$$\zeta = e^{i\omega}, \quad \sin\omega = \frac{\zeta-\zeta^{-1}}{2i}, \quad \mathrm{d}\zeta = i\zeta\mathrm{d}\omega, \quad \mathrm{d}\omega = -i\frac{\mathrm{d}\zeta}{\zeta};$$

$$e^{iA_i\sin\omega}\mathrm{d}\omega = \frac{1}{i\zeta}e^{A_i(\zeta-\zeta^{-1})/2}\mathrm{d}\zeta.$$

$$c_0^i = \frac{1}{2\pi i}\oint\frac{1}{\zeta}e^{A_i(\zeta-\zeta^{-1})/2}\mathrm{d}\zeta.$$

$$e^{A_i(\zeta-\zeta^{-1})/2} = 1 + A_i\frac{\zeta-\zeta^{-1}}{2} + \frac{A_i^2}{2!}\left(\frac{\zeta-\zeta^{-1}}{2}\right)^2 + \frac{A_i^3}{3!}\left(\frac{\zeta-\zeta^{-1}}{2}\right)^3 + \ldots.$$

$$\left(\zeta-\zeta^{-1}\right)^n = \sum_{r=0}^{n}\zeta^{n-2r}\left(\begin{array}{c} n \\ r \end{array}\right)(-1)^r,$$

[20]@ *The original manuscript alludes here to the unidentified Ref. 11.05.*

$$
\begin{aligned}
\left(\zeta-\zeta^{-1}\right)^{2n} &= \sum_{r=0}^{2n}(-1)^r\zeta^{2n-2r}\begin{pmatrix}2n\\ r\end{pmatrix}\\
&= \sum_{s=-n}^{n}(-1)^n\begin{pmatrix}2n\\ n+s\end{pmatrix}\zeta^{-2s}(-1)^s(-1)^n\begin{pmatrix}2n\\ n\end{pmatrix}\\
&= \frac{(2n)!}{n!^2}(-1)^n.
\end{aligned}
$$

$$
\boxed{c_0^i = 1-\frac{A_i^2}{1\cdot 2^2}+\frac{A_i^4}{2!^2\cdot 2^4}-\ldots = \mathcal{I}_0(A_i).}
$$

2.14. RETARDED FIELDS

The possibility is considered, in the following pages, of introducing an intrinsic constant *time delay τ (or an intrinsic space constant $\varepsilon = c\tau$) in the expressions for the electromagnetic retarded fields, generically denoted with $f(x,y,z,t)$.*

$f = f(x,y,z,t).$

$$
\varphi(x,y,z,t) = f\left(x,y,z,t-\frac{r}{c}\right) = \overline{f(x,y,z,t)}.
$$

$$
\begin{aligned}
\varphi_x'(x,y,z,t) &= f_x'\left(x,y,z,t-\frac{r}{c}\right)-\frac{x}{rc}f_t'\left(x,y,z,t-\frac{r}{c}\right)\\
&= \overline{f_x'(x,y,z,t)}-\frac{x}{rc}\overline{f_t'(x,y,z,t)},\\
\varphi_x''(x,y,z,t) &= f_{x^2}''\left(x,y,z,t-\frac{r}{c}\right)-\frac{2x}{rc}f_{xt}''\left(x,y,z,t-\frac{r}{c}\right)\\
&\quad +\frac{x^2}{r^2c^2}f_{tt}''\left(x,y,z,-\frac{r}{c}\right)-\frac{r^2-x^2}{r^3c}f_t'\left(x,y,z,t-\frac{r}{c}\right)\\
&= \overline{f_{xx}''(x,y,z,t)}-\frac{2x}{rc}\overline{f_{xt}''(x,y,z,t)}+\frac{x^2}{r^2c^2}\overline{f_{tt}''(x,y,z,t)}\\
&\quad -\frac{r^2-x^2}{r^3c}\overline{f_t'(x,y,z,t)}.
\end{aligned}
$$

$$
\begin{aligned}
\varphi_t'(x,y,z,t) &= f_t'\left(x,y,z,t-\frac{r}{c}\right) = \overline{f_t'(x,y,z,t)},\\
\varphi_{tt}''(x,y,z,t) &= f_t''\left(x,y,z,t-\frac{r}{c}\right) = \overline{f_{tt}''(x,y,z,t)}.
\end{aligned}
$$

$\Box = \boldsymbol{\nabla}^2 - \dfrac{1}{c^2}\dfrac{\partial^2}{\partial t^2}$:

$$\begin{aligned}\Box\varphi(x,y,z,t) &= \overline{\boldsymbol{\nabla}^2 f\left(x,y,z,t-\frac{r}{c}\right)} - \frac{2}{c}\overline{\frac{\partial^2}{\partial r\partial t}(x,y,z,t)} \\ &\quad -\frac{2}{rc}\overline{f_t'(x,y,z,t)},\end{aligned}$$

$$\begin{aligned}\frac{\partial}{\partial r}\varphi(x,y,z,t) &= \sum_x \frac{x}{r} f_x'\left(x,y,z,t-\frac{r}{c}\right) - \frac{1}{c} f_t'\left(x,y,z,t-\frac{r}{c}\right) \\ &= \overline{\frac{\partial}{\partial z} f(x,y,z,t)} - \overline{\frac{1}{c} f_t'(x,y,z,t)}, \\ \frac{\partial^2}{\partial r\partial t}\varphi(x,y,z,t) &= \frac{\partial^2}{\partial r\partial t}\overline{f(x,y,z,t)} - \frac{1}{c}\overline{f_t''(x,y,z,t)}.\end{aligned}$$

$$\begin{aligned}\Box\varphi + \frac{2}{c}\frac{\partial^2}{\partial z\partial t}\varphi &= \overline{\boldsymbol{\nabla}^2 f} - \frac{2}{c^2}\overline{f_t''} - \frac{2}{rc}\overline{f_t'} \\ &= \overline{\Box f} - \overline{\frac{1}{c^2} f_t''} - \overline{\frac{2}{rc} f_t'}.\end{aligned}$$

$$\boxed{\overline{\Box f} = \boldsymbol{\nabla}^2\varphi + \frac{2}{rc}\varphi' + \frac{2}{c}\frac{\partial^2}{\partial z\partial t}\varphi.}$$

$$\varphi(x,y,z,t) = f\left(x,y,z,t-\frac{\sqrt{r^2+\varepsilon^2}}{c}\right) = \widetilde{f}(x,y,z,t).$$

$$\begin{aligned}f(x,y,z,t) &= \varphi\left(x,y,z,t-\frac{\sqrt{r^2+\varepsilon^2}}{c}\right), \\ f_x'(x,y,z,t) &= \varphi_x'\left(x,y,z,t-\frac{\sqrt{r^2+\varepsilon^2}}{c}\right) \\ &\quad + \frac{x}{c\sqrt{r^2+\varepsilon^2}}\varphi_t'\left(x,y,z,t+\frac{\sqrt{r^2+\varepsilon^2}}{c^2}\right),\end{aligned}$$

$$
\begin{aligned}
f''_{xx}(x,y,z,t) &= \varphi''_x\left(x,y,z,t+\frac{\sqrt{r^2+\varepsilon^2}}{c}\right)\\
&+\frac{2x}{c\sqrt{r^2+\varepsilon^2}}\varphi''_{xt}\left(x,y,z,t+\frac{\sqrt{r^2+\varepsilon^2}}{c^2}\right)\\
&+\frac{r^2+\varepsilon^2-x^2}{c(r^2+\varepsilon^2)^{3/2}}\varphi'_t\left(x,y,z,t+\frac{\sqrt{r^2+\varepsilon^2}}{c}\right)\\
&+\frac{x^2}{c^2(r^2+\varepsilon^2)}\varphi''_{tt}\left(x,y,z,t+\frac{\sqrt{r^2+\varepsilon^2}}{c}\right),
\end{aligned}
$$

$$
f''_{tt}(x,y,z,t) = \varphi''_{tt}\left(x,y,z,t+\frac{\sqrt{r^2+\varepsilon^2}}{c}\right).
$$

$$
\begin{aligned}
\Box f''_{tt}(x,y,z,t) &= \boldsymbol{\nabla}^2\,\varphi\left(x,y,z,t+\frac{\sqrt{r^2+\varepsilon^2}}{c}\right)\\
&-\frac{\varepsilon^2}{c^2(r^2+\varepsilon^2)}\varphi''_{tt}\left(x,y,t+\frac{\sqrt{r^2+\varepsilon^2}}{c}\right)\\
&+\frac{2r^2+3\varepsilon^2}{c(\sqrt{r^2+\varepsilon^2})^3}\varphi'_t\left(x,y,z,t+\frac{\sqrt{r^2+\varepsilon^2}}{c}\right)\\
&+\frac{2r}{c\sqrt{r^2+\varepsilon^2}}\frac{\partial^2}{\partial r\partial t}\varphi\left(x,y,z,t+\frac{\sqrt{r^2+\varepsilon^2}}{c}\right).
\end{aligned}
$$

$$
\boxed{\widetilde{\Box} f = \boldsymbol{\nabla}^2\,\varphi - \frac{\varepsilon^2}{c^2(r^2+\varepsilon^2)}\ddot{\varphi} + \frac{2r^2+3\varepsilon^2}{c(r^2+\varepsilon^2)^{3/2}}\dot{\varphi} + \frac{2z}{c\sqrt{r^2+\varepsilon^2}}\frac{\partial^2}{\partial r\partial t}\varphi.}
$$

2.14.1 Time Delay

With the introduction of a time delay τ, *which is a* universal constant (classically $\tau = 0$), *by setting*

$$
\boxed{\varepsilon = \tau c\,,}
$$

we get:

$$\Phi = \int \frac{1}{\sqrt{r^2+\varepsilon^2}} S\left(t - \frac{\sqrt{z_2+\varepsilon^2}}{c},\, x,\, y,\, z\right) \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z,$$

and, for $\varepsilon \to 0$:

$$\begin{aligned}\Phi \;=\; & \int \frac{1}{r} S\left(t - \frac{r}{c},\, x, y,\, z\right) \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \\ & -\varepsilon^2 \left\{ \int \frac{1}{2r^3} S\left(t - \frac{r}{c},\, x,\, y,\, z\right) \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \right. \\ & \qquad \left. + \int \frac{1}{2r^2 c} \dot{S}\left(t - \frac{r}{c},\, x,\, y,\, z\right) \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \right\} + \ldots.\end{aligned}$$

2.15. MAGNETIC CHARGES

A modification of the classical Maxwell equations was considered in the following pages, in order to include also the effect of magnetic charges.

$$A(q) = -\frac{1}{4\pi} \int \frac{\boldsymbol{\nabla} \cdot \boldsymbol{g}(q')}{r} \,\mathrm{d}q'.$$

$$\boxed{\begin{aligned}\boldsymbol{g}^0 &= -\frac{1}{4\pi}\, \boldsymbol{\nabla} \int \frac{\boldsymbol{\nabla} \cdot \boldsymbol{g}(q')}{r} \,\mathrm{d}q', \\ \boldsymbol{g}^1 &= \boldsymbol{g} - \boldsymbol{g}^0.\end{aligned}}$$

$$\boldsymbol{g} = (\delta(q - q_0);\, 0;\, 0),$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{g} = \delta'(x - x_0)\ \delta(y - y_0)\ \delta(z - z_0).$$

$r = |q' - q|$:

$$\begin{aligned}& \int \frac{\delta'(x' - x_0)\ \delta(y - y_0)\ \delta(z - z_0)}{r} \,\mathrm{d}q' \\ &= \int \frac{\delta'(x' - x_0)}{\sqrt{(y_0 - y)^2 + (z_0 - z)^2 + (x' - x)^2}} \,\mathrm{d}x' \\ &= \int \delta(x' - x_0) \frac{x' - x}{[(y_0 - y)^2 + (z_0 - z)^2 + (x' - x)^2]^{3/2}} \,\mathrm{d}x' \\ &= -\frac{x - x_0}{[(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2]^{3/2}} = -\frac{x - x_0}{R^3}.\end{aligned}$$

$$g_1^0 = \frac{3(x-x_0)^2 - R^2}{R^5}, \qquad g_2^0 = \frac{3(x-x_0)(y-y_0)}{R^5},$$

$$g_3^0 = \frac{3(x-x_0)(z-z_0)}{R^5};$$

$$g_1^1 = \delta(q-q_0) - \frac{3(x-x_0)^2 - R^2}{R^5}, \qquad g_2^1 = -\frac{3(x-x_0)(y-y_0)}{R^5},$$

$$g_3^1 = -\frac{3(x-x_0)(z-z_0)}{R^5}.$$

$$\boldsymbol{E} = \frac{\boldsymbol{E}' + \boldsymbol{E}''}{2}, \qquad \boldsymbol{H} = \frac{\boldsymbol{H}' + \boldsymbol{H}''}{2}.$$

$$4\pi\boldsymbol{\mathcal{I}} \int + \frac{1}{c}\frac{\partial \boldsymbol{E}'}{\partial t} = \boldsymbol{\nabla} \times \boldsymbol{H}', \qquad 4\pi\boldsymbol{\mathcal{I}} + \frac{1}{c}\frac{\partial \boldsymbol{E}''}{\partial t} = \boldsymbol{\nabla} \times \boldsymbol{H}'',$$

$$-4\pi\boldsymbol{\mathcal{I}} - \frac{1}{c}\frac{\partial \boldsymbol{H}'}{\partial t} = \boldsymbol{\nabla} \times \boldsymbol{E}', \qquad 4\pi\boldsymbol{\mathcal{I}} - \frac{1}{c}\frac{\partial \boldsymbol{H}''}{\partial t} = \boldsymbol{\nabla} \times \boldsymbol{E}'',$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{E}' = 4\pi\rho, \qquad \boldsymbol{\nabla} \cdot \boldsymbol{E}'' = 4\pi\rho,$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{H}' = 4\pi\rho, \qquad \boldsymbol{\nabla} \cdot \boldsymbol{H}'' = -4\pi\rho.$$

$$\left\{ \begin{array}{l} 4\pi\boldsymbol{\mathcal{I}}\ (1-i) + \dfrac{1}{c}\dfrac{\partial(\boldsymbol{E}' - i\boldsymbol{H}')}{\partial t} = i\,\boldsymbol{\nabla} \times (\boldsymbol{E}' - i\boldsymbol{H}'), \\[2ex] \qquad\qquad \boldsymbol{\nabla} \cdot (\boldsymbol{E}' - i\boldsymbol{H}') = 4\pi\rho\ (1-i), \end{array} \right.$$

$$\left\{ \begin{array}{l} 4\pi\boldsymbol{\mathcal{I}}\ (1+i) + \dfrac{1}{c}\dfrac{\partial(\boldsymbol{E}'' - i\boldsymbol{H}'')}{\partial t} = i\,\boldsymbol{\nabla} \times (\boldsymbol{E}'' - i\boldsymbol{H}''), \\[2ex] \qquad\qquad \boldsymbol{\nabla} \cdot (\boldsymbol{E}'' - i\boldsymbol{H}'') = 4\pi\rho\ (1+i), \end{array} \right.$$

$$\begin{cases} 4\pi\boldsymbol{\mathcal{I}}\ (1+i) + \dfrac{1}{c}\dfrac{\partial(\boldsymbol{E}' + i\boldsymbol{H}')}{\partial t} = -i\,\boldsymbol{\nabla}\times(\boldsymbol{E}' + i\boldsymbol{H}'), \\ \boldsymbol{\nabla}\cdot(\boldsymbol{E}' + i\boldsymbol{H}') = 4\pi\rho\ (1+i), \end{cases}$$

$$\begin{cases} 4\pi\boldsymbol{\mathcal{I}}\ (1-i) + \dfrac{1}{c}\dfrac{\partial(\boldsymbol{E}'' + i\boldsymbol{H}'')}{\partial t} = -i\,\boldsymbol{\nabla}\times(\boldsymbol{E}'' + i\boldsymbol{H}''), \\ \boldsymbol{\nabla}\cdot(\boldsymbol{E}'' + i\boldsymbol{H}'') = 4\pi\rho\ (1-i), \end{cases}$$

For $\boldsymbol{E}' = -\boldsymbol{H}''$, $\boldsymbol{H}' = \boldsymbol{E}''$ *we re-obtain the* Maxwell equations:

$$\boldsymbol{E} = \frac{\boldsymbol{E}' + \boldsymbol{H}'}{2}, \qquad \boldsymbol{H} = \frac{\boldsymbol{H}' - \boldsymbol{E}'}{2}.$$

[21]

Appendix:
Potential experienced by an electric charge: a particular case

For a charge-1 particle:

$$\boxed{\frac{\mathrm{d}V}{\mathrm{d}t} = -\frac{1}{2(a^2+t)(a^2+t)(c^2+t)}} = -\frac{1}{2(a^2+t)\sqrt{c^2+t}},$$

[21]@ *The page ended with an attempt to generalize the above results to arbitrary linear combinations of the* $\boldsymbol{E}$ *and* $\boldsymbol{H}$ *fields (with space-time dependent coefficients), in the case of Maxwell equations without sources:*

$$\boldsymbol{E}' = \alpha\boldsymbol{E} + \beta\boldsymbol{H}, \qquad \boldsymbol{H}' = -\beta\boldsymbol{E} + \alpha\boldsymbol{H};$$

$\alpha = \alpha(q,t)$, $\beta = \beta(q,t)$;

$$\frac{1}{c}\frac{\partial\boldsymbol{E}}{\partial t} = \boldsymbol{\nabla}\times\boldsymbol{H}, \qquad -\frac{1}{c}\frac{\partial\boldsymbol{H}}{\partial t} = \boldsymbol{\nabla}\times\boldsymbol{E},$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{E} = 0, \qquad \boldsymbol{\nabla}\cdot\boldsymbol{H} = 0;$$

$$\begin{aligned} \boldsymbol{\nabla}\cdot\boldsymbol{E}' &= \boldsymbol{\nabla}\alpha\cdot\boldsymbol{E} + \boldsymbol{\nabla}\beta\cdot\boldsymbol{H}, \\ \boldsymbol{\nabla}\cdot\boldsymbol{H}' &= -\boldsymbol{\nabla}\beta\cdot\boldsymbol{E} + \boldsymbol{\nabla}\alpha\cdot\boldsymbol{H}. \end{aligned}$$

$$
\begin{aligned}
-\frac{1}{c} = V &= \int_0^\infty \frac{\mathrm{d}t}{2(a^2+t)\sqrt{(c^2+t)}} \\
&= \int_c^\infty \frac{\mathrm{d}z}{z^2+(a^2-c^2)} = \frac{1}{\sqrt{a^2-c^2}}\left(\frac{\pi}{2} - \arctan\frac{c}{\sqrt{a^2-c^2}}\right) \\
&= \frac{1}{\sqrt{a^2-c^2}}\arctan\frac{\sqrt{a^2-c^2}}{c}.
\end{aligned}
$$

$$
\begin{aligned}
z &= \sqrt{c^2+t}, \\
z^2 &= c^2+t, \\
\mathrm{d}t &= 2z\,\mathrm{d}z, \\
t &= z^2-c^2, \\
a^2+t &= z^2+(a^2-c^2).
\end{aligned}
$$

$$
c = a\sqrt{1-\beta^2}, \quad a^2-c^2 = a^2\beta^2.
$$

$$
\frac{1}{c} = V = \frac{1}{a\beta}\arctan\frac{\beta}{\sqrt{1-r^2}} = \frac{1}{a\beta}\arcsin\beta.
$$

$$
\boxed{\boxed{c = a\frac{\beta}{\arcsin\beta};\ V = \frac{1}{c} = \frac{1}{a}\frac{\arcsin\beta}{\beta}.}}
$$

$$
\begin{aligned}
&\left(\frac{\partial C_x}{\partial z} - \frac{\partial C_z}{\partial C_x}\right)^2 + \left(\frac{\partial C_y}{\partial x} - \frac{\partial C_x}{\partial y}\right)^2 + \left(\frac{\partial C_z}{\partial y} - \frac{\partial C_y}{\partial z}\right)^2 \\
&= |\boldsymbol{\nabla}\, C_x|^2 + |\boldsymbol{\nabla}\, C_y|^2 + |\boldsymbol{\nabla}\, C_z|^2 - \sum_{xy}\frac{\partial C_x}{\partial y}\frac{\partial C_y}{\partial x}.
\end{aligned}
$$

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