

Chapter 2

The Problem

The system we wish to study is a binary mixture of non reactive dilute, electrically charged system of particles. Their masses will be labelled m_a and m_b with charges e_a and e_b where $e_a = -e_b = e$. The ions could have a positive charge Ze but we shall keep $Z = 1$ for simplicity. The number densities of the species are n_a and n_b where $n_a + n_b = n$ so that the total mass density ρ is given by

$$\rho = \rho_a + \rho_b = m_a n_a + m_b n_b \quad (2.1)$$

Following the standard notation of the kinetic theory of gases, the single particle distribution functions for each species is denoted by $f_i(\vec{r}, \vec{v}_i, t)$ where \vec{v}_i is the velocity of the particle of species i , $i = a, b$. If we now assume that in general the system is acted upon by an electric field \vec{E} measured in volts m^{-1} and a magnetic induction \vec{B} in teslas, the Boltzmann equation determining the time evolution of the distribution function f_i is given by

$$\frac{\partial f_i}{\partial t} + \vec{v}_i \cdot \frac{\partial f_i}{\partial \vec{r}} + \frac{1}{m_i} \left(\vec{F}_i + e_i \vec{v}_i \times \vec{B} \right) \cdot \frac{\partial f_i}{\partial \vec{v}_i} = \sum_{i,j=a}^b J(f_i f_j) \quad (2.2)$$

Here,

$$\vec{F}_i = F_i^{(e)} + e_i \vec{E} \quad \text{for } i, j = a, b \quad (2.3)$$

$F_i^{(e)}$ denoting an external conservative force, the electric and magnetic fields, \vec{E} and \vec{B} respectively, are the self consistent fields generated by the plasma as determined from Maxwell's equations, and

$$J(f_i f_j) = \int \cdots \int \left\{ f(\vec{v}_i) f(\vec{v}_j) - f(\vec{v}_i) f(\vec{v}_j) \right\} \sigma \left(\vec{v}_i \vec{v}_j \rightarrow \vec{v}_i' \vec{v}_j' \right) g_{ij} d\vec{v}_j d\vec{v}_i' d\vec{v}_j' \quad (2.4)$$

In Eq. (2.4) we recall the reader that the \vec{r}, t dependence of the f_i 's has been omitted. The primes denote the values of v_i after the binary collision takes place, $\sigma(\vec{v}_i \vec{v}_j \rightarrow \vec{v}'_i \vec{v}'_j) d\vec{v}'_i d\vec{v}'_j$ is the cross section, namely, the number of molecules per unit time of species i colliding with a molecule of species j such that after the collision the molecules have velocities \vec{v}'_i in the range $d\vec{v}'_i$ and \vec{v}'_j in the range $d\vec{v}'_j$; $g_{ij} \equiv |\vec{v}_i - \vec{v}_j| = |\vec{v}'_i - \vec{v}'_j|$. For collisions between molecules of the same species $\vec{v}_i \rightarrow \vec{v}$ and $\vec{v}_j \rightarrow \vec{v}_1$ to distinguish the two velocities. A caution note has to be mentioned with respect of Eq. (2.2). The magnetic induction \vec{B} is taken to be the average magnetic field, determined from Maxwell's equations where the current density will depend on the distribution functions f_i . In fact one should write $\vec{B} = \vec{B}_{av} + \vec{B}_e$ where \vec{B}_e is an external field which may or may not be taken as a constant field.¹ We also recall the reader that the cross section σ satisfies the principle of microscopic reversibility, namely, it is invariant upon spatial and temporal reflections, so that,

$$\sigma(\vec{v}_i \vec{v}_j \rightarrow \vec{v}'_i \vec{v}'_j) = \sigma(\vec{v}'_i \vec{v}'_j \rightarrow \vec{v}_i \vec{v}_j) \quad \text{for } i, j = a, b \quad (2.5)$$

thus guaranteeing the existence of inverse collisions.

As well known in kinetic theory, two general results may be derived from Eq. (2.4) regardless of the specific form of the cross section that is, without specifying the details of the interaction potential between the particles. Such results are the conservation equations and the H theorem. In our case this will require particular care since collisions do not exist for Coulomb interactions which as well known is a long range repulsive potential. Advancing the fact that this will be appropriately taken care of using the Debye-Hückel approximation we assume that σ is well defined and finite. We proceed to discuss the first of two general results namely, the conservation equations. Section 2.2 will be devoted to the H-theorem.

2.1 Conservation Equations

As usual, we define the local particle densities as,

$$n_i(\vec{r}, t) = \int f_i(\vec{r}, \vec{v}_i, t) d\vec{v}_i \quad (2.6)$$

¹For a thorough discussion of this question see [7].

and denote by $\psi_i(\vec{r}, \vec{v}_i, t)$ any dynamical variable whose local value is given by

$$\langle \psi_i \rangle \equiv \psi_i(\vec{r}, t) = \frac{1}{n_i} \int \psi_i(\vec{r}, \vec{v}_i, t) f_i(\vec{r}, \vec{v}_i, t) d\vec{v}_i \quad (2.7)$$

Moreover, we define the thermal or chaotic velocity \vec{c}_i as

$$\vec{c}_i = \vec{v}_i - \vec{u}(\vec{r}, t) \quad (2.8)$$

and \vec{u} is the barycentric velocity given by

$$\rho \vec{u}(\vec{r}, t) = \sum_i \rho_i \vec{u}_i(\vec{r}, t) \quad (2.9)$$

where,

$$\vec{u}_i(\vec{r}, t) = \frac{1}{n_i} \int f_i(\vec{r}, \vec{v}_i, t) \vec{v}_i d\vec{v}_i \quad (2.10)$$

is the local hydrodynamic velocity for species i . Notice here that contrary to what occurs in the case of a single species, $\langle \vec{c}_i \rangle \neq 0$ whereas

$$m_a n_a \langle \vec{c}_a \rangle + m_b n_b \langle \vec{c}_b \rangle = \rho_a \vec{u}_a + \rho_b \vec{u}_b - \rho \vec{u} = 0$$

or

$$\sum_i \rho_i \langle \vec{c}_i \rangle = 0 \quad (2.11)$$

This expression is important because the mass diffusion flux of the i^{th} species is defined as

$$\vec{J}_i = m_i \int \vec{c}_i f_i(\vec{r}, \vec{v}_i, t) d\vec{v}_i = m_i n_i \langle \vec{c}_i \rangle \quad (2.12)$$

so that by Eq. (2.11)

$$\sum_i \vec{J}_i = 0 \quad (2.13)$$

or $\vec{J}_a = -\vec{J}_b$.

With these definitions, the flow of charge is readily expressed in a convenient way. In fact, the numerical charge density Q is defined as

$$Q = n_a e_a + n_b e_b = (n_a - n_b) e \quad (2.14a)$$

and the charge current

$$\vec{J}_T = \sum_i n_i e_i \langle \vec{v}_i \rangle \quad (2.14b)$$

which, with the aid of Eq. (2.8) reads $\vec{J}_T = Q\vec{u} + \vec{J}_c$, \vec{J}_c the conduction current being given by

$$\vec{J}_c = \sum_i n_i e_i \langle \vec{c}_i \rangle \quad (2.14c)$$

which in turn can be written with the aid of Eqs. (2.12) and (2.13) as

$$\vec{J}_c = \frac{m_a + m_b}{m_a m_b} e \vec{J}_a \quad (2.15)$$

a result often ignored by authors of this subject.

Returning to our quest, we now derive the equivalent of Maxwell-Enskog's transport equation by taking $\psi_i = (m_i, m_i \vec{v}_i$ and $\frac{1}{2} m_i \vec{v}_i^2)$. We first notice that from Eq. (2.4)

$$\sum_{i,j=a}^b \int \psi_i J(f_i f_j) d\vec{v}_i = 0 \quad (2.16)$$

a result which follows from the standard transformation of the collision kernels using Eq. (2.5) and the fact that i and j are dummy indices in Eq. (2.16).

So let $\psi_i = m_i$. Multiplying (2.2) by m_i and integrating over $d\vec{v}_i$ using (2.16) one gets

$$\frac{\partial \rho_i}{\partial t} + \text{div} (\rho_i \vec{u}_i) = \int (\vec{v}_i \times \vec{B}) \cdot \frac{\partial f_i}{\partial \vec{v}_i} d\vec{v}_i$$

In the right hand term, for any component $\frac{\partial f_i}{\partial \vec{v}_i}$ the cross product $(\vec{v}_i \times \vec{B})$ does not contain such component so that the integration by parts yields zero whence

$$\frac{\partial \rho_i}{\partial t} + \text{div} (\rho_i \vec{u}_i) = 0 \quad (2.17a)$$

and summation over i yields

$$\frac{\partial \rho}{\partial t} + \text{div} (\rho \vec{u}) = 0 \quad (2.17b)$$

Using Eqs. (2.8) and (2.12) in Eq. (2.17a) we may also write that

$$\frac{\partial \rho_i}{\partial t} + \text{div} (\rho_i \vec{u}) = -\text{div} \vec{J}_i \quad (2.17c)$$

Eqs. (2.17a)-(2.17c) are thus the several alternative expressions for mass conservation.

Take now $\psi_i = m_i \vec{v}_i = m_i(\vec{c}_i + \vec{u})$. Multiply Eq. (2.2) by it using Eq. (2.11), after integrating a couple of terms by parts and summing over i , one readily gets that

$$\frac{\partial}{\partial t}(\rho \vec{u}) + \text{div}(\overleftarrow{\tau}^k + \rho \vec{u} \vec{u}) = \sum_i n_i \vec{F}_i - \sum_i e_i \int \vec{v}_i (\vec{v}_i \times \vec{B}) \cdot \frac{\partial f_i}{\partial \vec{v}_i} d\vec{v}_i$$

where the kinetic part of the stress tensor $\overleftarrow{\tau}^k$ is defined as

$$\overleftarrow{\tau}^k = \sum_{i=a}^b m_i \int f_i \vec{c}_i \vec{c}_i d\vec{v}_i \quad (2.18)$$

Integration by parts of the last term reduces to $\sum_i e_i n_i \langle \vec{v}_i \rangle \times \vec{B}$ so that we reach the result that

$$\frac{\partial}{\partial t}(\rho \vec{u}) + \text{div}(\overleftarrow{\tau}^k + \rho \vec{u} \vec{u}) = \sum_i n_i \vec{F}_i + (\vec{J}_T \times \vec{B}) \quad (2.19)$$

the conservation equation for momentum. If the external force is zero using the definition of \vec{J}_T we readily find that

$$\frac{\partial}{\partial t}(\rho \vec{u}) + \text{div}(\overleftarrow{\tau}^k + \rho \vec{u} \vec{u}) = Q(\vec{E} + (\vec{u} \times \vec{B})) + \vec{J}_c \times \vec{B} \quad (2.20)$$

Here $\vec{E}' = \vec{E} + \vec{u} \times \vec{B}$ can be interpreted as the effective electric field as viewed by an observer moving in the mixture with the barycentric velocity \vec{u} . Also, it should be pointed out that often Eqs. (2.17a)-(2.17c) and (2.20) are referred to as the equations of magnetohydrodynamics for isothermal fluids in the absence of external fields $\vec{F}^e = 0$.

We finally take $\psi_i = \frac{1}{2} m_i v_i^2$ and repeat the procedure as in the previous case. After summation over i and use of Eq. (2.11) we get that,

$$\begin{aligned} & \frac{1}{2} \frac{\partial}{\partial t}(\rho u^2) + \frac{\partial}{\partial t} \sum_i \frac{1}{2} m_i \int f_i d\vec{v}_i c_i^2 + \sum_i \frac{1}{2} m_i \int \vec{v}_i \cdot \frac{\partial f_i}{\partial \vec{r}} v_i^2 d\vec{v}_i + \\ & \sum_i \frac{1}{2} \int \vec{F}_i \cdot v_i^2 \frac{\partial f_i}{\partial \vec{v}_i} d\vec{v}_i + \frac{1}{2} \sum_i e_i \int (\vec{v}_i \times \vec{B}) \cdot \frac{\partial f_i}{\partial \vec{v}_i} v_i^2 d\vec{v}_i = 0 \end{aligned} \quad (2.21)$$

We define the internal energy density of the mixture as,

$$\rho e(\vec{r}, t) = \sum_i \frac{1}{2} \rho_i \langle c_i^2 \rangle \quad (2.22)$$

In the third term we set $\vec{v}_i = \vec{c}_i + \vec{u}$, expand, use Eq. (2.11) and find that it reduces to

$$\text{div}(\vec{J}_q + \vec{u} \cdot \overleftarrow{\tau}^k + \rho e \vec{u} + \frac{1}{2} \rho \vec{u} u^2)$$

where

$$\vec{J}_q = \sum_i \frac{1}{2} \rho_i \langle \vec{c}_i c_i^2 \rangle \quad (2.23)$$

is the total heat flux in the mixture. After a first integration by parts, use of (2.14c), the definition of \vec{J}_T and assuming $\vec{F}^e = \vec{0}$, the fourth term simply reduces to $-\vec{J}_T \cdot \vec{E}$. Finally integration by parts clearly shows that the last term vanishes, so that collecting all terms we find that,

$$\frac{1}{2} \frac{\partial}{\partial t} (\rho u^2) + \frac{\partial \rho e}{\partial t} + \text{div} (\vec{J}_q + \vec{u} \cdot \overleftarrow{\tau}^k + \rho \vec{u} e + \frac{1}{2} \rho \vec{u} u^2) - \vec{J}_T \cdot \vec{E} = 0$$

Using Eq. (2.20) and following the standard steps to combine the first three terms in this equation we are finally lead to the balance equation for the internal energy namely,

$$\rho \frac{d}{dt} e + \text{div} \vec{J}_q + \overleftarrow{\tau}^k : \text{grad} \vec{u} - \vec{J}_c \cdot \vec{E}' = 0 \quad (2.24)$$

where as introduced above,

$$\vec{E}' = \vec{E} + \vec{u} \times \vec{B}$$

Eqs. (2.17a)-(2.17c), (2.20) and (2.24) are the sought result for the conservation equations. Clearly the unknowns \vec{J}_i , $\overleftarrow{\tau}^k$ and \vec{J}_q have to be determined by seeking solutions to Eq. (2.2), a task to be dealt with later.

2.2 The H Theorem and Local Equilibrium

Before discussing these important properties of the Boltzmann equation we need to specify clearly the domain of its applicability. In the absence of a magnetic field Eq. (2.2) is valid in the so called kinetic regime characterized by time $t \sim \tau$ the mean free time where $\tau \gg t_c$ the duration of a collision time. However, in the presence of a magnetic field we have two characteristic frequencies competing in the mixture, the collision frequency $\omega_c \sim 1/\tau$ and the Larmor frequencies $\omega_i = \frac{|e|B}{m_i}$. For electrons $\omega_e \sim 1.76 \times 10^{11} B$ whereas

for ions $\omega_i = \omega_e \frac{m_e}{m_i}$. If the field is weak enough $\omega_i \tau$ is of the order of 1 for both cases implying that the field does not interfere in the collisional regime of the mixture. We shall limit ourselves to this case. When $\omega_i \tau \gg 1$ radical modifications have to be made to the whole approach to the problem and we shall not discuss it here at all (see however Ref. [7]). Once this is clarified we proceed with our discussion. If we multiply Eq. (2.2) by $\ln f_i$ integrate over $d\vec{v}_i$ and sum over i the left hand side vanishes since the only extra term, $\int (\vec{v}_i \times \vec{B}) \cdot \frac{\partial f_i}{\partial \vec{v}_i} \ln f_i d\vec{v}_i$ vanishes after integration by parts. Therefore, using the same procedure for the right hand side as in the single component case remembering Eq. (2.2) and Klein's inequality one obtains that for

$$H \equiv \sum_i \int f_i \ln f_i d\vec{v}_i, \quad (2.25)$$

$$\frac{\partial H(\vec{r}, t)}{\partial t} \leq 0 \quad (2.26)$$

for all binary collisions and their corresponding inverses $(i, j \rightleftharpoons i', j')$. Remember that in Eq. (2.5), $H \equiv H(\vec{r}, t)$ is still function of \vec{r} and t . So the irreversibility criteria imposed by Eq. (2.26) is still valid in the weak field approximation and moreover, the quantity usually associated with the entropy production $\sigma(\vec{r}, t)$ is always positive definite for all exact solutions to Eq. (2.4)

$$\sigma = -k \sum_{i,j} \int \ln f_i J(f_i f_j) d\vec{v}_i \quad (2.27)$$

This result will be used later on. We also notice that the solution to the homogenous Boltzmann equation, namely,

$$J(f_i^{(0)} f_j^{(0)}) + J(f_i^{(0)} f_j^{(0)}) = 0 \quad \text{for } i, j = a, b$$

is a local Maxwellian distribution function. This arises from the well known argument stating that $\frac{\partial H}{\partial t} = 0$ for every binary collision. By the standard argument of kinetic theory this implies that

$$f_i^{(0)} = n_i(\vec{r}, t) \left(\frac{m_i}{2\pi kT(\vec{r}, t)} \right)^{\frac{3}{2}} e^{-\frac{m(\vec{v}_i - \vec{u}(\vec{r}, t))^2}{2kT(\vec{r}, t)}} \quad (2.28)$$

provided we define

$$n_i(\vec{r}, t) = \int f_i^{(0)} d\vec{v}_i \quad (2.29a)$$

$$\rho \vec{u} = \sum_i \rho_i \int f_i^{(0)} \vec{v}_i d\vec{v}_i \quad (2.29b)$$

$$\rho e(\vec{r}, t) = \frac{3}{2} n k T = \sum_i \frac{1}{2} \rho_i \langle c_i^2 \rangle \quad (2.29c)$$

Nevertheless Eq. (2.28) is still not a solution to the full Boltzmann equation since it is necessary that

$$\left(\frac{\partial}{\partial t} + \vec{v}_i \cdot \frac{\partial}{\partial \vec{r}} + \frac{\vec{F}_i}{m_i} \cdot \frac{\partial}{\partial \vec{v}_i} + \frac{e_i}{m_i} (\vec{v}_i \times \vec{B}) \cdot \frac{\partial}{\partial \vec{v}_i} \right) \ln f_i^{(0)} = 0 \quad (2.30)$$

is satisfied for $i = a, b$. The procedure is, once more, the standard one [1]-[2]. We write

$$\ln f_i^{(0)} = \nu(\vec{r}, t) + \vec{k}(\vec{r}, t) \cdot \vec{v}_i - h(\vec{r}, t) v_i^2 \quad (2.31)$$

where $\nu = \ln A - \frac{m_i \beta}{2} u^2$, $\vec{k} = \beta m_i \vec{u}$; $h = \frac{m_i \beta}{2}$; $A = n_i \left(\frac{m_i \beta}{2\pi} \right)^{\frac{3}{2}}$ with $\beta = (kT)^{-1}$.

Substitution of (2.31) into (2.30) and noticing that $(\vec{v}_i \times \vec{B}) \cdot \vec{v}_i = 0$ we get that,

$$\begin{aligned} & \frac{\partial \nu}{\partial t} - v_i^2 \vec{v}_i \cdot \frac{\partial h}{\partial \vec{r}} + \left(-v_i^2 \frac{\partial h}{\partial t} + \vec{v}_i \cdot \left(\vec{v}_i \cdot \frac{\partial \vec{k}}{\partial \vec{r}} \right) \right) + \\ & \vec{v}_i \cdot \left(\frac{\partial \vec{k}}{\partial t} + \frac{\partial \nu}{\partial \vec{r}} - \frac{2h}{m_i} \vec{F}_i \right) + \vec{k} \cdot (\vec{v}_i \times \vec{B}) \frac{e_i}{m_i} + \frac{\vec{F}_i}{m_i} \cdot \vec{k} = 0 \end{aligned}$$

which must hold for all values of \vec{v} . The coefficients of order v_i^3 and v_i^2 do not depend on \vec{B} so by the standard procedure $h = h(t)$ and $\vec{k} = \vec{r} \frac{\partial h}{\partial t} + \vec{r} \times \vec{\Omega}(t) + \vec{k}_0(t)$. For conservative forces (including $\vec{F} = -e \text{ grad } \phi$) the linear coefficient in \vec{v}_i yields

$$\frac{\partial \vec{k}}{\partial t} + \text{grad} \left(\nu + \frac{2he_i}{m_i} \phi_i \right) - \frac{e_i}{m_i} \vec{k} \times \vec{B} = 0$$

and ϕ_i is the electrical potential. Scalar multiplication by \vec{B} , yields in turn that

$$\vec{B} \cdot \left(\frac{\partial \vec{k}}{\partial t} + \text{grad} \left(\nu + \frac{2he_i}{m_i} \phi_i \right) \right) = 0$$

which for $\vec{B} \neq \vec{0}$ and ignoring the possible but unlikely occurrence that \vec{B} is perpendicular to the term in parenthesis,

$$\frac{\partial \vec{k}}{\partial t} + \text{grad} \left(\nu + \frac{2he_i}{m_i} \phi_i \right) = 0$$

This implies $\frac{\partial}{\partial t} \text{rot} \vec{k} = \vec{0}$ or $\vec{\Omega}$ is a constant vector and, once more by the argument for a one component system, and non-pathological external forces,

$$f_i^{eq} = n_i \left(\frac{m_i}{2\pi kT} \right)^{\frac{3}{2}} \exp \left\{ -\beta \left(\frac{m_i v_i^2}{2} + \phi_i(\vec{r}) \right) \right\} \quad \text{for } i = a, b \quad (2.32)$$

where the potential energy is $\phi_i = \phi_{ext} + e_i \phi$. Thus equilibrium is achieved and characterized by the Maxwell distribution function Eq. (2.32).

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