Chapter 2

S-N or Wöhler Field Models

Contents

E. Castillo, A. Fernández-Canteli, *A Unified Statistical Methodology for Modeling Fatigue Damage*,

2.1 Introduction

In the evaluation and prediction of the fatigue lifetime of machines and structures the role of mathematical and statistical models is crucial, due to the high complexity of the fatigue problem, in which the consideration of the stress range, stress level and the size effect, together with an efficient estimation of the corresponding parameters represents one of the most difficult and attracting challenges, which have not yet been satisfactorily solved. As a consequence, reliable failure prediction, engineering design and risk analysis in fatigue are not possible without the help of statistical models.

The use of some existing models, together with a good knowledge of physical or metallurgical aspects of the lifetime phenomenon, could represent an adequate approach for tackling fatigue problems, and for a limited judgment or interpretation on the experimental results obtained. However, it would neither allow for extrapolation of results outside the testing range considered, nor contribute to providing an overview about the general treatment of fatigue evaluation and prediction, indispensable progression towards a better understanding of this complex phenomenon or to developing appropriate strategies and adequate test planning, as an alternative to a simple data fitting such as those commonly used in the past. Additionally, the possible shortage of data, which represents a common feature in the case of fatigue experimentation due to economic and/or time reasons, and the physical impossibility of testing specimens over a certain size, must be taken into consideration.

Critical points in this modeling are: (a) the general applicability of a model to all possible cases of lifetime problems (fatigue, creep, corrosion fatigue, dielectric stress breakdown, dock damage, road surface damage, etc.) and to different types of materials (metallic, cementitious, polymers, etc.), irrespective of the failure mechanisms (dominant crack or generalized microcracks growth), (b) the parameter and quantile estimation and the determination of confidence intervals from data and finally, (c) the extrapolation of S-N fields to out-of-range cases, both in duration and size-effect. All these questions are of practical paramount significance because they exert a strong influence on the quality of fatigue analysis and prediction, as well as on the structural or mechanical design of elements under fatigue loads.

In this chapter, our aim is to build fatigue models to be used in the stress-based approach. To this end, we proceed from the simplest to the most complex and general cases for constant stress range and level.

1. *Fatigue lifetime for constant stress range and stress level.* The basic and simplest step in fatigue modeling consists of reproducing the fatigue behavior of materials when they are subject to alternating stresses ranging from a fixed value σ_m to a fixed value σ_M, i.e. when subjected to tests associated with a given stress range and stress level.

   Due to the random character of fatigue lifetime, if several specimens were subjected to this type of tests with the same values of σ_m and σ_M we would not obtain the same, but different lifetimes. Thus, since the fatigue life of the specimen is random, the model must be statistical in nature, so, from the very beginning we treat fatigue lifetime N as a random variable. As we shall see later in this chapter, based on the weakest-link principle, the Weibull or Gumbel distributions seem to be the most adequate and theoretically justified distributions to reproduce fatigue lifetime from a statistical point of view (see Castillo et al. (1987a)). In other words, the weakest-link principle tells us that the Weibull or Gumbel distributions are the natural and adequate distributions to reproduce fatigue lifetime.

2. *Fatigue lifetime for constant stress level and different constant stress ranges.* Next, it is necessary to consider a second problem, which is the case of varying stress ranges and constant stress level. In other words, in the next step our intention is to model the fatigue lifetime changes with the stress range for constant stress level. We consider a constant reference stress level either σ_m or σ_M, and we attempt to model how the fatigue lifetime N changes with $\Delta\sigma = \sigma_M - \sigma_m$. We also make the simplifying assumption that the same parametric model is valid for constant stress level σ_M or σ_m, though with different parameter values. As in the previous case, we will propose a model, which will turn out to be the unique solution which satisfies a set of compatibility conditions and statistical assumptions, that is, the model is obtained without arbitrary assumptions about its functional form.

3. *Fatigue lifetime for any constant stress range and level.* Finally, the influence of the stress level on fatigue lifetime is studied as the third problem for deriving a general model able to reproduce any combination of stress ranges and levels. Again some compatibility conditions will allow us to derive this model without arbitrary assumptions.

In this chapter, we deal only with general models that are consistent with physical and statistical considerations and exclude all other types of models violating these compatibility conditions.

The organization of the chapter is as follows. In Sect. 2.2 we identify the main variables involved in the considered problems of fatigue, and use the well known Buckingham Π-theorem of dimensional analysis to obtain a smaller equivalent set of dimensionless variables. This enables us to reduce the complexity of our models and work with dimensionless variables and parameters, a convenient way of avoiding inconsistencies and problems. In Sect. 2.3 the Weibull and Gumbel models, which are systematically used in the following sections, are selected as the most adequate for dealing with fatigue problems, and some of their properties are discussed. Section 2.4 describes the fatigue models for constant stress range and level together with the corresponding estimation methods. Section 2.5 derives a fatigue model for different constant stress ranges and constant stress level, justifies the model based on physical and engineering considerations, and provides some estimation methods and one example of application. In Sect. 2.6 a general fatigue model for any constant stress range and level is derived, several submodels are analyzed, the parameter estimation is dealt with and one example of application is given. In Sect. 2.7 the dimensional form of the models is recovered. Section 2.8 is devoted to discussing the properties of the proposed models. Finally, we end the chapter with a section devoted to conclusions, and some appendices where the details of the mathematical derivations are included.

2.2 Dimensional analysis

As indicated in Chap. 1, dimensional analysis is a very useful tool in building mathematical and physical models aimed at reproducing engineering problems, and ignoring dimensional analysis techniques can lead to invalid models. In this section we make use of the Buckingham theorem to solve the first two problems indicated in the previous section, that is, to determine the relation between lifetime N and stress range $\Delta\sigma = \sigma_M - \sigma_m$.

As indicated, in the first step for developing a model for fatigue we must identify the relevant variables involved. From experience accumulated in the study of the fatigue phenomenon we know that the five variables initially involved in the fatigue problem under fixed stress range and level are those in the set:

$$\mathcal{V} \equiv \{p, N, N_0, \Delta\sigma, \Delta\sigma_0\},$$

where p is the probability of fatigue failure of a piece when subject to N cycles at a stress range $\Delta\sigma$, N_0 is the threshold value for N, i.e. the minimum lifetime for any $\Delta\sigma$, and $\Delta\sigma_0$ is the fatigue limit, which as already indicated is defined in this book as the $\Delta\sigma$ value leading to a lifetime of 1×10^6 cycles for the median $(p = 0.5)$ S-N curve.

Table 2.1: Dimensional analysis of the initial set of variables involved in the fatigue problem.

	N	N_0	$\Delta\sigma$	$\Delta\sigma_0$	p
M	0	0	1	1	0
L	0	0	-1	-1	0
T	1	1	-2	-2	0

Fortunately, the Π-Theorem allows us to represent any relation between these variables in terms of a reduced set of dimensionless variables. In fact, a dimensional analysis of the initial set $\mathcal{V}$ of five variables leads to a matrix of dimensions, the determinant of which has rank two, given in Table 2.1, where M, L and T refer to the fundamental magnitudes mass, length and time, respectively.[1] Thus, the initial set of five variables reduces to a set of three dimensionless variables. Though there are other alternatives, it seems convenient to choose N_0 and $\Delta\sigma_0$ as the normalizing variables, leading to the reduced set

[1]In Sect. 2.6, where the fatigue model is written in terms of σ_m and σ_M, a different dimensional analysis is required.

of variables $\mathcal{V}^* \equiv \{N/N_0, \Delta\sigma/\Delta\sigma_0, p\}$. This means that any existing relation $r(N, N_0, \Delta\sigma, \Delta\sigma_0, p) = 0$ among the independent five variables in set $\mathcal{V}$ can be written in terms of only these three dimensionless variables:

$$r(N, N_0, \Delta\sigma, \Delta\sigma_0, p) = 0 \quad \Leftrightarrow \quad f\left(\frac{N}{N_0}, \frac{\Delta\sigma}{\Delta\sigma_0}, p\right) = 0. \tag{2.1}$$

Since we are interested in p, (2.1) can be written as:

$$p = q\left(\frac{N}{N_0}, \frac{\Delta\sigma}{\Delta\sigma_0}\right), \tag{2.2}$$

where $q()$ is a function to be determined.

The important result is that only the dimensionless quotients N/N_0 and $\Delta\sigma/\Delta\sigma_0$ have any influence on the probability of failure p, so that either N/N_0 and $\Delta\sigma/\Delta\sigma_0$, or any monotone functions of them, such as $h(N/N_0)$ and $g(\Delta\sigma/\Delta\sigma_0)$ have to be considered. All authors agree upon the use of logarithmic scales for N, but there is no agreement for the $\Delta\sigma$ scale (some authors use an arithmetic and some a logarithmic scale). Thus, function $h(\cdot)$ is the natural logarithm, and function $g(\cdot)$ can be either the log or the identity function. In this book we assume $g(x) = x$ for some cases and $g(x) = \log x$ in other cases.

For the sake of notation simplicity, in the following sections we denote

$$N^* = h(N/N_0); \;\; \Delta\sigma^* = g(\Delta\sigma/\Delta\sigma_0), \tag{2.3}$$

where the asterisks refer to dimensionless variables.

Depending on the specific research program undertaken and the S-N field region covered by the experimentation, different intuitive models (parabolic, hyperbolic, linear, piecewise linear, etc.) have been proposed in the literature (see, for example, Refs [7–18] and references in Castillo et al. (1985)) to fit experimental data.

Some of these models are shown in Table 2.2. However, unfortunately, not all are physically valid models. For a model to be valid it must have the same functional form irrespective of the units of measurements of the different variables involved. For example the model

$$\log N = A - B\log\Delta\sigma$$

requires all summands to have the same units of measurement. Since N is lifetime and $\Delta\sigma$ is stress, the constants A and B must have dimensions and they need to be different, and hence, if the units of the data are changed the values of the constants A and B must be modified accordingly. Thus, this model is not a physically valid model. The same can be said for the first eight models in Table 2.2. Only Models 9, 10 and 11, with dimensionless variables inside the logarithms, are physically valid. In this book we consider only this type of models. Note that a better fit to data could be obtained using other models, but further use of the model and mainly extrapolation outside the fitted region can be a problem if physical conditions are violated. Table 2.3 shows modified versions of the models in Table 2.2, where the dimensionality problems have been solved.

Table 2.2: Models proposed in the literature for the S-N curves.

Model	Functional Form
Wöhler (1870)	$\log N = A - B\Delta\sigma;\ \Delta\sigma \geq \Delta\sigma_0$
Basquin (1910)	$\log N = A - B\log\Delta\sigma;\ \Delta\sigma \geq \Delta\sigma_0$
Strohmeyer (1914)	$\log N = A - B\log(\Delta\sigma - \Delta\sigma_0)$
Palmgren (1924)	$\log(N + D) = A - B\log(\Delta\sigma - \Delta\sigma_0)$
Palmgren (1924)	$\log N = A - B\log(\Delta\sigma - \Delta\sigma_0)$
Weibull (1949)	$\log(N + D) = A - B\log((\Delta\sigma - \Delta\sigma_0)/(\Delta\sigma_{st} - \Delta\sigma_0))$
Stüssi (1955)	$\log N = A - B\log((\Delta\sigma - \Delta\sigma_0)/(\Delta\sigma_{st} - \Delta\sigma))$
Bastenaire (1972)	$(\log N - B)(\Delta\sigma - \Delta\sigma_0) = A\exp\left[-C(\Delta\sigma - \Delta\sigma_0)\right]$
Spindel–Haibach (1981)	$\log(N/N_0) = A\log(\Delta\sigma/\Delta\sigma_0) - B\log(\Delta\sigma/\Delta\sigma_0) + B\left\{(1/\alpha)\log\left[1 + (\Delta\sigma/\Delta\sigma_0)^{-2\alpha}\right]\right\}$
Castillo et al. (1985)	$\log(N/N_0) = \dfrac{\lambda + \delta(-\log(1-p))^{1/\beta}}{\log(\Delta\sigma/\Delta\sigma_0)}$
Kohout and Vechet (2001)	$\log\dfrac{\Delta\sigma}{\Delta\sigma_\infty} = \log\left(\dfrac{N + N_1}{N + N_2}\right)^b$
Pascual and Meeker (1999)	$\log N = A - B\log(\Delta\sigma - \Delta\sigma_0)$

Table 2.3: Models proposed in the literature for the S-N curves, after correction.

Modified Model	Dimensionless Functional Form
Wöhler (1870)	$\log(N/N_0) = A - C\dfrac{\Delta\sigma}{\Delta\sigma_0};\ \Delta\sigma \geq \Delta\sigma_0$
Basquin (1910)	$\log(N/N_0) = A - B\log\dfrac{\Delta\sigma}{\Delta\sigma_0};\ \Delta\sigma \geq \Delta\sigma_0$
Strohmeyer (1914)	$\log(N/N_0) = A - B\log\left(\dfrac{\Delta\sigma}{\Delta\sigma_0} - 1\right)$
Palmgren (1924)	$\log(N/N_0 + D) = A - B\log\left(\dfrac{\Delta\sigma}{\Delta\sigma_0} - 1\right)$
Palmgren (1924)	$\log(N/N_0) = A - B\log\left(\dfrac{\Delta\sigma}{\Delta\sigma_0} - 1\right)$
Weibull (1949)	$\log(N/N_0 + D) = A + B\left[\log\left(\dfrac{\Delta\sigma}{\Delta\sigma_0} - 1\right) - \log\left(\dfrac{\Delta\sigma_{st}}{\Delta\sigma_0} - 1\right)\right]$
Stüssi (1955)	$\log N/N_0 = A - B\left[\log\left(\dfrac{\Delta\sigma}{\Delta\sigma_0} - 1\right) - \log\left(\dfrac{\Delta\sigma_{st}}{\Delta\sigma_0} - \dfrac{\Delta\sigma}{\Delta\sigma_0}\right)\right]$
Bastenaire (1972)	$\log(N/N_0) = \dfrac{A\exp\left[-C\left(\dfrac{\Delta\sigma}{\Delta\sigma_0} - 1\right)\right]}{\dfrac{\Delta\sigma}{\Delta\sigma_0} - 1}$
Spindel–Haibach (1981)	$\log(N/N_0) = A\log(\Delta\sigma/\Delta\sigma_0) - B\log(\Delta\sigma/\Delta\sigma_0) + B\left\{(1/\alpha)\log\left[1 + (\Delta\sigma/\Delta\sigma_0)^{-2\alpha}\right]\right\}$
Castillo et al. (1985)	$\log(N/N_0) = A/\log(\Delta\sigma/\Delta\sigma_0)$
Kohout and Vechet (2001)	$\log(\Delta\sigma/\Delta\sigma_\infty) = b\log\left(\dfrac{1 + N_1/N}{1 + N_2/N}\right)$
Pascual and Meeker (1999)	$\log N/N_0 = A - B\log\left(\dfrac{\Delta\sigma}{\Delta\sigma_0} - 1\right)$

2.3 Extreme models in fatigue

In this section we introduce the Weibull and reverse Gumbel extreme models used in fatigue, which are systematically used in the following sections and have a solid theoretical justification from a statistical point of view, and some of their properties, especially those related to lifetime problems.

2.3.1 The Weibull model

The cumulative distribution function (cdf) of the three-parameter Weibull family of distributions is given by:

$$\begin{aligned} F(x;\lambda,\delta,\beta) &= 1-\exp\left[-\left(\frac{x-\lambda}{\delta}\right)^{\beta}\right] \\ & x\geq\lambda;\ -\infty<\lambda<\infty,\ \delta>0,\ \beta>0, \end{aligned} \tag{2.4}$$

where, $F(x;\lambda,\delta,\beta)$ represents the probability of the event $X \leq x$, and δ, λ and β are the scale, the location (minimum possible value of the random variable X), and the shape parameter, respectively. When X has the cumulative distribution function in (2.4) we write $X \sim W(\lambda,\delta,\beta)$ where the W refers to Weibull.

Its mean and variance are:

$$\begin{aligned} \mu &= \lambda+\delta\Gamma[1+1/\beta], \\ \sigma^2 &= \delta^2\left[\Gamma[1+2/\beta]-\Gamma^2[1+1/\beta]\right], \end{aligned} \tag{2.5}$$

and its percentiles:

$$x_p=\lambda+\delta\left[-\log(1-p)\right]^{1/\beta},\ \ 0\leq p\leq 1. \tag{2.6}$$

Two important properties of the Weibull family are:

1. It is stable with respect to location and scale transformations. More precisely:
$$X\sim W(\lambda,\delta,\beta)\Leftrightarrow\frac{X-a}{b}\sim W\left(\frac{\lambda-a}{b},\frac{\delta}{b},\beta\right), \tag{2.7}$$
where the new location and scale parameters are given in terms of the old parameters and the transformation constants a and b, and the shape parameter remains the same. This means that if a Weibull random variable is transformed by location and scale transformations, the resulting variable is also a Weibull random variable, but with different location and scale parameters.

2. It is stable with respect to minimum operations, that is, if the random variables $X_i; i=1,2,\ldots,m$ are independent and identically distributed, then
$$X_i\sim W(\lambda,\delta,\beta)\Rightarrow\min(X_1,X_2,\ldots,X_m)\sim W(\lambda,\delta m^{-1/\beta},\beta) \tag{2.8}$$

In other words, if the random variables in a set are identically distributed Weibull random variables and independent, its minimum is also a Weibull random variable.

Since the cdf $F_{min}(x)$ of the minimum of a set of independent and identically distributed random variables $X_1, X_2, \ldots, X_m$, with common cdf $F(x)$ is:

$$F_{min}(x) = 1 - [1 - F(x)]^m, \tag{2.9}$$

it follows that:

$$\begin{aligned} F_{min}(x) &= 1 - \left\{1 - \left(1 - \exp\left[-\left(\frac{x-\lambda}{\delta}\right)^\beta\right]\right)\right\}^m \\ &= 1 - \exp\left[-\left(\frac{x-\lambda}{\delta m^{-1/\beta}}\right)^\beta\right], \end{aligned} \tag{2.10}$$

which proves (2.8).

2.3.2 The minimal Gumbel model

The cumulative distribution function (cdf) of the minimal or reverse Gumbel family of distributions is given by:

$$\begin{aligned} F(x; \lambda, \delta) = 1 - \exp\left[-\exp\left(\frac{x-\lambda}{\delta}\right)\right] \\ x \in \mathbb{R};\ -\infty < \lambda < \infty,\ \delta > 0, \end{aligned} \tag{2.11}$$

where δ and λ are the scale and the location parameters, respectively. When X has the cumulative distribution function in (2.11) we write $X \sim G(\lambda, \delta)$ where the G refers to Gumbel.

Its mean and variance are:

$$\begin{aligned} \mu &= \lambda - 0.57772\delta, \\ \sigma^2 &= \pi^2\delta^2/6, \end{aligned} \tag{2.12}$$

and its percentiles:

$$x_p = \lambda + \delta\left[\log(-\log(1-p))\right], \quad 0 \le p \le 1. \tag{2.13}$$

Two important properties of the minimal Gumbel family are:

1. It is stable with respect to location and scale transformations. More precisely:

$$X \sim G(\lambda, \delta) \Leftrightarrow \frac{X-a}{b} \sim G\left(\frac{\lambda-a}{b}, \frac{\delta}{b}\right), \tag{2.14}$$

   where the new location and scale parameters are given in terms of the old parameters and the transformation constants a and b, and the shape parameter remains the same. This means that if a reverse Gumbel random variable is transformed by location and scale transformations, the resulting variable is also a reverse Gumbel random variable, but with different location and scale parameters.

2. It is stable with respect to minimum operations, that is, if the random variables $X_i; i = 1, 2, \ldots, n$ are independent and identically distributed, then

$$X_i \sim G(\lambda, \delta) \Rightarrow \min(X_1, X_2, \ldots, X_n) \sim G(\lambda - \delta \log n, \delta). \qquad (2.15)$$

In other words, if a set of independent and identically distributed random variables are reverse Gumbel, its minimum is also a reverse Gumbel random variable.

Since the cdf $F_{min}(x)$ of the minimum of a set of independent and identically distributed random variables $X_1, X_2, \ldots, X_n$, with common cdf $F(x)$ is:

$$F_{min}(x) = 1 - [1 - F(x)]^n, \qquad (2.16)$$

it follows that:

$$\begin{aligned} F_{min}(x) &= 1 - \left\{1 - \left(1 - \exp\left[-\exp\left(\frac{x-\lambda}{\delta}\right)\right]\right)\right\}^n \\ &= 1 - \exp\left[-n\exp\left(\frac{x-\lambda}{\delta}\right)\right] \qquad (2.17) \\ &= 1 - \exp\left[-\exp\left(\frac{x-(\lambda-\delta\log n)}{\delta}\right)\right], \end{aligned}$$

which proves (2.15).

2.4 A fatigue model for constant stress range and level

As indicated in the introduction, the first step in fatigue modeling consists in reproducing the fatigue behavior of materials for a given stress range and stress level, that is, for fixed σ_m and σ_M. We note that this model is needed because the fatigue phenomenon is random; otherwise, any specimen when subject to constant σ_m and σ_M would lead to the same fixed lifetime N. It is the random behavior of this lifetime that we want to model in this section.

2.4.1 Derivation of the model

Following the methodology proposed by Castillo et al. (1985), the selection of the Weibull model for the fatigue lifetime of specimens subject to alternating stresses ranging from constant σ_m to σ_M, is based on the following considerations:

1. **Weakest link principle**: This principle establishes that the fatigue lifetime of a longitudinal element is the minimum fatigue life of its constituting pieces. Thus, we are dealing with a minimum model. As shown in

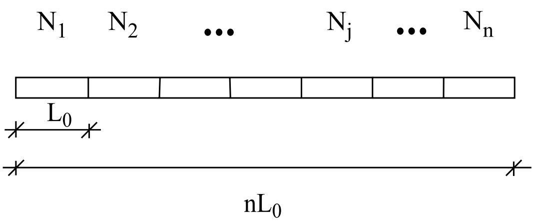


Figure 2.1: Pieces in which a longitudinal element can be supposedly subdivided, and corresponding lifetimes $N_1, N_2, \ldots, N_n$.

Fig. 2.1 the actual longitudinal element of length $L = n\ell$ can be supposedly subdivided in n pieces of length ℓ. Thus, taking into account (2.9), we have:

$$F_{min}(x) = F_{n\ell}(x) = 1 - [1 - F_\ell(x)]^n \,, \tag{2.18}$$

where $F_\ell(x)$ is the cumulative distribution function of the fatigue lifetime of an element of length ℓ.

2. **Stability**: The selected family of distributions for lifetime must hold for different lengths. If a parametric family of cdfs $F(x; \lambda(\ell), \delta(\ell), \beta(\ell))$ is used to represent the cdf for fatigue lifetime of a longitudinal element of length ℓ, then, according to (2.18), the cdf for an element of length $n\ell$ must be

$$F(x; \lambda(n\ell), \delta(n\ell), \beta(n\ell)) = 1 - [1 - F(x; \lambda(\ell), \delta(\ell), \beta(\ell))]^n \,. \tag{2.19}$$

This is a functional equation, where the unknowns are the functions $\lambda(\ell)$, $\delta(\ell)$, $\beta(\ell)$ and $F(x; \lambda, \delta, \beta)$, which only some families of distributions, such as the Weibull and Gumbel families, satisfy.

3. **Limit behavior**: To include the extreme case of the size of the supposed pieces constituting the element going to zero, or the number of pieces going to infinity, it is convenient for the distribution function family to be an asymptotic family. It is well known that in the case of independence, there are only three asymptotic distributions, namely, Weibull and Gumbel and Frechet (see Castillo (1988) and Castillo et al. (2005a)).

4. **Limited range**: Experience shows that the selected dimensionless variables N^* and $\Delta\sigma^*$ have a finite lower end, which must coincide with the theoretical end of the selected cdf. This implies that the Weibull distribution is the only one satisfying this requirement. If the variable were unlimited in the left tail, the Gumbel and the Frechet models would still be possible, but the Frechet model is ruled out because of physical reasons.

   Since $\Delta\sigma \geq \Delta\sigma_0$, we have for the dimensionless variable $\Delta\sigma/\Delta\sigma_0 \geq 1$, and then $\Delta\sigma^* = \log(\Delta\sigma/\Delta\sigma_0) \geq 0$. Thus, selection of either $g(x) = x$, or $g(x) = \log x$ leads to a limited variable in the lower tail, and then Weibull is the only adequate family, so that the relation (2.2) for constant $\Delta\sigma^*$ becomes (see (2.6))

$$N^* = \lambda^* + \delta^* \left[-\log(1-p)\right]^{1/\beta^*}, \quad \lambda^* \le N^*, \tag{2.20}$$

or since without loss of generality N^* can be replaced by N, because the role of the location parameter λ^* can be played by $\Delta\sigma_0$, we get the dimensional model

$$\boxed{\textbf{Weibull model :} \quad p = 1 - \exp\left[-\left(\frac{N-\lambda}{\delta}\right)^{\beta}\right]; \quad N \ge \lambda.} \tag{2.21}$$

where λ, δ and β are parameters of the corresponding Weibull law.

Since the reverse Gumbel distribution is the limit of a Weibull distribution, we can also consider the Gumbel model

$$\boxed{\textbf{Gumbel model :} \quad p = 1 - \exp\left[-\exp\left(\frac{N-\lambda}{\delta}\right)\right]; \quad N \in \mathbb{R}.} \tag{2.22}$$

2.4.2 Parameter estimation

Several methods have been proposed for estimating the parameters of the Weibull distribution (see references in Castillo and Hadi (1994)). Jenkinson (1969) uses the method of sextiles. The maximum likelihood method (ML) has been considered by Jenkinson (1969) and Prescott and Walden (1980, 1983). Smith (1985) considers the applicability of ML and discusses non-regular cases. The maximum likelihood estimates (MLE) require numerical solutions, and for some samples the likelihood may not have a local maximum. Furthermore, for $\beta < 1$, the likelihood can become infinite and hence the MLE does not exist. Hosking et al. (1985) suggest estimating the parameters and quantiles by the probability-weighted moments (PWM), introduced by Greenwood et al. (1979). They find that the PWM outperform the ML in many cases. Hosking et al. (1985), however, consider only cases where the shape parameter β lies in the range $\beta < 2$ because it has been observed in practice that β usually lies in this range. While the PWM performs quite admirably within the above restricted range of β, it presents problems outside this range.

The ML estimators

Given a sample $(N_1, N_2, \ldots, N_n)$, the maximum likelihood estimates of the parameters of the Weibull model (2.21) are obtained by maximizing the log-likelihood

$$L = -\sum_{i \in I_1 \cup I_0} \left(\frac{N_i - \lambda}{\delta}\right)^{\beta} + (\beta - 1)\sum_{i \in I_1} \log\left(\frac{N_i - \lambda}{\delta}\right) + \sum_{i \in I_1} \log\frac{\beta}{\delta} \tag{2.23}$$

with respect to λ, δ and β, where I_0 and I_1 are the set of run-outs and non-runots, respectively.

The log-likelihood for the Gumbel model (2.22) is

$$L = -\sum_{i \in I_1 \cup I_0} \exp\left(\frac{N_i - \lambda}{\delta}\right) + \sum_{i \in I_1} \log\left(\frac{N_i - \lambda}{\delta}\right) - \sum_{i \in I_1} \log \delta. \tag{2.24}$$

The PWM estimators

The PWM estimators for the Weibull model (2.21) are given by

$$\hat{\beta}_{PWM} = (7.859c + 2.9554c^2)^{-1}, \tag{2.25}$$

$$\hat{\delta}_{PWM} = \frac{(\bar{v} - 2b_1)}{\Gamma(1 + 1/\hat{\beta}_{PWM})(1 - 2^{-1/\hat{\beta}_{PWM}})}, \tag{2.26}$$

$$\hat{\lambda}_{PWM} = \bar{v} - \hat{\delta}_{PWM}\Gamma(1 + 1/\hat{\beta}_{PWM}), \tag{2.27}$$

where $\bar{v}$ is the sample mean,

$$c = \frac{2b_1 - \bar{v}}{3b_2 - \bar{v}} - \frac{log\ 2}{log\ 3},$$

and

$$b_j = n^{-1} \sum_{i=1}^{n} \frac{(i-1)(i-2)\dots(i-j)}{(n-1)(n-2)\dots(n-j)}\ v_{n-i+1:n}, \quad j = 1, 2.$$

The Castillo-Hadi estimators

Castillo and Hadi (1994) proposed a method based on a two-stage procedure for estimating the parameters and quantiles of the Weibull distribution. First, a set of initial estimates are obtained by equating the cdf evaluated at the observed order statistics to their corresponding percentile values (first stage). Next, these estimates are combined to obtain a statistically more efficient parameter estimate (second stage).

The First Stage: Initial Estimates Let $v_{i:n} \leq v_{2:n} \leq \dots \leq v_{n:n}$ be the order statistics obtained from a random sample from $W(\lambda, \delta, \beta)$. Let $I = \{i, j, r\}$ be a set of three distinct indices, where $i < j < r \in \{1, 2, \dots, n\}$. Then, using (2.6), we write

$$\begin{aligned} v_{i:n} &\cong \lambda + \delta \left[-\log(1 - p_{i:n})\right]^{1/\beta}, \\ v_{j:n} &\cong \lambda + \delta \left[-\log(1 - p_{j:n})\right]^{1/\beta}, \\ v_{r:n} &\cong \lambda + \delta \left[-\log(1 - p_{r:n})\right]^{1/\beta}, \end{aligned} \tag{2.28}$$

where

$$p_{s:n} = \frac{s - 0.35}{n} \tag{2.29}$$

are suitable plotting positions. The system in (2.28) is a set of three independent equations in three unknowns, λ, δ, and β. Estimates of λ, δ, and β can then be obtained by solving (2.28) for λ, δ, and β.

The solution of (2.28) can be obtained by the elimination method as follows. Eliminating λ and δ, we obtain

$$D_{ijr} = \frac{v_{j:n} - v_{r:n}}{v_{i:n} - v_{r:n}} = \frac{C_r^k - C_j^k}{C_r^k - C_i^k} = \frac{1 - A_{jr}^k}{1 - A_{ir}^k}, \tag{2.30}$$

where $k = 1/\beta$, $C_i = -log(1 - p_{i:n})$ and $A_{ir} = C_i/C_r$. An initial estimate $\hat{k}_{ijr}$ of k, which depends on $v_{i:n}, v_{j:n}$, and $v_{r:n}$, is obtained by solving (2.30). Equation (2.30) involves only one variable, hence it can be easily solved using the bisection method.

To this end, Castillo and Hadi (1994) show that:

1. If $D_{ijr} < \log(A_{jr})/\log(A_{ir})$, then $\hat{k}_{ijr}$ lies in the interval $(0, \frac{\log D_{ijr}}{\log A_{jr}})$.

2. If $D_{ijr} > \log(A_{jr})/\log(A_{ir})$, then $\hat{k}_{ijr}$ lies in the interval $(\frac{\log(1-D_{ijr})}{\log A_{jr}}, 0)$.

Once $\hat{k}_{ijr}$ is obtained, $\hat{\beta}_{ijr}$, $\hat{\lambda}_{ijr}$, and $\hat{\delta}_{ijr}$ are obtained in a closed form as:

$$\hat{\beta}_{ijr} = 1/\hat{k}_{ijr}, \tag{2.31}$$

$$\hat{\delta}_{ijr} = \frac{v_{i:n} - v_{r:n}}{C_i^{k_{ijr}} - C_r^{k_{ijr}}}, \tag{2.32}$$

$$\hat{\lambda}_{ijr} = v_{i:n} - \hat{\delta}_{ijr} C_{ijr}^{k_{ijr}}. \tag{2.33}$$

The Second Stage: Final Estimates The above initial estimates are based on only three order statistics. More statistically efficient estimates are obtained using other order statistics as follows.

1. Let $i = 1$ and $r = n$ and compute $\hat{\beta}_{1jn}, \hat{\lambda}_{1jn}$ and $\hat{\delta}_{1jn}, j = 2, 3, \ldots, n-1$.

2. Apply the robust median function to each of the above sets of estimates to obtain the corresponding overall estimates:

$$\begin{aligned} \hat{\beta}_{MED} &= median(\hat{\beta}_{1,2,n}, \hat{\beta}_{1,3,n}, \ldots, \hat{\beta}_{1,n-1,n}), \\ \hat{\lambda}_{MED} &= median(\hat{\lambda}_{1,2,n}, \hat{\lambda}_{1,3,n}, \ldots, \hat{\lambda}_{1,n-1,n}), \\ \hat{\delta}_{MED} &= median(\hat{\delta}_{1,2,n}, \hat{\delta}_{1,3,n}, \ldots, \hat{\delta}_{1,n-1,n}), \end{aligned} \tag{2.34}$$

   where $median(y_1, y_2, \ldots, y_n)$ is the median of $\{y_1, y_2, \ldots, y_n\}$.

The reason for setting $i = 1$ and $r = n$ in Step 1 is that the range of the random variable in this case depends on the parameters. We, therefore, have to ensure that $v_{1:n} > \lambda$. In this way we force parameter estimates to be consistent with the observed data.

The quantile estimates for any desired p are then obtained by substituting the above parameter estimates in (2.6).

Note that since the parameter and quantile estimates are well defined for all possible combinations of parameter and sample values, the variances of these estimates (hence, confidence intervals for the corresponding parameter or quantile values) can be obtained using sampling based methods such as the jackknife and the bootstrap methods (Efron (1979) and Diaconis and Efron (1983)).

2.4.3 Alternative methods for dealing with run-outs

Experimental programs in fatigue usually involve the presence of censored data, i.e., tests interrupted before failure of the specimen occurs, due to accidental causes or because the limit number of cycles has been reached. This type of data is called censored data or run-outs. In such cases it is possible to resort to specific statistical parameter estimation techniques, such as for instance the E-M algorithm, based on an iterative process to deal with these censored data in the statistical analysis.

Let t_0 be the limit number of cycles. Since the Weibull distribution for $N \geq t_0$ is:

$$F(N|N \geq t_0) = 1 - \exp\left[-\left(\frac{t_0-\lambda}{\delta}\right)^{\beta} - \left(\frac{N-\lambda}{\delta}\right)^{\beta}\right], \quad N|N \geq t_0, \tag{2.35}$$

and the expected value of the rth order statistic of a sample of size q from an uniform distribution $U(0,1)$ is $r/(q+1)$, the censored result t_0 can be replaced by the N solution, obtained from:

$$1 - \exp\left[-\left(\frac{t_0-\lambda}{\delta}\right)^{\beta} - \left(\frac{N-\lambda}{\delta}\right)^{\beta}\right] = \frac{r}{q+1};\ r = 1, 2, \ldots, q_i, \tag{2.36}$$

where q is the number of run-outs coinciding at the same t_0. Thus:

$$N = \lambda + \delta\left[\left(\frac{t_0-\lambda}{\delta}\right)^{\beta} - \log\left(1 - \frac{r}{q+1}\right)\right]; \quad r = 1, 2, \ldots, q. \tag{2.37}$$

In summary, this technique consists in:

1. Estimating the model parameters considering only the results associated with failures.

2. Assigning to the censored results their expected failure values, based on the estimated model parameters using (2.37).

3. Re-estimating the model parameters but considering the data associated with real failures plus the expected ones associated with the run-outs.

4. Repeating Steps 2 and 3 until convergence of the process takes place.

2.5 A fatigue model for varying stress range and given stress level

If some specimens are tested to fatigue failure with loading cycles ranging from σ_m to σ_M for three different constant values of σ_m and a given fixed value $\sigma_M = \sigma_{M_1}$, we obtain the data indicated by crosses in Fig. 2.2(a). If we repeat the same experiments for a different constant value $\sigma_M = \sigma_{M_2}$, we get the data indicated by circles in the same figure. The data suggest a family of percentiles of the form indicated in Fig. 2.2(a). Apart from showing that the results of the experiments (lifetimes N) with constant σ_M are random, this indicates that they are also dependent on the stress level σ_M, and finally that the trends of the percentile curves for different σ_M values are similar in shape.

If, alternatively, instead of fixing σ_M, σ_m is fixed, and the experiments for three stress range $\Delta\sigma$ levels are performed, the results in Fig. 2.2(b) are obtained, from which similar conclusions can be drawn, that is, the fatigue lifetime depends on the stress level selected (in this case σ_m). Again, the percentile lines appear with similar trends as the previous ones, but with a different inclination. This suggests that the same parametric family of percentile curves could be used to represent all these cases, but with different parameter values, which obviously will depend on the constant values of σ_M or σ_m, respectively.

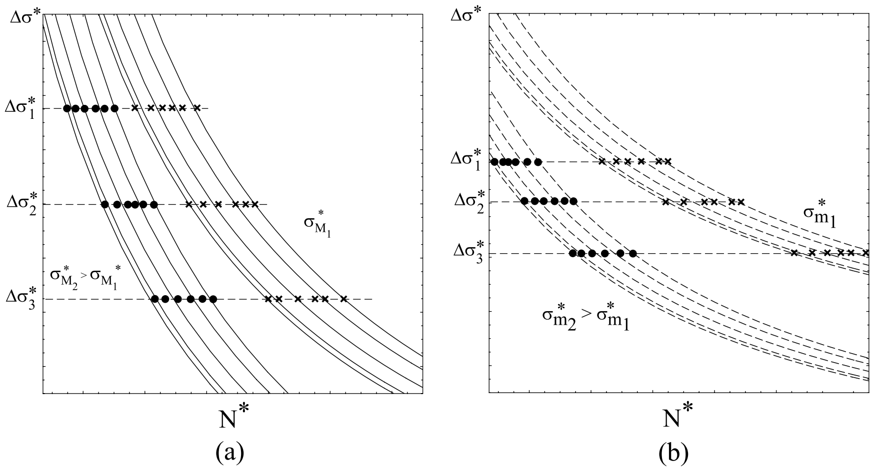


Figure 2.2: (**a**) Set of experiments run for two constant values of σ_M, and (**b**) set of experiments run for two constant values of σ_m.

2.5.1 Derivation of the model

In the following paragraphs the functional form of the model that gives the fatigue lifetime in terms of the stress range $\Delta\sigma^*$ for constant σ^*_M or σ^*_m is derived, i.e., the stress range $\Delta\sigma^*$ is incorporated into the model (2.21). Following the methodology proposed in this book (see Castillo et al. (1985); Castillo and Fernández-Canteli (1986)), this upgrade is done based on the following compatibility condition:

Compatibility condition: In the Wöhler field, the cumulative distribution function $E(N^*|\Delta\sigma^*)$ of lifetime N^* given stress range $\Delta\sigma^*$ should be compatible with the cumulative distribution function $F(\Delta\sigma^*|N^*)$ of stress range $\Delta\sigma^*$ given lifetime N^*.

The compatibility condition, which was illustrated in Fig. 1.3, can be written as the following functional equation:[2]

$$Q^*(N^*, \Delta\sigma^*) = E^*(N^*|\Delta\sigma^*) = F^*(\Delta\sigma^*|N^*) = q_{min}(N^*, \Delta\sigma^*), \tag{2.38}$$

where $q_{min}()$ is the cdf of a minimum law (weakest link principle). As has been indicated in Sect. 2.4, if the cdf belongs to a location and scale family of distributions, this functional equation can be written as

$$q_{min}\left(\frac{N^* - \mu_1^*(\Delta\sigma^*)}{\sigma_1^*(\Delta\sigma^*)}\right) = q_{min}\left(\frac{\Delta\sigma^* - \mu_2^*(N^*)}{\sigma_2^*(N^*)}\right), \tag{2.39}$$

where $\mu_1^*(\Delta\sigma^*), \sigma_1^*(\Delta\sigma^*)$ and $\mu_2^*(N^*), \sigma_2^*(N^*)$ are the location and scale parameters of N^* given $\Delta\sigma^*$ and $\Delta\sigma^*$ given N^*, respectively, leading to the functional equation:

$$\frac{N^* - \mu_1^*(\Delta\sigma^*)}{\sigma_1^*(\Delta\sigma^*)} = \frac{\Delta\sigma^* - \mu_2^*(N^*)}{\sigma_2^*(N^*)}. \tag{2.40}$$

Two general solutions of functional equation (2.40) are possible (see Castillo and Galambos (1987)), leading to the two fatigue models:

$$\textbf{MODEL I:} \quad Q^*(N^*, \Delta\sigma^*) = q_{min}\left[\frac{(N^* - B^*)(\Delta\sigma^* - C^*) - \lambda^*}{\delta^*}\right], \tag{2.41}$$

which together with the Weibull distribution leads to the model

$$Q^*(N^*, \Delta\sigma^*) = 1 - \exp\left\{-\left[\frac{(N^* - B^*)(\Delta\sigma^* - C^*) - \lambda^*}{\delta^*}\right]^{\beta^*}\right\}; \quad (N^* - B^*)(\Delta\sigma^* - C^*) \geq \lambda^*, \tag{2.42}$$

where $B^*, C^*, \lambda^*, \delta^*$ and β^* are the dimensionless model parameters, the physical meanings of which (see Fig. 2.3) are the following:

B^*: threshold value of lifetime N^*.

C^*: endurance limit for $\Delta\sigma$.

λ^*: Weibull location parameter (position of the corresponding zero-percentile hyperbola).

[2] This is the compatibility equation (1.8), where for the sake of simplicity we have removed σ_ℓ^* and L^*.

δ^*: Weibull scale parameter.

β^*: Weibull shape parameter of the whole cdf in the Wöhler field.

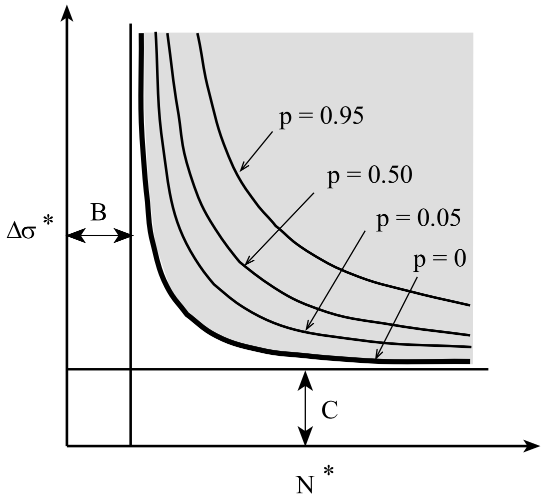


Figure 2.3: Percentile curves representing the relationship between lifetime, N^*, and stress range, $\Delta\sigma^*$, in the Wöhler field for Model I.

Since the normalizing variables N_0 and $\Delta\sigma_0$ can be merged in B and C,[3] model (2.42) can be written as the dimensional log-Weibull model

$$\boxed{Q(N, \Delta\sigma) = 1 - \exp\left\{-\left[\frac{(\log N - B)(g(\Delta\sigma) - C) - \lambda}{\delta}\right]^{\beta}\right\}; \quad (\log N - B)(g(\Delta\sigma) - C) \geq \lambda,} \tag{2.43}$$

where B, C, λ, δ and β are the corresponding dimensional parameters.[4] The resulting percentile curves are given by:

$$(\log N - B)(g(\Delta\sigma) - C) = \text{ constant.} \tag{2.44}$$

The zero-percentile curve represents the minimum possible number of cycles to fatigue failure for different values of $\Delta\sigma$, and happens to be a hyperbola (thick line in Fig. 2.3). For such a curve, the minimum number of cycles to fatigue failure decreases with increasing $\Delta\sigma$, in agreement with experimental results. This percentile can be interpreted as the end of the crack initiation phase and the start of the crack propagation phase.

[3]Then $B = \log N_0 + B^*$, $C = C^*$ and $\beta = \beta^*$.

[4]The dimensions of λ and δ depend on $g()$.

If the reverse Gumbel distribution is used, we get the log-Gumbel model

$$\boxed{Q(N,\Delta\sigma) = 1 - \exp\left[-\exp\left(\frac{(\log N - B)(g(\Delta\sigma) - C) - \lambda}{\delta}\right)\right],} \tag{2.45}$$

$$\textbf{MODEL II:} \quad q_{min}(N^*,\Delta\sigma^*) = q_{min}\left(\frac{(N^*)^{\beta}(\Delta\sigma^* - C^*)^{\gamma}}{\delta}\right), \tag{2.46}$$

which together with the Weibull distribution leads to the dimensional model

$$\boxed{\begin{aligned} Q(N,\Delta\sigma) = 1 - \exp\left[-\frac{(\log N - B)^{\beta}(g(\Delta\sigma) - C)^{\gamma}}{\delta}\right], \\ (\log N - B)(g(\Delta\sigma) - C) \geq 0, \end{aligned}} \tag{2.47}$$

where δ, β and γ are the dimensionless model parameters.

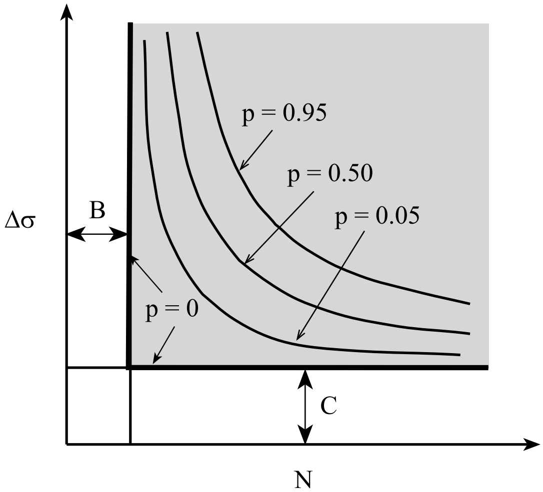


Figure 2.4: Percentiles curves representing the relationship between lifetime, N, and stress range, $\Delta\sigma$, in the S-N field for Model II.

The physical meaning of these parameters (see Fig. 2.4) is the following:

γ: shape parameter associated with $\Delta\sigma$.

δ: scale parameter.

β: shape parameter associated with N.

The corresponding percentile curves are given by:

$$(\log N - B)^{\beta}(g(\Delta\sigma) - C)^{\gamma} = \text{ constant.} \tag{2.48}$$

In contrast to Model I, in this model the zero-percentile curve degenerates to the two asymptotes (thick lines in Fig. 2.4). This means that the minimum number of cycles to fatigue failure remains constant with increasing $\Delta\sigma$, that is, a sufficiently large initial crack size is always possible for failure occurrence. Nevertheless, a very small percentile can play the role of the zero percentile if needed. So, from a practical point of view this model can be useful, but from a physical point of view model (2.42) is more justified.

As a limiting case of model (2.47) we obtain the Gumbel model

$$\boxed{\begin{array}{r} Q(N, \Delta\sigma) = 1 - \exp\left[-\exp\left(\dfrac{(\log N - B)^{\beta}(g(\Delta\sigma) - C)^{\gamma}}{\delta}\right)\right], \\ \log N \geq B, \quad (g(\Delta\sigma) - C) \geq 0, \end{array}} \tag{2.49}$$

The model (2.42) has been subsequently studied and successfully applied to different cases of lifetime problems, such as, for instance, prestressing wires and strands of different lengths, plain concrete, etc. (see Castillo et al. (1985)).

Expression (2.42) implies that:

$$V = (\log N - B)(g(\Delta\sigma) - C) \sim W(\lambda, \delta, \beta). \tag{2.50}$$

Models (2.43) and (2.47) reveal that the probability of failure of a piece subject to a stress range $\Delta\sigma$ during N cycles, depends only on the product $V = (\log N - B)(g(\Delta\sigma) - C)$ or $V = (\log N - B)^{\beta}(g(\Delta\sigma) - C)^{\gamma}$, showing that V is useful to compare fatigue strength at different, but constant, stress levels, and can be considered as a normalizing variable.

As a summary of the whole process undertaken to derive the proposed S-N field Weibull model, we include Fig. 2.5, where all the properties, on which it is based, are given. Note how the same properties hold for the random variables $N^*|\Delta\sigma^*$, that is, N^* for given $\Delta\sigma^*$, and $\Delta\sigma^*|N^*$, i.e. $\Delta\sigma^*$ for given N^*, and the important role played by the compatibility condition (2.38).

Note that the compatibility condition alone determines the hyperbolic nature of the S-N field, irrespective of the distributional assumptions, and that the physical, stability, limit and limited range conditions lead to the Weibull or reverse Gumbel distributions.

A point of interest is that the basic hypotheses of this model can also be established on the basis of microstructural properties of the material (see Bolotin (1998)).

2.5.2 Some weaknesses of the proposed model

Some of the above five properties that led to the proposed model can be questioned. In fact the following are pertinent comments:

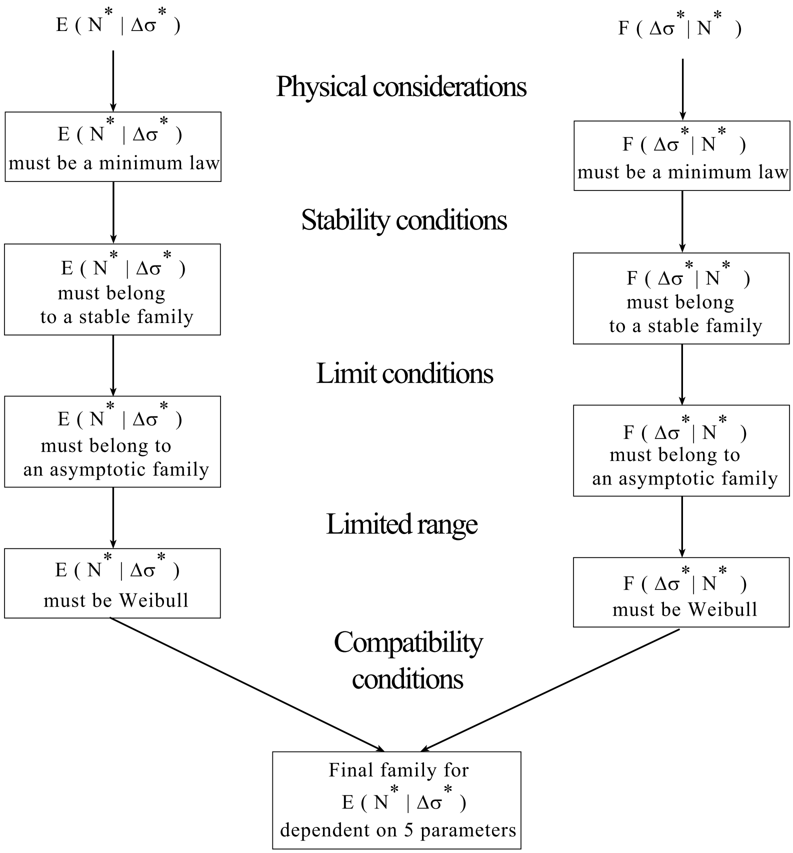


Figure 2.5: Illustration of the selection procedure for the cumulative distribution function of the lifetime.

1. The weakest-link assumption, stated in the form of Eq. (2.18), implies that the fatigue lives of its different pieces are assumed to be independent. This can hold approximately for long pieces, but can be not true for small sizes.

2. Though the limit behavior is a convenient property for the fatigue model, it is not necessary. Furthermore, weak convergence in increasing L is not concerned with preserving lower tail behavior in terms of relative error. So, other models, different from the one proposed here, can be obtained, as a consequence of dropping this assumption.

   Some researchers have found no "a priori"' reason to rule out model forms satisfying
   $$f(N;\Delta\sigma) = 1 - \exp\{-LG(N;\Delta\sigma)\},$$
   where G is an appropriate increasing function of lifetime N and stress $\Delta\sigma$, and L is length or volume of material. There are perfectly justifiable

models, from the point of view of random defects, micromechanical stress redistribution and catastrophic crack growth that are not Weibull in form and do not lead to Weibull distributions. A discussion of possible forms for G can be found in a classic paper by Coleman (1958b).

3. The limited range assumption implies that a fatigue limit exists. Since there is no universal agreement on the concept of a fatigue limit for steels and much less so for other materials such as aluminum, B and C can be viewed as scale constants with these ratios greater than or equal to zero (as for instance might occur when using two parameter Weibull distributions with $\lambda = 0$). In these cases, other alternatives are also possible.

4. Due to plasticity effects, a change in curvature occurs in the upper part of the S-N curves (low-cycle region). Thus, validity of the model is limited to a certain region.

In summary, the proposed model is a convenient and practical model that is actually the only model satisfying the five assumptions, but if some of these assumptions are relaxed, other models are possible.

2.5.3 Parameter estimation

The parameter estimation of model (2.43) can be divided in two steps: estimation of the threshold parameters B and C, and estimation of the Weibull λ, δ and β, or Gumbel λ, δ parameters.[5]

Estimation of the threshold values B and C for the Weibull and Gumbel models

The regression curve of N on $\Delta\sigma$ is given by the mean value of N as a function of $\Delta\sigma$. Since, as indicated in (2.5), the mean of a Weibull $W(\lambda, \delta, \beta)$ distribution is $\mu = \lambda + \delta\Gamma(1 + 1/\beta)$, and the mean of a reverse Gumbel $G(\lambda, \delta)$ distribution is $\mu = \lambda - 0.57772\delta$, from (2.42) and (2.45) we have:

$$E[\log N - B | g(\Delta\sigma) - C] = \frac{\mu}{g(\Delta\sigma) - C}, \tag{2.51}$$

which is equivalent to:

$$E[\log N | g(\Delta\sigma) - C] = B + \frac{\mu}{g(\Delta\sigma) - C}. \tag{2.52}$$

The regression equation (2.52) suggests estimating B and C by minimizing, with respect to B, C and μ:

$$Q = \sum_{i=1}^{n} \left(\log N_i - B - \frac{\mu}{g(\Delta\sigma_i) - C} \right)^2, \tag{2.53}$$

[5] We warn the reader that the parameter estimation of this model can lead to practical problems, especially when a reduced number of data is available, or outliers are present in the data.

where n is the sample size and N_i is the number of cycles to failure of the i-th specimen tested at stress range $\Delta\sigma_i$.

Initial estimates

Since the function in (2.52) is non-linear, it is convenient to obtain some initial estimates to avoid convergence problems. One possibility consists of using three different stress ranges (the first three for example) $(\Delta\sigma_i; i = 1, 2, 3)$, obtaining the corresponding means μ_i of $\log N$ and choosing the values of B, C, and μ such that the regression curve (2.52) passes through those mean points, as the initial estimates, i.e., solving in B, C, and μ the system of equations

$$\mu_i = \frac{1}{n}\sum_{i=1}^{n} \log N_i = B + \frac{\mu}{g(\Delta\sigma_i) - C}; \quad i = 1, 2, 3. \tag{2.54}$$

The Weibull and Gumbel parameter estimates

Once B and C have been estimated, all the data points can be pooled together by calculating the values of

$$V_i = (\log N_i - B)(g(\Delta\sigma_i) - C), \tag{2.55}$$

to estimate δ, λ and β of a Weibull distribution $W(\lambda, \delta, \beta)$, or δ and λ of a Gumbel distribution. To this end we use the methods explained in Sect. 2.4.2.

2.5.4 Use of the model in practice

In this section the use of the model for practical applications in fatigue design is described, and in particular how the parameters of the model can be estimated and how it can be used to predict lifetimes under other testing conditions when the parameters are known.

To this end, we can use the log-Weibull model (2.43) or the log-Gumbel model (2.45).

Next, we proceed as follows:

Step 1: *Design the testing strategy and obtain data.* The stress level, say σ_M, for which the model is to be fitted is selected. Next, the set of $\Delta\sigma$ or σ_m values for the tests is chosen, leading to the test stress pairs:

$$\{(\sigma_{m_i}, \sigma_{M_i}) \mid i = 1, 2, \ldots, n\},$$

which must cover the desired region, where the regression equation is to be used.[6]

Step 2: *Estimate the B and C values.* Minimize (2.53) to estimate B and C.

[6] Though, in accordance with the international system the data will normally be given in MPa, the user is free to use any units. However, the resulting estimate of C in step 2 will be consequently given in the same units.

Step 3: *Obtain the values of V.* Using Eq. (2.55) calculate the dimensionless values of V, that is

$$V_i = (\log N_i - B)(g(\Delta\sigma_i) - C). \tag{2.56}$$

Step 4: *Estimate the model parameters.* Use one of the estimation methods discussed in Sect. 2.5.3 to estimate the parameters λ, δ and β (this parameter only for the Weibull model) in (2.43) or (2.45).

Step 5: *Obtain the model expressions.* Replace the parameters values in models (2.43) or (2.45) to get the model expressions.

Step 6: *Extrapolate to other testing conditions.* Use the model (2.43) or (2.45) to get the percentiles, mean or variance associated with any other testing condition, for example $\Delta\sigma$.

2.5.5 Example of application

In this section we present one example of application to illustrate the methods proposed in previous sections. We note that, thanks to the generality of the imposed compatibility, physical and statistical conditions, the same general model is applicable to completely different materials, with a unique dominant crack or with generalized microcracking failure.

The Holmen data

This example is based on the Holmen (1979) data. To illustrate the process indicated in Sect. 2.5.4, we choose to represent $\Delta\sigma$ on a logarithmic scale ($g(x) = \log x$), and we follow all the steps indicated there.

Step 1: *Design the testing strategy and obtain data.* In this case we do not need to design a testing strategy, because the data are given. These data consist of 75 fatigue tests, at 5 different stress levels as shown in Table 2.4.

Thus, the stress pairs $\{(\sigma_{m_i}, \sigma_{M_i}) \mid i = 1, 2, \ldots, n\}$ can be immediately obtained from this table.

Step 2: *Estimate the B and C values.* To estimate B and C, we have to minimize (2.53). However, we first obtain the initial estimates using the method given in Sect. 2.5.3, taking as selected stress ranges $\Delta\sigma = 0.95, 0.90$ and 0.825, which leads to the system of equations

$$\begin{aligned}
\tfrac{1}{n_1}\sum_{j=1}^{n_1} \log N_{1j} &= -2.18364 = B + \frac{\mu}{\log\Delta\sigma_1 - C};\\
\tfrac{1}{n_2}\sum_{j=1}^{n_2} \log N_{2j} &= -0.99939 = B + \frac{\mu}{\log\Delta\sigma_2 - C};\\
\tfrac{1}{n_3}\sum_{j=1}^{n_3} \log N_{3j} &= 1.01688 = B + \frac{\mu}{\log\Delta\sigma_3 - C},
\end{aligned} \tag{2.57}$$

Table 2.4: The Holmen data.

$\Delta\sigma_i$	Lifetime N_{ij} (thousands of cycles)							
0.95	0.257	0.217	0.206	0.203	0.143	0.123	0.120	0.109
	0.105	0.085	0.083	0.076	0.074	0.072	0.037	
0.90	1.129	0.680	0.540	0.509	0.457	0.451	0.356	0.342
	0.311	0.295	0.257	0.252	0.226	0.216	0.201	
0.825	5.598	5.560	4.820	4.110	3.847	3.590	3.330	2.903
	2.590	2.410	2.400	1.492	1.460	1.258	1.246	
0.75	67.340	50.090	48.420	36.350	27.940	26.260	24.900	20.300
	18.620	17.280	16.190	15.580	12.600	9.930	6.710	
0.675	11748	11748	3295	1459	1400	1330	1250	1242
	896	659	486	367	340	280	103	

where $n_i = 15$; $i = 1, 2, 3$, and N_{ij} is the lifetime of sample j tested at stress range $\Delta\sigma_i$.

The solution of system (2.57) is:

$$B = -57.4174; \quad \mu = 142.27; \quad C = -2.62708.$$

Now using these estimates we minimize (2.52) and obtain the final estimates:

$$B = -20.7843; \quad \mu = 19.731; \quad C = -1.10607. \tag{2.58}$$

Step 3: *Obtain the dimensionless values of V.* Once the threshold values have been calculated, using Eq. (2.55) we can pool the sample together by calculating the values of $V_i = (\log N_i - B)(g(\Delta\sigma_i) - C)$ for all sample data points, thus, getting a sample of size 75, which is shown in Fig. 2.6 on a Weibull probability plot. The linear trend of the cumulative distribution function guarantees that the Weibull law assumption is reasonable.

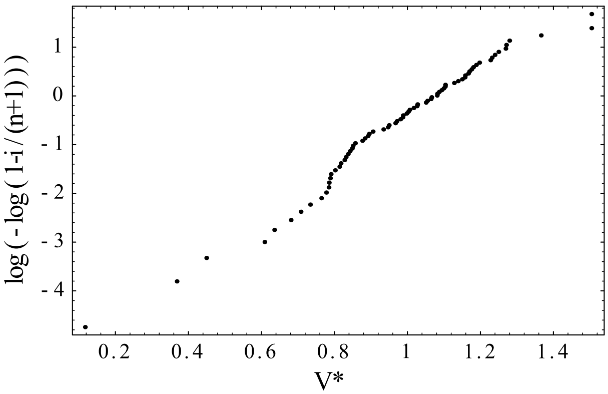


Figure 2.6: Pooled sample of v_i values on a Weibull probability paper.

Step 4: *Estimate the model parameters.* Now, using the PWM estimates (see Sect. 2.4.2) of the Weibull parameters we get:

$$\beta = 2.70123; \quad \delta = 1.68844; \quad \lambda = 18.2305,$$

and using the Castillo-Hadi estimates (see Sect. 2.4.2) we finally obtain:

$$\beta = 3.40031; \quad \delta = 2.42772; \quad \lambda = 17.5225. \tag{2.59}$$

Step 5: *Obtain the model expressions.* We replace the parameter values in model (2.45) to get the model:

$$\boxed{\begin{array}{r} Q(N,\Delta\sigma) = 1 - \exp\left\{ - \left[\dfrac{(\log N + 20.78)(\log \Delta\sigma + 1.106) - 17.52}{2.428} \right]^{3.4} \right\}; \\ (N + 20.7843)(\Delta\sigma + 1.11) \geq 17.52, \end{array}} \tag{2.60}$$

Step 6: *Extrapolate to other testing conditions.* Use the model (2.60) to get the percentiles, mean or variance associated with any other testing condition, for example (σ_m, σ_M). For instance, Fig. 2.7 shows the data and the S-N field in terms of $N^* = \log N - B$ and $\Delta\sigma^* = \log \Delta\sigma - C$, for these last estimates.

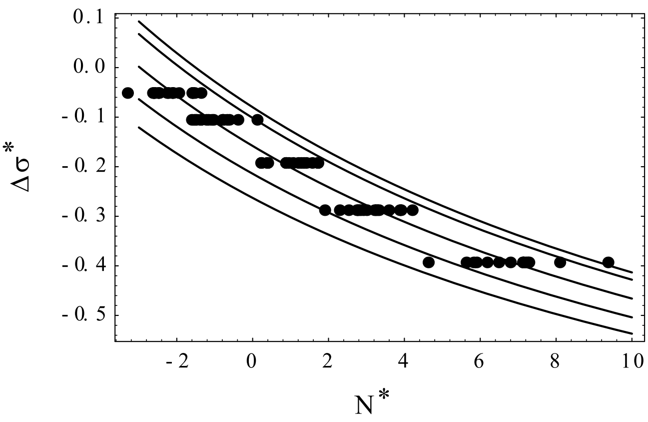


Figure 2.7: Data and fitted S-N field in terms of N^* and $\Delta\sigma^*$.

2.6 A fatigue model for any varying stress range and level

In Sect. 2.5 we have derived, based on functional equations, the mathematical structure of the S-N field, for constant σ_M^* or constant σ_m^*, and we have shown that based on physical and statistical considerations together with compatibility conditions we were able to derive a particular and very concrete parametric model, without using arbitrary assumptions on the functional form of the percentiles.

In this section, we show that the parameters of the models for constant σ^*_M and constant σ^*_m are not independent, but related. Furthermore, based on some compatibility conditions, we will derive a common model which shows the real degrees of freedom of their parameters.

Consider four different series of tests corresponding to two constant $\sigma^*_{M_1}$ and $\sigma^*_{M_2}$ values of σ^*_M, and two constant $\sigma^*_{m_1}$ and $\sigma^*_{m_2}$ values of σ^*_m. In Fig. 2.8 we illustrate the percentile curves associated with these four series of tests and show one important compatibility condition: the four families of curves must intersect, two by two, as horizontal lines (see figure). This is true because to each common intersection of $\Delta\sigma^*$ we have a test that can be seen as a test pertaining to the series with constant σ^*_M or to a series with constant σ^*_m. More precisely, the horizontal intersections correspond to the test stress pairs $(\sigma^*_{m_1}, \sigma^*_{M_1})$, $(\sigma^*_{m_1}, \sigma^*_{M_2})$, $(\sigma^*_{m_2}, \sigma^*_{M_1})$ and $(\sigma^*_{m_2}, \sigma^*_{M_2})$. As we shall see, this compatibility condition is going to play a very relevant role in deriving the general model for fatigue lifetime subject to arbitrary stress levels and ranges.

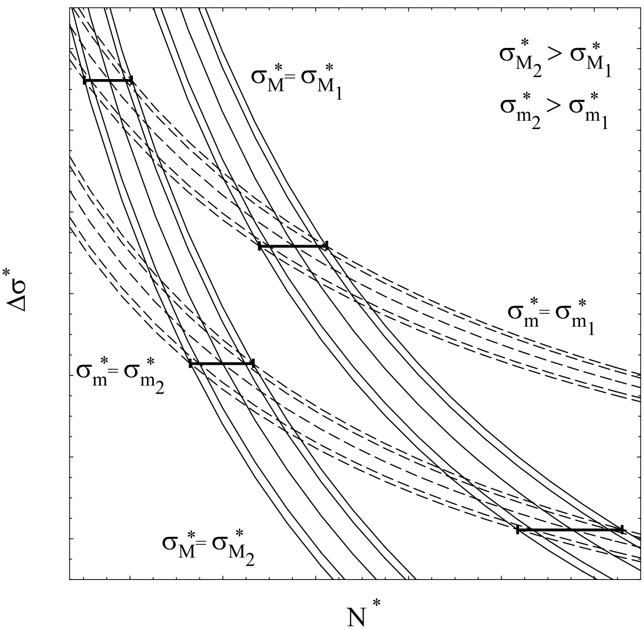


Figure 2.8: Schematic S-N curves for percentiles $\{0.01, 0.05, 0.5, 0.95, 0.99\}$ for constant $\sigma^*_{M_1}$ and $\sigma^*_{M_2}$, and constant $\sigma^*_{m_1}$ and $\sigma^*_{m_2}$, illustrating the compatibility condition. Dashed lines refer to S-N curves for constant σ^*_m, and continuous lines refer to S-N curves for constant σ^*_M.

In this section, a Weibull regression model for statistical analysis of stress life data for any possible loading situations in tension and compression is developed, thus facilitating its application to real loading spectra. The model enforces the compatibility condition of the S-N fields associated with constant σ_m and constant σ_M, which leads to a system of functional equations, the solution of which provides a model with the desired requirements.

The model depends on 9 parameters that can be estimated by maximum likelihood or non-linear regression methods, and supplies all the material basic probabilistic fatigue information to be used in a damage accumulation assessment for fatigue life prediction using practical load spectra.

The main achievements of the model proposed in this section are:

- Only dimensionless variables are used in the model and the corresponding regression equation. This implies, on one hand, fewer variables involved in the problem, i.e., a simpler but not less powerful model and, on the other hand, that the parameters or constants resulting in the model are dimensionless too, that is, their values are independent of the units being used.

- The model is not based on arbitrary considerations, but on sound physical and statistical properties exigeable to any fatigue model. Thus the model is the only one resulting from the selected constraints and not the direct consequence of an gratitious assumption.

- The model provides useful statistical information including not only mean values but also variability of the model, and permits probabilities to be calculated.

Consider as before a fatigue test conducted at constant σ_m^* and σ_M^*. Since one is interested in determining the probability of failure p of a randomly chosen specimen when subject to such a test, in this section a formula for p in terms of N^*, σ_m^* and σ_M^* is derived, using as few arbitrary assumptions as possible.

To this end, we use the Buckingham Π theorem, some knowledge from fatigue and extreme value theory and some compatibility assumptions.

From fatigue knowledge we conclude that our problem depends on the following set of six variables:

$$\{p, \sigma_m, \sigma_M, \Delta\sigma_0, N, N_0\}.$$

If we assume that there is a relationship between these variables

$$r(p, \sigma_m, \sigma_M, \Delta\sigma_0, N, N_0) = 0, \tag{2.61}$$

using the Buckingham Π theorem, we can select the 4 dimensionless variables $\sigma_m^* = \sigma_m/\Delta\sigma_0$, $\sigma_M^* = \sigma_M/\Delta\sigma_0$[7] and $N^* = \log(N/N_0)$ and p, already dimensionless, and then the relationship

$$g^*(p, N^*, \sigma_m^*, \sigma_M^*) = 0 \tag{2.62}$$

[7] A natural scale instead of a logarithmic scale (see the function $g(\cdot)$ in (2.3)) for the stress amplitude occurs as a natural requirement from the model. Otherwise, inconsistencies appear in the solution of the functional equation.

or

$$p = h^*(N^*, \sigma_m^*, \sigma_M^*) \tag{2.63}$$

is equivalent to (2.61). So, one of our aims in this section is to obtain the function $h^*(N^*, \sigma_m^*, \sigma_M^*)$.

With this purpose in mind, we proceed as follows: First, we use the fatigue model (2.42) for constant maximum stress σ_M^*:

$$p = 1 - \exp\left\{ - \left[\frac{(N^* - B^*)\,(\sigma_M^* - \sigma_m^* - C^*) - \lambda^*}{\delta^*} \right]^{\beta^*} \right\}, \tag{2.64}$$

where $\Delta\sigma^*$ has been replaced by $\sigma_M^* - \sigma_m^*$.

Since for different constant values of σ_M^* one must have different models of the form (2.64), the parameters $\beta^*, B^*, C^*, \delta^*$ and λ^* must be functions of σ_M^*.

Assuming that the model is valid not only for any constant value of σ_M^* but for constant values of σ_m^*, one has another family of models, where now the parameters $\beta^*, B^*, C^*, \delta^*$ and λ^* are functions of σ_m. The next goal is to obtain these functions using the following compatibility condition:

If a constant load fatigue test oscillating from σ_m^* to σ_M^* is run, the model can be derived as a particular case of (a) constant σ_m^* or (b) constant σ_M^*, but both models must be the same (compatibility condition), that is:

$$\begin{aligned} &\left[\frac{(N^* - B_m^*(\sigma_m^*))(\Delta\sigma^* - C_m^*(\sigma_m^*)) - \lambda_m^*(\sigma_m^*)}{\delta_m^*(\sigma_m^*)} \right]^{A_m^*(\sigma_m^*)} \\ &\quad = \left[\frac{(N^* - B_M^*(\sigma_M^*))(\Delta\sigma^* - C_M^*(\sigma_M^*)) - \lambda_M^*(\sigma_M^*)}{\delta_M^*(\sigma_M^*)} \right]^{A_M^*(\sigma_M^*)}. \end{aligned} \tag{2.65}$$

This was illustrated in Fig. 2.8, where the compatibility states that the set of percentiles must intersect at horizontally aligned points.

Equation (2.65) is a functional equation, in which the unknowns are the ten functions $A_m^*(\sigma_m^*)$, $B_m^*(\sigma_m^*)$, $C_m^*(\sigma_m^*)$, $\delta_m^*(\sigma_m^*)$, $\lambda_m^*(\sigma_m^*)$, $A_M^*(\sigma_M^*)$, $B_M^*(\sigma_M^*)$, $C_M^*(\sigma_M^*)$, $\delta_M^*(\sigma_M^*)$ and $\lambda_M^*(\sigma_M^*)$. The beauty of functional equations is that a single equation allows us to determine the solution for all the unknown functions involved.

For the functional equation (2.65) to be satisfied for any N^*, σ_m^* and σ_M^*, both models must have the same parameters. Writing the model in (2.65) as

$$\begin{aligned} &\left[\frac{N^* - \left[B_m^*(\sigma_m^*) + \dfrac{\lambda_m^*(\sigma_m^*)}{\Delta\sigma^* - C_m^*(\sigma_m^*)} \right]}{\dfrac{\delta_m^*(\sigma_m^*)}{\Delta\sigma^* - C_m^*(\sigma_m^*)}} \right]^{A_m^*(\sigma_m^*)} \\ &\quad = \left[\frac{N^* - \left[B_M^*(\sigma_M^*) + \dfrac{\lambda_M^*(\sigma_M^*)}{\Delta\sigma^* - C_M^*(\sigma_M^*)} \right]}{\dfrac{\delta_M^*(\sigma_M^*)}{\Delta\sigma^* - C_M^*(\sigma_M^*)}} \right]^{A_M^*(\sigma_M^*)} \quad ; \quad \forall N^* \end{aligned} \tag{2.66}$$

and forcing the Weibull parameters to coincide one gets $\forall \sigma_m^*, \sigma_M^*$:

$$A_m^*(\sigma_m^*) = A_M^*(\sigma_M^*); \tag{2.67}$$

$$\frac{\delta_m^*(\sigma_m^*)}{\delta_M^*(\sigma_M^*)} = \frac{\Delta\sigma^* - C_m^*(\sigma_m^*)}{\Delta\sigma^* - C_M^*(\sigma_M^*)} = \frac{(\sigma_M^* - \sigma_m^*) - C_m^*(\sigma_m^*)}{(\sigma_M^* - \sigma_m^*) - C_M^*(\sigma_M^*)}; \tag{2.68}$$

$$\begin{aligned} B_M^*(\sigma_M^*) &= B_m^*(\sigma_m^*) - \frac{\lambda_M^*(\sigma_M^*)}{\Delta\sigma^* - C_M^*(\sigma_M^*)} + \frac{\lambda_m^*(\sigma_m^*)}{\Delta\sigma^* - C_m^*(\sigma_m^*)} \\ &= B_m^*(\sigma_m^*) - \frac{\lambda_M^*(\sigma_M^*)}{(\sigma_M^* - \sigma_m^*) - C_M^*(\sigma_M^*)} + \frac{\lambda_m^*(\sigma_m^*)}{(\sigma_M^* - \sigma_m^*) - C_m^*(\sigma_m^*)}. \end{aligned} \tag{2.69}$$

The system of functional equations (2.67) to (2.69) deserves careful attention because it contains a deep knowledge of our problem. In particular, they are not simple equalities, but each a full collection of equalities, because they must hold for any feasible test stress pair (σ_m^*, σ_M^*).

Solving the system of functional equations (2.67) to (2.69) (see Appendix A) the following model is obtained:

$$\boxed{Q^*(N^*; \sigma_m^*, \sigma_M^*) = 1 - \exp\left\{ - \left[r^*(\sigma_m^*, \sigma_M^*) + s^*(\sigma_m^*, \sigma_M^*) N^* \right]^{\beta^*} \right\},} \tag{2.70}$$

where

$$r^*(\sigma_m^*, \sigma_M^*) = C_0^* + C_1^*\sigma_m^* + C_2^*\sigma_M^* + C_3^*\sigma_m^*\sigma_M^* \tag{2.71}$$

$$s^*(\sigma_m^*, \sigma_M^*) = C_4^* + C_5^*\sigma_m^* + C_6^*\sigma_M^* + C_7^*\sigma_m^*\sigma_M^* \tag{2.72}$$

and C_0^* to C_7^* are constants.[8]

Model (2.70) depends on nine parameters supplying all probabilistic information for any S-N curves of the material related to whichever given stress level, and where $p = F^*(N; \sigma_m^*, \sigma_M^*)$ is the cdf of N^* for given σ_m^* and σ_M^*.

Note that an important limiting case of the Weibull model is the Gumbel model which results for $\beta \to \infty$ (see Castillo et al. (2005a)). In addition, if all the arguments used to obtain the Weibull model are derived for the Gumbel model the same functional equations are obtained. This implies that a valid fatigue model is the following Gumbel model:

$$\boxed{Q^*(N^*; \sigma_m^*, \sigma_M^*) = 1 - \exp\left\{ - \exp\left[r^*(\sigma_m^*, \sigma_M^*) + s^*(\sigma_m^*, \sigma_M^*) N^* \right] \right\},} \tag{2.73}$$

which has the advantage of having one parameter less, and even more importantly, that the range of definition for N^* includes the range $(-\infty, \infty)$. This makes deciding whether or not data are in the allowable region.

[8] The reader is reminded that $N^* = \log(N/N_0)$.

2.7 Dimensional Weibull and Gumbel models

It is interesting to see what the models (2.70) and (2.73) look like when written in terms of the initial (dimensional) variables. To this end, the Weibull model (2.70) must be written as

$$\begin{aligned} p &= 1-\exp\left\{-\left[C_0^*+C_1^*\frac{\sigma_m}{\Delta\sigma_0}+C_2^*\frac{\sigma_M}{\Delta\sigma_0}+C_3^*\frac{\sigma_m}{\Delta\sigma_0}\frac{\sigma_M}{\Delta\sigma_0}\right.\right.\\ &\quad\left.\left.+\left(C_4^*+C_5^*\frac{\sigma_m}{\Delta\sigma_0}+C_6^*\frac{\sigma_M}{\Delta\sigma_0}+C_7^*\frac{\sigma_m}{\Delta\sigma_0}\frac{\sigma_M}{\Delta\sigma_0}\right)\log\frac{N}{N_0}\right]^{\beta}\right\} \quad (2.74)\\ &= 1-\exp\{-[C_0+C_1\sigma_m+C_2\sigma_M+C_3\sigma_m\sigma_M\\ &\quad+(C_4+C_5\sigma_m+C_6\sigma_M+C_7\sigma_m\sigma_M)\log N]^{\beta}\Big\} \quad (2.75)\end{aligned}$$

or the Gumbel model as

$$\begin{aligned} p = 1-\exp\{-\exp[&C_0+C_1\sigma_m+C_2\sigma_M+C_3\sigma_m\sigma_M\\ &+(C_4+C_5\sigma_m+C_6\sigma_M+C_7\sigma_m\sigma_M)\log N]\} \quad (2.76)\end{aligned}$$

where

$$\begin{aligned} C_0 &= C_0^*-C_4^*\log N_0\\ C_1 &= (C_1^*-C_5^*\log N_0)/\Delta\sigma_0\\ C_2 &= (C_2^*-C_6^*\log N_0)\Delta\sigma_0\\ C_3 &= (C_3^*-C_7^*\log N_0)\Delta\sigma_0^2\\ C_4 &= C_4^*\\ C_5 &= C_5^*/\Delta\sigma_0\\ C_6 &= C_6^*/\Delta\sigma_0\\ C_7 &= C_7^*/\Delta\sigma_0^2 \end{aligned} \quad (2.77)$$

which leads to the following inverse transformation

$$\begin{aligned} C_0^* &= C_0+C_4\log N_0\\ C_1^* &= C_1\Delta\sigma_0+C_5\Delta\sigma_0\log N_0\\ C_2^* &= C_2\Delta\sigma_0+C_6\Delta\sigma_0\log N_0\\ C_3^* &= C_3\Delta\sigma_0^2+C_7\Delta\sigma_0^2\log N_0\\ C_4^* &= C_4\\ C_5^* &= C_5\Delta\sigma_0\\ C_6^* &= C_6\Delta\sigma_0\\ C_7^* &= C_7\Delta\sigma_0^2. \end{aligned} \quad (2.78)$$

Note that the C_1^* to C_7^* parameters are dimensionless parameters, while the parameters C_1 to C_7 have dimensions.

Remark 2.1 *Equations (2.75) and (2.76) say that the fatigue model has the same functional structure no matter whether it is written in terms of the dimensional or the dimensionless variables. So, once the model can be stated in terms of dimensionless ratios, as stated by the Buckingham theorem, we can* *work with any of the two representations (dimensional or dimensionless) and the relationships between the resulting parameters are given by (2.77) and (2.78). Though from the point of view of numerical behavior, the dimensionless approach normally provides a better performance, the dimensional form is simpler to use.*

Equations (2.77) and (2.78) show that the model in (2.74) really depends on nine parameters, and that $\Delta\sigma_0$ and N_0 can be chosen arbitrarily. Thus, the normalizing values can be freely chosen. If, for example,

$$\log N_0 = -C_0/C_4 \tag{2.79}$$

$$\Delta\sigma_0 = 1/C_5, \tag{2.80}$$

are selected, $C_0^* = 0$ and $C_1^* = 1$ are obtained, showing that the normalization variables $\Delta\sigma_0$ and N_0 can replace these two parameters C_0^* and C_1^*.

This means that models (2.70) and (2.73) and expressions (2.71) and (2.72) are also valid if the asterisks are removed. More precisely, the resulting dimensional log-Weibull and log-Gumbel models are:

$$\boxed{Q(N;\sigma_m,\sigma_M) = 1 - \exp\left\{-\left[r(\sigma_m,\sigma_M) + s(\sigma_m,\sigma_M)\log N\right]^{\beta}\right\},} \tag{2.81}$$

and

$$\boxed{Q(N;\sigma_m,\sigma_M) = 1 - \exp\left\{-\exp\left[r(\sigma_m,\sigma_M) + s(\sigma_m,\sigma_M)\log N\right]\right\},} \tag{2.82}$$

where

$$r(\sigma_m,\sigma_M) = C_0 + C_1\sigma_m + C_2\sigma_M + C_3\sigma_m\sigma_M \tag{2.83}$$

$$s(\sigma_m,\sigma_M) = C_4 + C_5\sigma_m + C_6\sigma_M + C_7\sigma_m\sigma_M. \tag{2.84}$$

For simplicity, we recommend using these dimensional models.

2.8 Properties of the model

The graphs $(\log N, g(\Delta\sigma))$ of the percentiles for constant σ_M or σ_m are hyperbolas. We note that the hyperbolas arise not because of a reasonable, though nevertheless arbitrary, assumption, but as the only possible solution to the compatibility functional equation.

If C_5, C_6 and C_7 are not simultaneously null, the model has asymptotes. In this case, the two asymptotes of the hyperbolas can be calculated as follows. The asymptotic value of $\Delta\sigma$ for large N keeping constant σ_m is

$$g(\Delta\sigma_{m0}) = \lim_{N\to\infty} g(\Delta\sigma) = -\frac{C_4 + \sigma_m(C_5 + C_6 + C_7\sigma_m)}{C_6 + C_7\sigma_m} \tag{2.85}$$

and the asymptotic value of $\Delta\sigma$ for constant σ_M is

$$g(\Delta\sigma_{M0}) = \lim_{N\to\infty} g(\Delta\sigma) = \frac{C_4 + \sigma_M(C_5 + C_6 + C_7\sigma_M)}{C_5 + C_7\sigma_M}. \tag{2.86}$$

Similarly, the asymptotic value of N for large $\Delta\sigma$ keeping σ_m constant is

$$\log N_{m0} = \lim_{\Delta\sigma\to\infty} N = -\frac{C_2 + C_3\sigma_m}{C_6 + C_7\sigma_m}, \tag{2.87}$$

and the asymptotic value of N for constant σ_M is

$$\log N_{M0} = \lim_{\Delta\sigma\to\infty} N = -\frac{C_1 + C_3\sigma_M}{C_5 + C_7\sigma_M}. \tag{2.88}$$

It is interesting to see that the general model allows the asymptotes to be dependent on the constant σ_m and σ_M levels being considered.

Required constraints

For the model with asymptotes to be physically and statistically valid its parameters must satisfy the following constraints:

1. The cdf in (2.70) must be non-decreasing in N:

$$C_4 + C_5\sigma_m + C_6\sigma_M + C_7\sigma_m\sigma_M > 0; \quad \sigma_M \geq \sigma_m, \tag{2.89}$$

   which implies

$$C_7 = 0, \quad C_5 = -C_6, \quad C_4 \geq 0, \quad C_6 \geq 0. \tag{2.90}$$

2. The asymptotic value N_{m0} must be non-increasing in σ_m:

$$C_3C_6 - C_2C_7 \geq 0, \tag{2.91}$$

   which together with (2.90) implies $C_3 \geq 0$.

3. The asymptotic value of N_{M0} must be non-increasing in σ_M:

$$C_3C_5 - C_1C_7 \geq 0, \tag{2.92}$$

   which together with (2.91) leads to $C_3 = 0$.

4. The asymptotic value $g(\Delta\sigma_{m0})$ must be non-negative, i.e.

$$g(\Delta\sigma_{m0}) = -\frac{C_4 + \sigma_m(C_5 + C_6 + C_7\sigma_m)}{C_6 + C_7\sigma_m} \geq 0. \tag{2.93}$$

5. The asymptotic value $g(\Delta\sigma_{M0})$ must be non-negative, i.e.

$$g(\Delta\sigma_{M0}) = \frac{C_4 + \sigma_M(C_5 + C_6 + C_7\sigma_M)}{C_5 + C_7\sigma_M} \geq 0. \tag{2.94}$$

   Equations (2.93), (2.94) and (2.90) imply $C_4 = 0$.

6. The cdf in (2.70) must be non-increasing in σ_m:

$$C_1 + C_3\sigma_M + (C_5 + C_7\sigma_M)N \leq 0; \quad \sigma_{m0} \leq \sigma_M \leq \sigma_{M0}. \tag{2.95}$$

7. The cdf in (2.70) must be non-decreasing in σ_M:

$$C_2 + C_3\sigma_m + (C_6 + C_7\sigma_m)N \geq 0; \quad \forall N. \tag{2.96}$$

Equations (2.95) and (2.96) with (2.90) imply

$$\min_i N_i \geq \max\left(\frac{C_1}{C_6}, -\frac{C_1}{C_6}\right). \tag{2.97}$$

In summary, the set of constraints for the model with asymptotes reduces to

$$C_3 = C_4 = C_7 = 0, \quad C_5 = -C_6, \quad C_6 \geq 0 \quad \min_i N_i \geq \max\left(\frac{C_1}{C_6}, -\frac{C_1}{C_6}\right). \tag{2.98}$$

For the model without asymptotes this set of constraints reduces to

$$C_3 = C_5 = C_6 = C_7 = 0, \quad C_1 \leq 0, \quad C_2, C_4 \geq 0. \tag{2.99}$$

Inclusion of these constraints into the estimation method leads to valid models. This is an important fact to be taken into consideration because alternative methods do not take this into account sufficiently, and lack generality.

Resulting models

In this section, some particular and interesting submodels of the general log-Weibull model (2.81) and log-Gumbel model (2.82) are discussed:

Linear submodel: The simplest log-Weibull model with no asymptotes:

$$\boxed{\begin{array}{r} p = 1 - \exp\left\{-\left[C_0 + C_1\sigma_m + C_2\sigma_M + C_4\log N\right]^{\beta}\right\} \\ C_1 \leq 0; \quad C_2, C_4 \geq 0, \end{array}} \tag{2.100}$$

or the simplest log-Gumbel model with no asymptotes

$$\boxed{\begin{array}{r} p = 1 - \exp\left\{-\exp\left[C_0 + C_1\sigma_m + C_2\sigma_M + C_4\log N\right]\right\} \\ C_1 \leq 0; \quad C_2, C_4 \geq 0, \end{array}} \tag{2.101}$$

which results for $C_3 = C_5 = C_6 = C_7 = 0$, and in a semilog scale leads to a S-N field made of straight lines.

Model with fixed asymptotes: The log-Weibull model with $\Delta\sigma$ asymptotes independent on σ_m and σ_M:

$$\boxed{p = 1 - \exp\left\{- \left[C_0 + C_1\sigma_m + C_2\sigma_M + C_6(\sigma_M - \sigma_m)\log N\right]^\beta\right\},} \tag{2.102}$$

or the log-Gumbel model with $\Delta\sigma$ asymptotes independent on σ_m and σ_M:

$$\boxed{p = 1 - \exp\left\{- \exp\left[C_0 + C_1\sigma_m + C_2\sigma_M + C_6(\sigma_M - \sigma_m)\log N\right]\right\}.} \tag{2.103}$$

This model has asymptotes:

$$\Delta\sigma_{m0} = \Delta\sigma_{M0} = 0; \quad N_{m0} = -C_2/C_6; \quad N_{M0} = -C_1/C_5.$$

General Model The general log-Weibull model with $\log R$ and N asymptotes dependent on σ_m and σ_M can be used in some limited regions of σ_m and σ_M:

$$\boxed{\begin{aligned} p = 1 - \exp\{&- [C_0 + C_1\sigma_m + C_2\sigma_M + C_3\sigma_m\sigma_M \\ &+ (C_4 + C_5\sigma_m + C_6\sigma_M + C_7\sigma_m\sigma_M)\log N]^\beta\Big\}, \end{aligned}} \tag{2.104}$$

or the log-Gumbel model

$$\boxed{\begin{aligned} p = 1 - \exp\{&- \exp[C_0 + C_1\sigma_m + C_2\sigma_M + C_3\sigma_m\sigma_M \\ &+ (C_4 + C_5\sigma_m + C_6\sigma_M + C_7\sigma_m\sigma_M)\log N]\}, \end{aligned}} \tag{2.105}$$

subject to constraints (2.98) or (2.99) with some relaxation depending on the domain being considered for (σ_m, σ_M).

These constraints are very important in order to have a physically based model.

S-N curves for the general model

In particular, different parametric forms for the S-N field, as required by the user, can be selected from the Weibull model (2.102), such as for instance:

1. $\Delta\sigma$-$\log N$ for constant σ_M:

$$\log N = \frac{-C_0 + C_1\Delta\sigma - C_1\sigma_M - C_2\sigma_M + (-\log[1-p])^{1/\beta}}{C_6\Delta\sigma}. \tag{2.106}$$

2. $\Delta\sigma$-$\log N$ for constant σ_m:

$$\log N = \frac{-C_0 - C_2\Delta\sigma - C_1\sigma_m - C_2\sigma_m + (-\log[1-p])^{1/\beta}}{C_6\Delta\sigma}. \tag{2.107}$$

3. $\Delta\sigma$-$\log N$ for constant σ_{mean}:

$$\log N = \frac{-4C_0 + 2C_1\Delta\sigma - 2C_2\Delta\sigma}{4C_4} + \frac{-4\sigma_{mean}(C_1 + C_2) + 4(-\log[1-p])^{1/\beta}}{4C_4}. \quad (2.108)$$

4. $\Delta\sigma$-$\log N$ for constant R:

$$\log N = \frac{-C_0(R-1)^2 + \Delta\sigma(C_1 + C_2R(1-R) - RC_1)}{\Delta\sigma C_6(R-1)^2} + \frac{(R-1)^2(-\log[1-p])^{1/\beta}}{\Delta\sigma C_6(R-1)^2}. \quad (2.109)$$

5. R-$\log N$ for constant σ_M:

$$\log N = -\frac{C_0 + C_2\sigma_M + C_1R\sigma_M - (-\log[1-p])^{1/\beta}}{C_6\sigma_M(1-R)}. \quad (2.110)$$

The corresponding Gumbel S-N curves can be obtained by replacing $(-\log \times [1-p])^{1/\beta}$ by $\log(-\log(1-p))$.

From (2.5) and (2.6) we can conclude that the regression line for the Weibull model can be obtained from the corresponding percentile curves by replacing $[-\log(1-p)]^{1/\beta}$ by $\Gamma\left(1+\frac{1}{\beta}\right)$, and from (2.12) and (2.13) we can conclude that the regression line for the Gumbel model can be obtained from corresponding percentile curves by replacing $\log(-\log(1-p))$ by $-\gamma$, where γ is the Euler-Mascheroni constant.

An illustration of these formulas is given in Sect. 2.8.3.

The S-N curves for the general model (2.104) are given in Appendix B on page 89.

Finally, the following important remark is included.

Remark 2.2 *The percentile and regression models in (2.106) to (2.110) written in terms of the dimensionless parameters and variables are those resulting by adding asterisks to them, and replacing* $\log N$ *by* N *in the corresponding expressions. In other words, the percentile and regression models in terms of dimensional and dimensionless variables and parameters have the same functional form.)*

2.8.1 Parameter estimation

The parameter estimation of the log-Weibull and log-Gumbel models can be done by several methods. Some of them are described below.

Maximum likelihood estimation

The best known method for estimating the parameters of a statistical model is the maximum likelihood method, which shows good statistical properties. Thus, it is one of the first possibilities to be considered.

The log-likelihood function of the Weibull model (2.104) is

$$L = \sum_{i\in I_1} [\log(\beta) + (\beta-1)\log(H(N_i))] - \sum_{i\in I_1\cup I_0} H^{\beta}(N_i) + \sum_{i\in I_1} [\log(C_4 + C_5\sigma_{m_i} + C_6\sigma_{M_i} + C_7\sigma_{m_i}\sigma_{M_i})], \tag{2.111}$$

where I_1 and I_0 are the set of non-run-outs and run-outs, respectively, N_i refers to the actual value of the fatigue life in number of cycles, or the limit number of cycles for run-outs, and

$$H(N_i) = C_0 + C_1\sigma_{m_i} + C_2\sigma_{M_i} + C_3\sigma_{m_i}\sigma_{M_i} + (C_4 + C_5\sigma_{m_i} + C_6\sigma_{M_i} + C_7\sigma_{m_i}\sigma_{M_i})\log N_i. \tag{2.112}$$

Similarly, for the log-Gumbel model (2.105), the log-likelihood becomes:

$$L = \sum_{i\in I_1} [H(N_i) + \log(C_4 + C_5\sigma_{m_i} + C_6\sigma_{M_i} + C_7\sigma_{m_i}\sigma_{M_i})] - \sum_{i\in I_1\cup I_0} \exp(H(N_i)). \tag{2.113}$$

Thus, to estimate the parameters of the model one can maximize (2.111) or (2.113) with respect to the parameters, but subject to the set of constraints (2.98) or (2.99) or the corresponding ones for simpler models. For the Weibull models one must add the condition $H(N_i) \geq 0; \quad \forall i$, which is a very disturbing set of constraints, because the C's are unknown. Thus, when possible, it is recommendable to use the Gumbel model instead of the Weibull model because of its simplicity and due to the fact that estimation is much easier. If the values of the β parameter are high, as happens with many materials, the Gumbel model is the most convenient option to choose.

The asymptotic covariance matrix of the estimates C_0, C_1, C_2, and C_6 if the constraints (2.98) or (2.99) are not active,[9] can be calculated using the well known formula[10]

$$Covar = \left(-\frac{\partial L^2}{\partial C_i \partial C_j}\right)\Bigg|_{\hat{\mathbf{C}}}^{-1} \tag{2.114}$$

where $\hat{\mathbf{C}}$ are the maximum likelihood parameter estimates. This matrix is the basic tool to determine confidence intervals of other related variables, such as percentiles for example.

Otherwise, and especially when the sample size is small, the covariance matrix of the estimates and confidence intervals can be obtained by the bootstrap method (see Efron and Tibshirami (1993) or Naess and Hungness (2002)).

[9] If the sample size is large enough and the data are reasonable, the set of constraints (2.98) or (2.99) should not be active.

[10] This is valid only if the constraints are inactive, that is, the optimum is not attained at the boundary of the feasible region.

Parameter estimation by regression

Another possibility consists of using a regression model, i.e., minimizing the following sum of squares for the log-Weibull model

$$Q = \sum_{i=1}^{n} \left(\log N + \frac{C_0 + C_2\sigma_M + C_1 R\sigma_M + C_3 R(\sigma_M)^2 - \Gamma\left(1 + \frac{1}{\beta}\right)}{C_4 + C_6\sigma_M + C_5 R\sigma_M + C_7 R(\sigma_M)^2} \right)^2 \tag{2.115}$$

where the parameter β must be estimated using other methods, or for the log-Gumbel model

$$Q = \sum_{i=1}^{n} \left(\log N + \frac{C_0 + C_2\sigma_M + C_1 R\sigma_M + C_3 R(\sigma_M)^2 + \gamma}{C_4 + C_6\sigma_M + C_5 R\sigma_M + C_7 R(\sigma_M)^2} \right)^2, \tag{2.116}$$

subject to the constraints (2.98) or (2.99).

To estimate the shape Weibull parameter β one can also use a sample with constant σ_M, σ_m or $R = \sigma_m/\sigma_M$, because the corresponding distribution is Weibull with the same β parameter. Once this has been estimated, one can minimize (2.115) to estimate the remaining parameters.

For large sample sizes one could avoid the constraints, and assume that the data already contain the necessary information about the constraints. However, this is risky, and one can face problems depending on the posterior use of the model.

The treatment of the run-out data can be handled by iteration. Initially, the run-out data are ignored in the first iteration, and once the parameters have been obtained, one assigns the run-outs to their expected values. Next, the process is repeated until convergence. In order to avoid repetition, we do not include the details here. The interested reader is referred to Sect. 2.4.3 or Castillo and Fernández-Canteli (2006).

There are many other estimation methods (see, for example, Castillo and Hadi (1995), Castillo et al. (1999), and the references in these two papers).

2.8.2 Use of the model in practice

In this section the use of the model for practical applications in fatigue design is described, and in particular how the parameters of the model can be estimated and how it can be used to predict lifetimes under other testing conditions when the parameters are known.

The suggested procedure is as follows:

Step 1: *Design the testing strategy and obtain data.* A set of testing cases encompassing several stress level conditions, i.e., varying σ_M and σ_m is selected, for example, the set of stress pairs[11]

$$\{(\sigma_{m_i}, \sigma_{M_i}) \mid i = 1, 2, \ldots, n\},$$

which must cover the desired region, where the fatigue model is to be used.

Step 2: *Estimate the model parameters.* Use one of the estimation methods discussed in Sect. 2.8.1 to estimate the parameters $C_0, C_1, C_2, C_3, C_4, C_5, C_6, C_7$ and β if the model is Weibull.

Step 3: *Obtain the fatigue model.* Replace the parameter values C_0 to C_7 in (2.104) or (2.105) to obtain the model.

Step 4: *Extrapolate to other testing conditions.* Use the models (2.104) and (2.105) for any other testing conditions.

2.8.3 Example of applications

In this section two examples are used to illustrate the proposed methods.

The MIL-HDBK-5G (1994) data example

In this example fatigue sample data from specimens made of notched Inconel 718 bars including three stress ratios ($R = -0.50, 0.10, 0.50$) extracted from the MIL-HDBK-5G (1994) are used (see Castillo et al. (2008a)). Because the numerical values concerning σ_M and numbers of cycles to failure were not explicitly supplied in this reference, they were directly estimated from the graphic representations.

In the evaluation of fatigue results for different materials, a regression model proposed by the MIL-HDBK-5G (1994) of the form

$$\log N = A_1 + A_2 \log_{10}(S_M(1-R)^{A_3} - A_4), \tag{2.117}$$

is considered, where A_1, A_2 and A_4 are constants with dimensions, and A_3 is a dimensionless constant. However, the bases for selecting this model are not given.

Alternatively, the regression model of N on σ_M for different stress ratios R resulting from the model (2.105) becomes

$$\log N = \frac{C_0 + C_2\sigma_M + C_1 R\sigma_M + \gamma}{C_4 + C_6\sigma_M(1-R)}, \tag{2.118}$$

where $\gamma = 0.57772$ is the Euler-Mascheroni number.

The relevant issue of Eq. (2.118) is that they have been derived from all the indicated properties, and not arbitrarily chosen. Thus, this regression model is selected to fit the experimental data.

[11] Note that not every set of stress pairs is sufficient for estimation purposes. It must cover a wide region of values to permit the estimation of all parameters.

Since in Sect. 2.8.2 a methodology has been proposed to deal with practical cases, it is applied here step by step, as follows.

Step 1: *Design the testing strategy and obtain data.* This is not required here because the data are obtained from MIL-HDBK-5G (1994).

The following cases have been considered:

Case 0. It corresponds to the regression model (2.117) without constraints fitted to all data excluding run-outs.

Case 1. It corresponds to the general 9 parameter regression model (2.73) without constraints fitted to all data excluding run-outs.

Case 2. It corresponds to the Gumbel version of the model (2.102) with fixed asymptotes including all the constraints and fitted with all data excluding the run-outs.

Case 3. It corresponds to the Gumbel version of the model (2.102) with fixed asymptotes including all the constraints and fitted with all data including the run-outs.

Case 4. It corresponds to the Gumbel version of the model (2.102) with fixed asymptotes including all the constraints and fitted with all data but the outlier and including the run-outs.

Step 2: *Estimate the model parameters.* We have used the maximum likelihood method as discussed in Sect. 2.8.1 to estimate the parameters C_0, C_1, C_2, C_3, C_4, C_5, C_6 and C_7 of the log-Gumbel models (2.103) and (2.105). The resulting parameter estimates are shown in Table 2.5.

Step 3: *Obtain the fatigue model.* Replace the parameter values C_0 to C_7 in (2.104) or (2.105) to obtain the model.

Table 2.5: Parameter estimates for different cases and models for Cases 1 to 4.

Case	Parameters			
	C_0	C_1	C_2	C_3
1	−46748.8	11631.7	−20502.9	−10430.8
2	−10.1959	44.7721	−39.1862	0
3	−10.8552	42.8311	−36.9141	0
4	−12.1504	49.975	−42.9144	0
	C_4	C_5	C_6	C_7
1	2457.73	−5134.	5737.14	3070.4
2	0	−5.66951	5.66951	0
3	0	−5.56015	5.56015	0
4	0	−6.34962	6.34962	0

Then we have:

Case 0. The data and the corresponding curves provided in the MIL-HDBK-5G are plotted in the top graph of Fig. 2.9. The fit is reasonably good, but the quality of extrapolations based on this model is not guaranteed by a physically justified regression equation.

Case 1. The intermediate graph of Fig. 2.9 shows the real data classified by R values, together with the estimated (regression) curves using the proposed method without constraints. The fit is better than in the previous case, especially in the lower region. In addition, since the regression model has been derived based on physical and statistical bases, the extrapolation can be done with a higher reliability. However, taking into account that the constraints were not imposed and normally not all will be satisfied, extrapolation must be done with care.

Case 2. The proposed model is plotted in the lower part of Fig. 2.9. Since the model used has been constrained by a high number of constraints or conditions, the fit of the model to the data is not as good as in the previous case. However, this is not a shortcoming but, in contrast, can be considered as an advantage. In fact, the plot reveals that the data point with the smallest number of cycles to failure appears to be an outlier, and corresponds to the low cycle fatigue region. In addition, the curvature of the data points for $R = -0.5$ points out the possibility of a plastic failure. Note that the outlier character of this point was hidden in the two previous cases.

Case 3. The model appears in the upper part of Fig. 2.10. In this case the run-outs were not removed, but taken into consideration in the estimation process, using expression (2.113). Note that the run-outs do not contain exact information about the lifetime, but contain some information, which is also valuable, and must not be ignored. A comparison of the plots of cases 2 and 3 show that they are very similar, and that the main differences occur in the lower right region, as expected. Note that again the data point with the least lifetime appears as a clear outlier, which was not the case for cases 0 and 1.

Case 4. The model appears in the lower part of Fig. 2.10. In this case since the data point with the smallest lifetime appears as an outlier and there are physical reasons to justify it, this data point has been removed and the model re-estimated. A comparison with the plots of case 3 reveals that the resulting models are very similar, and then one can conclude that the outlier has a negligible influence due to the strong constraints imposed on the model. So, the robustness of the proposed estimation regression method has been shown by comparing the results obtained when removing one outlier suspected of belonging to low-cycle fatigue, as shown in Fig. 2.10, which shows practically the same resulting evaluation results.

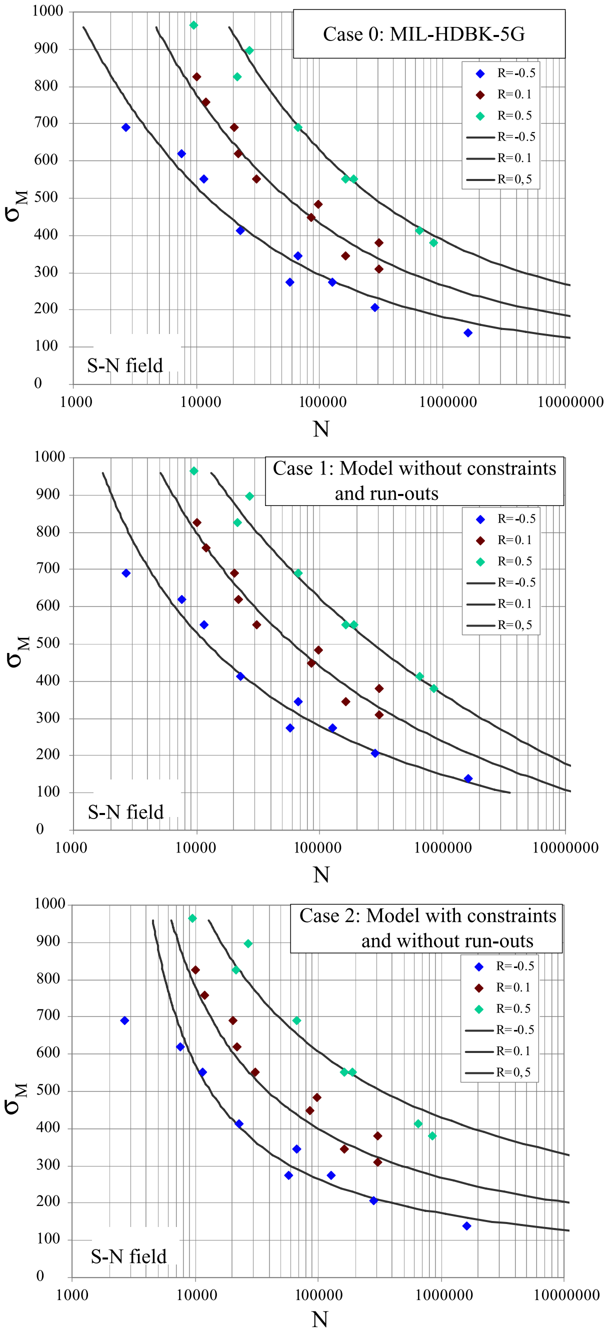


Figure 2.9: S-N curves for notched ($K_t = 3.3$), AISI 4340 alloy steel bars fitted by three different methods. The upper figure corresponds to the MIL-HDBK-5G model, the intermediate to the proposed model without constraints and the lower to the proposed model including all the constraints.

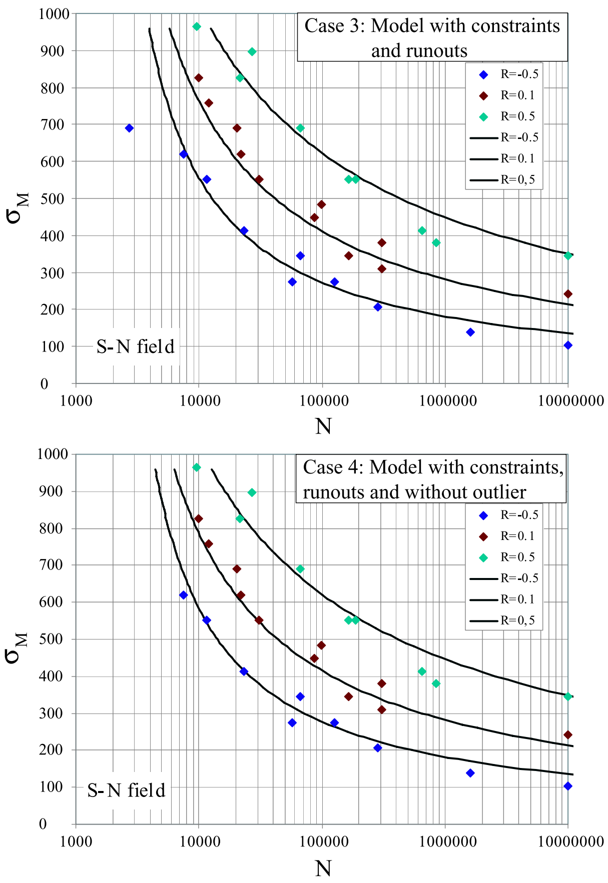


Figure 2.10: S-N curves for notched ($K_t = 3.3$), AISI 4340 alloy steel bars fitted using the proposed regression model with constraints and including the run-outs. In the lower figure the outlier has been removed.

It seems reasonable to use this model as the most adequate to represent the material fatigue strength corresponding to the given data. Then, the variance matrix of the Gumbel parameter estimates C_1, C_2, C_6, C_0) has been calculated using the bootstrap method with 1000 simulations for Case 4, and the following covariance matrix has been obtained

$$\begin{pmatrix} 74.208 & -63.381 & 9.281 & -17.897 \\ -63.381 & 55.267 & -7.988 & 15.001 \\ 9.281 & -7.988 & 1.187 & -2.335 \\ -17.897 & 15.001 & -2.335 & 4.983 \end{pmatrix}.$$

Step 4: *Extrapolate to other testing conditions.* Once the parameter estimates are available, models (2.103) or (2.105) can be used to extrapolate to other test-

Table 2.6: Estimated percentile values associated with the different data points using the Gumbel fitted model.

Data	p	Data	p	Data	p
1	0.004	6	0.161	11	0.734
2	0.356	7	0.664	12	0.638
3	0.632	8	0.249	13	0.912
4	0.512	9	0.082	14	0.618
5	0.933	10	0.049	15	0.516
16	0.959	21	0.860	26	0.312
17	0.639	22	0.259	27	0.369
18	0.874	23	0.876	28	0.144
19	0.220	24	0.385	29	0.092
20	0.223	25	0.547	30	0.376

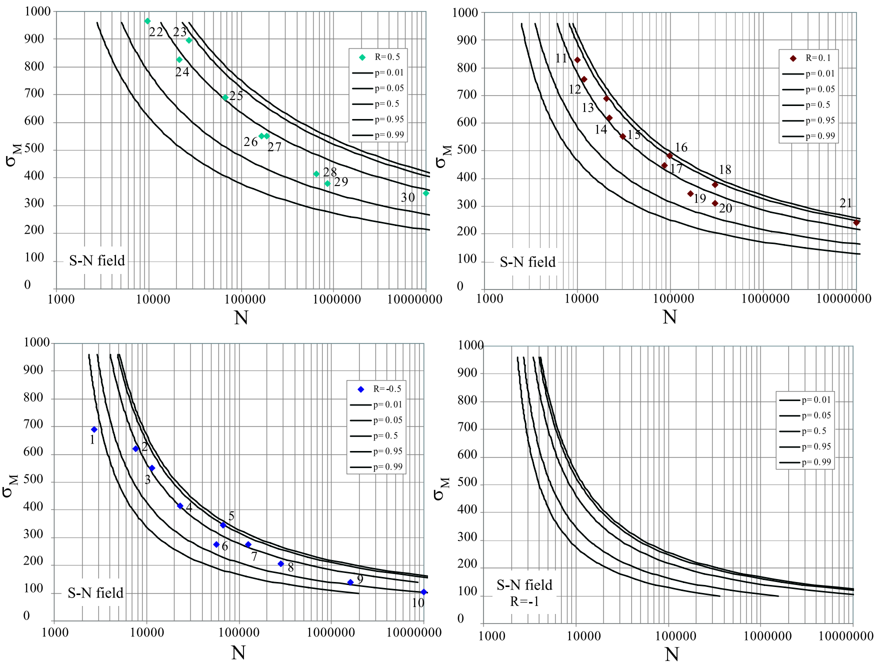


Figure 2.11: S-N curves for constant $R = 0.5, 0.1, -0.5$ and -1 (from top to bottom and left to right). The percentiles 0.01, 0.05, 0.50, 0.95 and 0.99 are represented.

ing conditions. One can predict the expected lifetimes associated with other R values, plot the percentiles curves, etc. For example, in Table 2.6 the estimated percentile values associated with the different data points in Fig. 2.11 are shown. They have been determined using the Gumbel fitted model. It is interesting to

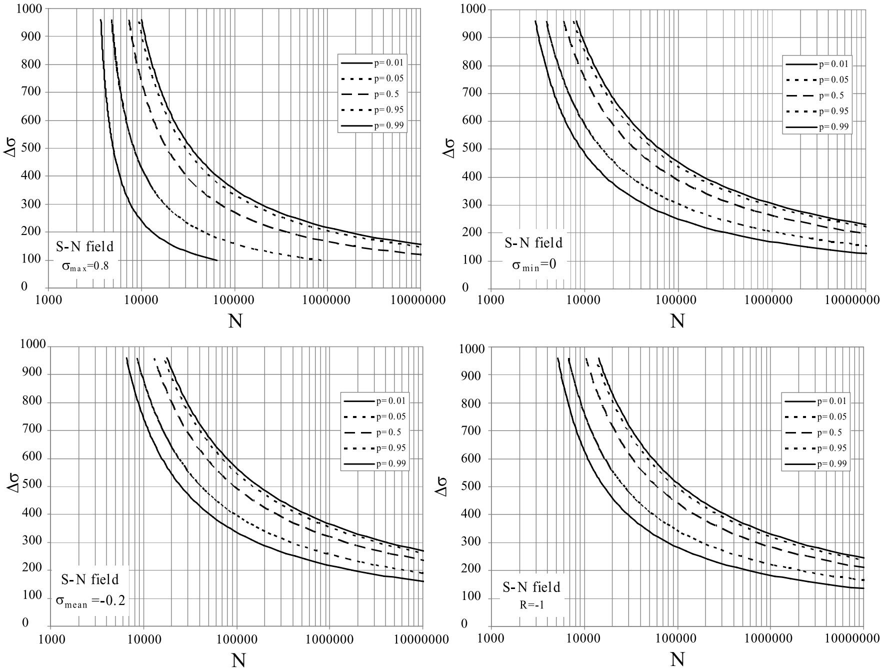


Figure 2.12: S-N curves for constant $\sigma_M = 0.8$, $\sigma_m = 0$, $\sigma_{mean} = 0$ and $R = -1$ (from top to bottom and left to right). The percentiles 0.01, 0.05, 0.50, 0.95 and 0.99 are represented.

see that the first data point has an associated value of 0.004, which reveals its outlier character.

Furthermore, the model can supply all the desired information corresponding to any possible testing condition alternative. For example, in Fig. 2.11 the S-N curves for constant $R = 0.5, 0.1, -0.5$ and -1 (from top to bottom and left to right) including the percentiles 0.01, 0.05, 0.50, 0.95 and 0.99 are represented.

Similarly, in Fig. 2.12 the $\Delta\sigma$-log N S-N fields for $\sigma_M = 0.8$, $\sigma_m = 0$, $R = -1$ and $\sigma_{mean} = -0.20$ are shown. They have been plotted using equations (2.106) to (2.109), respectively.

The Empa data example

In this second example, two different materials are considered corresponding to an experimental fatigue program launched in the Empa (Swiss Federal Laboratories for Testing and Research at Dübendorf (Switzerland)) presented by Koller et al. (2009):

1. A low-alloy steel 42CrMo4 (material number coded DIN-1.7225) with a nominal value of the ultimate strength $R_m = 1067 MPa$ and of the yield strength, $R_y = 975.3 MPa$.

2. An aluminium alloy AlMgSi1 (material number coded DIN-3.2315) with a nominal value of the ultimate strength $R_m = 391.7MPa$ and of the 0.2% yield strength, $R_{p_{0.2}} = 364.3MPa$.

Other characteristics of the two materials can be seen in Koller et al. (2009) The proposed methodology for this case is as follows:

Step 1: *Design the testing strategy and obtain data.* In the tests, 50 cylindrical specimens, as shown in Fig. 2.13, were used (27 specimens for $42CrMo4$ and 23 for $AlMgSi1$). The test length of the specimens was L_2 mm, and they were 8 mm in diameter. The total length was L_1 mm and the radius of transition to the test section of the specimen was 55 mm. The different lengths used for each material are shown in Table 2.7.

In the case of the 42CrMo4 steel, all the tests were conducted using a servo hydraulic testing machine, 160 kN load capacity with a steel alloy grip based on ASTM (2005) at frequencies ranging from 1 to 10 Hz. For the AlMgSi1 alloy, all the tests were conducted using a high frequency Vibrophore machine running at a maximum frequency of 80 Hz. In this way, only 34.7 hours were needed to complete a test with a limit number of cycles of 10 million. The specimen temperature was continuously monitored to keep it sufficiently low in order not to influence the failure mechanism during testing. All the tests were run at 25C degrees.

All testing programs were conducted under four constant σ_M levels corresponding to given percentages of the yield strength and different values of σ_m. In the case of the 42CrMo4, the values of these levels correspond to 0.98, 0.9,

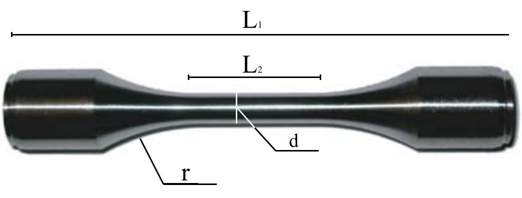
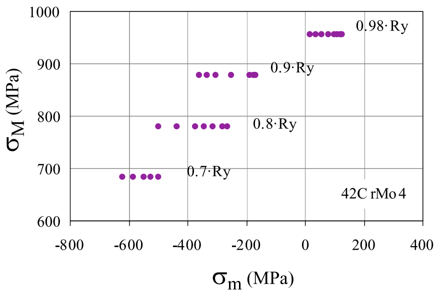


Figure 2.13: Geometry of the testing specimen.

Table 2.7: Specimen's dimensions for each material (see Fig. 2.13).

Material	L_1 (mm)	L_2 (mm)	d (mm)	r (mm)
42CrMo4	130	30	8	55
AlMgSi1	110	10	8	55



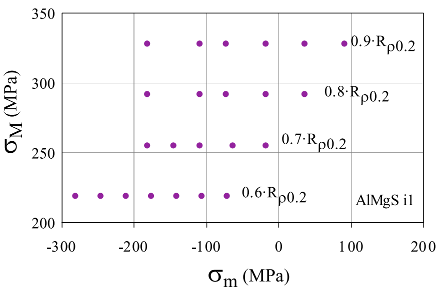


Figure 2.14: Distribution of the different test loads. The left figure corresponds to the 42CrMo4 steel, and the right figure to the AlMgSi1 alloy.

0.8 and 0.7 of the yield strength. For the AlMgSi1, the values of these levels correspond to 0.9, 0.8, 0.7 and 0.6 of the yield strength (See Fig. 2.14).

Due to the lack of any exact knowledge of the fatigue limit for both materials, the minimum testing amplitudes of these materials were estimated from the literature (see Boller and Seeger (1987a,b)): 450 MPa for the steel alloy and 80 MPa for the aluminium alloy. The different levels were chosen to optimize the test times trying to avoid run-outs.

The distributions of the different test loads are shown in Fig. 2.14. In addition, cycles of five and ten million were fixed as run-out values, respectively, for the 42CrMo4 and AlMgSi1 alloys.

The resulting lifetimes are shown in Table 2.8.

Step 3: *Obtain the fatigue model.* Replace the parameter values C_0 to C_7 in (2.104) or (2.105) to obtain the model.

We have used the two different estimation methods, maximization of the log-likelihood (2.113) and minimization of the regression equation (see (2.106)):

$$Q=\sum_{i=1}^{n}\left(\log N_i-\frac{-C_0+C_1\Delta\sigma_i-C_1\sigma_{M_i}-C_2\sigma_{M_i}+C_3\Delta\sigma_i\sigma_{M_i}-C_3\sigma_{M_i}^2-\gamma}{C_4-C_5\Delta\sigma_i+C_5\sigma_{M_i}+C_6\sigma_{M_i}-C_7\Delta\sigma_i\sigma_{M_i}+C_7\sigma_{M_i}^2}\right)^2.$$

The parameters of all the cases considered have been estimated for both materials and the results are shown in Tables 2.9 and 2.10 for 42CrMo4 and AlMgSi1 materials, respectively.

Figures 2.15 and 2.16 show the experimental data and the resulting median curves according to the fitted mode for both materials. They show a reasonable fit.

The variance-covariance matrix of the Gumbel parameter estimates C_1, C_2, C_6, C_0 cannot be calculated using formula (2.114) because there are active constraints in (2.98), so the bootstrap method has been used with 1000 replications,

Table 2.8: Resulting lifetimes for 42CrMo4 and AlMgSi1.

Material	42CrMo4			AlMgSi1		
Nr. Test	σ_m	σ_M	N (cycles)	σ_m	σ_M	N (cycles)
1	−250.00	877.80	17281	−182.15	327.87	19100
2	−190.00	877.80	48787	−109.29	327.87	34000
3	−175.00	877.80	81244	−72.86	327.87	42800
4	−360.00	877.80	3373	36.43	327.87	153100
5	−305.00	877.80	21812	−18.22	327.87	63800
6	−332.50	877.80	7265	91.08	327.87	360400
7	−167.50	877.80	125800	−182.15	291.44	28700
8	−500.00	780.27	11439	−72.86	291.44	71700
9	−437.50	780.27	14973	−109.29	291.44	59900
10	−375.00	780.27	32055	−18.22	291.44	143500
11	−312.50	780.27	483000	36.43	291.44	326400
12	−343.75	780.27	36708	−281.42	218.58	37100
13	−265.63	780.27	532200	−246.42	218.58	54400
14	−281.25	780.27	123100	−211.42	218.58	80300
15	−550.00	682.73	183024	−176.42	218.58	96300
16	−620.00	682.73	19331	−141.42	218.58	175500
17	−525.00	682.73	347102	−106.42	218.58	172800
18	−585.00	682.73	33925	−71.42	218.58	526500
19	−500.00	682.73	381543	−182.15	255.01	48000
20	17.51	955.50	65277	−145.72	255.01	57900
21	35.24	955.50	23700	−109.29	255.01	113100
22	55.27	955.50	52700	−18.22	255.01	348300
23	78.43	955.50	40900	−63.75	255.01	172500
24	119.11	955.50	85900			
25	108.91	955.50	124300			
26	124.21	955.50	222900			
27	98.71	955.50	93500			

and the following matrix has been obtained for the 42CrMo4:

$$\begin{pmatrix} 7.229 & -19.485 & 0.204 & -0.204 & 20.605 \\ -19.485 & 79.197 & -1.775 & 1.775 & -93.011 \\ 0.204 & -1.775 & 0.075 & -0.075 & 2.461 \\ -0.204 & 1.775 & -0.075 & 0.075 & -2.461 \\ 20.605 & -93.011 & 2.461 & -2.461 & 113.368 \end{pmatrix}.$$

Finally, the variance-covariance matrix of the Gumbel parameter estimates C_0 to C_7 has been calculated using formula (2.114) and the following matrix

Table 2.9: Parameter estimates for different estimation methods for the 42CrMo4.

Case	Parameters							
	C_0	C_1	C_2	C_3	C_4	C_5	C_6	C_7
Max.Lik.	-46.30	-3.77	33.47	0.00	0.00	-1.43	1.43	0.00
L.Squares	-46.30	-3.770	32.94	0.00	0.00	-1.46	1.46	0.00

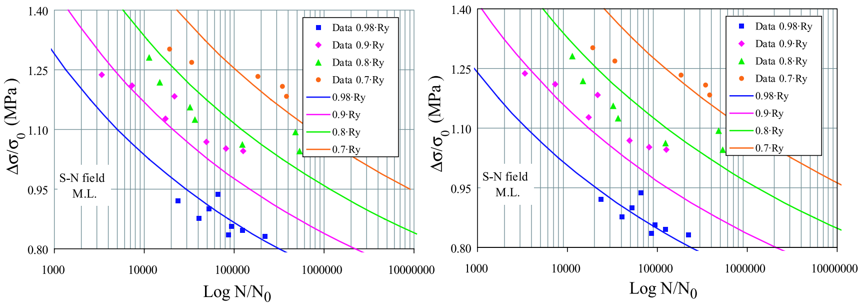


Figure 2.15: S-N curves for constant σ_M^* for the Gumbel model with constraints for the 42CrMo4 steel using different methods: least squares (left side) and maximum likelihood (right side).

Table 2.10: Parameter estimates for different estimation methods for the AlMgSi1.

Case	Parameters							
	C_0	C_1	C_2	C_3	C_4	C_5	C_6	C_7
Max. Lik.	-61.02	191.72	-108.87	0.00	0.00	-26.00	26.00	0.00
L.Squares	-13.71	43.67	-26.19	0.00	0.00	-5.86	5.86	0.00

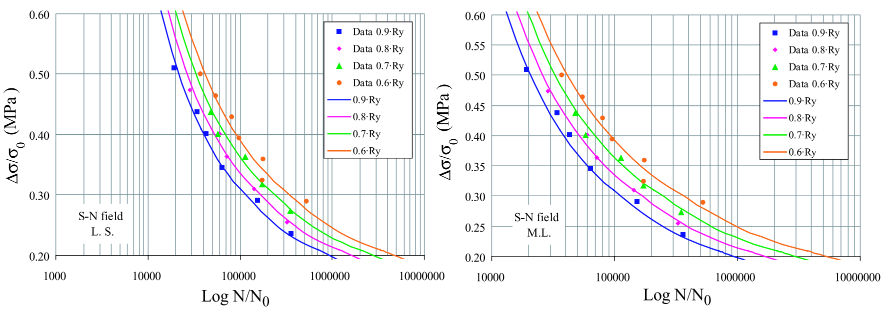


Figure 2.16: S-N curves for constant σ_M^* for the Gumbel Model with constraints for the AlMgSi1 alloy using different methods: least squares (left side) and maximum likelihood (right side).

has been obtained for the AlMgSi1 and the least squares method:

$$\begin{pmatrix} 39.835 & -23.254 & -4.516 & 4.516 & -8.657 \\ -23.254 & 52.363 & 1.900 & -1.900 & -8.491 \\ 4.516 & -1.900 & -0.704 & 0.704 & -1.988 \\ -8.657 & -8.491 & 1.988 & -1.988 & 9.823 \end{pmatrix}.$$

Step 4: *Extrapolate to other testing conditions.* Once the parameter estimates are available, the model (2.105) can be used to extrapolate to other testing conditions. For example, one can predict the expected lifetimes associated with other constant values of σ_M, plot the percentile curves, etc.

Table 2.11: Estimated percentile values associated with the different data points using the Gumbel fitted model for 42CrMo4.

Data	p	Data	p	Data	p	Data	p
1	0.883	8	0.921	15	0.940	22	0.441
2	0.113	9	0.241	16	0.547	23	0.433
3	0.383	10	0.485	17	0.154	24	0.330
4	0.134	11	0.964	18	0.087	25	0.855
5	0.459	12	0.329	19	0.041	26	0.896
6	0.606	13	0.492	20	0.899	27	0.647
7	0.256	14	0.738	21	0.054		

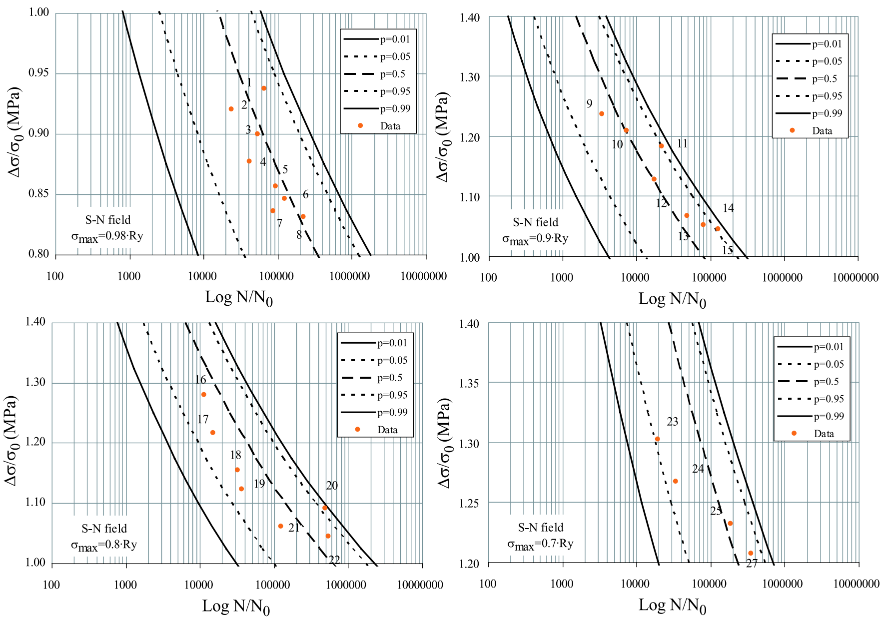


Figure 2.17: S-N curves representing constant $\sigma_M = 0.98, 0.9, 0.8$ and $0.7R_y$ for 42CrMo4 steel (from top to bottom and left to right). The percentiles 0.01, 0.05, 0.50, 0.95 and 0.99 are represented.

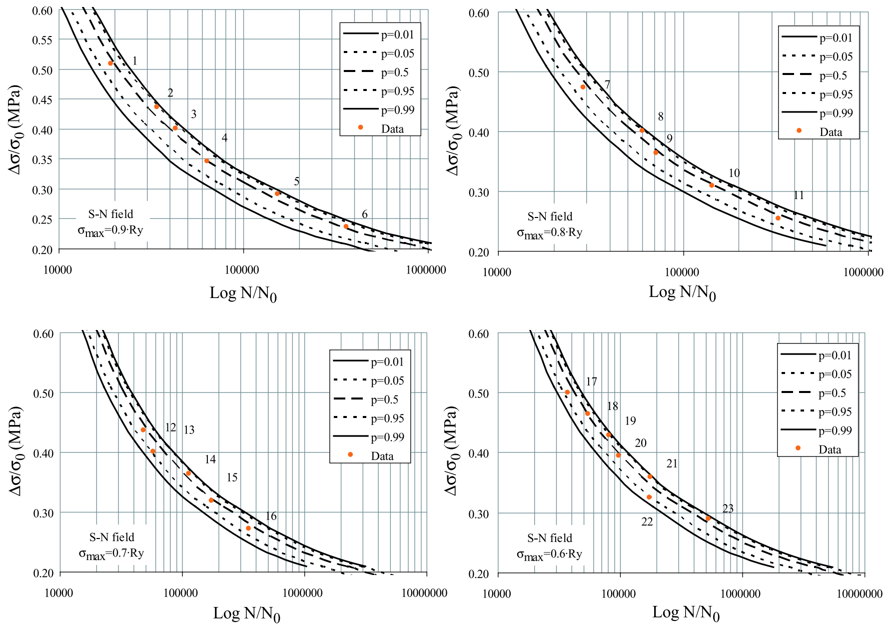


Figure 2.18: S-N curves representing constant $\sigma_M = 0.9, 0.8, 0.7$ and $0.6R_{p0.2}$ for AlMgSi1 alloy (from top to bottom and left to right). The percentiles 0.01, 0.05, 0.50, 0.95 and 0.99 are represented.

Figures 2.17 and 2.18 show the S-N curves for the 42CRMo4 steel and the ALMgSi1 alloy, respectively.

2.9 Concluding remarks

From the sections above we can conclude the following:

1. The use of dimensionless variables simplifies the problem under consideration and clarifies which are the minimal set of variables, or functions of them, relevant to the problem. It also makes it possible to work with dimensionless parameters what have many important advantages, such as independency of the set of selected units, and a better numerical behavior.

2. Physical and engineering considerations allow us to reject many models not satisfying the associated constraints. These considerations can be written, in many cases, in terms of functional equations, which lead to explicit forms for the mathematical and statistical models.

3. A Weibull based model for the S-N field has been obtained by solving a functional equation. This model is useful not only to fit fatigue data, but also to explain the fatigue behavior of longitudinal elements.

4. There are two types of parameters. One is related to the dimensionless variables, and used for normalization purposes, and includes the threshold parameters B and C. Other types of parameters are statistical parameters, such as the location parameter λ, the scale parameter δ, and the shape parameter β.

5. By relaxing some of the initial assumptions, apart from the suggested models for the S-N field, other models are possible.

6. General log-Weibull and log-Gumbel regression models for the statistical analysis of stress life data in the case where σ_M is tension has been developed. The models are based on statistical and physical considerations, and, in particular, on compatibility conditions in the S-N field that lead to a system of functional equations.

7. The general models (2.104) and (2.105) depend on 9 and 8 parameters, respectively, that can be estimated by maximum likelihood and also by non-linear regression. However, they are valid only on a restricted domain for (σ_m, σ_M). In contrast, models (2.102) and (2.103) are valid in any domain. They supply all the material basic probabilistic fatigue information to be used in a damage accumulation assessment for fatigue life prediction of structural and mechanical components under real loading spectra.

8. The model was satisfactorily applied to the evaluation of fatigue results from an external experimental program, and has shown to be very robust to outliers. In particular, for the illustrative example in the chapter, the method allows us to detect some data points corresponding presumably to low-cycle fatigue, so that, when they are removed, the resulting model remains practically unaltered from the original one. This is due to the large number of constraints that have been observed in order to obtain only physically and statistically valid models. This is not the case of other models commonly used in practice.

9. Once the parameters of the model have been estimated, the model allows us to obtain any kind of S-N field according to the testing condition chosen, as has been demonstrated in the example of application.

10. Finally, it is worthwhile mentioning that the model is the basic tool to develop a damage accumulation tool involving any load spectrum.

2.10 Appendix A: Derivation of the general model

The functional equation (2.68) can be written as:

$$(\sigma_M^* - C_M^*(\sigma_M^*))\delta_m^*(\sigma_m^*) - \delta_m^*(\sigma_m^*)\sigma_m^* - \delta_M^*(\sigma_M^*)\sigma_M^* + \delta_M^*(\sigma_M^*)(C_m^*(\sigma_m^*) + \sigma_m^*) = 0, \tag{2.119}$$

and solved as follows (see Aczél (1966) and Castillo et al. (1992, 2004)):

$$\begin{pmatrix} \sigma_M^* - C_M^*(\sigma_M^*) \\ 1 \\ \delta_M^*(\sigma_M^*)\sigma_M^* \\ \delta_M^*(\sigma_M^*) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ a_0 & b_0 \\ c_0 & d_0 \end{pmatrix} \begin{pmatrix} \sigma_M^* - C_M^*(\sigma_M^*) \\ 1 \end{pmatrix} \tag{2.120}$$

$$\begin{pmatrix} \delta_m^*(\sigma_M^*) \\ -\delta_m^*(\sigma_m^*)\sigma_m^* \\ -1 \\ C_m^*(\sigma_m^*) + \sigma_m^* \end{pmatrix} = \begin{pmatrix} m_0 & n_0 \\ p_0 & q_0 \\ -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ C_m^*(\sigma_m^*) + \sigma_m^* \end{pmatrix} \tag{2.121}$$

with

$$\begin{pmatrix} 1 & 0 & a_0 & c_0 \\ 0 & 1 & b_0 & d_0 \end{pmatrix} \begin{pmatrix} m_0 & n_0 \\ p_0 & q_0 \\ -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \tag{2.122}$$

from which one gets:

$$m_0 = a_0; \quad n_0 = -c_0; \quad p_0 = b_0; \quad q_0 = -d_0. \tag{2.123}$$

and replacing (2.123) into (2.120) and (2.121) and operating, one gets the solution of (2.68):

$$\delta_M^*(\sigma_M^*) = \frac{a_0 d_0 - b_0 c_0}{a_0 - c_0 \sigma_M^*} \tag{2.124}$$

$$C_M^*(\sigma_M^*) = \frac{b_0 - \sigma_M^*(d_0 - a_0 + c_0 \sigma_M^*)}{a_0 - c_0 \sigma_M^*} \tag{2.125}$$

$$\delta_m^*(\sigma_m^*) = \frac{a_0 d_0 - b_0 c_0}{d_0 + c_0 \sigma_m^*} \tag{2.126}$$

$$C_m^*(\sigma_m^*) = \frac{b_0 - \sigma_m^*(d_0 - a_0 + c_0 \sigma_m^*)}{d_0 + c_0 \sigma_m^*}. \tag{2.127}$$

where a_0, b_0, c_0 and d_0 are arbitrary constants.

Similarly, the functional equation (2.69) can be written as

$$\mathbf{A}^T \mathbf{B} = 0, \tag{2.128}$$

where

$$\mathbf{A} = \begin{pmatrix} B_M^*(\sigma_M^*)C_M^*(\sigma_M^*)\sigma_M^* - \lambda_M^*(\sigma_M *)\sigma_M^* - B_M^*(\sigma_M^*)(\sigma_M^*)^2 \\ B_M^*(\sigma_M^*)C_M^*(\sigma_M^*) - \lambda_M^*(\sigma_M^*) \\ B_M^*(\sigma_M^*)\sigma_M^* \\ C_M^*(\sigma_M^*)\sigma_M * -(\sigma_M^*)^2 \\ C_M^*(\sigma_M^*) \\ B_M^*(\sigma_M^*) \\ \sigma_M^* \\ 1 \end{pmatrix} \tag{2.129}$$

and

$$\mathbf{B} = \begin{pmatrix} 1 \\ -C_m^*(\sigma_m^*) - \sigma_m^* \\ C_m^*(\sigma_m^*) + 2\sigma_m^* \\ -B_m^*(\sigma_m^*) \\ B_m^*(\sigma_m^*)C_m^*(\sigma_m^*) - \lambda_m^*(\sigma_m^*) + B_m^*(\sigma_m^*)\sigma_m^* \\ -C_m^*(\sigma_m^*)\sigma_m^* - (\sigma_m^*)^2 \\ -B_m^*(\sigma_m^*)C_m^*(\sigma_m^*) + \lambda_m^*(\sigma_m^*) - 2B_m^*(\sigma_m^*)\sigma_m^* \\ B_m^*(\sigma_m^*)C_m^*(\sigma_m^*)\sigma_m^* - \lambda_m^*(\sigma_m^*)\sigma_m^* + B_m^*(\sigma_m^*)(\sigma_m^*)^2 \end{pmatrix}. \tag{2.130}$$

To solve this functional equation, one writes

$$\begin{pmatrix} B_M^*(\sigma_M^*)C_M^*(\sigma_M^*)\sigma_M^* - \lambda_M^*(\sigma_M *)\sigma_M^* - B_M^*(\sigma_M^*)(\sigma_M^*)^2 \\ B_M^*(\sigma_M^*)C_M^*(\sigma_M^*) - \lambda_M^*(\sigma_M^*) \\ B_M^*(\sigma_M^*)\sigma_M^* \\ C_M^*(\sigma_M^*)\sigma_M * -(\sigma_M^*)^2 \\ C_M^*(\sigma_M^*) \\ B_M^*(\sigma_M^*) \\ \sigma_M^* \\ 1 \end{pmatrix}$$
$$= \begin{pmatrix} \alpha & \beta & \gamma & \delta \\ \epsilon & \phi & C_0 & \eta \\ m_1 & n_1 & p_1 & q_1 \\ r_1 & s_1 & t_1 & u_1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} C_M^*(\sigma_M^*) \\ B_M^*(\sigma_M^*) \\ \sigma_M^* \\ 1 \end{pmatrix} \tag{2.131}$$

$$\begin{pmatrix} 1 \\ -C_m^*(\sigma_m^*) - \sigma_m^* \\ C_m^*(\sigma_m^*) + 2\sigma_m^* \\ -B_m^*(\sigma_m^*) \\ B_m^*(\sigma_m^*)C_m^*(\sigma_m^*) - \lambda_m^*(\sigma_m^*) + B_m^*(\sigma_m^*)\sigma_m^* \\ -C_m^*(\sigma_m^*)\sigma_m^* - (\sigma_m^*)^2 \\ -B_m^*(\sigma_m^*)C_m^*(\sigma_m^*) + \lambda_m^*(\sigma_m^*) - 2B_m^*(\sigma_m^*)\sigma_m^* \\ B_m^*(\sigma_m^*)C_m^*(\sigma_m^*)\sigma_m^* - \lambda_m^*(\sigma_m^*)\sigma_m^* + B_m^*(\sigma_m^*)(\sigma_m^*)^2 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ a & b & c & d \\ e & f & g & h \\ m & n & p & q \\ r & s & t & u \end{pmatrix} \begin{pmatrix} 1 \\ -C_m^*(\sigma_m^*) - \sigma_m^* \\ C_m^*(\sigma_m^*) + 2\sigma_m^* \\ B_m^*(\sigma_m^*) \end{pmatrix} \tag{2.132}$$

with

$$\begin{pmatrix} \alpha & \epsilon & m_1 & r_1 & 1 & 0 & 0 & 0 \\ \beta & \phi & n_1 & s_1 & 0 & 1 & 0 & 0 \\ \gamma & C_0 & p_1 & t_1 & 0 & 0 & 1 & 0 \\ \delta & \eta & q_1 & u_1 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ a & b & c & d \\ e & f & g & h \\ m & n & p & q \\ r & s & t & u \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{2.133}$$

from which one gets:

$$a = -\alpha; \ \ b = -\epsilon; \ \ c = -m_1; \ \ d = r_1; \ \ m = -\gamma; \ \ n = -C_0; \ \ p = -p_1; \ \ q = t_1; \tag{2.134}$$

$$e = -\beta; \ \ f = -\phi; \ \ g = -n_1; \ \ h = s_1; \ \ r = -\delta; \ \ s = -\eta; \ \ t = -q_1; \ \ u = u_1. \tag{2.135}$$

From the fourth row of (2.131) and using $C^*_M(\sigma^*_m)$ from (2.125) one deduces that $B^*_M(\sigma^*_M)$ must be of the following form:

$$B^*_M(\sigma^*_M) = \frac{m_2 + n_2\sigma^*_M + p_2(\sigma^*_M)^2}{a_0 - c_0\sigma^*_M}. \tag{2.136}$$

From the first and second rows of (2.131) and using again $C^*_M(\sigma^*_m)$ from (2.125) one deduces that $\lambda^*_M(\sigma^*_M)$ must be of the following for:

$$\lambda^*_M(\sigma^*_M) = \frac{m_3 + n_3\sigma^*_M + p_3(\sigma^*_M)^2 + q_3(\sigma^*_M)^3}{(\sigma^*_M - n_1)(a_0 - c_0\sigma^*_M)^2}. \tag{2.137}$$

Similarly, from the fifth and sixth rows of (2.132) and using $C^*_m(\sigma^*_m)$ from (2.127) one can derive the following two expressions for $B^*_m(\sigma^*_m)$ and $\lambda^*_m(\sigma^*_m)$, respectively:

$$B^*_m(\sigma^*_m) = \frac{m_4 + n_4\sigma^*_m + p_4(\sigma^*_m)^2}{d_0 + c_0\sigma^*_m} \tag{2.138}$$

$$\lambda^*_m(\sigma^*_m) = \frac{m_5 + n_5\sigma^*_m + p_5(\sigma^*_m)^2}{(d_0 + c_0\sigma^*_m)^2}. \tag{2.139}$$

We note that in order to deduce that the numerator of $\lambda^*_m(\sigma^*_m)$ is a second degree polynomial equation, one also has to use the seventh and eighth rows of (2.132) and compare the resulting expressions for $B^*_m(\sigma^*_m)$ and $\lambda^*_m(\sigma^*_m)$ with those in (2.138) and (2.139).

Replacing (2.136), (2.137), (2.138) and (2.139) into (2.69) one obtains a polynomial in σ^*_m and σ^*_M, which must be identically equal to zero, that is, all its coefficients must be null. This leads to $p_2 = 0$ and $p_4 = 0$, which, when substituted in (2.65) provides the model:

$$
\begin{aligned}
&\left[\frac{N^* - \left[B^*_m(\sigma^*_m) + \dfrac{\lambda^*_m(\sigma^*_m)}{\Delta\sigma^* - C^*_m(\sigma^*_m)}\right]}{\dfrac{\delta^*_m(\sigma^*_m)}{\Delta\sigma^* - C^*_m(\sigma^*_m)}}\right]^{A^*_m} \\
&\qquad = \left[\frac{N^* - \left[B^*_M(\sigma^*_M) + \dfrac{\lambda^*_M(\sigma^*_M)}{\Delta\sigma^* - C^*_M(\sigma^*_M)}\right]}{\dfrac{\delta^*_M(\sigma^*_M)}{\Delta\sigma^* - C^*_M(\sigma^*_M)}}\right]^{A^*_M} \\
&\qquad = 1 - \exp\left\{-\left[C^*_0 + C^*_1\sigma^*_m + C^*_2\sigma^*_M + C^*_3\sigma^*_m\sigma^*_M\right.\right. \\
&\qquad\quad \left.\left. + \left(C^*_4 + C^*_5\sigma^*_m + C^*_6\sigma^*_M + C^*_7\sigma^*_m\sigma^*_M\right)N^*\right]^{\beta^*}\right\} \qquad (2.140)
\end{aligned}
$$

in which the parameters have been redefined.

2.11 Appendix B: S-N curves for the general model

In particular, different parametric forms for the S-N field, as required by the user, can be selected from the Weibull model (2.104), as for instance:

1. $\Delta\sigma$-$\log N$ for constant σ_M:

$$
\begin{aligned}
\log N = & \frac{-C_0 + C_1\Delta\sigma - C_1\sigma_M - C_2\sigma_M + C_3\Delta\sigma\sigma_M - C_3\sigma_M^2}{C_4 - C_5\Delta\sigma + C_5\sigma_M + C_6\sigma_M - C_7\Delta\sigma\sigma_M + C_7\sigma_M^2} \\
& + \frac{(-\log[1-p])^{1/\beta}}{C_4 - C_5\Delta\sigma + C_5\sigma_M + C_6\sigma_M - C_7\Delta\sigma\sigma_M + C_7\sigma_M^2} \qquad (2.141)
\end{aligned}
$$

2. $\Delta\sigma$-$\log N$ for constant σ_m:

$$
\begin{aligned}
\log N = & \frac{-C_0 - C_2\Delta\sigma - C_1\sigma_m - C_2\sigma_m - C_3\Delta\sigma\sigma_m - C_3\sigma_m^2}{C_4 + C_6\Delta\sigma + C_5\sigma_m + C_6\sigma_m + C_7\Delta\sigma\sigma_m + C_7\sigma_m^2} \\
& + \frac{(-\log[1-p])^{1/\beta}}{C_4 + C_6\Delta\sigma + C_5\sigma_m + C_6\sigma_m + C_7\Delta\sigma\sigma_m + C_7\sigma_m^2} \qquad (2.142)
\end{aligned}
$$

3. $\Delta\sigma$-$\log N$ for constant σ_{mean}:

$$
\begin{aligned}
\log N = & \frac{-4C_0 + 2C_1\Delta\sigma - 2C_2\Delta\sigma + C_3\Delta\sigma^2}{H} \\
& + \frac{-4\sigma_{mean}(C_1 + C_2 + C_3\sigma_{mean}) + 4(-\log[1-p])^{1/\beta}}{H} \qquad (2.143)
\end{aligned}
$$

   where

$$H = 4C_4 - 2C_5\Delta\sigma + 2C_6\Delta\sigma - C_7\Delta\sigma^2 + 4\sigma_{mean}(C_5 + C_6 + C_7\sigma_{mean}).$$

4. $\Delta\sigma$-$\log N$ for constant R:

$$\log N = \frac{-C_0(R-1)^2 + \Delta\sigma(C_1 + C_2R(1-R) - R(C_1 + C_3\Delta\sigma^2))}{C_4(R-1)^2 + \Delta\sigma(C_5(R-1) + R(C_6(R-1) + C_7\Delta\sigma))} + \frac{(R-1)^2(-\log[1-p])^{1/\beta}}{C_4(R-1)^2 + \Delta\sigma(C_5(R-1) + R(C_6(R-1) + C_7\Delta\sigma)} \tag{2.144}$$

5. R-$\log N$ for constant σ_M):

$$\log N = -\frac{C_0 + C_2\sigma_M + C_1R\sigma_M + C_3R(\sigma_M)^2 - (-\log[1-p])^{1/\beta}}{C_4 + C_6\sigma_M + C_5R\sigma_M + C_7R(\sigma_M)^2}. \tag{2.145}$$

The corresponding Gumbel S-N curves can be obtained by replacing $(-\log[1-p])^{1/\beta}$ by $\log(-\log(1-p))$.

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Castillo, E.; Fernandez-Canteli, A.

2009, XIV, 232 p., Hardcover
ISBN: 978-1-4020-9181-0