Chapter 2
Numerical semigroups with maximal embedding dimension

Introduction

Even though the study and relevance of maximal embedding dimension numerical semigroups arises in a natural way among the other numerical semigroups, they have become specially renowned due to the existing applications to commutative algebra via their associated semigroup ring (see for instance [1, 5, 15, 16, 99, 100]). They are a source of examples of commutative rings with some maximal properties. As we mentioned in the introduction of Chapter 1, this is partially due to the fact that the study of some attributes of an analytically unramified one-dimensional local domains can be performed via their value semigroups. Of particular interest are two subclasses of maximal embedding dimension numerical semigroups, which are those semigroups having the Arf property and saturated numerical semigroups. These two families are related with the problem of resolution of singularities in a curve.

Inspired by [3], Lipman in [47] introduces and motivates the study of Arf rings. The characterization of these rings via their value semigroups yields the Arf property for numerical semigroups. The reader can find in [5] a considerable amount of characterizations of this property for numerical semigroups. Arf numerical semigroups have gained lately a particular interest due to their applications to algebraic error correcting codes (see [18, 7] and the references given therein).

Saturated rings were introduced in three different ways by Zariski ([109]), Pham-Teissier ([50]) and Campillo ([17]), though their definitions coincide for algebraically closed fields of zero characteristic. As for the Arf property, saturated numerical semigroups come into the scene after a characterization of saturated rings in terms of their value semigroups (see [26, 49]).

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1 Characterizations

Let S be a numerical semigroup. We know that its embedding dimension, $\mathrm{e}(S)$, is less than or equal to its multiplicity, $\mathrm{m}(S)$. We say that S has maximal embedding dimension if $\mathrm{e}(S) = \mathrm{m}(S)$. In this section we give several characterizations of this property in terms of the notable elements presented in Chapter 1.

If x is a minimal generator of S, and $n \in S \setminus \{0,x\}$, then $x-n$ does not belong to S. This implies that $x \in \mathrm{Ap}(S,n)$.

Proposition 3.1. *Let S be a numerical semigroup minimally generated by $\{n_1 < n_2 < \cdots < n_e\}$. Then S has maximal embedding dimension if and only if* $\mathrm{Ap}(S,n_1) = \{0, n_2, \ldots, n_e\}$.

Proof. As we have pointed out above, $\{n_2,\ldots,n_e\} \subseteq \mathrm{Ap}(S,n_1) \setminus \{0\}$. We know that the cardinality of $\mathrm{Ap}(S,n_1)$ is n_1. Hence $e = n_1$ if and only if $\{0,n_2,\ldots,n_e\} = \mathrm{Ap}(S,n_1)$. □

As a consequence of Propositions 2.12 and 2.20 we obtain the following properties.

Corollary 3.2. *Let S be a numerical semigroup minimally generated by $\{n_1 < n_2 < \cdots < n_e\}$.*

1) If S has maximal embedding dimension, then $\mathrm{F}(S) = n_e - n_1$.
2) S has maximal embedding dimension if and only if $\mathrm{g}(S) = \frac{1}{n_1}(n_2 + \cdots + n_e) - \frac{n_1-1}{2}$.
3) S has maximal embedding dimension if and only if $\mathrm{t}(S) = n_1 - 1$.

Example 3.3. The numerical semigroup $S = \langle 4,5,11 \rangle$ has $\mathrm{F}(S) = 11 - 4 = n_e - n_1$, but it does not have maximal embedding dimension.

Remark 3.4. Let S be a numerical semigroup minimally generated by $\{n_1 < n_2 < \cdots < n_e\}$.

1) We already know (Corollary 2.23) that $\mathrm{t}(S) \leq \mathrm{m}(S) - 1$. So the numerical semigroups with maximal embedding dimension are those with maximal type (in terms of their multiplicities).
2) As by Selmer's formula (Proposition 2.12) $\mathrm{g}(S) \geq \frac{1}{n_1}(n_2 + \cdots + n_e) - \frac{n_1-1}{2}$, numerical semigroups with maximal embedding dimension can also be viewed as those with the least possible number of holes (in terms of their minimal generators).

Given a nonzero integer n and two integers a and b, we write $a \equiv b \bmod n$ to denote that n divides $a-b$. We denote by $b \bmod n$ the remainder of the division of b by n. The following result characterizes those subsets of positive integers that can be realized as Apéry sets of a numerical semigroup.

Proposition 3.5 ([57]). *Let $n \in \mathbb{N} \setminus \{0\}$ and let $C = \{w(0) = 0, w(1), \ldots, w(n-1)\} \subseteq \mathbb{N}$ be such that $w(i)$ is congruent with i modulo n for all $i \in \{1, \ldots, n-1\}$. Let S be the numerical semigroup $\langle \{n\} \cup C \rangle$. The following conditions are equivalent.*

1) $\mathrm{Ap}(S,n) = C$.
2) For all $i, j \in \{1, \ldots, n-1\}$, $w(i) + w(j) \geq w((i+j) \bmod n)$.

Proof. Note that $w(i) + w(j)$ and $w((i+j) \bmod n)$ are congruent modulo n for all $i, j \in \{1, \ldots, n-1\}$. Hence Condition 2) is equivalent to

2′) for all $i, j \in \{1, \ldots, n-1\}$, there exists $t \in \mathbb{N}$ such that $w(i) + w(j) = tn + w((i+j) \bmod n)$.

If $\mathrm{Ap}(S,n) = C$, then by Lemma 2.6, $w(i) + w(j) = kn + c$ for some $k \in \mathbb{N}$ and $c \in C$. Clearly $w(i) + w(j) \equiv c \bmod n$, and thus $c = w((i+j) \bmod n)$.

Now, assume that the second statement holds. Let us show that $\mathrm{Ap}(S,n) \subseteq C$. If $s \in \mathrm{Ap}(S,n) \subset S$, then there exist $c_1, \ldots, c_t \in C$ such that $s = \sum_{i=1}^{t} c_i$. By applying several times Condition 2′), we get that $s = kn + c$, with $c \in C$ and $k \in \mathbb{N}$. As $s \in \mathrm{Ap}(S,n)$, k must be zero and consequently $s = c \in C$.

In view of Lemma 2.4, the cardinality of $\mathrm{Ap}(S,n)$ is n. As the cardinality of C is also n and $\mathrm{Ap}(S,n) \subseteq C$, this forces $\mathrm{Ap}(S,n)$ to be equal to C. □

As we have seen in Proposition 3.1, the Apéry sets of the multiplicity in numerical semigroups with maximal embedding dimension have special shapes. This together with the last characterization of Apéry sets yields an alternative way to distinguish numerical semigroups with maximal embedding dimension by looking at the Apéry sets of their multiplicities.

Corollary 3.6. *Let S be a numerical semigroup with multiplicity m and assume that $\mathrm{Ap}(S,m) = \{w(0) = 0, w(1), \ldots, w(m-1)\}$ with $w(i) \equiv i \bmod m$ for all $i \in \{1, \ldots, m-1\}$. Then S has maximal embedding dimension if and only if for all $i, j \in \{1, \ldots, m-1\}$, $w(i) + w(j) > w((i+j) \bmod m)$.*

Proof. The necessity follows from Propositions 3.1 and 3.5.

By Lemma 2.6, we know that $S = \langle m, w(1), \ldots, w(m-1) \rangle$. From the condition $w(i) + w(j) > w((i+j) \bmod m)$, we deduce that $\{m, w(1), \ldots, w(m-1)\}$ is a minimal system of generators of S. Hence S has maximal embedding dimension. □

Proposition 3.5 and Corollary 3.6 can be used to construct maximal embedding numerical semigroups from an arbitrary numerical semigroup.

Corollary 3.7. *Let S be a numerical semigroup and let n be a positive integer in S. Then $\langle n, w(1) + n, \ldots, w(n-1) + n \rangle$ is a maximal embedding dimension numerical semigroup, where for all $i \in \{1, \ldots, n-1\}$, $w(i)$ is the element in $\mathrm{Ap}(S,n)$ congruent with i modulo n.*

Example 3.8. Let a and b be two positive integers greater than one with $\gcd\{a, b\} = 1$. We already know that $\mathrm{Ap}(\langle a, b \rangle, a) = \{0, b, 2b, \ldots, (a-1)b\}$. By Corollary 3.7,

$$\langle a, a+b, a+2b, \ldots, a+(a-1)b \rangle$$

has maximal embedding dimension.

A sort of converse operation can be performed on a numerical semigroup with maximal embedding dimension. The proof also follows from Proposition 3.5 and Corollary 3.6.

Corollary 3.9 ([57]). *Let S be a numerical semigroup with maximal embedding dimension and multiplicity m. For all $i \in \{1,\ldots,m-1\}$, write $w(i)$ for the unique element in $\mathrm{Ap}\,(S,m)$ congruent with i modulo m. Define $T = \langle m, w(1)-m, \ldots, w(m-1)-m \rangle$. Then T is a numerical semigroup with $\mathrm{Ap}\,(T,m) = \{0, w(1)-m, \ldots, w(m-1)-m\}$.*

From these two last results and Proposition 2.12, we obtain the following correspondence.

Corollary 3.10 ([57]). *There is a one to one correspondence between the set of numerical semigroups with multiplicity m and Frobenius number f, and the set of numerical semigroups with maximal embedding dimension, Frobenius number $f+m$, multiplicity m and the rest of minimal generators greater than $2m$.*

Remark 3.11. If we want to construct the set of all numerical semigroups, according to this last result, it suffices to construct those having maximal embedding dimension. In other words, maximal embedding dimension numerical semigroups can be used to represent the whole class of numerical semigroups.

The following characterization can be deduced from [5, Proposition I.2.9]. For an integer z and a subset A of integers, the set $\{\, z+a \mid a \in A \,\}$ is denoted by $z+A$.

Proposition 3.12. *Let S be a numerical semigroup. The following conditions are equivalent.*

1) S has maximal embedding dimension.
2) For all $x,y \in S^*$, $x+y-\mathrm{m}(S) \in S$.
3) $-\mathrm{m}(S)+S^*$ is a numerical semigroup.

Proof. *1) implies 2).* If either $x-\mathrm{m}(S) \in S$ or $y-\mathrm{m}(S) \in S$, then 2) follows trivially. So assume that both x and y are in $\mathrm{Ap}\,(S,\mathrm{m}(S))$. The result now follows by Corollary 3.6.

2) implies 3). Trivial.

3) implies 1). Denote by $w(i)$ the unique element in $\mathrm{Ap}\,(S,\mathrm{m}(S))$ congruent with i modulo m, $1 \le i \le m-1$. We use Corollary 3.6 again. If $w(i)+w(j) = w((i+j) \bmod \mathrm{m}(S))$ for some $i,j \in \{1,\ldots,\mathrm{m}(S)-1\}$, then $w(i)-\mathrm{m}(S)+w(j)-\mathrm{m}(S) = w((i+j) \bmod \mathrm{m}(S)) - 2\mathrm{m}(S) \notin \{\, x-\mathrm{m}(S) \mid x \in S^* \,\}$, contradicting that this set is a numerical semigroup. $\square$

If in the last proposition we use T to denote the semigroup $-\mathrm{m}(S)+S^*$, then $S = (\mathrm{m}(S)+T) \cup \{0\}$. From this proposition it is not hard to prove the following characterization (see also Exercise 2.9).

Corollary 3.13 ([63]). *Let S be a numerical semigroup. Then S has maximal embedding dimension if and only if there exists a numerical semigroup T and $t \in T \setminus \{0\}$ such that $S = (t+T) \cup \{0\}$.*

Example 3.14. Let $S = \langle 4,5,7 \rangle = \{0,4,5,7,\rightarrow\}$. Then $T = (9+S) \cup \{0\} = \{0,9,13,14,16,\rightarrow\}$ is a maximal embedding dimension numerical semigroup. Note that $T = \langle 9,13,14,16,17,19,20,21,24 \rangle$.

Lemma 3.15. *Let S and T be numerical semigroups. Let $s \in S^*$ and $t \in T^*$. Then $(s+S) \cup \{0\} = (t+T) \cup \{0\}$ if and only if $S = T$ and $s = t$.*

Proof. Assume that $(s+S) \cup \{0\} = (t+T) \cup \{0\}$. Note that $\mathrm{m}((s+S) \cup \{0\}) = s$ and $\mathrm{m}((t+T) \cup \{0\}) = t$. Hence $s = t$. Moreover, $S = -s + (s+S) = -s + (t+T) = -t + (t+T) = T$. The other implication is trivial. □

If S is a numerical semigroup and s is a nonzero element of S, then $s+S$ is an ideal of S (see Exercise 2.13). These ideals are called *principal* ideals of S. Numerical semigroups of the form $(x+S) \cup \{0\}$ with S a numerical semigroup and x a nonzero element of S are called in [63] *pi-semigroups* (where *pi* is an acronym of principal ideal). For a given numerical semigroup S define

$$\mathscr{P}I(S) = \{\, (x+S) \cup \{0\} \mid x \in S^* \,\}.$$

If $x \neq 1$, it can be shown that $\mathrm{F}((x+S) \cup \{0\}) = \mathrm{F}(S) + x$ and that $\mathrm{g}((x+S) \cup \{0\}) = \mathrm{g}(S) + x - 1$ (clearly, the multiplicity of $(x+S) \cup \{0\}$ is x; see Exercises 2.9 and 3.6). From this it easily follows that two elements S_1 and S_2 in $\mathscr{P}I(S)$ coincide if and only if they have the same Frobenius number, or equivalently, they have the same genus.

Proposition 3.16. *The set $\{\, \mathscr{P}I(S) \mid S$ is a numerical semigroup$\}$ is a partition of the set of numerical semigroups with maximal embedding dimension.*

Proof. Follows from Corollary 3.13 and Lemma 3.15. □

This result is telling us that from a fixed numerical semigroup we obtain infinitely many maximal embedding dimension numerical semigroups, and that different numerical semigroups produce different maximal embedding dimension numerical semigroups. All maximal embedding dimension numerical semigroups are constructed in this way.

2 Arf numerical semigroups

A numerical semigroup S is *Arf* if for all $x,y,z \in S$, with $x \geq y \geq z$, $x+y-z$ is in S. In this section we present some characterizations of this property. For a numerical semigroup we will show how to compute the least Arf numerical semigroup containing it.

From Proposition 3.12 it follows that an Arf numerical semigroup has maximal embedding dimension.

Example 3.17. If m is a positive integer, then the numerical semigroup $\{0,m,\rightarrow\}$ is a numerical semigroup with the Arf property. Note that the semigroup T of Example 3.14 has maximal embedding dimension, while it is not Arf, because $14+14-13=15\notin T$.

Given a positive integer x in a numerical semigroup S, the numerical semigroup $(x+S)\cup\{0\}$ is Arf if and only if S is Arf. This follows easily from the definition.

Proposition 3.18 ([63]). *Let S be a numerical semigroup and let $x\in S^*$. Then S is Arf if and only if $S'=(x+S)\cup\{0\}$ is Arf.*

In particular, S is Arf if and only if all the elements in $\mathscr{P}I(S)$ are Arf.

Let S be an Arf numerical semigroup. Then S has maximal embedding dimension. By Corollary 3.13 there exists a numerical semigroup S' and $x\in S'\setminus\{0\}$ such that $S=(x+S')\cup\{0\}$. If $S\neq\mathbb{N}$, then $S\subsetneq S'$. In view of Proposition 3.18, S' is also an Arf numerical semigroup. We can repeat this argument with S', and obtain an Arf numerical semigroup S'' and $y\in S''\setminus\{0\}$ such that $S'=(y+S'')\cup\{0\}$. As $\mathbb{N}\setminus S$ has finitely many elements, this process is finite, obtaining in this way a stationary ascending chain of Arf numerical semigroups: $S_0=S\subsetneq S_1\subsetneq\cdots\subsetneq S_n=\mathbb{N}$, with $S_i=(x_{i+1}+S_{i+1})\cup\{0\}$ for some $x_{i+1}\in S_{i+1}\setminus\{0\}$. The following statement can be derived from this idea.

Corollary 3.19. *Let S be a proper subset of $\mathbb{N}$. Then S is an Arf numerical semigroup if and only if there exist positive integers $x_1,\ldots,x_n$ such that*

$$S=\{0,x_1,x_1+x_2,\ldots,x_1+\cdots+x_{n-1},x_1+\cdots+x_n,\rightarrow\}$$

and $x_i\in\{x_{i+1},x_{i+1}+x_{i+2},\ldots,x_{i+1}+\cdots+x_n,\rightarrow\}$ for all $i\in\{1,\ldots,n\}$.

Proof. Necessity. Follows from the construction of the chain $S=S_0\subsetneq S_1\subsetneq\cdots\subsetneq S_n=\mathbb{N}$, with $S_i=(x_{i+1}+S_{i+1})\cup\{0\}$ and $x_{i+1}\in S_{i+1}\setminus\{0\}$.

Sufficiency. Note that

$$S=(x_1+(x_2+(\cdots+((x_n+\mathbb{N})\cup\{0\})\cdots)\cup\{0\}.$$

As $\mathbb{N}$ is Arf, by applying Proposition 3.18 several times, we obtain that S is Arf. □

Example 3.20. Take $x_1=7$, $x_2=4$ and $x_3=2$. This sequence fulfills the condition of Corollary 3.19. Then $S=\{0,7,11,13,\rightarrow\}$ is a numerical semigroup with the Arf property. In view of Proposition 3.18, $(7+S)\cup\{0\},(11+S)\cup\{0\},(13+S)\cup\{0\},\ldots$ are Arf numerical semigroups as well. Proposition 3.18 also states that $T=-7+S^*$ is an Arf numerical semigroup, because so is $S=(7+T)\cup\{0\}$.

Recall (see Exercise 2.2) that the intersection of finitely many numerical semigroups is a numerical semigroup.

Example 3.21. It can be easily seen that

$$\bigcap_{n\in\mathbb{N}} \langle n, n+1\rangle = \{0\}.$$

Hence the above result does not extend to the intersection of arbitrary families of numerical semigroups.

From the definition it also follows that the intersection of finitely many Arf numerical semigroups is again Arf.

Proposition 3.22. *The intersection of finitely many Arf numerical semigroups is an Arf numerical semigroup.*

Let S be a numerical semigroup. Since the complement of S in $\mathbb{N}$ is finite, the set of Arf numerical semigroups containing S is also finite. Proposition 3.22 ensures that the intersection of these semigroups is again an Arf numerical semigroup (it is actually one of them). We will denote this intersection by $\mathrm{Arf}(S)$ and we will refer to it as the *Arf closure* of S. Observe that the Arf closure of S is the smallest (with respect to set inclusion) Arf numerical semigroup containing S.

If X is a nonempty subset of nonnegative integers with $\gcd(X) = 1$, then $\langle X\rangle$ is a numerical semigroup. Any Arf numerical semigroup containing X must contain $\langle X\rangle$. So it makes sense to talk about the *Arf closure of* X, and define it as $\mathrm{Arf}(\langle X\rangle)$. We make an abuse of notation and will write $\mathrm{Arf}(X)$ to denote $\mathrm{Arf}(\langle X\rangle)$.

Computing the set of numerical semigroups that contain a given numerical semigroup can be tedious. Even more if one has to decide which are Arf among them, and then either compute the intersection of them all or decide which is the smallest. We now describe an alternative way introduced in [88] to compute the Arf closure that is much more efficient.

Lemma 3.23. *Let S be a submonoid of $\mathbb{N}$. Then*

$$S' = \{x+y-z \mid x,y,z \in S, x \geq y \geq z\}$$

is a submonoid of $\mathbb{N}$ and $S \subseteq S'$.

Proof. Let $x \in S$. Then $x+x-x \in S'$, whence $S \subseteq S'$. Clearly $S' \subseteq \mathbb{N}$. Now take $a, b \in S'$. By the definition of S', there exist $x_1, x_2, y_1, y_2, z_1, z_2 \in S$, such that $x_i \geq y_i \geq z_i$, $i \in \{1,2\}$, and $a = x_1+y_1-z_1$, $b = x_2+y_2-z_2$. Hence, $a+b = (x_1+x_2)+(y_1+y_2)-(z_1+z_2)$. Clearly $x_1+x_2, y_1+y_2, z_1+z_2 \in S$ and $x_1+x_2 \geq y_1+y_2 \geq z_1+z_2$. This proves that $a+b \in S'$. $\square$

For a given submonoid S of $\mathbb{N}$ and $n \in \mathbb{N}$, define S^n recurrently as follows:

- $S^0 = S$,
- $S^{n+1} = (S^n)'$.

We see that this becomes stationary at the Arf closure of S.

Proposition 3.24. *Let S be a numerical semigroup. Then there exists $k \in \mathbb{N}$ such that $S^k = \mathrm{Arf}(S)$.*

Proof. By using induction on n, it can be easily proved that $S^n \subseteq \mathrm{Arf}(S)$ for all $n \in \mathbb{N}$. By Lemma 3.23, $S^n \subseteq S^{n+1}$ and $S \subseteq S^n$ for all $n \in \mathbb{N}$. As we pointed out before, the number of numerical semigroups containing S is finite, whence $S^k = S^{k+1}$ for some $k \in \mathbb{N}$. It follows that S^k is an Arf numerical semigroup. As $S^k \subseteq \mathrm{Arf}(S)$ and $\mathrm{Arf}(S)$ is the smallest Arf numerical semigroup containing S, we conclude that $S^k = \mathrm{Arf}(S)$. □

Although this is a nice characterization, we have not yet shown how to compute S^k. So more effort is needed to find an effective way to compute the Arf closure of a numerical semigroup.

Lemma 3.25. *Let* $m, r_1, \ldots, r_p, n \in \mathbb{N}$ *such that* $\gcd(\{m, r_1, \ldots, r_p\}) = 1$. *Then*

$$m + \langle m, r_1, \ldots, r_p \rangle^n \subseteq \mathrm{Arf}(m, m+r_1, \ldots, m+r_p).$$

Proof. We use once more induction on n. For $n = 0$ we have to prove that $m + \langle m, r_1, \ldots, r_p \rangle \subseteq \mathrm{Arf}(m, m+r_1, \ldots, m+r_p)$. Let $i, j \in \{1, \ldots, p\}$. Then $m, m+r_i, m+r_j \in \mathrm{Arf}(m, m+r_1, \ldots, m+r_p)$, whence $m + r_i + r_j = (m+r_i) + (m+r_j) - m \in \mathrm{Arf}(m, m+r_1, \ldots, m+r_p)$. Now, for $k \in \{1, \ldots, p\}$, $m, m+r_i+r_j, m+r_k \in \mathrm{Arf}(m, m+r_1, \ldots, m+r_k)$, and therefore $m+r_i+r_j+r_k = (m+r_i+r_j) + (m+r_k) - m \in \mathrm{Arf}(m, m+r_1, \ldots, m+r_k)$. By repeating this argument we obtain that for every $a, a_1, \ldots, a_p \in \mathbb{N}$, we have that $(a+1)m + a_1 r_1 + \cdots + a_p r_p \in \mathrm{Arf}(m, m+r_1, \ldots, m+r_p)$, and thus $m + \langle m, r_1, \ldots, r_p \rangle \subseteq \mathrm{Arf}(m, m+r_1, \ldots, m+r_p)$.

Now assume that $m + \langle m, r_1, \ldots, r_p \rangle^n \subseteq \mathrm{Arf}(m, m+r_1, \ldots, m+r_p)$ and let us prove that $m + \langle m, r_1, \ldots, r_p \rangle^{n+1} \subseteq \mathrm{Arf}(m, m+r_1, \ldots, m+r_p)$. Let $a \in m + \langle m, r_1, \ldots, r_p \rangle^{n+1}$. Then $a = m + b$ with $b \in \langle m, r_1, \ldots, r_p \rangle^{n+1}$. Hence there exist $x, y, z \in \langle m, r_1, \ldots, r_p \rangle^n$ such that $x \geq y \geq z$ and $x + y - z = b$. In this way $a = m + b = m + x + y - z = (m+x) + (m+y) - (m+z) \in \mathrm{Arf}(m, m+r_1, \ldots, m+r_p)$, since by induction hypothesis $m+x, m+y, m+z \in m + \langle m, r_1, \ldots, r_p \rangle^n \subseteq \mathrm{Arf}(m, m+r_1, \ldots, m+r_p)$. □

From this we can give a procedure to compute Arf closures that has an extended Euclid's algorithm taste.

Proposition 3.26. *Let* $m, r_1, \ldots, r_p$ *be nonnegative integers with greatest common divisor one. Then*

$$\mathrm{Arf}(m, m+r_1, \ldots, m+r_p) = (m + \mathrm{Arf}(m, r_1, \ldots, r_p)) \cup \{0\}.$$

Proof. By using Proposition 3.24 and Lemma 3.25, we obtain that $(m + \mathrm{Arf}(m, r_1, \ldots, r_p)) \cup \{0\} \subseteq \mathrm{Arf}(m, m+r_1, \ldots, m+r_p)$. For the other inclusion observe that $m, m+r_1, \ldots, m+r_p \in (m + \mathrm{Arf}(m, r_1, \ldots, r_p)) \cup \{0\}$, which by Proposition 3.18 is an Arf numerical semigroup. It follows that $\mathrm{Arf}(m, m+r_1, \ldots, m+r_p) \subseteq (m + \mathrm{Arf}(m, r_1, \ldots, r_p)) \cup \{0\}$. □

The Frobenius number of the Arf closure can then be computed as follows.

Corollary 3.27. *Let $m, r_1, \ldots, r_p$ be nonnegative integers with greatest common divisor one. Then*

$$\mathrm{F}(\mathrm{Arf}(m, m+r_1, \ldots, m+r_p)) = m + \mathrm{F}(\mathrm{Arf}(m, r_1, \ldots, r_p)).$$

We have now all the ingredients needed to give a recursive way of calculating the elements of the Arf closure of any subset of nonnegative integers with greatest common divisor one. Let $X \subseteq \mathbb{N} \setminus \{0\}$ be such that $\gcd(X) = 1$. Define recursively the following sequence of subsets of $\mathbb{N}$:

- $A_1 = X$,
- $A_{n+1} = (\{x - \min A_n \mid x \in A_n\} \setminus \{0\}) \cup \{\min A_n\}$.

As a consequence of Euclid's algorithm for the computation of $\gcd(X)$, we obtain that there exists $q = \min\{k \in \mathbb{N} \mid 1 \in A_k\}$.

Proposition 3.28. *With the above notation, we have that*

$$\mathrm{Arf}(X) = \{0, \min A_1, \min A_1 + \min A_2, \ldots, \min A_1 + \cdots + \min A_{q-1}, \rightarrow\}.$$

Proof. Since $1 \in A_q$, $\mathrm{Arf}(A_q) = \mathbb{N}$. Hence by Proposition 3.26, $\mathrm{Arf}(A_{q-1}) = (\min A_{q-1} + \mathbb{N}) \cup \{0\}$. This implies

$$\mathrm{Arf}(A_{q-1}) = \{0, \min A_{q-1}, \rightarrow\}.$$

Assume as induction hypothesis that

$$\mathrm{Arf}(A_{q-i}) = \{0, \min A_{q-i}, \min A_{q-i} + \min A_{q-i+1}, \ldots, \\ \min A_{q-i} + \cdots + \min A_{q-1}, \rightarrow\}.$$

We must prove now that

$$\mathrm{Arf}(A_{q-i-1}) = \{0, \min A_{q-i-1}, \min A_{q-i-1} + \min A_{q-i}, \ldots, \\ \min A_{q-i-1} + \cdots + \min A_{q-1}, \rightarrow\}.$$

By Proposition 3.26, we know that $\mathrm{Arf}(A_{q-i-1}) = (\min A_{q-i-1} + \mathrm{Arf}(A_{q-i})) \cup \{0\}$. By using now the induction hypothesis and Corollary 3.27, we obtain the desired result. □

Example 3.29 ([88]). Let us compute $\mathrm{Arf}(7, 24, 33)$.

$A_1 = \{7, 24, 33\}$, $\min_{\leq} A_1 = 7$,
$A_2 = \{7, 17, 26\}$, $\min_{\leq} A_2 = 7$,
$A_3 = \{7, 10, 19\}$, $\min_{\leq} A_3 = 7$,
$A_4 = \{7, 3, 12\}$, $\min_{\leq} A_4 = 3$,
$A_5 = \{4, 3, 9\}$, $\min_{\leq} A_5 = 3$,
$A_6 = \{1, 3, 6\}$,

whence $\mathrm{Arf}(7, 24, 33) = \{0, 7, 14, 21, 24, 27, \rightarrow\}$.

3 Saturated numerical semigroups

A numerical semigroup S is *saturated* if the following condition holds: if $s, s_1, \ldots, s_r \in S$ are such that $s_i \leq s$ for all $i \in \{1, \ldots, r\}$ and $z_1, \ldots, z_r \in \mathbb{Z}$ are such that $z_1 s_1 + \cdots + z_r s_r \geq 0$, then $s + z_1 s_1 + \cdots + z_r s_r \in S$.

Example 3.30. The semigroup $S = \langle 7, 11, 13, 15, 16, 17, 19 \rangle$ appearing in Example 3.20 is an Arf semigroup but it is not saturated. Note that $7, 11 \in S$ and $12 = 11 + 2 \times 11 - 3 \times 7 \notin S$.

From the definition it easily follows that every saturated numerical semigroup is Arf, and thus it has maximal embedding dimension.

Lemma 3.31. *Every saturated numerical semigroup has the Arf property.*

Next we describe a characterization of this kind of semigroup that appears in [89].

For $A \subseteq \mathbb{N}$ and $a \in A \setminus \{0\}$, set

$$\mathrm{d}_A(a) = \gcd\{x \in A \mid x \leq a\}.$$

Lemma 3.32. *Let S be a saturated numerical semigroup and let $s \in S$. Then $s + \mathrm{d}_S(s) \in S$.*

Proof. Let $\{s_1, \ldots, s_r\} = \{x \in S \mid x \leq s\}$. By Bézout's identity, there exist integers $z_1, \ldots, z_r$ such that $z_1 s_1 + \cdots + z_r s_r = \mathrm{d}_S(s)$. As S is saturated, we get $s + \mathrm{d}_S(s) \in S$. □

We are going to see that this property characterizes saturated numerical semigroups. First we need some previous lemmas.

Lemma 3.33. *Let A be a nonempty subset of positive integers such that* $\gcd(A) = 1$ *and $a + \mathrm{d}_A(a) \in A$ for all $a \in A$. Then $a + k\mathrm{d}_A(a) \in A$ for all $k \in \mathbb{N}$, and $A \cup \{0\}$ is a numerical semigroup.*

Proof. We proceed by induction on $\mathrm{d}_A(a)$.

Note that $\mathrm{d}_A(a) > 0$. We show that if $\mathrm{d}_A(a) = 1$, then $a + k \in A$ for all $k \in \mathbb{N}$. To this end we use induction on k. For $k = 0$, the result is clear. Assume that $a + k \in A$. Since $0 \neq \mathrm{d}_A(a+k) \leq \mathrm{d}_A(a) = 1$, we have that $\mathrm{d}_A(a+k) = 1$. Hence $a + k + 1 = a + k + \mathrm{d}_A(a+k) \in A$.

Now assume that if $a' \in A$ and $\mathrm{d}_A(a') < \mathrm{d}_A(a)$, then $a' + k\mathrm{d}_A(a') \in A$ for all $k \in \mathbb{N}$. Thus, suppose that $\mathrm{d}_A(a) \geq 2$ and let us prove that $a + k\mathrm{d}_A(a) \in A$ for all $k \in \mathbb{N}$. Since $\gcd(A) = 1$, there exists $b \in A$ such that $\mathrm{d}_A(b) = 1$. If $\mathrm{d}_A(a + k\mathrm{d}_A(a)) = \mathrm{d}_A(a)$ and $a + k\mathrm{d}_A(a) \in A$, then $a + (k+1)\mathrm{d}_A(a) = a + k\mathrm{d}_A(a) + \mathrm{d}_A(a + k\mathrm{d}_A(a)) \in A$. From these two remarks, we deduce that there exists a least positive integer t such that $a + t\mathrm{d}_A(a) \in A$ and $\mathrm{d}_A(a + t\mathrm{d}_A(a)) < \mathrm{d}_A(a)$. As $\mathrm{d}_A(a + t\mathrm{d}_A(a)) < \mathrm{d}_A(a)$, by applying the induction hypothesis, we obtain that $(a + t\mathrm{d}_A(a)) + k\mathrm{d}_A(a + t\mathrm{d}_A(a)) \in A$ for all $k \in \mathbb{N}$. Clearly, $\mathrm{d}_A(a + t\mathrm{d}_A(a))$ divides $\mathrm{d}_A(a)$, whence $\mathrm{d}_A(a) = l\mathrm{d}_A(a + t\mathrm{d}_A(a))$ for some positive integer l. Consequently, $a + t\mathrm{d}_A(a) + \frac{k}{l}\mathrm{d}_A(a) \in A$ for all $k \in \mathbb{N}$, and thus $a + (t+n)\mathrm{d}_A(a) \in A$ for all $n \in \mathbb{N}$. From the definition of t, it follows that $a + k\mathrm{d}_A(a) \in A$ for all $k \in \{0,\ldots,t\}$. We conclude that $a + k\mathrm{d}_A(a) \in A$ for all $k \in \mathbb{N}$.

Finally, let us prove that $A \cup \{0\}$ is a numerical semigroup. Since $\gcd(A) = 1$, it suffices to prove that for any $a,b \in A$, $a+b \in A$. Assume that $a \leq b$. Then $\mathrm{d}_A(b)$ divides $\mathrm{d}_A(a)$ and thus there exists $\lambda \in \mathbb{N}$ such that $\mathrm{d}_A(a) = \lambda \mathrm{d}_A(b)$. Note also that $\mathrm{d}_A(a)$ divides a, whence $a = \mu \mathrm{d}_A(a)$ for some $\mu \in \mathbb{N}$. Therefore $a + b = \mu \mathrm{d}_A(a) + b = \mu\lambda \mathrm{d}_A(b) + b$, which, as we have just proven, belongs to A. □

Proposition 3.34. *Let A be a nonempty subset of $\mathbb{N}$ such that $0 \in A$ and $\gcd(A) = 1$. The following conditions are equivalent.*

1) A is a saturated numerical semigroup.
2) $a + \mathrm{d}_A(a) \in A$ for all $a \in A \setminus \{0\}$.
3) $a + k\mathrm{d}_A(a) \in A$ for all $a \in A \setminus \{0\}$ and $k \in \mathbb{N}$.

Proof. 1) implies 2). Follows from Lemma 3.32.

2) implies 3). Follows from Lemma 3.33.

3) implies 1). By Lemma 3.33, we know that A is a numerical semigroup. Let $a, a_1, \ldots, a_r \in A$ with $a_i \leq a$ for all $i \in \{1,\ldots,r\}$, and let $z_1,\ldots,z_r$ be integers such that $z_1a_1 + \cdots + a_rz_r \geq 0$. Since $a_i \leq a$, it follows that $\mathrm{d}_A(a)$ divides a_i for all $i \in \{1,\ldots,r\}$. Hence, there exists $k \in \mathbb{N}$ such that $z_1a_1 + \cdots + z_ra_r = k\mathrm{d}_A(a)$, and thus $a + z_1a_1 + \cdots + z_ra_r = a + k\mathrm{d}_A(a) \in A$. This proves that A is saturated. □

We now focus on obtaining a similar characterization as the one given in Proposition 3.18 and Corollary 3.19 for Arf numerical semigroups. As we will see in this setting the characterization is not so generous.

Proposition 3.35 ([63]). *Let S be a numerical semigroup. The following conditions are equivalent.*

1) S is saturated.
2) There exists $x \in S^*$ such that $(x+S) \cup \{0\}$ is a saturated numerical semigroup.

Proof. 1) implies 2). Assume that $S = \{0 < s_1 < s_2 < \cdots < s_n < \cdots\}$. We prove that $(s_1 + S) \cup \{0\} = \{0 < s_1 < s_1 + s_1 < s_1 + s_2 < \cdots < s_1 + s_n < \cdots\}$ is saturated. In view of Proposition 3.34, it suffices to show that for all $n \in \mathbb{N}$, the element $s_1 + s_n + \gcd\{0, s_1, s_1+s_1, \ldots, s_1+s_n\}$ lies in $(s_1+S) \cup \{0\}$. Since S is saturated, $s_n + \gcd\{0, s_1, \ldots, s_n\} \in S$. Moreover $\gcd\{0, s_1, s_1+s_1, s_1+s_2, \ldots, s_1+s_n\} = \gcd\{0, s_1, s_2, \ldots, s_n\}$, whence $s_1 + s_n + \gcd\{0, s_1, s_1+s_1, \ldots, s_1+s_n\} \in (s_1+S) \cup \{0\}$.

2) implies 1). If $S = \{0 < s_1 < \cdots < s_n < \cdots\}$, then $(x+S) \cup \{0\} = \{0 < x < s_1 + x < \cdots < s_n + x < \cdots\}$. Since $\gcd\{0, x, x+s_1, \ldots, x+s_n\} = \gcd\{0, x, s_1, \ldots, s_n\}$, we have that $\gcd\{0, x, x+s_1, \ldots, x+s_n\}$ divides $\gcd\{0, s_1, \ldots, s_n\}$, namely, there exists $k \in \mathbb{N}$ such that $k(\gcd\{0, x, x+s_1, \ldots, x+s_n\}) = \gcd\{0, s_1, \ldots, s_n\}$. By Proposition 3.34, if we want to prove that S is saturated, it suffices to show that $s_n + \gcd\{0, s_1, \ldots, s_n\} \in S$ for all n. As $(x+S) \cup \{0\}$ is saturated, by Proposition 3.34, we have that $x + s_n + k(\gcd\{0, x, x+s_1, \ldots, x+s_n\}) \in (x+S) \cup \{0\}$ and thus $s_n + \gcd\{0, s_1, \ldots, s_n\} \in S$. □

From the proof of this result we obtain the following consequence.

Corollary 3.36. *Let S be a numerical semigroup. Then S is saturated if and only if* $(\mathrm{m}(S)+S)\cup\{0\}$ *is saturated.*

Example 3.37. The semigroup $S=\langle 5,7,8,9,11\rangle$ is a saturated numerical semigroup. From Corollary 3.36 we have that both $(5+S)\cup\{0\}$ and $-5+S^*$ are saturated.

Corollary 3.38. *Let S be a proper subset of $\mathbb{N}$. Then S is a saturated numerical semigroup if and only if there exist positive integers $x_1,\ldots,x_n$ such that*

$$S=\{0,x_1,x_1+x_2,\ldots,x_1+\cdots+x_n,\rightarrow\}$$

and

$$\gcd\{x_1,\ldots,x_k\}\in\{x_{k+1},x_{k+1}+x_{k+2},\ldots,x_{k+1}+\cdots+x_n,\rightarrow\}$$

for all $k\in\{1,\ldots,n\}$.

Proof. Necessity. Since S is a saturated numerical semigroup, S is also Arf, whence by Corollary 3.19 there exist positive integers $x_1,\ldots,x_n$ such that

$$S=\{0,x_1,x_1+x_2,\ldots,x_1+\cdots+x_n,\rightarrow\}.$$

As S is saturated, for all $k\in\{1,\ldots,n\}$, $(x_1+\cdots+x_k)+\gcd\{0,x_1,x_1+x_2,\ldots,x_1+\cdots+x_k\}\in S$ and since $\gcd\{0,x_1,x_1+x_2,\ldots,x_1+\cdots+x_k\}=\gcd\{x_1,\ldots,x_k\}$, we have that $(x_1+\cdots+x_k)+\gcd\{x_1,\ldots,x_k\}\in\{0,x_1,x_1+x_2,\ldots,x_1+\cdots+x_n,\rightarrow\}$, or equivalently, $\gcd\{x_1,\ldots,x_k\}\in\{x_{k+1},x_{k+1}+x_{k+2},\ldots,x_{k+1}+\cdots+x_n,\rightarrow\}$.

Sufficiency. By using Proposition 3.34, it suffices to show that $(x_1+\cdots+x_k)+\gcd\{0,x_1,x_1+x_2,\ldots,x_1+\cdots+x_k\}\in S$ for all $k\in\{1,\ldots,n\}$. As pointed out above, this is equivalent to prove that $(x_1+\cdots+x_k)+\gcd(x_1,\ldots,x_k)\in S$, and this follows from the hypothesis. □

As for Arf numerical semigroups, the intersection of finitely many saturated numerical semigroups is again saturated. This follows easily from the definition.

Proposition 3.39. *The intersection of finitely many saturated numerical semigroups is a saturated numerical semigroup.*

This allows us to define the saturated closure of a numerical semigroup (or of a subset of nonnegative integers with greatest common divisor one), as we did for Arf numerical semigroups. Given a numerical semigroup S, we denote by $\mathrm{Sat}(S)$ the intersection of all saturated numerical semigroups containing S, or in other words, the smallest (with respect to set inclusion) saturated numerical semigroup containing S. We call this semigroup the *saturated closure* of S.

The saturated closure of a semigroup (or of any set of nonnegative integers with greatest common divisor one) can be computed as follows.

Proposition 3.40. *Let $n_1 < n_2 < \cdots < n_e$ be positive integers such that* $\gcd(n_1,\ldots,n_e) = 1$. *For every $i \in \{1,\ldots,e\}$, set $d_i = \gcd(n_1,\ldots,n_i)$ and for all $j \in \{1,\ldots,p-1\}$ define $k_j = \max\{k \in \mathbb{N} \mid n_j + kd_j < n_{j+1}\}$. Then*

$$\mathrm{Sat}(n_1,\ldots,n_e) = \{0, n_1, n_1+d_1,\ldots,n_1+k_1d_1, n_2, n_2+d_2,\ldots,n_2+k_2d_2,\\ \ldots,n_{e-1},n_{e-1}+d_{e-1},\ldots,n_{e-1}+k_{e-1}d_{e-1},n_e,n_e+1,\rightarrow\}.$$

Proof. Let

$$A = \{0, n_1, n_1+d_1,\ldots,n_1+k_1d_1, n_2, n_2+d_2,\ldots,n_2+k_2d_2,\\ \ldots,n_{e-1},n_{e-1}+d_{e-1},\ldots,n_{p-1}+k_{e-1}d_{e-1},n_e,n_e+1,\rightarrow\}.$$

Clearly A is not empty, $0 \in A$, $\gcd(A) = 1$ and $a + \mathrm{d}_A(a) \in A$ for all $a \in A$. By Proposition 3.34, A is a saturated numerical semigroup, and as $\{n_1,\ldots,n_e\} \subset A$, we get that $\mathrm{Sat}(n_1,\ldots,n_e) \subseteq A$. For the other inclusion, take $a \in A$. Then there exists $i \in \{1,\ldots,e\}$ and $k \in \mathbb{N}$ such that $a = n_i + kd_i$ (note that $d_e = 1$). Since $\{n_1,\ldots,n_e\} \subset \mathrm{Sat}(n_1,\ldots,n_e)$, we have that $\mathrm{d}_{\mathrm{Sat}(n_1,\ldots,n_e)}(n_i)$ divides d_i, whence there exists $l \in \mathbb{N}$ such that $d_i = l\mathrm{d}_{\mathrm{Sat}(n_1,\ldots,n_e)}(n_i)$. From Proposition 3.34, we know that $n_i + t\mathrm{d}_{\mathrm{Sat}(n_1,\ldots,n_e)}(n_i) \in \mathrm{Sat}(n_1,\ldots,n_e)$ for all $t \in \mathbb{N}$ and thus $a = n_i + kd_i = n_i + kl\mathrm{d}_{\mathrm{Sat}(n_1,\ldots,n_e)}(n_i) \in \mathrm{Sat}(n_1,\ldots,n_e)$. □

Example 3.41. $\mathrm{Sat}(\{12,20,26,35\}) = \{0,12,20,24,26,28,30,32,34,35,\rightarrow\}$.

Exercises

Exercise 3.1 ([45, 87]). Let m be an integer greater than or equal to two, and let $(k_1,\ldots,k_{m-1}) \in \mathbb{N}^{m-1}$. Prove that $\{0, k_1m+1,\ldots,k_{m-1}m+m-1\}$ is the Apéry set of m in a numerical semigroup with multiplicity m if and only if $(k_1,\ldots,k_{m-1})$ is a solution to the system of inequalities

$$\begin{array}{ll} x_i \geq 1 & \text{for all } i \in \{1,\ldots,m-1\},\\ x_i + x_j - x_{i+j} \geq 0 & \text{for all } i,j \text{ with } 1 \leq i \leq j \leq m-1, i+j \leq m-1,\\ x_i + x_j - x_{i+j-m} \geq -1 & \text{for all } i,j \text{ with } 1 \leq i \leq j \leq m-1, i+j > m. \end{array}$$

Exercise 3.2 ([87]). Let m be an integer greater than or equal to two, and let $(k_1,\ldots,k_{m-1}) \in \mathbb{N}^{m-1}$. Show that $S = \langle m, k_1m+1,\ldots,k_{m-1}m+m-1\rangle$ is a numerical semigroup with multiplicity m and maximal embedding dimension if and only if $(k_1,\ldots,k_{m-1})$ is a solution to the system of inequalities

$$\begin{array}{ll} x_i \geq 1 & \text{for all } i \in \{1,\ldots,m-1\},\\ x_i + x_j - x_{i+j} \geq 1 & \text{for all } i,j \text{ with } 1 \leq i \leq j \leq m-1, i+j \leq m-1,\\ x_i + x_j - x_{i+j-m} \geq 0 & \text{for all } i,j \text{ with } 1 \leq i \leq j \leq m-1, i+j > m. \end{array}$$

Exercise 3.3. Let m be a positive integer. Prove that the intersection of finitely many maximal embedding dimension numerical semigroups with multiplicity m is again a maximal embedding dimension numerical semigroup with multiplicity m.

Show with an example that the intersection of finitely many maximal embedding dimension numerical semigroups might not have maximal embedding dimension.

Exercise 3.4. Let S be a numerical semigroup and let m be a nonzero element of S. Denote by $\mathrm{Ap}_0(S,m) = \{w \in \mathrm{Ap}(S,m) \mid w \text{ is even}\}$ and by $\mathrm{Ap}_1(S,m) = \{w \in \mathrm{Ap}(S,m) \mid w \text{ is odd}\}$. Prove that

a) if m is even, then the cardinalities of $\mathrm{Ap}_0(S,m)$ and $\mathrm{Ap}_1(S,m)$ are the same and equal to $\frac{m}{2}$,
b) if m is odd and $\mathrm{Ap}_1(S,m) = m-1$, then S is a maximal embedding dimension of multiplicity m.

Exercise 3.5. Let S be a numerical semigroup. Show that if S has maximal embedding dimension, has the Arf property or it is saturated, then so is $S \cup \{\mathrm{F}(S)\}$.

Exercise 3.6. Let S be numerical semigroup and let $m \in S \setminus \{0,1\}$. Set $T = (m + S) \cup \{0\}$ (see Exercise 2.9). Prove that

a) $\mathrm{F}(T) = \mathrm{F}(S) + m$,
b) $\mathrm{g}(T) = \mathrm{g}(S) + m - 1$.

Exercise 3.7. Let S be a numerical semigroup with maximal ideal M (see Exercise 2.13) and multiplicity m. Prove that S has maximal embedding dimension if and only if $M - M = -m + M$. Show that this is also equivalent to $m + M = M + M (= \{x + y \mid x, y \in M\})$.

Exercise 3.8. Let S be a maximal embedding dimension numerical semigroup with minimal system of generators $\{n_1 < n_2 < \cdots < n_e\}$. Prove that $\mathrm{g}(S) \geq \frac{n_e - 1}{2}$.

Exercise 3.9. Let $S = \langle 5, 7, 9 \rangle$. Compute the smallest (with respect to set inclusion) maximal embedding dimension numerical semigroup with multiplicity 5 containing S.

Exercise 3.10. Prove that $\{\langle 2, 2k+1 \rangle \mid k \in \mathbb{N}\}$ is the set of all maximal embedding dimension numerical semigroups of type 1. Which is the set of all Arf numerical semigroups of type 1? And that of saturated numerical semigroups?

Exercise 3.11. Check that $\langle 4, 9, 10, 11 \rangle$ is the smallest (with respect to set inclusion) Arf numerical semigroup containing $\langle 4, 9, 11 \rangle$.

Exercise 3.12. Show that $\langle 10, 14, 16, 18, 22, 27, 29, 31, 33, 35 \rangle$ is the smallest (with respect to set inclusion) saturated numerical semigroup containing $\langle 10, 14, 27 \rangle$.

Exercise 3.13 ([8]). Prove that every Arf numerical semigroup that is not a half-line is acute (see Exercise 2.8).

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