Chapter 2

BASICS AND LOGIC FUNCTIONS

In order to understand and to solve the problems in this chapter, it is recommended to read Chaps. 1, 2 and 3 of [18] and to follow carefully the introduction into the XBOOLE system. Make sure that you know the respective definitions and concepts and refer to the given examples.

1. Combinatorial Considerations in B and B^n

The use of the definitions for the relation $\leq$ and the operations $\wedge$ and $\vee$ will solve the following problem using the respective tables.

Exercise 2.1 (Relations in B). 1 Show that $x \leq (x \vee y)$ and $x \geq xy$.

2 Show that if $x_1 \leq y_1$ and $x_2 \leq y_2$ then $x_1x_2 \leq y_1y_2$ and $(x_1 \vee x_2) \leq (y_1 \vee y_2)$.

3 Show that $x \leq (y \vee z)$ if $x \leq y$ or $x \leq z$.

4 Show that $x \leq yz$ is equivalent to $(x \leq y)$ and $(x \leq z)$.

We remember that the positions from the right to the left of a binary vector can be considered as the values $x_0, x_1, \ldots$ of a binary number which have to be multiplied by $2^0, 2^1, \ldots,$ and the products have to be added. Because of the values 0 and 1, only the values for the positions with the value 1 have to be added.

Exercise 2.2 (Binary Vectors). 1 Find the decimal equivalent $\mathrm{dec}(\mathbf{x})$ for the vectors $(1001) \in B^4$, $(01101) \in B^5$, $(110010) \in B^6$.

2 Find the vector $\mathbf{x} \in B^6$ with $\mathrm{dec}(\mathbf{x}) = 19$. Find the vector $\mathbf{x}$ for $\mathrm{dec}(\mathbf{x}) = 19$ in B^8.

3 Find the binary vectors $\mathbf{x}$ with $2^{n-1} \leq \mathrm{dec}(\mathbf{x}) < 2^n$.

Bernd Steinbach, Christian Posthoff, *Logic Functions and Equations, Examples and Exercises,*

4 Let be given the vector $\mathbf{x} = (10010101) \in B^8$.

(a) Find all $\mathbf{y} \in B^8$ with $\mathbf{x} \le \mathbf{y}$.
(b) Find all $\mathbf{y} \in B^8$ with $\mathbf{y} \le \mathbf{x}$.

5 Let be given two vectors $\mathbf{x}$, $\mathbf{y} \in B^n$ with $\mathbf{x} \le \mathbf{y}$. Find all vectors $\mathbf{z}$ with $\mathbf{x} \le \mathbf{z} \le \mathbf{y}$.

This small example can also be used to find a solution based on *Logic Equations*. We remember that the relation $x \le z$ can be equivalently translated into the equation $x \wedge \overline{z} = 0$. For two vectors $\mathbf{x}$ and $\mathbf{z}$ the equation must hold in each component, hence we get the equivalent system of equations

$$x_1 \wedge \overline{z}_1 = 0, \quad \ldots, \quad x_n \wedge \overline{z}_n = 0,$$

and in the same way

$$z_1 \wedge \overline{y}_1 = 0, \quad \ldots, \quad z_n \wedge \overline{y}_n = 0.$$

Now we use, for instance, $\mathbf{x} = (x_1, \ldots, x_6) = (010101)$ and $\mathbf{y} = (y_1, \ldots, y_6) = (110111)$ and insert the constants which results in

$$\begin{array}{lll} 0 \wedge \overline{z}_1 = 0, & 1 \wedge \overline{z}_2 = 0, & 0 \wedge \overline{z}_3 = 0, \\ 1 \wedge \overline{z}_4 = 0, & 0 \wedge \overline{z}_5 = 0, & 1 \wedge \overline{z}_6 = 0 \end{array}$$

and

$$\begin{array}{lll} z_1 \wedge 0 = 0, & z_2 \wedge 0 = 0, & z_3 \wedge 1 = 0, \\ z_4 \wedge 0 = 0, & z_5 \wedge 0 = 0, & z_6 \wedge 0 = 0. \end{array}$$

These two sets of equations define the solution immediately, and we get

$$z_1 = -, \quad z_2 = 1, \quad z_3 = 0, \quad z_4 = 1, \quad z_5 = -, \quad z_6 = 1,$$

i.e.

$$\mathbf{z} = (-101-1).$$

The two values $z_1 = -$ and $z_5 = -$ are based on the fact that the coefficients of z_1 and z_5 in both equations are equal to 0 which means that the equation is identically satisfied without binding the value of z_1 and z_5. This can also be seen from $x_1 = 0$, $y_1 = 1$ and $x_5 = 0$, $y_5 = 1$.

The function $\|\mathbf{x}\|$ counts the number of values 1 in a given vector.

Exercise 2.3 (Binary Vectors). Let $\mathbf{x}$ and $\mathbf{y}$ be two elements of B^n. Show that

1 $\|\overline{\mathbf{x}}\| = n - \|\mathbf{x}\|$;

2 $\|\mathbf{x} \vee \mathbf{y}\| = \|\mathbf{x}\| + \|\mathbf{y}\| - \|\mathbf{x}\mathbf{y}\|$;

3 $\|\mathbf{xy}\| = \|\mathbf{x}\| + \|\mathbf{y}\| - \|\mathbf{x} \vee \mathbf{y}\|$.

Now we consider the *shell* of a sphere which is given by $S_i(\mathbf{x}, \mathbf{c}) = \{\mathbf{x} \mid h(\mathbf{x}, \mathbf{c}) = i\}$, for $i = 0, \ldots, n$, with h as the HAMMING-metric and $\mathbf{c}$ as the center of the sphere. The *open* and the *closed spheres with center* $\mathbf{c}$ themselves are given by $K_i(\mathbf{x}, \mathbf{c}) = \{\mathbf{x} \mid h(\mathbf{x}, \mathbf{c}) < i\}$ and $\overline{K}_i(\mathbf{x}, \mathbf{c}) = \{\mathbf{x} \mid h(\mathbf{x}, \mathbf{c}) \leq i\}$, resp. (see [18], pp. 33, 34). For closed spheres, the radius i will also have the values $0, \ldots, n$, for open spheres we can set $K_0 = \emptyset$ and allow $i = n + 1$ since $\overline{K}_n = K_{n+1}$.

Exercise 2.4 (Shells and Spheres). 1 Use $\mathbf{c} = (0000)$ and find $S_i(\mathbf{x}, \mathbf{c})$ for $i = 0, \ldots, 4$ with regard to this center.

2 Now use $\overline{\mathbf{c}} = (1111)$ and find $S_i(\mathbf{x}, \overline{\mathbf{c}}), i = 0, \ldots, 4$ with regard to this new center.

3 Confirm that $S_i(\mathbf{x}, \mathbf{c}) = S_{n-i}(\mathbf{x}, \overline{\mathbf{c}})$.

4 Let $n = 4, \mathbf{c} = (0000)$. Show that $K_0 = \emptyset$, $K_1 = S_0, \ldots,$ $K_5 = S_0 \cup \cdots \cup S_4$. Generalize this relation to any value of n.

5 Let $n = 4, \mathbf{c} = (0000)$. Show that $\overline{K}_0 = S_0, \ldots, \overline{K}_4 = S_0 \cup \cdots \cup S_4$. Generalize this relation to any value of n.

6 What is the relation between spheres with center $\mathbf{c}$ and spheres with center $\overline{\mathbf{c}}$? Base your considerations on the analogous relation for shells.

2. Logic Functions, Formulas and Expressions

The definition of a logic function is quite easy and fully supported by the XBOOLE Monitor. After starting the XBOOLE Monitor we use the sequence `Objects` – `Define Space` – `Create TVL` – `Append Ternary Vector(s)`.

And now we simply write down all the (ternary) vectors for which the function to be defined has the value 1. When we are using the variables $x1$ and $x2$ and the only ternary vector is selected as (11), then $f = x1 \wedge x2$. By clicking on `K` the representation can change to the Karnaugh map for this function, the letter changes to `T`, and when we click on this letter now, we go back to the representation as TVL with the character `K` on the key.

A second XBOOLE possibility is the use of the concept of a *Logic Expression* or a *Logic Formula*. The menu point `Extras` offers the possibility `Solve Boolean Equation` which is, according to the philosophy of the book, one of the most powerful and sophisticated options of the monitor. After clicking on this topic a new window opens, and there the respective expression (formula) can be typed. Hint: the system is using as a standard that the right side of an equation is equal to 1. We type, for instance, $x3 \,\&\, x4$ and get the solution vector (11) and an object number for this solution set. The view is exactly the same as before for the definition based on TVL. Hence, the two possibilities are completely equivalent. In this way any formula can be used for the input of the respective function.

If we want to emphasize that an equation has a value of 1, then we can type $f = 1$. The system understands the $=$ as the equivalence function $\sim$, and, since we know that $f \sim 1$ is equal to f, we get the correct solution. In the case of an equation $f = 0$ the system solves $(f \sim 0) = 1$, and since $f \sim 0$ is equal to $\overline{f}$, we solve actually $\overline{f} = 1$ which means $f = 0$, again without any problem. The best way to avoid any confusion: type the equation to be solved without the value 0 or 1 on the right side and solve it. If a solution for the value 0 on the right side is required, then use additionally the set operation *complement*.

Exercise 2.5 (Definition of Functions). 1 Define the functions

$$f_0(x) = \overline{x}, \quad f_1(x,y) = x \wedge y, \quad f_2(x,y) = x \vee y,$$
$$f_3(x,y) = x \oplus y, \quad f_4(x,y) = x \sim y, \quad f_5(x,y) = x \rightarrow y,$$
$$f_6(x,y) = x|y = \overline{x \wedge y}, \quad f_7(x,y) = x \downarrow y = \overline{x \vee y}$$

using TVLs.

2 Define the same set of functions using `Solve Boolean Equations ...`

3 Which functions are represented by solving the equations $(x \wedge y) = 0$ and $(x \vee y) = 0$, resp. Use the Karnaugh map to define an appropriate expression for these functions.

4 Show that the expressions $(x \oplus y) \oplus z = 1$ and $x \oplus (y \oplus z) = 1$ define the same function?

5 Answer the same question for $\wedge$, $\vee$, $\sim$, $\rightarrow$.

Very often the question arises how many functions of a given type can be found. Problems of this kind are dealt with mostly by methods from combinatorics. As an example the following question should be considered.

Exercise 2.6 (Combinatorial Properties). 1 How many functions exist with $f(x_1, \ldots, x_n) = f(\overline{x}_1, \ldots, \overline{x}_n)$?

2 Two binary vectors $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots, y_n)$ are called neighbored if $h(\mathbf{x}, \mathbf{y}) = 1$. How many functions exist with $f(\mathbf{x}) = \overline{f(\mathbf{y})}$ for neighbored vectors $\mathbf{x}$ and $\mathbf{y}$?

3 How many functions of n variables exist with less than k values 1, $k \geq 1$?

Sometimes logic functions can be defined in a given context based on verbal descriptions. In a very general understanding this can be understood as "logic modeling" and sometimes be very difficult. We start here with a very simple example.

Exercise 2.7 (Logic Modeling). Find the Karnaugh map, a TVL and a disjunctive form for the following functions!

1 The function $f(x, y, z)$ has the value 1 either for $x = 1$ or if $y \neq z$ and the value of x is less than the value of z, otherwise the value of the function is equal to 0.

2 $f(x_1, x_2, x_3, x_4) = 0$ for such vectors satisfying $x_1 + x_2 > x_3 + 2x_4$. Hint: understand 0 and 1 as integers and $+$ as the addition for integers.

Exercise 2.8 (Definition of Functions by Formulas). Which functions are defined by the following formulas (equations):

1 $(x \to y) \oplus ((y \to z) \oplus (z \to x))$;

2 $\overline{(\overline{x} \vee y) \vee (x\overline{z})} \downarrow (x \sim y)$;

3 $\overline{x} \to (\overline{z} \sim (y \oplus xz))$;

4 $(((x \mid y) \downarrow z) \mid y) \downarrow z$.

Give the disjunctive, conjunctive, antivalence and equivalence normal forms for these functions.

Exercise 2.9 (Normal Forms). Find the disjunctive and the antivalence normal form of the following functions:

1 $f = ((x_1 \vee x_2\overline{x}_3x_4)((\overline{x}_2 \vee x_4) \to x_1\overline{x}_3\overline{x}_4) \vee x_2x_3)(\overline{x}_1 \vee x_4)$;

2 $f = ((x_1 \to x_2x_3)(x_2x_4 \oplus x_3) \to x_1\overline{x}_4) \vee \overline{x}_1$;

3 $f = (x_1 \vee x_2 \vee x_3 \vee \cdots \vee x_9 \vee x_{10})(\overline{x}_1 \vee \overline{x}_2 \vee \overline{x}_3 \vee \cdots \vee \overline{x}_9 \vee \overline{x}_{10})$;

4 $f = (x_1 \vee x_2 \vee x_3)(\overline{x}_1 \vee \overline{x}_2 \vee \overline{x}_3) \oplus x_4 \oplus x_5 \oplus x_6 \oplus x_7 \oplus x_8 \oplus x_9 \oplus x_{10}$.

Exercise 2.10 (Normal Forms). Find the conjunctive and the equivalence normal forms of the functions given in the previous question.

Exercise 2.11 (Normal Forms). Generalize the two last items of the previous question to larger values of n:

1 $f = (x_1 \vee x_2 \vee \cdots \vee x_n)(\overline{x}_1 \vee \overline{x}_2 \vee \cdots \vee \overline{x}_n)$;

2 $f = (x_1 \vee x_2 \vee x_3)(\overline{x}_1 \vee \overline{x}_2 \vee \overline{x}_3) \oplus x_4 \oplus \cdots \oplus x_n$.

Exercise 2.12 (Antivalence Polynomial). Find the antivalence polynomial and the equivalence polynomial for the following functions:

1 $f = (x_1|x_2) \downarrow x_3$;

2 $f = (x_1 \rightarrow x_2(x_2 \downarrow x_3))$;

3 $f = ((x_1 \rightarrow x_2) \vee \overline{x}_3)|x_1$.

Not very often the transformation of a logic function into a "normal" polynomial is used. This polynomial only uses addition, subtraction and multiplication. A given elementary conjunction (containing all variables) will be translated in the following way: each variable remains unchanged, for negated variables x we write $1-x$, and then all variables are combined by multiplication. The use of elementary conjunctions ensures that at most only one conjunction will be equal to 1 for a given set of values, hence, the expressions for the different conjunctions can be added to form such a polynomial. As an example see, for instance, $C = x_1\overline{x}_2x_3\overline{x}_4 = x_1(1-x_2)x_3(1-x_4) = x_1x_3 - x_1x_3x_4 - x_1x_2x_3 + x_1x_2x_3x_4$. The value of C will be equal to 1 only for the vector (1010), for all the other vectors of B^4 it will be equal to 0.

Exercise 2.13 (Arithmetic Representation). Transform the following functions into the respective arithmetic polynomials:

1 $f = x_1 \oplus x_2 \oplus x_3$;

2 $f = (x_1 \rightarrow x_2) \rightarrow x_3$;

3 $f = x_1x_2x_3 \vee \overline{x}_1\overline{x}_2\overline{x}_3$.

Sometimes the question arises whether a given formula represents a *tautology*, or as also can be said whether the function represented by a given formula is always equal to 1. Again the use of the XBOOLE monitor makes this kind of questions very easy. The formula has to be entered, and the solution of the respective equation answers this question immediately. Since, however, many different TVLs can represent a tautology, it is elegant and efficient to use the complement, because this complement must be the empty set, independent on the representation of the tautology.

Exercise 2.14 (Tautologies). Which one of the following formulas defines a tautology?

1 $(x \rightarrow y) \rightarrow ((x \vee z) \rightarrow (y \vee z))$;

2 $((x \oplus y) \sim z)(x \rightarrow yz)$;

3 $((\overline{x} \vee \overline{y}) \downarrow (x \oplus \overline{y})) \oplus (\overline{(x \to \overline{y})} \to (\overline{x} \vee y))$;

4 $((x \vee \overline{y})z \to ((x \sim z) \oplus y))(x(yz))$.

Very often we find simple or complex identities with the meaning that one side of such an identity defines the same function as the other side. In this case the two formulas can be considered as *equivalent*. A simple example is the expression

$$a(b \vee c) = ab \vee ac,$$

the function defined by $a(b \vee c)$ is supposed to be the same function as the function defined by $ab \vee ac$. The solution of these problems can be achieved in two ways:

- We take the XBOOLE Monitor facility for solving equations and solve the two equations *left side* $= 1$ and *right side* $= 1$ and store the solution sets. Then we compare the solution sets using the *symmetric difference*. If the result is the *empty set*, then the identity is correct, otherwise not.
- We type the suspected identity as it is. The system understands the $=$ as the equivalence, and since the equivalence is equal to 1 for $0 = 0$ and $1 = 1$, we also get all the vectors for which the identity holds, and if the set of all these vectors is equal to the respective B^n, the identity is valid. This means that we ask the question whether the given expression represents a tautology.

Exercise 2.15 (Identities). Are the following pairs of formulas equivalent – try to prove this equivalence by building the disjunctive normal form of the two formulas.

1 $f_1 = (x \vee y \vee z) \to (x \vee y)(x \vee z)$; $f_2 = x \sim z$;

2 $f_1 = (x \to y) \to z$; $f_2 = x \to (y \to z)$;

3 $f_1 = [(x \oplus y) \to (x \vee y)][(\overline{x} \to y) \to (x \oplus y)]$; $f_2 = x \mid y$;

4 $f_1 = \overline{(x \to y) \vee (x \to z)y}$; $f_2 = (x\overline{y})(\overline{y} \to x\overline{z})$.

Correct identities (or *equivalent formulas*) can be used as a possibility for the transformation (simplification, normalization) of formulas and should be well known for this purpose. They can be used from the left to the right or from the right to the left (dependent on the respective effect).

Exercise 2.16 (Transformation Rules). Can the following rules be used?

1 $x \vee (y \sim z) = (x \vee y) \sim (x \vee z)$;

2 $x \rightarrow (y \sim z) = (x \rightarrow y) \sim (x \rightarrow z)$;

3 $x \wedge (y \sim z) = (x \wedge y) \sim (x \wedge z)$;

4 $x \rightarrow (y \vee z) = (x \rightarrow y) \vee (x \rightarrow z)$;

5 $x \rightarrow (y \wedge z) = (x \rightarrow y) \wedge (x \rightarrow z)$;

6 $x \oplus (y \rightarrow z) = (x \oplus y) \rightarrow (x \oplus z)$;

7 $x \rightarrow (y \rightarrow z) = (x \rightarrow y) \rightarrow (x \rightarrow z)$.

The composition of functions is a powerful mechanism that can and will be used very often. The basic idea is the *implementation of "smaller" functions* and its combination by other functions. As an illustration answer the following question.

Exercise 2.17 (Composition of Functions). Let be given $f(a, b) = a \vee \overline{b}$ and $g(x_3, x_4) = x_3 \sim x_4$.

1 Find all vectors (x_2, x_3, x_4) with $h(x_2, x_3, x_4) = f(g(x_3, x_4), x_2)$.

2 Find all vectors (x_1, x_2, x_3, x_4) with $h(x_1, x_2, x_3, x_4) = f(x_1, x_2) \vee g(x_3, x_4)$.

3 Find all vectors (x_1, x_2, x_3, x_4) with $h(x_1, x_2, x_3, x_4) = f(x_1, x_2) \wedge g(x_3, x_4)$.

3. Special Functions and Representations

One of the most interesting problems is the question whether a function (given by a formula, a description, a table) is an element of a special class of functions (is *linear* or *monotone* or *linearly degenerated* etc.). This question will be dealt with in Chapter 4 using operations of the Boolean Differential Calculus. Here we will give some examples for these functions using the XBOOLE Monitor.

Exercise 2.18 (Special Formulas). Show the TVL and the Karnaugh map of the following functions:

1 $f_1 = x_1 x_2 x_3 x_4 x_5 x_6 x_7 x_8 x_9 x_{10}$;

2 $f_2 = x_1 \vee x_2 \vee x_3 \vee x_4 \vee x_5 \vee x_6 \vee x_7 \vee x_8 \vee x_9 \vee x_{10}$;

3 $f_3 = x_1 \oplus x_2 \oplus x_3 \oplus x_4 \oplus x_5 \oplus x_6 \oplus x_7 \oplus x_8 \oplus x_9 \oplus x_{10}$;

4 $f_4 = x_1 \sim x_2 \sim x_3 \sim x_4 \sim x_5 \sim x_6 \sim x_7 \sim x_8 \sim x_9 \sim x_{10}$.

Exercise 2.19 (Implication). 1 Find the TVLs for the functions $f = (x \to y) \to z$ and $g = x \to (y \to z)$.

2 Which function is the XBOOLE Monitor using for $x \to y \to z$.

3 Discuss the result received for $f_1 = x_1 \to x_2 \to x_3 \to x_4 \to x_5 \to x_6 \to x_7 \to x_8 \to x_9 \to x_{10}$.

Exercise 2.20 (NAND and NOR). 1 Compare the two functions $f = (x|y)|z = \overline{\overline{(x \wedge y)} \wedge z}$ and $g = x|(y|z) = \overline{x \wedge \overline{(y \wedge z)}}$.

2 Compare $f = (x \downarrow y) \downarrow z = \overline{\overline{(x \vee y)} \vee z}$ with $g = x \downarrow (y \downarrow z) = \overline{x \vee \overline{(y \vee z)}}$.

3 What can be said about $f = x_1|x_2|x_3|x_4$ and $g = x_1 \downarrow x_2 \downarrow x_3 \downarrow x_4$?

The construction of *disjunctive* and *conjunctive forms* and *normal forms* is easy when the XBOOLE Monitor will be used. Again a given overlap with the concept of *logic equations* exists and can be successfully used.

Exercise 2.21 (Conjunctive and Disjunctive Normal Forms). Find a disjunctive and a conjunctive form for the following functions:

1 $f_1 = (x \vee y\overline{z})(x \vee z)$;

2 $f_2 = ((x_1 \vee x_2\overline{x}_3x_4)((\overline{x}_2 \vee x_4) \to x_1\overline{x}_3\overline{x}_4) \vee x_2x_3) \vee (\overline{x}_1 \vee x_4)$;

3 $f_3 = ((x_1 \to x_2x_3)(x_2x_4 \oplus x_3) \to x_1\overline{x}_4) \vee \overline{x}_1$.

In order to find the disjunctive normal form, any ternary vector with the value – at a given position will be replaced by two vectors with the values 0 and 1 at this position, resp. Sometimes the function might be given by means of a vector of the function values, such as (01101100) or (10001110). In this case the assignment of the argument vectors to the positions of the vector of the function values must be known. The given functions can, for instance, be represented by the following table:

x	y	z	f_1	f_2
0	0	0	0	1
0	0	1	1	0
0	1	0	1	0
0	1	1	0	0
1	0	0	1	1
1	0	1	1	1
1	1	0	0	1
1	1	1	0	0

Very often this order of the assignment of the vector components of f to the vectors for the variables (xyz) from the left to the right is assumed, more or less as a standard, any other assignment of the variables to the positions of the vector of the function values must be mentioned appropriately. This arrangement is very popular because the decimal equivalent of the respective vectors corresponds to the integer numbers $0, 1, \ldots, 7$ or to $0, 1, \ldots, 2^n - 1$ for n variables. Therefore we can also write $f_1 = (01101100)$, $f_2 = (10001110)$.

Exercise 2.22 (Function Vectors). 1 Find the disjunctive and the conjunctive normal forms for the two given function vectors.

2 Find shorter disjunctive and conjunctive forms using the Karnaugh map.

3 Use the items `OBB Orthogonal Block Building` and `OBBC Orthogonal Block Building and Change` of the XBOOLE Monitor in order to find shorter versions.

Exercise 2.23 (Special Normal Forms). How many disjunctions (conjunctions) will be used for the conjunctive (disjunctive) normal forms of the following functions:

1 $f = x_1 \oplus x_2 \oplus \cdots \oplus x_n$;

2 $g = (x_1 \vee x_2 \vee \cdots \vee x_n)(\overline{x}_1 \vee \overline{x}_2 \vee \cdots \vee \overline{x}_n)$;

3 $h = (x_1 \vee x_2 \vee x_3)(\overline{x}_1 \vee \overline{x}_2 \vee \overline{x}_3) \oplus x_4 \oplus x_5 \oplus \cdots \oplus x_n$. Start your considerations with $n = 4$, $n = 5$, $n = 6$ and try to find a general rule.

Since the property of orthogonality is one of the fundamental principles of the solution process for Boolean equations, it is very easy to find antivalence forms of functions. We simply solve the equation $f = 1$ for any function f. Since the solution is given as a set of conjunctions $K_1, K_2, \ldots$ with the additional property that $K_i \wedge K_j = 0$ for $i \neq j$ (orthogonality), we can simply calculate the disjunctive form of the solution set and replace the $\vee$ by $\oplus$.

Exercise 2.24 (Antivalence Normal Forms). Find an antivalence form for the following functions:

1 $f_1 = (x \vee y\overline{z})(x \vee z)$;

2 $f_2 = ((x_1 \vee x_2\overline{x}_3x_4)((\overline{x}_2 \vee x_4) \rightarrow x_1\overline{x}_3\overline{x}_4) \vee x_2x_3) \vee (\overline{x}_1 \vee x_4)$;

3 $f_3 = ((x_1 \rightarrow x_2x_3)(x_2x_4 \oplus x_3) \rightarrow x_1\overline{x}_4) \vee \overline{x}_1$.

In order to find antivalence forms without complemented variables (Shegalkin polynomials), we remember the two rules $\overline{x} = 1 \oplus x$ and

$x(1 \oplus y) = x \oplus xy$. That means that each value 0 in a conjunction of an antivalence form results in two conjunctions, one without this variable and the other one with the value 1.

Exercise 2.25 (Shegalkin Polynomials). 1 Transform the antivalence forms of the previous problem into Shegalkin polynomials.

2 Find the Shegalkin polynomial for $f = x_1 \vee x_2 \vee x_3$.

The concept of a subfunction can be interesting if only some parts of a function are important in a given context. Any subfunction can be found by specifying the values of some variables. One very clear way is the definition of the function and the intersection with the respective value, the other possibility inserts the values of some variables as a constant into the formula and calculates the respective subfunction. Observe that the subfunctions do not depend on the variables that have now a fixed value.

Exercise 2.26 (Subfunctions). 1 For $f = ((x_1 \rightarrow x_2\overline{x}_3) \oplus x_2)\overline{x}_2$, find subfunctions $f_1(x_2, x_3) = f(x_1 = 1, x_2, x_3)$, $f_2(x_1, x_3) = f(x_1, x_2 = 1, x_3)$, $f_3(x_3) = f(x_1 = 1, x_2 = 0, x_3)$ using the intersection and the insertion of constants.

2 Which functions $f(x_1, x_2)$ do not change the value when x_1 and x_2 are exchanging their positions?

When we speak about special logic functions, then very often three questions have to be answered, mostly in a different context:

- how many functions of this special nature can be found (*combinatorial considerations*);
- find some (all) functions with this property (*constructive aspect*);
- check whether a given function is an element of this class (*analytical aspect*).

Exercise 2.27 (Degeneration of Functions). 1 Find all conjunctively degenerated functions of three and four variables. Explain the construction! What is the common property of these functions?

2 Find all disjunctively degenerated functions of three and four variables. Explain the construction! What is the common property of these functions?

3 Find all linearly degenerated functions of three and four variables using the antivalence. Explain the construction! What is the common property of these functions?

4 Find all linearly degenerated functions of three and four variables using the equivalence. Explain the construction! What is the common property of these functions?

Exercise 2.28 (Dual and Self-Dual Functions). Let be given the following functions:

- $f_1 = (11100111)$;
- $f_2 = (01110001)$;
- $f_3 = (11001101)$;
- $f_4 = x_1x_2 \vee \overline{x}_2(x_3 \oplus x_4)$.

Find for each function its dual function! Is one of these functions a self-dual function?

Exercise 2.29 (Symmetric Functions). 1 Find functions $f(x_1, x_2, x_3)$ symmetric in (x_1, x_2).

2 Find functions $f(x_1, x_2, x_3)$ symmetric in (x_2, x_3).

3 What can be said about the intersection of these two sets?

Exercise 2.30 (Symmetric Functions). 1 Find the number of functions $f(x_1, \ldots, x_n)$ which are symmetric in (x_1, x_2), $n \geq 2$.

2 Find all functions that do not change their values for any permutation of the variables.

Exercise 2.31 (Monotone Functions). Which functions of the given set are monotone? Check the increasing as well as the decreasing possibility.

1 $f_1 = (x \to (x \to y)) \to (y \to z)$;

2 $f_2 = x_1x_2 \oplus x_1x_3 \oplus x_1x_4 \oplus x_2x_3 \oplus x_2x_4 \oplus x_3x_4$;

3 $f_3 = (0000000010111111)$;

4 $f_4 = (0001010101010111)$.

Monotone functions are in a very strict sense considered as *monotonely increasing* and *monotonely decreasing*. However, this difference is not so important since it is easy to see that a function $f(\mathbf{x})$ is monotonely increasing if and only if $\overline{f(\mathbf{x})}$ is monotonely decreasing and vice versa. Therefore most of the time it is sufficient to deal with one of these properties or to assume the consideration of increasing functions more or less as a standard.

Exercise 2.32 (Monotone Functions). For which value of n are the following functions monotone?

1 $f_1(x_1, \ldots, x_n) = x_1x_2 \vee x_1x_3 \vee \cdots x_{n-1}x_n$ (the disjunctions of all conjunctions consisting of two non-negated variables);

2 $f_2(x_1, \ldots, x_n) = x_1x_2 \ldots x_{n-1}x_n \to (x_1 \oplus x_2 \oplus \cdots \oplus x_{n-1} \oplus x_n)$.

Exercise 2.33 (Monotone Functions). For each monotone function, we have

$$f(\mathbf{x}) = x_i f(x_i = 1) \vee f(x_i = 0),$$

and

$$f(\mathbf{x}) = (x_i \vee f(x_i = 0)) \wedge f(x_i = 1).$$

Prove these identities.

Exercise 2.34 (Monotone Functions). Let be given $f(x_1, x_2, x_3, x_4)$ such that $f(0, 1, 1, 0) = 1$, $f(1, 1, 0, 0) = 1$, $f(1, 0, 1, 0) = 0$, $f(0, 0, 1, 1) = 1$, $f(0, 1, 0, 1) = 0$.

1 Can this definition be used to build monotone functions with these values?

2 How many different monotone functions with these values can be built?

3 Represent these functions by disjunctive forms without negated variables.

Exercise 2.35 (Functional Constraints). Find all functions satisfying the following conditions:

1 $f(1, 0, 0, 0) = 1$, $f(0, 1, 1, 1) = 0$;

2 $f(1, 0, 0, 0) = 1$, f linear in one or more variables;

3 $f(0, 1, 0, 0) \neq f(1, 0, 1, 1)$, f symmetric (consider all possibilities);

4 $f(1, 0, 0, 1) = 0$, f self-dual.

4. Minimization

In Chap. 2, Sect. 4 of [18] the Method of BLAKE and the Algorithm of QUINE-MCCLUSKEY are represented; they give the prime implicants of a function and all irredundant disjunctive forms for a given function f.

We remember that an *implicant* of a function f is a conjunction K with the property $K \leq f$, i.e. $K \vee f = f$ and $K \wedge f = K$. A *prime implicant* is an implicant where this property will be lost if one of the variables in the conjunction is deleted.

Exercise 2.36 (Prime Implicants). Find the prime implicants of the following functions:

1 $f(x, y, z) = (00101111)$;

2 $f(x, y, z) = (01111110)$;

3 $f(x_1, x_2, x_3, x_4) = (1010111001011110)$.

Exercise 2.37 (Minimized Disjunctive Normal Form). What can be said about the minimization of the following functions:

1 $f_1 = x_1 \oplus x_2 \oplus x_3 \oplus x_4 \oplus x_5$;

2 $f_2 = (x_1 \vee x_2 \vee x_3)(\overline{x}_1 \vee \overline{x}_2 \vee \overline{x}_3) \oplus x_4 \oplus x_5$;

3 $f_3 = (x_1 \vee x_2 \vee x_3)(\overline{x}_1 \vee \overline{x}_2 \vee \overline{x}_3)(x_4 \oplus x_5)$;

4 $f_4 = (x_1 \oplus x_2 \oplus x_3)(x_4 \oplus x_5 \oplus x_6)$;

5 $f_5 = (x_1 \vee x_2 \vee x_3 \vee x_4 \vee x_5 \vee x_6)(x_1 \vee x_2 \vee x_3 \vee \overline{x}_4 \vee \overline{x}_5 \vee \overline{x}_6)$?

The XBOOLE Monitor is very flexible with regard to the variables which are used for a function. We can type, for instance, $f = a \vee b$ and $g = b \vee c$ as equations to be solved and consider $f \vee g$ which is solved by the union of the sets of ternary vectors for f and g, respectively. And when we check the solution sets then we see that the result depends on a, b and c. What has been happening in the background, is the extension of functions by variables or the embedding of functions in larger spaces. For the variables (a, b) and (b, c), we would need a space (a, b, c) in order to work with the two functions simultaneously. In the sense of the formulas this can be done in the following way:

$$f = (a \vee b) = (a \vee b)(c \vee \overline{c}) \quad \text{and} \quad g = (a \vee \overline{a})(b \vee c).$$

Now both f and g depend on (a, b, c). It can be seen that $f(a, b, c = 0) = f(a, b, c = 1)$ and $g(a = 0, b, c) = g(a = 1, b, c)$. The variables c or a, respectively, are *not essential*.

Sometimes it is also desirable to find out which variables are not essential, because if this is the case, then it is possible to find a formula without this variable which makes the respective expressions shorter. This will be done most efficiently using the first derivative, after the knowledge of the Boolean Differential Calculus is available (see Chap. 4). Here we will only use the two subfunctions directly. We build, for instance, the subfunction $f(x_i = 1)$ by means of the intersection of f and the ternary vector for $x_i = 1$, the subfunction $f(x_i = 0)$ in the same way and check the equality by using the symmetric difference. The two subfunctions are equal to each other, if the symmetric difference is equal to the empty set.

Sometimes it might even be possible to get (by using algebraic transformations of formulas or minimization algorithms) expressions without a variable; then this variable is not an essential variable. The equality of the subfunctions can be seen directly.

Exercise 2.38 (Essential Variables). Find the essential or non-essential variables of the following functions and find formulas without non-essential variables:

1 $f(x_1, x_2, x_3) = (x_1 \to (x_1 \vee x_2)) \to x_3$;

2 $f(x_1, x_2) = ((x_1 \vee x_2) \to x_2)$;

3 $f(x_1, x_2, x_3, x_4) = (x_1 \vee x_2 \vee \overline{x}_2 x_3 \vee \overline{x}_1 \overline{x}_2 \overline{x}_3) x_4$;

4 $f(x_1, x_2) = (x_1 \oplus x_2)(x_1 \downarrow x_2)$;

5 $f(x_1, x_2, x_3) = (((x_3 \to x_2) \vee x_1)(x_2 \to x_1) x_3 \overline{x}_1) \oplus x_3$;

6 $f(x_1, x_2, x_3) = (((x_1 \vee x_2)(x_1 \vee \overline{x}_3) \to \overline{x}_1) \to x_2 \overline{x}_3) x_2$;

7 $f(x_1, x_2, x_3, x_4) = (1011100111001010)$;

8 $f(x_1, x_2, x_3, x_4) = (0011110011000011)$.

5. Complete Systems of Functions

We already know that the systems consisting of *conjunction*, *disjunction* and *negation* are complete systems of functions, i.e. each function can be represented by using these functions only. This can be confirmed when we remember that each function can be represented as a *conjunctive* or *disjunctive normal form*. But it can be seen that there are also other complete systems of logic functions, sometimes consisting only of one single function.

Exercise 2.39. Show that the following systems of functions are complete systems:

1 $\{x \downarrow y\}$;

2 $\{xy \oplus z, (x \sim y) \oplus z\}$;

3 $\{x \to y, \overline{x \oplus y \oplus z}\}$;

4 $\{x \to y, (1100001100111100)\}$;

5 $\{0, xy \vee xz \vee yz, 1 \oplus x \oplus y \oplus z\}$;

6 $\{(1011), (1111110011000000)\}$.

Since the functions of the previous exercise are complete systems, it must be possible that each function can be expressed by the other functions.

Exercise 2.40. Express every function that appears in Exercise 2.39 by all the other complete systems of functions of this exercise.

Exercise 2.41. Represent every function of this exercise using all the complete systems given in Exercise 2.39.

1 $f_1 = x_1 \oplus x_2 \oplus x_3 \oplus x_4 \oplus x_5$;

2 $f_2 = (x_1 \vee x_2 \vee x_3)(\overline{x}_1 \vee \overline{x}_2 \vee \overline{x}_3) \oplus x_4 \oplus x_5$;

3 $f_3 = (x_1 \vee x_2 \vee x_3)(\overline{x}_1 \vee \overline{x}_2 \vee \overline{x}_3)(x_4 \oplus x_5)$;

4 $f_4 = (x_1 \oplus x_2 \oplus x_3)(x_4 \oplus x_5 \oplus x_6)$;

5 $f_5 = (x_1 \vee x_2 \vee x_3 \vee x_4 \vee x_5 \vee x_6)(x_1 \vee x_2 \vee x_3 \vee \overline{x}_4 \vee \overline{x}_5 \vee \overline{x}_6)$?

6 $f_6 = \overline{x}_2 x_4 x_5 \vee x_2 \overline{x}_4 x_5 \vee (x_4 \oplus x_5)(\overline{x}_1 x_2 x_3 \vee x_1 x_2 \overline{x}_3) \vee x_1 \overline{x}_2 \overline{x}_3 (x_4 \sim x_5)$.

7 $f_7 = (x_1 \oplus x_2) \wedge (x_3 \sim x_4 \sim x_5)$.

8 $f_8 = x_1 \overline{x}_2 \vee x_2 \overline{x}_3 \vee x_3 \overline{x}_4 \vee x_4 \overline{x}_1$.

9 $f_9 = (x_1 | x_2 | x_3) \downarrow (x_4 \downarrow x_5)$.

10 $f_{10} = (x_1 \rightarrow x_2) \sim (x_2 \rightarrow x_3) \sim (x_3 \rightarrow x_4) \sim (x_4 \rightarrow x_5) \sim (x_5 \rightarrow x_1)$.

Sometimes some functions can be allowed to be used in a given context. They will not be a complete system, but they can be used to build new functions. As an example we consider the system of functions $\{0, \overline{x}\}$. It can be assumed that these two functions are given and can be used several times, according to our convenience. Which functions can be built by using these two functions?

x	$f_1(x)$	$f_2(x)$
0	0	1
1	0	0

By using the function f_2 two times, it is possible to build

$$f_3(x) = f_2(f_2(x)) = x \quad \text{and} \quad f_4(x) = f_2(f_1(x)) = 1.$$

Therefore the whole system of functions that is available consists of the functions $\{0, 1, x, \overline{x}\}$, since $f_1(f_1(x)) = f_1(f_2(x)) = 0$. Here the given function had only one argument; when several arguments have to be considered, then the space B^n also has to be given. We will simply say that the arguments x_1, x_2, x_3 or x_1, x_2, x_3, x_4 can be used – see the next example. It can be seen later that this approach is also very typical for the design of circuits. A special function is given, and the problem has to be solved what can be done with this circuit or this gate, and how can it be done. The complete systems are particularly interesting because in this case it is known that each function can be implemented by means of this system.

Exercise 2.42. 1 Let be given the set of functions $\{1, x \oplus y\}$. Which functions can be built using these two functions for $M_1 = \{x_1, x_2, x_3\}$ and for $M_2 = \{x_1, x_2, x_3, x_4\}$.

2 Which functions can be built using $\{x \vee y, xyz\}$?

3 Which system of functions can be built by $\{x \vee y, xyz, x \vee yz, (x \vee y)z\}$?

4 Can xy be built using the set $\{0, 1, x, y, \overline{x}, \overline{y}\}$?

5 Which set can be built using the functions of the previous item?

6 Which set can be built using the given set $\{0, 1, x, y, \overline{x}, \overline{y}\}$ together with $f_1(x, y, z) = (11101000)$ or $f_2(x, y, z) = (01111111)$ or $f_3(x, y, z) = (10011001)$.

6. Partially Defined Functions

Very often the values of a function are not completely specified, they are given only for a given set of values of the argument vectors. We will see this property very often when the design of circuits or finite-state machines will be considered.

Such a function always can be seen as a ternary vector where only some of the values are given, the remaining positions are equal to – and can be replaced by values 0 or 1 (as we have seen already very often).

Exercise 2.43. Let be given the function(s) $(10--010-11000101)$.

1 Find the set of functions that can be derived from this partial definition.

2 Find for every function the antivalence normal form and compare the different representations.

3 Find for every function the conjunctive normal form and compare the different representations.

The range of values for the variables with specified values can be described by a function $\varphi(\mathbf{x})$ which is equal to 0 for the vectors of variables with specified values of the function and equal to 1 otherwise, i.e. for vectors with unspecified values. Later two more functions will be used in this context: the function $q(\mathbf{x})$ is equal to 1 for all vectors with $f = 1$ (the ON-set) and 0 otherwise. The function $r(\mathbf{x})$ is equal to 1 for all vectors with $f = 0$ (the OFF-set) and 0 otherwise.

Exercise 2.44. Find the function $\varphi(x_1, x_2, x_3, x_4)$ for the following definition ranges:

1 The function $f(x_1, x_2, x_3, x_4)$ is defined only for vectors with one or three values 1.

2 The function $f(x_1, x_2, x_3, x_4)$ is not defined for all vectors with two values 1.

3 The function $f(x_1, x_2, x_3, x_4)$ is not defined for the vectors (1110) and (1111).

Exercise 2.45. Let be given the function $f(x_1, x_2, x_3, x_4) = x_1\overline{x}_2 \vee x_2\overline{x}_3 \vee x_3\overline{x}_4 \vee x_4\overline{x}_1$. However, it is not defined for (1110) and (1111).

1 Find the set of four functions that meet this partial specification.

2 Find the function $\varphi(x_1, x_2, x_3, x_4)$ for this situation.

3 Represent the four possible functions by means of $f(x_1, x_2, x_3, x_4)$ and $\varphi(x_1, x_2, x_3, x_4)$.

7. Solutions

Exercise 2.1.

1

x	y	$x \vee y$	$x \le x \vee y$	$x \wedge y$	$x \ge x \wedge y$
0	0	0	yes	0	yes
0	1	1	yes	0	yes
1	0	1	yes	0	yes
1	1	1	yes	1	yes

2 Here the same table will help: use four columns for x_1, x_2, y_1 and y_2. Mark the rows where $x_1 \le y_1$ and $x_2 \le y_2$ and check the relations for the conjunctions and disjunctions.

3 Use a table with the same procedure.

4 Here the transformation of $\le$ into an equation can be used: $x \le (yz) = x\,\overline{yz} = x(\overline{y} \vee \overline{z}) = x\overline{y} \vee x\overline{z} = 0$. The disjunction is equal to 0 if and only if both conjunctions are equal to 0: $x\,\overline{y} = 0$ and $x\,\overline{z} = 0$ which is equivalent to $x \le y$ and $x \le z$.

Exercise 2.2.

1 9, 13 and 50.

2 $(010011) \in B^6, (00010011) \in B^8$.

3 The vector $\mathbf{x} = (10\ldots0) \in B^n$ represents the number 2^{n-1}. The first bit with the value 1 is kept, all the other bits are successively replaced by values 1, possibly in lexicographic order: $(10\ldots00)$, $(10\ldots01)$, $(10\ldots10)$, $(10\ldots11)$ etc. This results in 2^{n-1} different numbers from 2^{n-1} until $2^n - 1$. The number 2^n itself would require one more bit, i.e. it will be represented by $\mathbf{x} = (10\ldots0) \in B^{n+1}$.

An elegant way to describe the set of all these vectors is the use of ternary vectors. We keep the 1 in the first position and replace all the values 0 by $-$: $(1--\ldots-)$ represents the set of all these vectors $\mathbf{x}$ with $2^{n-1} \leq \mathrm{dec}(\mathbf{x}) < 2^n$ since every $-$ can be replaced by 0 or 1. When only the value 0 is used, then we get the original number 2^{n-1}, when only the value 1 is selected, then the resulting number is equal to $2^n - 1$.

4 In the first case the positions with $x_i = 1$ remain unchanged. In order to increase the given vector, the values 0 can change to 1, hence, the ternary vector $(1--1-1-1)$ represents all the desired vectors. Actually we build the interval between the given vector $\mathbf{x} = (10010101)$ and the largest vector $(11\ldots11)$.

In the second case the positions with $x_i = 0$ remain unchanged, the values 1 can change to 0, hence, the ternary vector $(-00-0-0-)$ represents the desired vectors, the interval $(00000000) \leq \mathbf{y} \leq \mathbf{x}$.

5 Let $\mathbf{x} = (010101)$, $\mathbf{y} = (110111)$. Using the considerations of the previous item, all vectors $\mathbf{z}$ with $\mathbf{z} \geq \mathbf{x}$ are given by $(-1-1-1)$, the ternary vector $(--0---)$ describes the set of all vectors $\mathbf{z}$ with $\mathbf{z} \leq \mathbf{y}$. The intersection of these two sets (intervals) will be the desired set which is given by $\{(-101-1)\}$.

(a) We click on `Objects` and select `Define Space`. This is our first space, and 32 variables are sufficient because we need only six.

(b) Now we go back to `Objects` and define two TVLs as objects 1 and 2, and we assign the variables $x1, \ldots, x6$ to both of them. As `Form` predicate we select `ODA`; this is an assumption for the application of set operations.

(c) For the next step we use `Objects` again, followed by the item `Append Ternary Vector(s)`. The vector $(-1-1-1)$ will be appended to `TVL1`, the vector $(--0---)$ to `TVL2`.

(d) The final step is the intersection of these two ternary vectors which can be understood as the intersection of the respective sets of binary vectors. We select `Sets` and `ISC Intersection` and get the desired result after specifying the two operands.

Exercise 2.3.

1 The complement $\overline{\mathbf{x}}$ exchanges the values 0 and 1. Hence, if $\mathbf{x}$ has k values 1 and $n-k$ values 0, then the complement must have $n-k$ values 1 and k values 0.

2 The subtraction is necessary because some of the values 1 are counted twice on the right side.

3 This relation follows from 2 by transformation of the equation.

This norm function which is accessing the components of a (binary or ternary) vector is not available in the XBOOLE monitor. If it is required very often, then you can rely on the XBOOLE library, or you write your own C-routines to be added to the existing system.

Exercise 2.4.

1 $\mathbf{c} = (0000)$

(a) $h = 0$: $S_0 = \{(0000)\}$

(b) $h = 1$: $S_1 = \{(1000), (0100), (0010), (0001)\}$

(c) $h = 2$: $S_2 = \{(1100), (1010), (1001), (0110), (0101), (0011)\}$

(d) $h = 3$: $S_3 = \{(1110), (1101), (1011), (0111)\}$

(e) $h = 4$: $S_4 = \{(1111)\}$.

2 This problem will now start with $S_0 = \{(1111)\}$, followed by $S_1 = \{(0111), (1011), (1101), (1110)\}$ etc.

3 This can be seen immediately by inspecting the previous results.

4 $K_0 = \emptyset$, $K_i = \bigcup_{k=0}^{i-1} S_k$, $K_{n+1} = \bigcup_{k=0}^{n} S_k$.

5 $\overline{K}_0 = S_0$, $\overline{K}_i = \bigcup_{k=0}^{i} S_k$, $\overline{K}_n = \bigcup_{k=0}^{n} S_k$.

6 This follows directly from item 3, 4 and 5.

Exercise 2.5.

1 This item simply repeats the definition of the elementary functions. Ensure that you remember these functions well and use the steps mentioned above.

2 Use the sequence `Objects`, `Define Space...`, `Extras` and `Solve Boolean Equation`. Then you type the respective formulas for f_1 to f_7 and solve these equations which will give the same results.

3 $f(x, y) = \overline{x} \vee \overline{y} = \overline{x \wedge y}$. $f(x, y) = \overline{x} \wedge \overline{y} = \overline{x \vee y}$.

4 We solve the two equations $(x \oplus y) \oplus z$ and $x \oplus (y \oplus z)$ – the right sides are equal to 1 – and compare the solution sets. When we store these two solutions as two different sets, then their symmetric difference would be empty (which means that the two sets are equal).

5 When we explore, for instance, the two expressions $(x \to y) \to z$ and $x \to (y \to z)$, then the symmetric difference of the solution sets will show that the functions are different for the two vectors (000) and (011).

Exercise 2.6.

1 Instead of 2^n argument vectors of the function now only 2^{n-1} pairs of arguments have to be considered. Therefore we get $2^{2^{n-1}}$ functions with this property.

2 The construction of these functions can start at any point. We use $\mathbf{x} = (0, \ldots, 0)$ with $f(0, \ldots, 0) = 0$. Then for all vectors $\mathbf{x}$ with $\|\mathbf{x}\| = 1$ the value of the function must be equal to 1, the value for all vectors $\mathbf{x}$ with $\|\mathbf{x}\| = 2$ must be equal to 0 etc. This means that the value for $\mathbf{x} = (0, \ldots, 0)$ defines the value for all other vectors in a unique way. Since $f(0, \ldots, 0) = 1$ is the second possibility, we only have two such functions.

3 The definition of the function indicates that the value 1 can be assigned to vectors $\mathbf{x}$ with $\|\mathbf{x}\| = 0, \ldots, k-1$. It is known from combinatorics that m vectors with the value 1 can be selected from 2^n possibilities, using the binomial coefficient $\binom{2^n}{m} = \frac{2^n!}{m!(2^n-m)!}$. Therefore, the number of functions with this property is equal to $\binom{2^n}{0} + \binom{2^n}{1} + \cdots + \binom{2^n}{k-1}$.

Exercise 2.7.

1 The condition $x = 1$ can be expressed by the vector $(1--)$, $y \neq z$ defines the two possibilities $y = 0, z = 1$ and $y = 1, z = 0$, but only in the first case $x = 0$ is less than $z = 1$. Therefore we have the two ternary vectors $(1--)$ and (001) as the result. The Karnaugh map follows from the change of the view after the TVL with these two vectors has been created. $f = x \vee \overline{x}\,\overline{y}z$ is the corresponding disjunctive form.

2 The given assumption changes the point of view and introduces or uses different algebraic concepts: the addition and the calculation of $2x_4$ takes the values 0 and 1 simply as numbers, and for those vectors where the values of the variables satisfy the condition the value 0 will be entered. By checking all 16 vectors, we get the following four vectors with $f = 0$: $(0100), (1000), (1100)$ and (1110). In order to find a disjunctive form, a TVL with these four vectors can be created. The complement of this set results in a set that can be used for the disjunctive form, and we get, for instance, $f = x_4 \vee \overline{x}_1\overline{x}_2 \vee \overline{x}_1 x_3 \vee \overline{x}_2 x_3$.

Exercise 2.8.

1 By using the BOOLE Monitor we get $f = 1$ for the two vectors (000) and (111), hence, $f = \overline{x}\,\overline{y}\,\overline{z} \vee xyz$. It will also be a good exercise to find the disjunctive or conjunctive normal form by algebraic transformations of the given formula. Use, for instance, the following sequence of transformations:

$x \rightarrow y = \overline{x} \vee y = \overline{x} \oplus y \oplus \overline{x}y = 1 \oplus x \oplus y \oplus (1 \oplus x)y = 1 \oplus x \oplus y \oplus y \oplus xy = 1 \oplus x \oplus xy$.
We can also use the disjunctive form of this function: $f = \overline{x}\,\overline{y}\,\overline{z} \vee xyz = \overline{x}\,\overline{y}\,\overline{z} \oplus xyz = 1 \oplus x \oplus y \oplus z \oplus xy \oplus xz \oplus yz$. An equivalent conjunctive form can be found when we are using the complement of the set $\{(000), (111)\}$. After using the XBOOLE Monitor for the original formula, the solution set is stored as an object. We can also create a TVL and enter the two vectors directly. The use of the items `Sets` and `CPL Complement` results in the set of ternary vectors $\{(100), (011), (-10), (-01)\}$. The corresponding disjunctions can be used for a conjunctive form of this function: $f = (\overline{x} \vee y \vee z)(x \vee \overline{y} \vee \overline{z})(\overline{y} \vee z)(y \vee \overline{z})$. Since the set operations are always based on orthogonal sets of vectors, we get the equivalence normal form immediately: $f = (\overline{x} \vee y \vee z) \sim (x \vee \overline{y} \vee \overline{z}) \sim (\overline{y} \vee z) \sim (y \vee \overline{z})$. The conjunctive form $f = (\overline{x} \vee y)(\overline{y} \vee z)(x \vee \overline{z})$ would be shorter.

2 Since the operator $\downarrow$ has not been implemented in the XBOOLE Monitor, we must use the definition $x \downarrow y = \overline{x \vee y} = \overline{x} \wedge \overline{y}$ before the input of the equation. This is easy to do: we replace the $\downarrow$ by $\vee$ and the complement:

$$\begin{aligned} \overline{(\overline{x} \vee y) \vee (x\overline{z})} \downarrow (x \sim y) &= \overline{\overline{(\overline{x} \vee y) \vee (x\overline{z})} \vee (x \sim y)} \\ &= \overline{\overline{(\overline{x} \vee y) \vee (x\overline{z})}} \wedge \overline{(x \sim y)} = (\overline{x} \vee y \vee x\overline{z})(x \oplus y). \end{aligned}$$

$f = 1$ for the two ternary vectors $(01-)$ and (100), hence, $f = \overline{x}y \vee x\overline{y}\,\overline{z} = (\overline{x} \vee \overline{z})(x \vee y)(\overline{x} \vee \overline{y})$ etc.

3 $f = x \vee y\overline{z} \vee \overline{y}z = (x \vee \overline{y} \vee \overline{z})(x \vee y \vee z)$.

4 Here $x \mid y = \overline{xy} = \overline{x} \vee \overline{y}$ must be used as well, and we get after some steps the expression $xy\overline{z}$ which already shows that $f = 1$ only for the vector (110).

This type of formulas is also a good possibility to use the XBOOLE Monitor, particularly the possibility of manipulating sets of ternary vectors. $\alpha|\beta$ can be calculated by creating objects for α and β, respectively, followed by the intersection of these two objects and the complement of the intersection. The NOR-operation will be implemented by the union of the two parts and the complement of the result.

Exercise 2.9.

1 The set of ternary vectors with $f = 1$ is equal to $\{(0110), (0111), (1111)\}$ (given by the XBOOLE Monitor) or, after using the orthogonal block building, equal to $\{(011-), (1111)\}$, and this results in the disjunctive normal form $f = \overline{x}_1 x_2 x_3 \vee x_1 x_2 x_3 x_4$.

2 This expression describes the function $1(x_1, x_2, x_3, x_4)$ which is constant equal to 1 for all vectors (x_1, x_2, x_3, x_4).

3 When the expression is checked carefully, then it can be seen that this is already the conjunctive normal form of the function f. It is equal to 0 only for the two vectors (0000000000) and (1111111111). For the remaining 2^{10}–2 vectors the function would be equal to 1, hence, it would be hardly possible to write down the disjunctive or antivalence normal forms. However, the XBOOLE Monitor needs (only) 18 vectors for the representation of this function.

4 Using the definitions involved in this expression, two parts of the formula can be considered: $f_{\text{left}} = (x_1 \vee x_2 \vee x_3)(\overline{x}_1 \vee \overline{x}_2 \vee \overline{x}_3)$ and $f_{\text{right}} = x_4 \oplus x_5 \oplus x_6 \oplus x_7 \oplus x_8 \oplus x_9 \oplus x_{10}$, and $f = f_{\text{left}} \oplus f_{\text{right}}$. According to the definition of the antivalence, $f = 1$ if $f_{\text{left}} = 0$ and $f_{\text{right}} = 1$ and vice versa. Therefore, the solution vectors have the vectors (000) and (111) for (x_1, x_2, x_3) together with all vectors $(x_4, x_5, x_6, x_7, x_8, x_9, x_{10})$ having an odd number of components with the value 1. All the other vectors (x_1, x_2, x_3) can be paired with those vectors $(x_4, x_5, x_6, x_7, x_8, x_9, x_{10})$ having an even number of components with the value 1. The XBOOLE Monitor shows 384 ternary vectors for the representation of this function.

Exercise 2.10.

The simplest way is the use of the set of ternary vectors that is already known followed by the correct interpretation of the vectors as disjunctions of a conjunctive normal form. We create, for instance, a list of ternary vectors with the vectors $(0110), (0111), (1111)$. The complement of this set followed by an orthogonal minimization results in three vectors $(-0--), (-10-), (1110)$, and the respective orthogonal conjunctive normal form can be written as $f = x_2(\overline{x}_2 \vee x_3)(\overline{x}_1 \vee \overline{x}_2 \vee \overline{x}_3 \vee x_4)$. The respective forms for the other functions can be found by using the same approach.

Exercise 2.11.

This more general problem uses the same ideas as the problem before. Since n is not specified, XBOOLE cannot be used immediately. However, the following considerations can be used for any special n after the value has been defined.

1 This conjunctive normal form shows that $f = 0$ for two vectors. For all the other vectors we get $f = 1$. The easiest way to get these vectors is the input of these two vectors followed by the complement.

2 For any $n \geq 4$ the two vectors (000) and (111) will be combined with all the vectors $x_4 \dots x_n$ with an odd number of values 1, all the other vectors for (x_1, x_2, x_3) will be combined with all the vectors with an even number of values 1. The number

of vectors with $f = 1$ is equal to the number of vectors with $f = 0$, i.e. equal to 2^{n-1}.

Exercise 2.12.

1 $f = x_1 x_2 \overline{x}_3 = x_1 x_2 (1 \oplus x_3) = x_1 x_2 \oplus x_1 x_2 x_3$;

2 $f = \overline{x}_1 = 1 \oplus x_1$;

3 $f = 1$ for the two vectors $(0--), (101)$, hence $f = \overline{x}_1 \oplus x_1 \overline{x}_2 x_3 = 1 \oplus x_1 \oplus x_1 x_3 \oplus x_1 x_2 x_3$.

Exercise 2.13.

1 $f = x_1 x_2 x_3 + (1 - x_1)(1 - x_2) x_3 + (1 - x_1) x_2 (1 - x_3) + x_1 (1 - x_2)(1 - x_3) = 4 x_1 x_2 x_3 + x_1 + x_2 + x_3 - 2 x_1 x_2 - 2 x_1 x_3 - 2 x_2 x_3$.
Check the value of f, for instance, for $x_1 = 1, x_2 = 1, x_3 = 1$.

2 We use `Solve Boolean Equation...` and deal with each elementary conjunction as in the previous item. In order to avoid ternary vectors, the Karnaugh map could be used.

3 Here we have two conjunctions that can be transformed as before.

Exercise 2.14.

1 The solution of this equation shows that this is a tautology.

2 We get $f = 1$ only for the vectors (000) and (011).

3 Tautology.

4 This function is always equal to 0, i.e. it is the negation of a tautology.

Exercise 2.15.

1 We use again the option of `Solve Boolean Equation`, for the first expression we get three solution vectors $(1--), (011), (000)$, for the second formula, however, we get two solution vectors $(00), (11)$. When we now check the variables then we can easily see that for the first formula the variables allocated are x, y and z, for the second, however, only x and z are used. It might be useful to know a possibility to adjust the variables in such a way that both formulas depend on the same sets of variables.
The first possibility is a bit tricky. We use the knowledge about the functions: $(x \vee \overline{x}) = 1$ and $(x \sim 1) = x$ and replace the equation $x \sim z$ by $x \sim z \sim (y \vee \overline{y})$ which obviously has the same set of solutions, but depends now on x, y and z. The resulting solution set is now equal to $\{(0-0), (1-1)\}$.
The second possibility corresponds more to the theoretical understanding of binary functions and equations. We use `Create TVL` and `Append Ternary Vector(s)`, and we append the vector $(---)$ only. This vector naturally describes the whole B^3. The intersection of this object with the solution vectors (00) and (11) considers the correct allocation of variables and results in the correct solutions $(0-0)$ and $(1-1)$.
By comparing now the two resulting solution sets, it can be seen that these two formulas are not equivalent. This can be done in the most correct way (particularly for large sets depending on many variables) by using the `Symmetric Difference`

which would be empty if the two solution sets are equal. In this example the solution sets are different, hence, the formulas are not equivalent.

2 The two formulas are not equivalent.

3 The two formulas are equivalent.

4 The two formulas are not equivalent. The two solution vectors $(10-)$ and (100) describe different solution sets.

Exercise 2.16.
The solution of the respective equations shows that only the cases 3 and 6 cannot be used as a rule.

Exercise 2.17.

1 In the times of *cut and paste* it is the easiest way to insert the expressions for a and b (first and second argument) into the formula:

$$f = a \vee \overline{b} = (x_3 \sim x_4) \vee \overline{x}_2 .$$

The resulting set of orthogonal ternary vectors is equal to $\{(0--), (100), (111)\}$ for (x_2, x_3, x_4). Later on we will deal with the application of logic equations in many different places. It might be useful to see that already for the composition of logic functions the equations can be a very useful tool. The problem shows us that the first argument a has to be equal to $x_3 \sim x_4$, hence, we write $a = (x_3 \sim x_4)$, or, in the language of the XBOOLE Monitor, $a = (x3 = x4)$. We are adding $b = x2$ and also $f = a \vee \overline{b}$, i.e. $a+!b$ as a formula of the XBOOLE Monitor. By using the topics `Extras` and `Solve Boolean Equation` of the XBOOLE Monitor we create the solution sets of these three equations, and the intersection of the three respective objects results in solution vectors for (a, b, x_2, x_3, x_4), and if we are interested only in values for x_2, x_3, x_4, then the components for a and b can be omitted, and only the different vectors for (x_2, x_3, x_4) must be used. In spite of the small example the methodology should be studied carefully, because many much larger applications can be handled in the same way.

2 Use the same approach and get successively $(1-), (00)$ for (x_1, x_2), $(00), (11)$ for (x_3, x_4) and use the union. Watch that the lists of ternary vectors have the predicate `ODA`; this can be achieved by using `Matrices` – `Orthogonalization`. Even when the vectors remain unchanged, still the predicate changes, and `ODA` is the assumption for the application of the set operations. The orthogonal minimization can be used directly to get simpler representations, such as $\{(1---),$ $(00--), (0100), (0111)\}$ in this case.

3 This problem can be solved in the same way. Here the intersection of the two solution sets has to be used and results in $\{(0011), (0000), (1-00), (1-11)\}$.

Exercise 2.18.

1 Only one vector (out of 1024) satisfies this conjunction, $\mathbf{x} = (1111111111)$.

2 The TVL of the solution set shows very nicely the advantages of the orthogonal representation. Ten ternary vectors are sufficient for the representation of the solution set, each solution vector is represented by precisely one ternary vector.

3 The last two cases show the problem of representing linear functions. The antivalence as well as the equivalence expression represent two sets of 512 vectors. Each vector must be represented by a full conjunction or disjunction with all ten variables. No simplification is possible.

4 Already covered by the previous items.

Exercise 2.19.

1 By solving the respective equations directly, we get $f = z \vee x\overline{y}$ with 5 solution vectors for $f = 1$ and $g = \overline{x} \vee \overline{y} \vee z$ with 7 solution vectors for $g = 1$. By comparing the function values, it can be seen that $f < g$.

2 The XBOOLE Monitor uses the $\rightarrow$ from the left to the right.

3 The solution is given as follows:

$$\begin{pmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & x_9 & x_{10} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ - & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ - & - & - & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ - & - & - & - & - & 1 & 0 & 0 & 0 & 0 \\ - & - & - & - & - & - & - & 1 & 0 & 0 \\ - & - & - & - & - & - & - & - & - & 1 \end{pmatrix} .$$

Exercise 2.20.

1 It holds that $(x|y)|z = \overline{z} \vee xyz = \overline{z} \vee xy$ whereas $x|(y|z) = \overline{x} \vee xyz = \overline{x} \vee yz$.

The transformation into the respective disjunctive forms can be based on algebraic transformations such as $\overline{x \wedge y} = \overline{x} \vee \overline{y}$, but it is also a good opportunity to get accustomed to the consideration of solution sets. The following steps must be performed, they are more or less trivial at this point of time, but a good possibility to get a better understanding for the transformation of a logic function into (solution) sets of binary vectors.

- We create three objects (TVLs) containing the vectors $(1--), (-1-), (--1)$ representing x, y and z, resp. Each TVL must be orthogonal by using `Matrices – Orthogonalization`. The matrices do not change because they contain only one vector, but they are now marked as `ODA` which is an assumption for the application of set operations.
- The sequence $1 \cap 2 \rightarrow 4$, $\overline{4} \rightarrow 5$, $5 \cap 3 \rightarrow 6$, $\overline{6} \rightarrow 7$ represents $4 = x \wedge y$, $5 = \overline{x \wedge y}$, $6 = \overline{(x \wedge y)} \wedge z$, and $\overline{6} = \overline{\overline{(x \wedge y)} \wedge z}$ is the final solution represented by the two vectors (111) and $(--0)$.

2 This problem uses the same approach for $x \downarrow y = \overline{x \vee y}$.

3 Representations like this should be avoided according to the previous considerations because they would need a definition of the order in which the operations have to be applied. It is better (more readable, easier to understand) when brackets are used for the definition of the sequence of the operations, such as $((x1 \downarrow x2) \downarrow x3) \downarrow x4$ or $(x1 \downarrow (x2 \downarrow x3)) \downarrow x4$ etc.

Exercise 2.21.

1 This function simplifies to $f_1 = x$.

2 For f_2 it is important to consider the structure of the formula in a correct way: $(x_1 \vee x_2\overline{x}_3x_4)$ is one disjunction: the next part comprises $((\overline{x}_2 \vee x_4) \rightarrow x_1\overline{x}_3\overline{x}_4)$, and these two parts must be combined by $\wedge$, or, if we have already the TVLs for the two parts, by intersection. Finally $x_2x_3 \vee \overline{x}_1 \vee x_4$ must be considered. It can also be seen from these considerations that the leftmost bracket and the closing bracket after x_2x_3 are not really necessary. The solution by using set operations or the solution by solving the equation shows that the vector (1010) for $(x_1x_2x_3x_4)$ is the only vector with $f = 1$.

3 The calculations for f_3 follow the same procedure.

Exercise 2.22.

1 The binary argument vectors for $f_1 = 1$ and $f_1 = 0$ can be translated directly into the respective conjunctions and disjunctions as we have seen before:
$f_1 = \overline{x}\,\overline{y}z \vee \overline{x}y\overline{z} \vee x\overline{y}\,\overline{z} \vee x\overline{y}z$; $f_2 = \overline{x}\,\overline{y}\,\overline{z} \vee x\overline{y}\,\overline{z} \vee x\overline{y}z \vee xy\overline{z}$;
$f_1 = (x\vee y\vee z)(x\vee\overline{y}\vee\overline{z})(\overline{x}\vee\overline{y}\vee z)(\overline{x}\vee\overline{y}\vee\overline{z})$; $f_2 = (x\vee y\vee\overline{z})(x\vee\overline{y}\vee z)(x\vee\overline{y}\vee\overline{z})(\overline{x}\vee\overline{y}\vee\overline{z})$.

2 This can be done by creating a TVL, appending the respective binary vectors and using the representation as a Karnaugh map. From here we get, for instance $f_1 = x\overline{y} \vee \overline{y}z \vee \overline{x}y\overline{z}$, $f_2 = x\overline{z} \vee x\overline{y} \vee \overline{y}\,\overline{z}$.

3 The results of the orthogonal block building are not as short as a full minimization, however, the results are already rather good and orthogonal, i.e. no double solutions have to be considered. For f_1 we get, for instance, $f_1 = \overline{x}y\overline{z} \vee x\overline{y}\,\overline{z} \vee \overline{y}z$, the result of `OBBC` is equal to $f_1 = \overline{x}y\overline{z} \vee x\overline{y} \vee \overline{x}\,\overline{y}z$ which has the same size; $f_2 = xy\overline{z} \vee \overline{y}\,\overline{z} \vee x\overline{y}z$, $f_2 = x\overline{z} \vee \overline{x}\,\overline{y}\,\overline{z} \vee x\overline{y}z$.

Exercise 2.23.

1 The value $f = 1$ appears for argument vectors with an odd number of values 1; therefore we get 2^{n-1} vectors with the value 1, the other half has the value $f = 0$ for an even number of values 1 in the argument vector.

2 The expression for g is already in conjunctive normal form. Therefore we get the value $g = 0$ for the two vectors $(0\ldots0)$ and $(1\ldots1)$, for all the other vectors we have $g = 1$. Therefore the desired number is equal to $2^n - 2$.

3 The two disjunctions $(x_1 \vee x_2 \vee x_3)(\overline{x}_1 \vee \overline{x}_2 \vee \overline{x}_3)$ result in two values 0 and six values 1. These values have to be combined with the second part of the formula; this part includes $n-3$ variables, i.e. it describes 2^{n-3} binary vectors, half of them result in 0, the other half results in 1. Therefore we get $6 \times 2^{n-4}$ values 1 and $2 \times 2^{n-4}$ values 0 for the given function.

Exercise 2.24.

1 $f = x$ as a border case contains only one variable. However, the solution set in the XBOOLE monitor is characterized as `ODA` which means *orthogonal – disjunctive or antivalence form.*

2 The result shows originally ten ternary vectors which could be replaced by the respective conjunctions and combined by $\oplus$. The orthogonal block building (minimization) results in four vectors representing $f = \overline{x}_1x_3\overline{x}_4 \oplus x_3x_4 \oplus x_1x_2x_3\overline{x}_4 \oplus \overline{x}_3$.

3 The solution of the given equation (right side equal to 1) shows that this function only has the value 1 which means that $f = 1$ is the respective antivalence form.

Exercise 2.25.

1 We replace the complemented variables $\overline{x}_i$ by $1 \oplus x_i$ and apply the distributive law as often as necessary. Two identical conjunctions can be deleted. For the second function we get, for instance, $f = 1 \oplus x_1x_3 \oplus x_1x_2x_3 \oplus x_1x_3x_4 \oplus x_1x_2x_3x_4$.

2 The solution of $x_1 \vee x_2 \vee x_3 = 1$ results in the orthogonal form $f = x_1 \vee \overline{x}_1x_2 \vee \overline{x}_1\overline{x}_2x_3 = x_1 \oplus \overline{x}_1x_2 \oplus \overline{x}_1\overline{x}_2x_3$, and from here we get $f = x_1 \oplus x_2 \oplus x_3 \oplus x_1x_2 \oplus x_1x_3 \oplus x_2x_3 \oplus x_1x_2x_3$.

Exercise 2.26.

1 The solution of the equation $f(x_1, x_2, x_3) = 1$ results in $f(x_1, x_2, x_3) = \overline{x}_1\overline{x}_2$ which does not depend on x_3. Therefore, all three required subfunctions are identically equal to 0. The replacement of the respective variables by the constants and the solution of these equations will naturally show the same result.

2 The solution can easily be found by stating that in this case $f(0, 1)$ must be equal to $f(1, 0)$. The values for $f(0, 0)$ and $f(1, 1)$ are not restricted. When we are using ternary vectors, then the two vectors $(-00-)$ and $(-11-)$ for the function f describe all these functions.

Exercise 2.27.

1 Without loss of generality the representation $f(x_1, x_2, x_3) = x_1 \wedge f'(x_2, x_3)$ will be considered. For $x_1 = 0$ the function f is always equal to 0, independent on x_2 and x_3. For $x_1 = 1$ any function of two variables can be used. Hence, we have 16 functions of three variables which are conjunctively degenerated (in x_1):

x_1	x_2	x_3	f_0	f_1	...	f_{14}	f_{15}
0	0	0	0	0	...	0	0
0	0	1	0	0	...	0	0
0	1	0	0	0	...	0	0
0	1	1	0	0	...	0	0
1	0	0	0	0	...	1	1
1	0	1	0	0	...	1	1
1	1	0	0	0	...	1	1
1	1	1	0	1	...	0	1

Using some of the functions of two variables, we get, for instance, $f_0 = x_1 \wedge 0$, $f_1 = x_1(x_2x_3)$, $f_{14} = x_1(\overline{x}_2 \vee \overline{x}_3)$, $f_{15} = x_1 \wedge 1$. The characteristic property is the value 0 for f if $x_1 = 0$. There are 256 functions of three variables, only 16 are conjunctively degenerated if x_1 is the selected variable.

2 The construction of disjunctively degenerated functions follows the same idea. Only the role of 0 and 1 has to be exchanged.

3 For the construction of linearly dependent functions the representation

$$f(x_1, x_2, x_3) = x_1 \oplus f'(x_2, x_3)$$

will be used. For $x_1 = 0$ any function $f'(x_2, x_3)$ can be used; $x_1 = 1$ results in $f(x_1, x_2, x_3) = 1 \oplus f'(x_2, x_3) = \overline{f'(x_2, x_3)}$. By using subfunctions, this property can be expressed by the following equation: $f(0, x_2, x_3) = \overline{f(1, x_2, x_3)}$. You can check, for instance, that for $f'(x_2, x_3) = x_2 x_3$ you get $\overline{x_2 x_3} = \overline{x}_2 \vee \overline{x}_3$, and this results in $f = x_1 \oplus x_2 x_3$.

For four variables the same considerations can be applied: $f(x_1, x_2, x_3, x_4) = x_1 \oplus f'(x_2, x_3, x_4)$. It can also be seen that each linear function is linearly degenerated in each variable that appears in the formula. The same considerations can also be applied to the equivalence.

Generally a test of this property can use the relation

$$f(x_1, \ldots, x_i = 0, \ldots, x_n) = \overline{f(x_1, \ldots, x_i = 1, \ldots, x_n)}$$

directly. The easiest way to do this is the checking of the equality:

- create the TVL for f, intersect (ISC) with $x_i = 0 \Rightarrow$ object 1;
- create the TVL for f, intersect (ISC) with $x_i = 1 \Rightarrow$ object 2;
- CPL (complement) of object 2 $\Rightarrow$ object 3;
- symmetric difference SYD of object 1 and object 3 must be equal to $\emptyset$.

4 The solution of this problem uses the same approach. Watch the different meaning of 0, 1 and the equivalence.

Exercise 2.28.

For the given function vectors use the reverse of the vector and the following negation. When we are using $f_4 = x_1 x_2 \vee \overline{x}_2(x_3 \oplus x_4)$, then we can transform the formula into $f_4 = x_1 x_2 \vee \overline{x}_2 \overline{x}_3 x_4 \vee \overline{x}_2 x_3 \overline{x}_4$, and the mutual exchange of $\wedge$ and $\vee$ results in ${f_4}^* = (x_1 \vee x_2)(\overline{x}_2 \vee \overline{x}_3 \vee x_4)(\overline{x}_2 \vee x_3 \vee \overline{x}_4)$.

Exercise 2.29.

1 The definition of a symmetric function results for x_1 and x_2 in $f(0, 1, x_3) = f(1, 0, x_3)$, or by inserting values for x_3, in

$$f(0, 1, 0) = f(1, 0, 0), \quad f(0, 1, 1) = f(1, 0, 1).$$

2 For x_2 and x_3 we get in the same way $f(x_1, 0, 1) = f(x_1, 1, 0)$, or by inserting values for x_1, in

$$f(0, 0, 1) = f(0, 1, 0), \quad f(1, 0, 1) = f(1, 1, 0).$$

3 This means that the function which has both properties is constant for argument vectors with the same weight. Therefore, for three variables we do not have eight degrees of freedom, but only four, as can be seen from the following table:

x_1	x_2	x_3	$f(x_1, x_2, x_3)$
0	0	0	α_1
0	0	1	α_2
0	1	0	α_2
0	1	1	α_3
1	0	0	α_2
1	0	1	α_3
1	1	0	α_3
1	1	1	α_4

The parameters α_1, α_2, α_3, α_4, can take any value, therefore we have 16 such functions.

Exercise 2.30.

1 The property of symmetry reduces the degrees of freedom from four to three since $f(0,1) = f(1,0)$ is required. Hence, the number of functions with this property is equal to $2^{\frac{3}{4}2^n} = 2^{3\cdot 2^{n-2}}$.

2 The invariance against all permutations generalizes the property we already have seen for three variables. For each vector with a given weight the function must be constant (either equal to 0 or equal to 1). Since there are weights from 0 to n, their number is equal to 2^{n+1}.

Exercise 2.31.

1 By solving the equation, we get the two vectors (-00) and $(--1)$ as solution vectors. By using the Karnaugh-map it can be seen that $f_1(000) = 1$, but $f_1(010) = 0$, therefore the function is not monotonely increasing. It can also not be monotonely decreasing, because $f_1(111) = 1$ and $f_1(110) = 0$, but $f_1(100) = 1$.

2 We have $f_2(1111) = 0$, but $f_2(x_1, x_2, x_3, x_4) = 1$ for several smaller vectors. Hence, $f_2(x_1, x_2, x_3, x_4)$ is not monotone.

3 $f_3(x_1, x_2, x_3, x_4)$ is not monotone because of $f_3(1000) = 1$ and $f_3(1001) = 0$.

4 $f_4(x_1, x_2, x_3, x_4)$ is monotone. Take, for instance, the Karnaugh-map for this function and draw the graph of the partial order from (0000) to (1111) with the respective values of the function.

Exercise 2.32.

1 We take $n \geq 2$. Then we have the value 0 for f_1 for all vectors with one component equal to 1 and for $(000\ldots000)$. These vectors are the two lowest levels of the graph of the partial order, all the other values of f_2 are equal to 1, therefore f_1 is monotone.

2 We transform the given formula: $f_1 = \overline{x}_1 \vee \overline{x}_2 \vee \cdots \overline{x}_{n-1} \vee \overline{x}_n \vee (x_1 \oplus x_2 \oplus \cdots \oplus x_n)$. The value of the disjunction of the complemented variables is 0 if and only if all the values of $x_1 \ldots x_n$ are equal to 1. Now it depends: if n is odd, then we get $f_2(x_1, \ldots, x_n)$ is constant equal to 1 because then the antivalence is equal to 1. If n is even, then the value of $f(1, 1, \ldots, 1, 1) = 0$, and the function is monotonely decreasing.

Exercise 2.33.

1 We remember the theorem that monotone functions have a disjunctive (normal) form without negated variables. We can split the set of all conjunctions of this normal form into two orthogonal subsets: one subset with conjunctions with the variable x_i, the other subset with all conjunctions without this variable. The inverse use of the distributive law transforms the disjunctive form already into the desired format.

2 This is the equivalent format derived by transformation into the equivalent conjunctive form.

Exercise 2.34.

1 We can use the properties of monotone functions in the following way: if there is the value 1 for a given vector, then all larger vectors must also have the value 1. Therefore, $f(0,1,1,0) = 1$ means that $f = 1$ also for the vectors (1110), (0111) and (1111). It can be derived from $f(1,1,0,0) = 1$ that $f = 1$ also for the vectors (1110), (1101) and (1111). On the other side the value 0 for a given vector implies this value for all smaller vectors. Therefore, $f(1,0,1,0) = 0$ means that $f = 0$ also for the vectors (0010), (1000) and (0000). For the last two values we get $f = 1$ for the vectors (1011), (0111) and (1111) and $f = 0$ for the vectors (0001), (0100) and (0000). When we put all these values together, then it can be seen that only the value $f(1,0,0,1)$ has not been defined. The graph of the function shows that it can be set to 0 or to 1, in both cases we get a monotone function.

2 See the previous item.

3 The creation of the two respective TVLs and minimization show the respective disjunctive forms. When we set, for instance, $f(1,0,0,1) = 0$, then we get $f(x_1,x_2,x_3,x_4) = x_1x_2 \vee x_2x_3 \vee x_1x_3x_4$.

Exercise 2.35.

1 This constraint is not very demanding. Two values of a function f have been set. The remaining 14 positions of the vector of the function values (16 bits) can be set arbitrarily (which results in 2^{14} different possibilities).

2 Here the value 1 is given for the vector (1000). Take, for instance, the nice view that the values of a linear function look like a chessboard, then you will see that this setting defines $f = x_1 \oplus x_2 \oplus x_3 \oplus x_4$. the definition $f(1,0,0,0) = 0$ would result in the complement $f = 1 \oplus x_1 \oplus x_2 \oplus x_3 \oplus x_4$.

3 As an example we explore the symmetry with regard to (x_1,x_2) (other symmetries can be checked in the same way). The problem requires that $f(0,1,0,0) \neq f(1,0,1,1)$. This will be achieved by setting $f(0,1,0,0) = \alpha$, $f(1,0,1,1) = \overline{\alpha}$. Because of the symmetry we get two more values: $f(1,0,0,0) = \alpha$ and $f(0,1,1,1) = \overline{\alpha}$. The symmetry implies two more equalities: $f(0,1,0,1) = f(1,0,0,1) = \gamma$, $f(0,1,1,0) = f(1,0,1,0) = \delta$. Up to now there are three parameters α, β and γ in the solution which results in 2^3 possibilities. These eight possibilities can be combined with the 2^8 possibilities that exist for $(x_1,x_2) = (00)$ and $(x_1,x_2) = (11)$. Therefore we get 2^{11} different solutions of this problem.

4 The condition $f(1,0,0,1) = 0$ implies for a self-dual function that $f(0,1,1,0) = 1$. For the remaining seven pairs of vectors always the setting $f(x_1,x_2,x_3,x_4) = \overline{f(\overline{x}_1,\overline{x}_2,\overline{x}_3,\overline{x}_4)}$ has to be used.

Exercise 2.36.

1 See item 3.

2 See item 3.

3 We will mention here a special possibility that will be based on the properties of the XBOOLE Monitor. Instead of starting with the elementary conjunctions and a comprehensive use of the Method of Blake we create a TVL and enter the respective vectors with the value 1; for the third problem we would enter

$$(0000), (0010), (0100), (0101), (0110), (1001), (1011), (1100), (1101), (1110).$$

Then do not forget to use the topic `Matrices` and there the item `Orthogonalization`. This will not change the TVL itself, but the predicate of the list from D to ODA. And now OBB `Orthogonal Block Building` or `OBBC Orthogonal Block Building and Change` can be used. In this case it will not really make a difference. We get $(00-0), (10-1), (-110), (-10-)$, and the Method of Blake can start here, and the procedure is much shorter and results in $f = \overline{x}_1\overline{x}_4 \vee x_2\overline{x}_3 \vee x_2\overline{x}_4 \vee x_1\overline{x}_2x_4$.

Exercise 2.37.

1 The input of this expression and the finding of the values for $x_1 \oplus x_2 \oplus x_3 \oplus x_4 \oplus x_5 = 1$ shows 16 Vectors, i.e. 16 conjunctions, and no further simplification is possible. This is a very typical property of *linear functions*.

2 The use of the XBOOLE Monitor shows 12 ternary vectors, the orthogonal minimization does not change the number of ternary vectors. Again the Method of BLAKE will show that there are no further simplifications. Check it out!

Here would also be a possibility to check whether a conjunction C is a prime implicant or not. The conjunction C is a prime implicant if $C \vee f = f$. Now we take, for instance, $C = \overline{x}_1x_3x_4x_5$ and omit x_3 which results in $C = \overline{x}_1x_4x_5$. In order to check the relation, we create a TVL with $(0--11)$ as the only vector, change the predicate to ODA (using orthogonalization for `Matrices`) and perform the union of f and this single vector. The sets for f and $f \vee C$ can be compared by using the symmetric difference. This difference must be equal to $\emptyset$ if the equality holds which is not the case for the situation considered. Therefore $C = \overline{x}_1x_4x_5$ is not a prime implicant.

The general idea can be understood as follows: the omission of one variable creates larger intervals with the value 1 for the function. In order to make this shorter conjunction an implicant and even more a prime implicant f must not be extended (falsified), the relation $C \leq f$ must still hold.

3 Here we get 8 solution vectors, no simplifications.

4 The solution vectors will have an even number of values 1, because $x_1 \oplus x_2 \oplus x_3 = 1$ only holds for an odd number of values 1, and the same holds for $x_4 \oplus x_5 \oplus x_6 = 1$, and the addition of the odd number gives an even number (2, 4 or 6).

5 The homogeneous orthogonal structure of the XBOOLE solution can be simplified and results in $f(x_1, x_2, x_3, x_4, x_5, x_6) = x_1 \vee x_2 \vee x_3 \vee x_4\overline{x}_5 \vee x_4\overline{x}_6 \vee \overline{x}_4x_5 \vee \overline{x}_4x_6 \vee \overline{x}_5x_6 \vee x_5\overline{x}_6$.

Exercise 2.38.

The solutions of these problems is not difficult, however, they should be studied carefully to get a good understanding of the underlying concepts.

1 $(x_1 \to (x_1 \vee x_2) \to x_3 = (\overline{x}_1 \vee x_1 \vee x_2) \to x_3 = 1 \to x_3 = 0 \vee x_3 = x_3$.

2 $(x_1 \vee x_2) \to x_2 = \overline{x}_1\overline{x}_2 \vee x_2 = \overline{x}_1 \vee x_2$.

3 This is a good possibility to see the application of BLAKE's rule. After some steps it can be seen that the expression in brackets is identically equal to 1; therefore we have $f = x_4$.

4 f is identically equal to 0.

5 The use of the XBOOLE monitor for this function shows directly the result $f(x_1, x_2, x_3) = x_3$, therefore x_1 and x_2 are non-essential variables. The sequence

of transformations shows the same result: $f(x_1, x_2, x_3) = (((x_3 \rightarrow x_2) \vee x_1)(x_2 \rightarrow x_1)x_3\overline{x}_1) \oplus x_3 = (x_1 \vee x_2 \vee \overline{x}_3)(x_1 \vee \overline{x}_2)\overline{x}_1 x_3) \oplus x_3 = x_3$. Another possibility is the checking of subfunctions; we insert the values $x_1 = 0$ and $x_1 = 1$ and compare the resulting subfunctions. In the same way we check the variables x_2 and x_3 and get finally the same result.

6 The same considerations result in $f(x_1, x_2, x_3) = x_1 x_2 \vee x_2 \overline{x}_3$, and the consideration of the respective subfunctions shows that all three variables are essential.

7 In order to find a solution for this problem, variables have to be assigned to the components of the vector with the values of the function. One possible coding is shown by the following representation:

f	1	0	1	1	1	0	0	1	1	1	0	0	1	0	1	0
x_1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1
x_2	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1
x_3	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1
x_4	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1

Now there are several possibilities. It would be, for instance, possible to enter the argument vectors with $f = 1$ into a list of ternary vectors, and to get a shorter description by using OBB or OBBC. Thereafter, we could again build the subfunctions $f(x_1 = 0)$ and $f(x_1 = 1)$, etc., and compare these subfunctions by means of the *symmetric difference SYD*.

But there is also another possibility for this kind of representation. In the first and second column, we have $x_2 = 0, x_3 = 0, x_4 = 0$, and x_1 changes its value from 0 to 1. The function value of f also changes from 1 to 0 which means that $f(x_1 = 1)$ cannot be equal to $f(x_1 = 0)$, and this means naturally that x_1 is an essential variable. The same can be seen by comparing the columns 2 and 4 (counted from the left) with regard to x_2, by comparing the columns 3 and 7 with regard to x_3, and finally by comparing the columns 3 and 11 with regard to x_4. All the variables are essential variables. This method is successfully working for small numbers of variables only, however, it shows very well the essence of these concepts.

8 This item can be dealt with in the same way as the previous one.

Exercise 2.39.
It is general practice to show that the given new functions are able to build other functions that are already known as contributing to a complete system of functions. Her we assume the knowledge that $\overline{x}$, $x \wedge y$ and $x \vee y$ are sufficient to implement any function (see, for instance, the concepts of *disjunctive* and *conjunctive normal form*).

1 In the first case the function $f(x, y) = x \downarrow y = \overline{x \vee y}$ will be used to implement these three functions:

$$x \downarrow x = \overline{x \vee x} = \overline{x},$$

$$(x \downarrow y) \downarrow (x \downarrow y) = \overline{\overline{x \vee y} \vee \overline{x \vee y}} = (x \vee y) \wedge (x \vee y) = x \vee y,$$

$$(x \downarrow x) \downarrow (y \downarrow y) = \overline{\overline{x \vee x} \vee \overline{y \vee y}} = (x \vee x) \wedge (y \vee y) = x \wedge y.$$

When the expression $h(x, y) = x \downarrow y$ is used, then the three functions can be expressed by using the function $h(x, y)$ alone:

$$\begin{aligned} \overline{x} &= h(x, x), \\ x \vee y &= h(h(x, y), h(x, y)), \\ x \wedge y &= h(h(x, x), h(y, y)). \end{aligned}$$

Only the function $h(x, y)$ has been used.

2 We use the same strategy: $h_1(x, y, z) = xy \oplus z$, $h_1(x, x, z) = x \oplus z$, $h_2(x, y, z) = (x \sim y) \oplus z = x \oplus y \oplus z \oplus 1$, $h_2(x, h_1(x, x, z), z) = 1$, $h_1(x, y, h_2(x, h_1(x, x, z), z)) = xy \oplus 1 = \overline{xy} = x|y$. This function is the NAND-function which itself is a complete system. Please, watch that only applications of h_1 and h_2 have been used.

3 We use $(x \rightarrow y) \rightarrow y = x \vee y$ and $\overline{x \oplus x \oplus x} = \overline{x}$, and by combination of these two functions $x \downarrow y$ can be implemented which has been explored before.

4 We use again $(x \rightarrow y) \rightarrow y = x \vee y$ and find a formula for the given vector of a function. When the variables x_1, x_2, x_3, x_4 are used together with the vectors (0000) to (1111) from the left to the right, then we get $f(x_1, x_2, x_3, x_4) = \overline{x}_1\, \overline{x}_2\, \overline{x}_3 \vee \overline{x}_1 x_2 x_3 \vee x_1 x_2 \overline{x}_3 \vee x_1 \overline{x}_2 x_3$. This function is, in fact, independent on x_4, and therefore the function $f(x_1, x_1, x_1) = \overline{x}_1$ can be used. Based on the considerations of the first item, the disjunction and the negation can be used to build the NOR-function which is complete.

5 For $f_1(x, y, z) = 0, f_2(x, y, z) = xy \vee xz \vee yz$ and $f_3(x, y, z) = 1 \oplus x \oplus y \oplus z$ we find $f_2(x, y, 0) = xy$ and $f_3(x, x, x) = \overline{x}$ which can be combined to build the NAND-function.

6 In this case we get $f_1(x_1, x_2) = x_1 \vee \overline{x}_2$ which can be used to build the function 1 by using $f_1(x_1, x_1) = 1$. The second function is given by $f_2(x_1, x_2, x_3, x_4) = \overline{x}_1\overline{x}_2 \vee \overline{x}_1\overline{x}_3 \vee \overline{x}_2\overline{x}_3$, and this function produces the NOR-function when the value 1 is inserted: $f_2(x_1, x_2, 1) = \overline{x}_1\overline{x}_2 = \overline{x_1 \vee x_2}$.

Exercise 2.40.
As an introduction we show the implemention of $x \oplus y$ by means of $x \downarrow y$.

$$\begin{aligned} \overline{x} &= \overline{x \vee x} = (x \downarrow x), \\ \overline{y} &= \overline{y \vee y} = (y \downarrow y), \\ f_1 &= x \wedge \overline{y} = \{(x \downarrow x) \downarrow [(y \downarrow y) \downarrow (y \downarrow y)]\}, \\ f_2 &= \overline{x} \wedge y = \{[(x \downarrow x) \downarrow (x \downarrow x)] \downarrow (y \downarrow y)\}, \\ x \oplus y &= f_1 \vee f_2 = (f_1 \downarrow f_2) \downarrow (f_1 \downarrow f_2). \end{aligned}$$

The solution of problems like this is not supported by XBOOLE. If they occur very often, then it would be advisable to write a special program. Sometimes it might be helpful to see that the left side of such a representation is equal to the right side – problems of this kind already have been solved very often.

When we need the constant functions $0(x)$ and $1(x)$, then this can be achieved by using $x \vee \overline{x} = 1(x)$ and $x \wedge \overline{x} = 0(x)$:

$$\begin{aligned} 0(x) &= (x \downarrow x) \downarrow ((x \downarrow x) \downarrow (x \downarrow x)), \\ 1(x) &= (x \downarrow (x \downarrow x)) \downarrow (x \downarrow (x \downarrow x)). \end{aligned}$$

Finally we can build the NAND by using NOR alone as follows:

$$x \wedge y = [(x \downarrow x) \downarrow (y \downarrow y)],$$
$$x|y = \overline{x \wedge y} = [(x \downarrow x) \downarrow (y \downarrow y)] \downarrow [(x \downarrow x) \downarrow (y \downarrow y)].$$

Several applications of the given equalities are sufficient to implement all the different functions of the previous exercise.

Exercise 2.41.
The solution of this problem uses the same principles that have been used in the previous item.

Exercise 2.42.

1 We set $h_1(x, y) = x \oplus y$, $h_2(x, y, z) = 1$ and get successively:

$$h_1(x_1, x_1) = 0, \quad h_1(x_1, h_2(x_1, x_2, x_3)) = x_1 \oplus 1 = \overline{x}_1, \quad h_1(x_1, 0) = x_1,$$
$$h_1(x_1, x_2) = x_1 \oplus x_2, \quad h_1(x_1, \overline{x}_2) = x_1 \oplus \overline{x}_2,$$
$$h_1(x_1, (x_2 \oplus x_3)) = x_1 \oplus x_2 \oplus x_3, \quad h_1(\overline{x}_1, (x_2 \oplus x_3)) = \overline{x}_1 \oplus x_2 \oplus x_3.$$

Some functions that can be built by only changing the name of a variable have not been mentioned explicitly, it is assumed and quite understandable that they can be found in the same way (see, for instance $h_1(x_2, x_3) = x_2 \oplus x_3$). In each step we only used the given function(s) and functions that have been constructed in previous steps.

When the set of variables is extended and the set $\{x_1, x_2, x_3, x_4\}$ will be used, then all the functions that have been constructed before can be built again; additionally we get the functions $h(x_1 \oplus x_2, x_3 \oplus x_4) = x_1 \oplus x_2 \oplus x_3 \oplus x_4$ and $h(\overline{x}_1 \oplus x_2, x_3 \oplus x_4) = \overline{x}_1 \oplus x_2 \oplus x_3 \oplus x_4$.

2 It can be seen immediately that single variables can be generated by $x_1 \vee x_1 = x_1$ (the same for x_2 and x_3), the disjunction of two variables as well, such as $x_1 \vee x_2$ and also $x_1 \vee x_2 \vee x_3$. The conjunction allows additionally the generation of $x_1 x_2$ (as well as $x_1 x_3$ and $x_2 x_3$) and $x_1 x_2 x_3$. Finally the combination of the conjunctions and disjunctions allows to build $x_1 \vee x_2 x_3$, $x_2 \vee x_1 x_3$, $x_3 \vee x_1 x_2$, $x_1 x_2 \vee x_1 x_3$, $x_1 x_2 \vee x_2 x_3$, $x_1 x_3 \vee x_2 x_3$ and finally $x_1 x_2 \vee x_1 x_3 \vee x_2 x_3$. All these functions are monotone functions. It is not possible to get the constant functions 0 and 1, complemented variables also cannot be achieved. The consideration of four variables follows the same ideas.

3 Here we use the same considerations as in the previous item.

4 By means of the given functions the conjunction cannot be implemented.

5 The application of the given functions will not produce new functions. There is no possibility to extend the given set of functions.

6 We find for the given function vectors $f_1(x, y, z) = \overline{x}\,\overline{y} \vee \overline{x}\,\overline{z} \vee \overline{y}\,\overline{z}$, $f_2(x, y, z) = x \vee y \vee z$ and $f_3(x, y, z) = xz \vee \overline{y}\,\overline{z}$. Now we follow the previous approach: We get, for instance, single negated variables as follows: $f(x_1, 0, 1) = \overline{x}$. Now the negation can be used to get the single variables x_1, x_2, x_3, and from here several disjunctive forms can be built. The two other functions and the extension to four variables follow the same ideas.

As we mentioned before, the XBOOLE Monitor can be used to check the correctness of some of the implemented relations.

Exercise 2.43.
We will use the variables x_1, x_2, x_3, x_4 and start with the vector (0000) from the left. This means that the function values are not defined for the vectors (0010), (0011) and (0111).

1 There are eight functions that can be derived from this incomplete specification. We replace successively the components with a – by all possible combinations of 0 and 1. The XBOOLE Monitor can be used as a tool in the following way: we define a TVL with the vectors $(0000), (0101), (1000), (1001), (1101), (1111)$. This is the basic structure for the following steps and describes already the function generated by using the value 0 for the free components. Thereafter we are adding to this list the vectors describing additional values 1: if, for instance, the value 1 is used for the vector (0011), then this vector is added to the original TVL. After the building of these eight TVL we change the predicate of the list by ORTH and reduce the size of the lists by an orthogonal block building OBB. This finally results in the following TVLs for the eight functions:

$$f_0(x_1,x_2,x_3,x_4) = \left(\begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{cccc} - & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ - & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{array}\right),$$

$$f_1(x_1,x_2,x_3,x_4) = \left(\begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ \hline 0 & 1 & 1 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{cccc} - & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ - & 1 & - & 1 \end{array}\right),$$

$$f_1(x_1,x_2,x_3,x_4) = x_2x_4 \vee \overline{x}_2\overline{x}_3\overline{x}_4 \vee x_1\overline{x}_2\overline{x}_3x_4 = x_2x_4 \oplus \overline{x}_2\overline{x}_3\overline{x}_4 \oplus x_1\overline{x}_2\overline{x}_3x_4 .$$

This is the function with the shortest orthogonal representation. This shows clearly that partially defined functions offer many optimization possibilities.

$$f_2(x_1,x_2,x_3,x_4) = \left(\begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ \hline 0 & 0 & 1 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{cccc} - & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ - & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right),$$

$$f_3(x_1,x_2,x_3,x_4) = \left(\begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ \hline 0 & 0 & 1 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{cccc} - & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ - & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right),$$

$$f_4(x_1,x_2,x_3,x_4) = \left(\begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ \hline 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right) \longrightarrow \begin{pmatrix} - & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ - & 1 & - & 1 \end{pmatrix},$$

$$f_5(x_1,x_2,x_3,x_4) = \left(\begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ \hline 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right) \longrightarrow \begin{pmatrix} - & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ - & 1 & - & 1 \end{pmatrix},$$

$$f_6(x_1,x_2,x_3,x_4) = \left(\begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ \hline 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right) \longrightarrow \begin{pmatrix} - & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ - & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & - \end{pmatrix},$$

$$f_7(x_1,x_2,x_3,x_4) = \left(\begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ \hline 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right) \longrightarrow \begin{pmatrix} - & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ - & 1 & - & 1 \\ 0 & 0 & 1 & - \end{pmatrix}.$$

2 Since the representation consists of orthogonal vectors (i.e. of orthogonal conjunctions), the antivalence and the disjunctive forms can be derived immediately. Further minimization possibilities have to be explored.

3 In order to find the conjunctive normal forms, we can use the vectors with $f = 0$ and build the respective disjunctions. The easiest way is the use of the previous matrix representations together with the complement, such us

$$\left(\begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ \hline 0 & 1 & 1 & 1 \end{array}\right) \longrightarrow \begin{pmatrix} - & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ - & - & 1 & 0 \\ - & 0 & 1 & 1 \end{pmatrix},$$

$$f_1(x_1, x_2, x_3, x_4) = (\overline{x}_2 \vee x_3 \vee x_4)(x_1 \vee x_2 \vee x_3 \vee \overline{x}_4)(\overline{x}_3 \vee x_4)(x_2 \vee \overline{x}_3 \vee \overline{x}_4).$$

Exercise 2.44.
The function $\varphi(x_1, x_2, x_3, x_4)$ will be equal to 1 for the vectors with defined values and equal to 0 for the vectors with undefined values.

1 $\varphi(x_1, x_2, x_3, x_4) = x_1 \oplus x_2 \oplus x_3 \oplus x_4$.

2 $\varphi(x_1, x_2, x_3, x_4) = 0$ for the vectors (0011), (0101), (1001), (0110), (1010) and (1100). Now there are two possibilities: we use these six vectors to build a conjunctive form; in this way we get a conjunctive form with six disjunctions of four variables. The other possibility would be the use of these vectors as a set, build the complement and try to simplify the result as a disjunctive form.

3 We get $\varphi(x_1, x_2, x_3, x_4) = \overline{x}_1 \vee \overline{x}_2 \vee \overline{x}_3$.

Exercise 2.45.

1 The function is equal to 0 only for the vector (0000) and not defined for the two vectors (1110) and (1111). Therefore the conjunctive form is very appropriate to represent the four different functions. When we replace the two empty positions by the four possibilities 00, 01, 10 and 11, then we get successively

$$\begin{aligned}
f_1(x_1, x_2, x_3, x_4) &= (x_1 \vee x_2 \vee x_3 \vee x_4), \\
f_2(x_1, x_2, x_3, x_4) &= (x_1 \vee x_2 \vee x_3 \vee x_4)(\overline{x}_1 \vee \overline{x}_2 \vee \overline{x}_3 \vee x_4), \\
f_3(x_1, x_2, x_3, x_4) &= (x_1 \vee x_2 \vee x_3 \vee x_4)(\overline{x}_1 \vee \overline{x}_2 \vee \overline{x}_3 \vee \overline{x}_4), \\
f_4(x_1, x_2, x_3, x_4) &= (x_1 \vee x_2 \vee x_3 \vee x_4)(\overline{x}_1 \vee \overline{x}_2 \vee \overline{x}_3).
\end{aligned}$$

2 $\varphi(x_1, x_2, x_3, x_4) = \overline{x}_1 \vee \overline{x}_2 \vee \overline{x}_3$.

3 We get four functions f^* by using the relation

$$\begin{aligned}
&f^*(x_1, x_2, x_3, x_4) \\
&\quad = f(x_1, x_2, x_3, x_4)\varphi(x_1, x_2, x_3, x_4) \vee g(x_1, x_2, x_3, x_4)\overline{\varphi(x_1, x_2, x_3, x_4)};
\end{aligned}$$

$g(x_1, x_2, x_3, x_4)$ can be any function of four variables. Therefore many choices will result in the same function $f^*(x_1, x_2, x_3, x_4)$. Only different values for the two vectors (1110) and (1110) are important.

http://www.springer.com/978-1-4020-9594-8

Logic Functions and Equations
Examples and Exercises
Steinbach, B.; Posthoff, C.
2009, XXII, 232 p., Hardcover
ISBN: 978-1-4020-9594-8